Duality for convex composed programming problems

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Dipl.-Math. Emese Tünde Vargyas
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Prof. Dr. Kathrin Klamroth
Conf. Dr. Gábor Kassay

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Emese Tünde Vargyas

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Report

The theory of duality represents an important research area in optimization. The goal of this work is to present a conjugate duality treatment of composed programming as well as to give an overview of some recent developments in both scalar and multiobjective optimization.

In order to do this, first we study a single-objective optimization problem, in which the objective function as well as the constraints are given by composed functions. By means of the conjugacy approach based on the perturbation theory, we provide different kinds of dual problems to it and examine the relations between the optimal objective values of the duals. Given some additional assumptions, we verify the equality between the optimal objective values of the duals and strong duality between the primal and the dual problems, respectively. Having proved the strong duality, we derive the optimality conditions for each of these duals. As special cases of the original problem, we study the duality for the classical optimization problem with inequality constraints and the optimization problem without constraints.

The second part of this work is devoted to location analysis. Considering first the location model with monotonic gauges, it turns out that the same conjugate duality principle can be used also for solving this kind of problems. Taking in the objective function instead of the monotonic gauges several norms, investigations concerning duality for different location problems are made.

We finish our investigations with the study of composed multiobjective optimization problems. In doing like this, first we scalarize this problem and study the scalarized one by using the conjugacy approach developed before. The optimality conditions which we obtain in this case allow us to construct a multiobjective dual problem to the primal one. Additionally the weak and strong duality are proved. In conclusion, some special cases of the composed multiobjective optimization problem are considered. Once the general problem has been treated, particularizing the results, we construct a multiobjective dual for each of them and verify the weak and strong dualities.
Keywords

composed functions, convex programming, perturbation theory, conjugate duality, optimality conditions, duality in multiobjective optimization, Pareto efficient and properly efficient solutions, gauges, norms, location problems, Weber problems, minmax problems
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Chapter 1

Introduction

1.1 Convex composed programming: A survey of the literature

In the last years convex composed programming (CCP) has received considerable attention since it offers a unified framework for solving different types of optimization problems. By (CCP) we mean a class of optimization problems in which the objective function as well as the constraints are convex composed functions. Problems of this form occur, for instance, when finding a feasible point of the system of inequalities \( F_i(x) \leq 0, \ i = 1, ..., m \), by minimizing the norm \( \|F(x)\| \), where \( F = (F_1, ..., F_m)^T : \mathbb{R}^n \to \mathbb{R}^m \) is a vector function. Similar problems arise when solving the Weber problem with infimal distances by minimizing \( \sum_{i=1}^{m} w_i d(x, A_i) \), where \( d(x, A_i) = \inf_{a_i \in A_i} \gamma_i(x - a_i) \), \( A = \{A_1, ..., A_m\} \) is a family of convex sets, \( \gamma_i \) are the gauges of the sets \( A_i \) and \( w_i, i = 1, ..., m \), are positive weights. All these examples can be cast within the structure of a convex composed optimization problem.

There are many papers on composed optimization problems both in finite and infinite dimensions. Among the many contributors to the study of these problems we mention A. D. Ioffe, who provided in 1979 (see [29], [30], [31]) the theoretical foundation for the composed problem

\[
(P^c) \quad \min_{x \in \mathbb{R}^n} f(F(x)),
\]

where \( F : \mathbb{R}^n \to \mathbb{R}^m \) is a differentiable function and \( f : \mathbb{R}^m \to \mathbb{R} \) is a sublinear function. In [7], J. V. Burke extended this theory to the case where \( f \) is convex. Later, V. Jeyakumar and X. Q. Yang provide in [38] first-order Lagrangian conditions and second-order optimality conditions for \( (P^c) \), in case when \( f \) is a lower semicontinuous convex function and \( F \) is a locally Lipschitzian and (Gâteaux) differentiable function. Further optimality conditions under twice continuously differentiability hypotheses can be found in [31] and [7].
Recently, G. WANKA, R. I. BOT and E. VARGYAS treated in [73] the composed problem with inequality constraints

\[
(P_{c}^i) \inf_{x \in \mathcal{A}} f(F(x)),
\]

where

\[
\mathcal{A} = \left\{ x \in X : g(G(x)) \leq 0 \right\},
\]

\( X \subseteq \mathbb{R}^n, \ F = (F_1, \ldots, F_m)^T : X \to \mathbb{R}^m, \ G = (G_1, \ldots, G_l)^T : X \to \mathbb{R}^l, \ f : \mathbb{R}^m \to \mathbb{R} \) and \( g = (g_1, \ldots, g_k)^T : \mathbb{R}^l \to \mathbb{R}^k \). The authors showed the existence of a solution to this problem via conjugate duality. Under some convexity assumptions and requiring a quite general constraint qualification they proved several duality results and derived the corresponding optimality conditions.

