Abstract

A short introduction into valuating derivatives in stochastic volatility markets using the p.d.e. approach is given. Concentrating on the solution of the p.d.e. with the finite difference method some suggestion to improve the accuracy of the numerical solution are provided.
1. Stochastic volatility models

1.1. The model (Lipton)

- stochastic process of the underlying is given by a stochastic differential equation similar to Black-Scholes

\[
\frac{dS_t}{S_t} = (r_d - r_f) \, dt + \sigma_L(S_t, t) \sqrt{v_t} \, dW_t^{(1)}
\]

- the variance is a stochastic mean reverting process

\[
dv_t = \kappa(\theta - v_t) \, dt + \xi \sqrt{v_t} \, dW_t^{(2)}
\]

- the Wiener processes might be correlated

\[
dW_t^{(1)} \, dW_t^{(2)} = \rho \, dt
\]
1.2. Sample path

http://www.tu-chemnitz.de/~tkluge/tools/sharesim/heston.php
1.3. Valuation of options using the p.d.e. approach

- value of an option denoted by $u(s, v, t)$
- hedging portfolio consists of: underlying, money market account, another (liquid) option
- value of the option and the hedge portfolio are equal
- comparing instantaneous changes of the value of the option $du(S_t, v_t, t)$ and the hedging strategy leads to a partial differential equation in $u$

$$
\begin{align*}
    u_t &+ \frac{1}{2}v \left( s^2 \sigma_L(s, t)^2 u_{ss} + \xi^2 u_{vv} + 2s\sigma_L(s, t)\rho \xi u_{sv} \right) + \\
    (r_d - r_f)u_s + (\kappa(\theta - v) - \lambda(s, v, t))u_v - r_d u &= 0
\end{align*}
$$

- $v \in (0, \infty)$, $s \in I \subset (0, \infty)$
• \( \lambda(s, v, t) \) is called market price of volatility risk and illustrates that we are faced with an incomplete market

• we only want to obtain the value of \( u \) in exactly one point but we have to solve the entire function
1.4. The Heston model

- $\sigma_L(s, t) = 1$ and $\lambda(s, v, t) = 0$

- transformation: time to maturity $t \rightarrow T - t$, log spot $x = \log(s)$

- we can then write the p.d.e. in convection diffusion form $u_t = \text{div}(A \nabla u) - \text{div}(bu) + f$

$$A = \frac{1}{2}v \begin{pmatrix} 1 & \rho \xi \\ \rho \xi & \xi^2 \end{pmatrix}$$

$$b = v \begin{pmatrix} \frac{1}{2} \kappa \\ \frac{1}{2} \rho \xi + r_f - r_d \end{pmatrix} + \left( \frac{1}{2} \rho \xi + r_f - r_d \right)$$

$$f = (\kappa - r_d)u$$

- diffusion vanishes at $v = 0$, makes analysis difficult

- for typical market parameters diffusion dominates
2. Fundamental solution

- $G(x, x', t)$ is called fundamental solution of a p.d.e. if

$$u(x, t) = \int_{\mathbb{R}^d} G(x, x', t)u(x', 0) \, dx'$$

- for $u_t = c \Delta u$, $x \in \mathbb{R}^d$, it is known

$$G(x, x', t) := (4\pi ct)^{-d/2} \exp \left( -\frac{\|x - x'\|^2}{4ct} \right)$$

- for $u_t = c \frac{\partial}{\partial v}(vu_v) - \frac{c}{2}u_v$, $v > 0$, one can show

$$G(v, v', t) = \frac{1}{\sqrt{4\pi cv't}} \exp \left( -\frac{(\sqrt{v} - \sqrt{v'})^2}{ct} \right).$$
3. **Finite difference method (f.d.m.)**

**p.d.e.:** let $L$ an elliptic operator in space, we then consider

\[
\frac{\partial u}{\partial t} = Lu
\]

**idea:** chose a grid, discretise space, then time derivatives

**space discretisation:** introduce $\varphi_{i,j}(t) := u(x_i, v_j, t)$ and approximate derivatives (in an uniform grid)

\[
\frac{\partial u(x_i, v_j, t)}{\partial x} \approx \frac{\varphi_{i+1,j}(t) - \varphi_{i-1,j}(t)}{2\Delta x}
\]

\[
\frac{\partial^2 u(x_i, v_j, t)}{\partial x^2} \approx \frac{\varphi_{i+1,j}(t) - 2\varphi_{i,j}(t) + \varphi_{i-1,j}(t)}{\Delta x^2}
\]

\[
\frac{\partial^2 u(x_i, v_j, t)}{\partial x \partial v} \approx \frac{\varphi_{i+1,j+1} - \varphi_{i-1,j+1} - \varphi_{i+1,j-1} + \varphi_{i-1,j-1}}{4\Delta x \Delta v}
\]
non uniform meshes: Consider the general approach

\[ \frac{\partial u}{\partial x}(x_i, v_j, t) \approx \sum_{k,l=-1}^1 \alpha_{k,l}(i, j) \cdot \varphi_{i+k,j+l}(t) \]

and determine the factors \( \alpha_{k,l}(i, j) \) so that one obtains second order consistency.

semi discrete system: We obtain a linear system of o.d.e.s with a time dependent matrix \( L_h(t) \) of the order \( n_x \cdot n_v \). The space discretisation parameter is denoted by \( h > 0 \).

\[ \frac{d}{dt} \varphi(t) = L_h(t)\varphi(t) \]
time discretisation: $\varphi^{(k)} := \varphi(t_k)$

$$
\frac{\varphi^{(k+1)} - \varphi^{(k)}}{\Delta t} = L_h(t) \left( \theta \varphi^{(k+1)} + (1 - \theta) \varphi^{(k)} \right)
$$

$$
\uparrow
$$

$$(I - \theta \Delta t L_h(t)) \varphi^{(k+1)} = (I + (1 - \theta) \Delta t L_h(t)) \varphi^{(k)}
$$

$\theta = 0$ fully explicit scheme

$\theta = \frac{1}{2}$ Crank-Nicholson scheme

$\theta = 1$ fully implicit scheme
3.1. Consistency, stability and convergence

**Theorem 3.1**
The Crank-Nicholson method is second order consistent and unconditionally stable in $L_2$ for Dirichlet boundary conditions given appropriate space discretisation. This statement remains true even if the parabolic p.d.e. degenerates, i.e. if the diffusion matrix $A$ is only positive semidefinite.

**Remark 3.2**
It is well known that for uniform parabolic p.d.e.s (diffusion dominated) the Crank-Nicholson method with centred difference approximation in space yields accurate results for derivatives.
4. Improvement of the numerical results

- choose artificial boundaries sufficiently far away
- sensitive boundary $v = 0$: no boundary condition, instead discretise p.d.e. (if $\kappa \theta > \frac{1}{2} \xi^2$)
- choose non uniform grid appropriately
5. Numerical results

5.1. Calculation times

- Intel Pentium II 300 MHz, C++ (gcc -O3)

<table>
<thead>
<tr>
<th>space</th>
<th>time</th>
<th>matrix size</th>
<th>CN (SuperLU)</th>
<th>ADI (LU)</th>
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<td>0.1s</td>
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<td>400 × 120</td>
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<td>48000</td>
<td>440s</td>
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5.2. Example market

Consider exchange rate USD/JPY:

<table>
<thead>
<tr>
<th>$r_d$</th>
<th>$\log(1.0005)$</th>
<th>$r_f$</th>
<th>$\log(1.0375)$</th>
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<tr>
<td>$S_0$</td>
<td>123.4</td>
<td>$\nu_0$</td>
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<tr>
<td>$\kappa$</td>
<td>1.98937</td>
<td>$\theta$</td>
<td>0.011876</td>
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<tr>
<td>$\rho$</td>
<td>0.0258519</td>
<td>$\xi$</td>
<td>0.33147</td>
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<tr>
<td></td>
<td></td>
<td>$\lambda$</td>
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5.3. Plain vanilla option
123.4 call, time to maturity 0.50137
we know analytical solution of p.d.e. so we are able to examine the error:

<table>
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<th>rel. error</th>
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<tbody>
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</table>
5.3.1. Numerical error, Neumann conditions on $v = 0$
5.3.2. Numerical error, zero second derivatives at $v = 0$
5.3.3. Numerical error, discretisation of the p.d.e. at $v = 0$
5.4. **Reverse barrier**

120.0 call, time to maturity 0.50137, 127.0 up and out
Black-Scholes value      0.167 ¥
Heston value (p.d.e.)    0.25 ¥
market price             0.234–0.296 ¥