Extended real-valued composed problems of the form

\[
(P_{c}^e) \min_{x \in \mathbb{R}^n, \ F(x) \in \text{dom}(f)} f(F(x)),
\]

where \( F : \mathbb{R}^n \to \mathbb{R}^m \) is a differentiable function and \( f : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\} \) is a convex function, have been studied by J. V. BURKE and R. A. POLIQUIN in [8]. The authors derived optimality conditions for these problems by reducing them to real-valued minimization problems and requiring a constraint qualification. Similar problems have also been studied by R. T. ROCKAFELLAR in [54] and [55], in case when \( F \) is twice continuously differentiable and \( f \) is piecewise linear quadratic function.

Multiobjective composed problems arise in many applications, subsuming most of the problem models used in mathematical programming. Problems of the form

\[
(P_{v}^c) \ v \text{-min } x \in \mathcal{A} f(F(x)),
\]

where

\[
\mathcal{A} = \left\{ x \in X : g(G(x)) \leq 0 \right\},
\]

\( X \) is a convex subset of \( \mathbb{R}^n, \ F = (F_1, \ldots, F_m)^T : X \to \mathbb{R}^m, \ G = (G_1, \ldots, G_k)^T : X \to \mathbb{R}^k, \ f = (f_1, \ldots, f_m)^T : \mathbb{R}^m \to \mathbb{R}^m, \ g = (g_1, \ldots, g_k)^T : \mathbb{R}^k \to \mathbb{R}^k \), \( f_i, \ g_j \) are real-valued convex functions and \( F_i, \ G_j \) are locally Lipschitz and differentiable functions, were studied by V. JEYAKUMAR and X. Q. YANG in [39], [75] and by C. J. GOH and X. Q. YANG in [22], respectively. In [39], using the Clarke subdifferential, the authors gave first-order optimality conditions and duality results for them. In [22] and [75], second-order optimality conditions are given for a special case of the problem \((P_{v}^c)\).

In what follows we briefly outline the contents of this work.
1.2 A description of the contents

This thesis provides some new duality results concerning different types of optimization problems. It is divided into three main parts, the first one being devoted to single-objective composed optimization problems, the second one to location problems and the last one to multiobjective optimization problems. Within this limitation we would like to have our results as general as possible. To fulfill this aim, first we consider the composed single-objective minimization problem

\[
(P) \quad \inf_{x \in A} f(F(x)),
\]

where \(X \subseteq \mathbb{R}^n\), \(F = (F_1, ..., F_m)^T : X \to \mathbb{R}^m\), \(G = (G_1, ..., G_l)^T : X \to \mathbb{R}^l\), \(f : \mathbb{R}^m \to \mathbb{R}\) and \(g = (g_1, ..., g_k)^T : \mathbb{R}^l \to \mathbb{R}^k\).

Because many interesting examples of optimization problems can be formulated in the above form, the suggested composed functions approach leads to a comprehensive theory that includes, as special cases, some former results in the literature. Examples we shall consider include the classical optimization problem with inequality constraints treated by G. WANKA and R. I. BÖT in [70], the optimization problem without constraints studied by G. WANKA, R. I. BÖT and E. VARGYAS in [72] and some variants of location- and multiobjective problems, respectively. In particular, we study the location model with gauges of closed convex sets introduced by Y. HINOJOSA and J. PUERTO in [27], the location problem involving sets as existing facilities treated by S. NICKEL, J. PUERTO and A. M. RODRIGUEZ-CHIA in [52] and some multiobjective extensions of these, such as the multiobjective Weber and minmax problems with infimal distances, treated in detail by G. WANKA, R. I. BÖT and E. VARGYAS in [71], respectively.

Throughout this work we address the standard questions of duality in constrained optimization: the formulation of dual problems, conditions ensuring the equality of primal and dual optimal objective values, attainment of the optimal objective values in the primal and dual problems, optimality conditions. There are numerous studies devoted to duality theory of optimization problems. The approach we adopt here, is based on conjugate duality, described for instance by I. EKELAND and R. TEMAM in [14].

After a short presentation of the idea presented in [14], we provide three different dual problems \((D_L)\), \((D_F)\) and \((D_{FL})\), respectively, for \((P)\). As we will see, \((D_L)\) is the well-known Lagrange dual problem, \((D_F)\) is the Fenchel dual problem, while \((D_{FL})\) is classified as a sort of mixed, so-called Fenchel-Lagrange dual problem. The new duals \((D_F)\) and \((D_{FL})\) have a compact form, and are defined in terms of the conjugates of the original functions \(f\), \(F\), \(g\) and \(G\).
CHAPTER 1. INTRODUCTION

This approach has the important property that the "weak duality" always holds, namely, that the optimal objective value of the primal problem is greater than or equal to the optimal objective values of the dual problems. We continue our study by comparing the three dual problems in order to analyze them in a unified framework and to assess the differences among them. As a first result, we establish in the general case ordering relations between their optimal objective values. In order to prove strong duality results for the introduced pairs of primal-dual problems, some generalized convexity assumptions and regularity conditions are made. Using these strong duality results, we derive the necessary and sufficient optimality conditions for each of the three primal-dual pairs.

Once the details for the general problem have been resolved, we focus our attention on some special cases of this composed problem. First, we consider the classical optimization problem with inequality constraints and then the optimization problem without constraints. Using the results obtained in the general case, we deduce a conjugate duality theory also for this class of problems. We mention that the convex analytic terminology we use here, is that of R. T. ROCKAFEL-LAR from [53].

The second part of this work is devoted to location analysis. After a short summary concerning some useful properties of the gauges of closed convex sets and its conjugates, we introduce the optimization problem

\[
(P_{nc}) \inf_{x \in X} \gamma_C^+(F(x)),
\]

where \( \gamma_C : \mathbb{R}^m \to \mathbb{R} \) is a monotonic gauge of a closed convex set \( C \) containing the origin, \( \gamma_C^+ : \mathbb{R}^m \to \mathbb{R} \), \( \gamma_C^+(t) := \gamma_C(t^+) \), with \( t^+ = (t_1^+, \ldots, t_m^+)^T \) and \( t_i^+ = \max\{0, t_i\} \), \( i = 1, \ldots, m \). As in the original composed problem, \( F = (F_1, \ldots, F_m)^T : X \to \mathbb{R}^m \) is a vector-valued function. This problem constitutes a general framework for location problems. Interestingly, the same conjugate duality principle as in the general case can be used in order to treat it. Applying the results obtained for the original problem, we determine its Fenchel-Lagrange dual and verify the weak and strong duality. Additionally, necessary and sufficient optimality conditions are derived.

Closely related to this case, we discuss the problem where the monotonic gauge \( \gamma_C \) is replaced by a monotonic norm \( l \). At the end of this part we study applications of these ideas to more concrete models, namely, to locations problems with unbounded unit balls. Within this topic we concentrate on two special problems: the Weber- and minmax problems with gauges of closed convex sets, which were introduced by Y. HINOJOSA and J. PUERTO in [27]. The authors give in [27] a geometrical description of the set of optimal solutions. Here we present, how can be treated the same problem via conjugate duality.

The last part of this thesis deals with duality for multiobjective optimization problems. Our purpose from here on is to extend the results from scalar to vector optimization. In order to keep our results as general as possible, we consider also
the multiobjective problem in the form of a composed optimization problem, namely,

\[
(P_v) \quad \text{v-min } \min_{x \in \mathcal{A}} f(F(x)),
\]

where \( X \subseteq \mathbb{R}^n, F = (F_1, \ldots, F_m)^T : X \to \mathbb{R}^m, G = (G_1, \ldots, G_l)^T : X \to \mathbb{R}^l, f = (f_1, \ldots, f_s)^T : \mathbb{R}^m \to \mathbb{R}^s \) and \( g = (g_1, \ldots, g_k)^T : \mathbb{R}^l \to \mathbb{R}^k \). Additionally, we assume that \( F_i, i = 1, \ldots, m, G_j, j = 1, \ldots, l \), are convex functions and \( f_i, i = 1, \ldots, s \), and \( g_j, j = 1, \ldots, k \), are convex and componentwise increasing functions.

In the multiobjective optimization there are different concepts of solutions for this problem. Throughout this work we are concerned with Pareto efficient and properly efficient solutions. The fruitful idea is to transform \((P_v)\) into a scalarized problem and then, based on conjugate duality information described in Chapter 2, to construct a dual problem to the last one. Analogously to the original primal-dual pair, weak and strong duality theorems as well as necessary and sufficient optimality conditions are derived for this scalarized problem and its dual. The optimality conditions obtained hereby are used later to construct a multiobjective dual problem \((D_v)\) to \((P_v)\). For the multiobjective primal and dual problems, the weak and strong duality are proved.

After we have considered the general multiobjective problem, we study some particular cases of it. First, we consider the classical multiobjective optimization problem with inequality constraints and then the multiobjective optimization problem without constraints. In fact, both these problems were already treated by G. WANKA and R. I. BOT in [69], and by G. WANKA, R. I. BOT and E. VARGYAS in [71], our intention hereby is to show how these results can be obtained as particular cases of the composed multiobjective problem.

In the last section of this third part, a new problem is introduced into the field of multicriteria location problems. At the beginning we consider the multiobjective problem, in which the components of the objective function are composites of some monotonic norms with a convex vector-valued function. After the formulation of the primal problem, a multiobjective dual is given. For this primal-dual pair weak and strong duality theorems are proved.

This multiobjective model with monotonic norms turns out to be very useful in the study of other location settings. In what follows, we study the duality for the multiobjective model involving sets as existing facilities. Finally, the biobjective Weber-minmax-, the multiobjective Weber- and the multiobjective minmax problems with infimal distances are discussed. These problem formulations were motivated by a paper of S. NICKEL, J. PUERTO and A. M. RODRIGUEZ-CHIA, [52], in which the authors give a geometrical characterization of the sets of optimal solutions. Embedding them into this unifying model with monotonic norms, we show how to solve they via duality.
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Chapter 2

Duality for a single-objective composed optimization problem

2.1 The composed optimization problem and its conjugate duals

There is a well-developed theory for the duality in convex optimization. One of the most fruitful duality ideas is based on conjugate functions, which concept was introduced by W. Fenchel [16]. Since then, many other authors have used it in their studies. Among the most important authors we mention R. T. Rockafellar [53], I. Ekeland and R. Temam [14], who gave an approach for constructing dual problems by using the perturbation theory. In [14] the authors have given a detailed description of this method, whose main idea is to embed the original problem into a family of perturbed problems, and then, by means of conjugate functions to associate a dual problem to it.

In order to study the duality for our single-valued composed minimization problem, which we call primal problem, we follow the same idea. Using different perturbation functions, we assign three dual problems to it and study the relations between the optimal objective values of the duals and then the relations between the optimal objective values of the primal and dual problems, respectively. In general, we denote the optimal objective value of the primal by \( v(P) \) and the optimal objective value of its dual by \( v(D) \). This notation does not automatically imply that the corresponding values are attained.

First, some ordering relations between the optimal objective values of the duals are obtained. Furthermore, we analyze the relations between the primal and the corresponding dual problems. By the construction of the dual problems, the weak duality (i.e. \( v(D) \leq v(P) \)) holds for each primal-dual pair. In order to ensure the strong duality (i.e. \( v(D) = v(P) \) and the dual problem has an optimal solution), we require some convexity assumptions and regularity conditions. Additionally, necessary and sufficient optimality conditions are derived.
The second part of this chapter is devoted to two special cases of the original problem. The first one is the classical optimization problem with inequality constraints, and the second one is the optimization problem without constraints. Applying the general results deduced from the first part, we obtain a conjugate duality theory also for these types of problems.

2.1.1 General notations and problem formulation

Let $p$ be a positive integer and let $x, y$ be two vectors of $\mathbb{R}^p$. Throughout this paper all vectors are supposed to be column vectors and we use superscripts for vectors, for example, $x^i$, and subscripts for components of vectors, for example, $x_i$. We denote by $x^T y = \sum_{i=1}^{p} x_i y_i$ the inner product of the vectors $x, y \in \mathbb{R}^p$ and by $\mathbb{R}_+^p$ the non-negative orthant of $\mathbb{R}^p$. For $x, y \in \mathbb{R}_+^p$, the inequality $x \leq y$ means that $y - x \in \mathbb{R}_+^p$, which is equivalent to $x_i \leq y_i$, for all $i = 1, ..., p$.

In what follows, let us consider a nonempty subset $X \subseteq \mathbb{R}^n$ and the functions $F = (F_1, ..., F_m)^T : X \rightarrow \mathbb{R}^m$, $G = (G_1, ..., G_l)^T : X \rightarrow \mathbb{R}^l$, $f : \mathbb{R}^m \rightarrow \mathbb{R}$ and $g = (g_1, ..., g_k)^T : \mathbb{R}^l \rightarrow \mathbb{R}^k$. Additionally, we extend $F$ and $G$ to $\tilde{F} = (\tilde{F}_1, ..., \tilde{F}_m)^T$ and $\tilde{G} = (\tilde{G}_1, ..., \tilde{G}_l)^T$, respectively, with

$$\tilde{F}_i : \mathbb{R}^n \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm \infty\}$$

and

$$\tilde{G}_j : \mathbb{R}^n \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm \infty\}$$

As a consequence we have to make now for the functions $f$ and $g_i$, $i = 1, ..., k$, the following conventions

$$f(y) = +\infty, \text{ if } y = (y_1, ..., y_m)^T \text{ with } y_i \in \mathbb{R} \cup \{+\infty\},$$
$$i = 1, ..., m, \text{ and } \exists \ j \in \{1, ..., m\} \text{ such that } y_j = +\infty, \hspace{1cm} (2.1)$$

and, for $i = 1, ..., k$,

$$g_i(z) = +\infty, \text{ if } z = (z_1, ..., z_l)^T \text{ with } z_i \in \mathbb{R} \cup \{+\infty\},$$
$$i = 1, ..., l, \text{ and } \exists \ j \in \{1, ..., l\} \text{ such that } z_j = +\infty. \hspace{1cm} (2.2)$$

The optimization problem which we investigate in this chapter is

$$(P) \inf_{x \in A} f(F(x)),$$

where

$$A = \left\{ x \in X : g(G(x)) \leq 0 \right\} \subseteq \mathbb{R}_+^k.$$
2.1 THE CONJUGATE DUALS OF THE COMPOSED PROBLEM

Here, \( g(G(x)) \leq 0 \) means that \( g_i(G(x)) \leq 0 \) for all \( i = 1, \ldots, k \). In the following we suppose that the feasible set \( \mathcal{A} \) is nonempty. The problem \((P)\) is said to be the primal problem and its optimal objective value is denoted by \( v(P) \).

**Definition 2.1** An element \( \bar{x} \in \mathcal{A} \) is said to be an optimal solution for \((P)\) if \( f(F(\bar{x})) = v(P) \).

The aim of this section is to construct different dual problems to \((P)\). To do this, we use an approach based on the theory of conjugate functions described by I. EKELAND and R. TEMAM in [14]. In order to reproduce it, let us consider first a general optimization problem without constraints

\[
(PG) \quad \inf_{x \in \mathbb{R}^n} h(x),
\]

with \( h \) a mapping from \( \mathbb{R}^n \) into \( \mathbb{R} \).

In what follows we give some definitions and remarks concerning the conjugate of a function and the conjugate relative to \( X \), if the function is defined only on a subset \( X \subseteq \mathbb{R}^n \).

**Definition 2.2** The function \( h^* : \mathbb{R}^n \to \mathbb{R} \), defined by

\[
h^*(x^*) = \sup_{x \in \mathbb{R}^n} \left\{ x^T x - h(x) \right\},
\]

is called the (Fenchel-Moreau) conjugate of \( h \).

**Definition 2.3** When \( X \) is a nonempty subset of \( \mathbb{R}^n \) and \( h : X \to \mathbb{R} \), let \( h_X^* : \mathbb{R}^n \to \mathbb{R} \) be the so-called conjugate of \( h \) relative to the set \( X \) defined by

\[
h_X^*(x^*) = \sup_{x \in X} \left\{ x^T x - h(x) \right\}.
\]

**Remark 2.1** Considering the extension of \( h : X \to \mathbb{R} \) to the whole space,

\[
\tilde{h} : \mathbb{R}^n \to \mathbb{R}, \quad \tilde{h}(x) = \begin{cases} h(x), & \text{if } x \in X, \\ +\infty, & \text{otherwise}, \end{cases}
\]

one can see that the conjugate of \( h \) relative to the set \( X \) is identical to the Fenchel-Moreau conjugate of \( \tilde{h} \).

**Definition 2.4** Let \( X \) be a subset of \( \mathbb{R}^n \). The function \( \delta_X : \mathbb{R}^n \to \mathbb{R} \) defined by

\[
\delta_X(x) = \begin{cases} 0, & \text{if } x \in X, \\ +\infty, & \text{otherwise}, \end{cases}
\]

is called the indicator function of the set \( X \).
Remark 2.2 By Definition 2.2 we have that
\[ \delta^*_X(-x^*) = - \inf_{x \in X} x^T x. \]

Following now the path of the approach described in [14], which is based on a perturbation method, we embed the problem \((PG)\) into a family of perturbed problems
\[ (PG_p) \inf_{x \in \mathbb{R}^n} \Phi(x, p), \]
where \(\Phi : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}\) is the so-called perturbation function and has the property that
\[ \Phi(x, 0) = h(x), \ \forall x \in \mathbb{R}^n. \] (2.3)
Here, \(\mathbb{R}^s\) is the space of the perturbation variables. The conjugate function of the perturbation function \(\Phi\) looks like
\[ \Phi^*(x^*, p^*) = \sup_{x \in \mathbb{R}^n, p \in \mathbb{R}^s} \left\{ x^T x + p^T p - \Phi(x, p) \right\}. \] (2.4)
The problem
\[ (DG) \sup_{p^* \in \mathbb{R}^s} \{-\Phi^*(0, p^*)\} \]
defines the dual problem of \((PG)\) and its optimal objective value is denoted by \(v(DG)\). This approach has the important property that between the primal and the dual problem weak duality holds, i.e. the value of the primal objective function at any primal feasible point is greater than or equal to the value of the dual objective function at any dual feasible point. The following theorem states this fact.

Theorem 2.1 ([14]) The relation
\[ -\infty \leq v(DG) \leq v(PG) \leq +\infty \] (2.5)
always holds.

Because of the basic significance of this assertion we want to recall here its proof.

Proof. Let \(p^* \in \mathbb{R}^s\). From (2.4) we have
\[
\Phi^*(0, p^*) = \sup_{x \in \mathbb{R}^n, p \in \mathbb{R}^s} \left\{ 0^T x + p^T p - \Phi(x, p) \right\} \\
= \sup_{x \in \mathbb{R}^n, p \in \mathbb{R}^s} \left\{ p^T p - \Phi(x, p) \right\} \\
\geq \sup_{x \in \mathbb{R}^n} \left\{ p^T 0 - \Phi(x, 0) \right\} = \sup_{x \in \mathbb{R}^n} \{-\Phi(x, 0)\},
\]
which means that
\[-\Phi^*(0, p^*) \leq \Phi(x, 0) = h(x), \ \forall x \in \mathbb{R}^n, \ \forall p^* \in \mathbb{R}^s,\]
and so, \( v(DG) \leq v(PG) \). \( \Box \)

In order to apply the approach described above we introduce the function
\[ h: \mathbb{R}^n \to \mathbb{R}, \]
\[ h(x) = \begin{cases} f(\hat{F}(x)), & \text{if } g(\hat{G}(x)) \leq 0, \\ +\infty, & \text{otherwise,} \end{cases} \]
and therefore (P) is rewritable as an optimization problem without constraints
\[ (P) \inf_{x \in \mathbb{R}^n} h(x). \]
Since the perturbation function \( \Phi: \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R} \) satisfies \( \Phi(x, 0) = h(x) \), for each \( x \in \mathbb{R}^n \), the assumptions (2. 1) and (2. 2) imply that
\[ \Phi(x, 0) = f(F(x)), \ \forall x \in \mathcal{A} \quad (2. 6) \]
and
\[ \Phi(x, 0) = +\infty, \ \forall x \in \mathbb{R}^n \setminus \mathcal{A}. \quad (2. 7) \]

In the following we construct three different perturbation functions, the corresponding dual problems to (P) and we study the relations between their optimal objective values.

**2.1.2 The Lagrange dual problem**

At first let us consider the perturbation function \( \Phi_L: \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R} \) defined by
\[ \Phi_L(x, q) = \begin{cases} f(\hat{F}(x)), & \text{if } g(\hat{G}(x)) \leq q, \\ +\infty, & \text{otherwise,} \end{cases} \]
with the perturbation variable \( q \in \mathbb{R}^k \). One may see that \( \Phi_L \) fulfills relations (2. 6) and (2. 7). For its conjugate we have
\[ \Phi^*_L(x^*, q^*) = \sup_{x \in \mathbb{R}^n, \ q \in \mathbb{R}^k} \left\{ x^*^T x + q^*^T q - \Phi_L(x, q) \right\} \]
\[ = \sup_{x \in \mathbb{R}^n, \ q \in \mathbb{R}^k, \ g(\hat{G}(x)) \leq q} \left\{ x^*^T x + q^*^T q - f(\hat{F}(x)) \right\} \]
\[ = \sup_{x \in \mathcal{X}, \ q \in \mathbb{R}^k, \ g(\hat{G}(x)) \leq q} \left\{ x^*^T x + q^*^T q - f(F(x)) \right\}. \]
In order to calculate this expression we introduce the variable \( a \) instead of \( q \), by
\[
\Phi^*_L(x^*,q^*) = \sup_{x \in X, a \in \mathbb{R}^k_+} \left\{ x^T x + q^T g(G(x)) + q^T a - f(F(x)) \right\}
\]
\[
= \sup_{x \in X} \left\{ x^T x + q^T g(G(x)) - f(F(x)) \right\} + \sup_{a \in \mathbb{R}^k_+} \left\{ q^T a \right\}
\]
\[
= \begin{cases} 
\sup_{x \in X} \left\{ x^T x + q^T g(G(x)) - f(F(x)) \right\}, & \text{if } q^* \in \mathbb{R}^k_+, \\
+\infty, & \text{otherwise}.
\end{cases}
\]

As we have seen, the dual of \((P)\) obtained by the perturbation function \( L \) is
\[
(D_L) \quad \sup_{q^* \in \mathbb{R}^k} \left\{ -\Phi^*_L(0,q^*) \right\}.
\]

Because
\[
\sup_{q^* \in \mathbb{R}^k} \left\{ -\Phi^*_L(0,q^*) \right\} = \sup_{q^* \in -\mathbb{R}^k_+} \left\{ -\sup_{x \in X} \left\{ q^T g(G(x)) - f(F(x)) \right\} \right\}
\]
\[
= \sup_{q^* \in -\mathbb{R}^k_+} \inf_{x \in X} \left\{ -q^T g(G(x)) + f(F(x)) \right\},
\]

denoting \( t := -q^* \in \mathbb{R}^k_+ \), the dual becomes
\[
(D_L) \quad \sup_{t \in \mathbb{R}^k_+} \inf_{x \in X} \left\{ f(F(x)) + t^T g(G(x)) \right\}. \tag{2.8}
\]

The problem \((D_L)\) is actually the well-known Lagrange dual problem. Its optimal objective value is denoted by \( v(D_L) \) and Theorem 2.1 implies that
\[
v(D_L) \leq v(P). \tag{2.9}
\]

### 2.1.3 The Fenchel dual problem

Let us consider the perturbation function \( \Phi_F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R} \) given by
\[
\Phi_F(x,p,q) = \begin{cases} 
f(\tilde{F}(x+p) + q), & \text{if } g(\tilde{G}(x)) \leq 0, \\
+\infty, & \text{otherwise},
\end{cases}
\]

with the perturbation variables \( p \in \mathbb{R}^n \) and \( q \in \mathbb{R}^m \). The relations (2.6) and (2.7) are also fulfilled and it holds
\[
\Phi^*_F(x^*,p^*,q^*) = \sup_{x, p \in \mathbb{R}^n, q \in \mathbb{R}^m} \left\{ x^T x + p^T p + q^T q - \Phi_F(x,p,q) \right\}
\]
\[
= \sup_{x, p \in \mathbb{R}^n, x+p \in X, \atop q \in \mathbb{R}^m, g(G(x)) \leq 0} \left\{ x^T x + p^T p + q^T q - f(\tilde{F}(x+p) + q) \right\}.
\]
2.1 THE CONJUGATE DUALS OF THE COMPOSED PROBLEM

Introducing the new variables $r = x + p \in X$ and $a = \tilde{F}(x + p) + q \in \mathbb{R}^m$, we obtain

$$\Phi_F^*(x^*, p^*, q^*) = \sup_{x, r \in X, a \in \mathbb{R}^m, g(G(x)) \leq 0} \left\{ x^T x + p^T r - p^T x + q^T a - q^T \tilde{F}(r) - f(a) \right\}$$

$$= \sup_{a \in \mathbb{R}^m} \left\{ q^T a - f(a) \right\} + \sup_{r \in X} \left\{ p^T r - q^T \tilde{F}(r) \right\} + \sup_{x \in A} \left\{ (x^* - p^*)^T x \right\}$$

$$= f^*(q^*) + (q^T F^*)_X^*(p^*) + \sup_{x \in A} \left\{ (x^* - p^*)^T x \right\} .$$

Denoting $p := p^*$ and $q := q^*$, the dual problem of $(P)$

$$(D_F) \sup_{p^* \in \mathbb{R}^n, q^* \in \mathbb{R}^m} \left\{ -\Phi_F^*(0, p^*, q^*) \right\}$$

can be written as

$$(D_F) \sup_{p \in \mathbb{R}^n, q \in \mathbb{R}^m} \left\{ -f^*(q) - (q^T F^*)_X^*(p) + \inf_{x \in A} p^T x \right\} .$$

Taking into consideration Remark 2.2, problem $(D_F)$ is equivalent to

$$(D_F) \sup_{p \in \mathbb{R}^n, q \in \mathbb{R}^m} \left\{ -f^*(q) - (q^T F^*)_X^*(p) - \delta_A^*(-p) \right\} . \quad (2.10)$$

Let us call $(D_F)$ the Fenchel dual problem and denote its optimal objective value by $v(D_F)$. Theorem 2.1 implies that

$$v(D_F) \leq v(P). \quad (2.11)$$

2.1.4 The Fenchel-Lagrange dual problem

Another dual problem can be obtained considering the perturbation function $\Phi_{FL} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \times \mathbb{R}^k \rightarrow \mathbb{R}$, defined by

$$\Phi_{FL}(x, p, q, p', q', t) = \begin{cases} f(\tilde{F}(x + p) + q), & \text{if } g(\tilde{G}(x + p') + q') \leq t, \\ +\infty, & \text{otherwise,} \end{cases}$$

with the perturbation variables $p, p' \in \mathbb{R}^n$, $q \in \mathbb{R}^m$, $q' \in \mathbb{R}^l$ and $t \in \mathbb{R}^k$. $\Phi_{FL}$ satisfies relations (2.6) and (2.7), therefore a dual problem to $(P)$ can be introduced as

$$(D_{FL}) \sup_{p^*, p'^* \in \mathbb{R}^n, q^* \in \mathbb{R}^m, q'^* \in \mathbb{R}^l, t \in \mathbb{R}^k} \left\{ -\Phi_{FL}^*(0, p^*, q^*, p'^*, q'^*, t^*) \right\} .$$
For the conjugate of $\Phi_{FL}$ we have

$$
\Phi_{FL}^*(x^*, p^*, q^*, p'^*, q'^*, t^*) = \sup_{x, p, p' \in \mathbb{R}^m, q, q' \in \mathbb{R}^k, q' \in \mathbb{R}^l, t \in \mathbb{R}^k, \text{ with } g(G(x+p') + q') \geq t} \left\{ x^T x + p^T p + q^T q + p'^T p' + q'^T q + t^T t - \Phi_{FL}(x, p, q, p', q', t) \right\}
$$

Introducing the new variables $r = x + p \in X$, $r' = x + p' \in X$, $a = F(x + p) + q \in \mathbb{R}^m$, $b = G(x + p') + q' \in \mathbb{R}^k$, and $c = t - g(G(x + p') + q') \in \mathbb{R}^k$, we have

$$
\Phi_{FL}^*(x^*, p^*, q^*, p'^*, q'^*, t^*) = \sup_{x \in \mathbb{R}^m, r, r' \in X, a \in \mathbb{R}^m, b \in \mathbb{R}^k, c \in \mathbb{R}^k} \left\{ x^T x + p^T r - p^T x + q^T a - q^T F(r) + p'^T r' - p'^T x + q'^T G(r') + t^T c + t^T g(b) - f(a) \right\} =
$$

$$
\sup_{a \in \mathbb{R}^m} \left\{ q^T a - f(a) \right\} + \sup_{b \in \mathbb{R}^k} \left\{ q'^T b + t^T g(b) \right\} + \sup_{r \in X} \left\{ p^T r - q^T F(r) \right\} + \sup_{r' \in X} \left\{ p'^T r' - q'^T G(r') \right\} + \sup_{x \in \mathbb{R}^m} \left\{ (x^* - p^* - p'^*)^T x \right\} + \sup_{c \in \mathbb{R}^k} \left\{ t^T c \right\}.
$$

Because

$$
\sup_{x \in \mathbb{R}^m} \left\{ -(p^* + p'^*)^T x \right\} = \begin{cases} 0, & \text{if } p^* + p'^* = 0, \\ +\infty, & \text{otherwise}, \end{cases}
$$

and

$$
\sup_{c \in \mathbb{R}^k} \left\{ t^T c \right\} = \begin{cases} 0, & \text{if } t^* \in -\mathbb{R}^k, \\ +\infty, & \text{otherwise}, \end{cases}
$$

follows that

$$
\Phi_{FL}^*(0, p^*, q^*, p'^*, q'^*, t^*) =
$$

$$
\begin{cases} f^*(q^*) + (-t^T g)^* (q'^*) + (q^T F)^* (p^*) + \left( q'^T G \right)^* (p'^*) , & \text{if } p^* + p'^* = 0 \text{ and } t^* \in -\mathbb{R}^k, \\ +\infty, & \text{otherwise}, \end{cases}
$$

Denoting $p := p^* = -p'^*$, $q := q^*$, $q' := q'^*$, $t := -t^*$, the dual is rewrites as

$$(D_{FL}) \sup_{p \in \mathbb{R}^m, q \in \mathbb{R}^k, q' \in \mathbb{R}^l, t \in \mathbb{R}^k} \left\{ -f^*(q) - (t^T g)^* (q') - (q^T F)^* (p) - (q'^T G)^* (t) \right\}.$$

(2.12)
2.2 RELATIONS BETWEEN THE DUALS’ OBJECTIVE VALUES

By Theorem 2.1 the weak duality
\[ v(D_{FL}) \leq v(P) \]  \hspace{1cm} (2.13)
is also true, where \( v(D_{FL}) \) is the optimal objective value of \( (D_{FL}) \).

2.2 The relations between the optimal objective values of the dual problems

In the previous section we have seen that the optimal objective values \( v(D_L) \), \( v(D_F) \) and \( v(D_{FL}) \) of the dual problems \( (D_L) \), \( (D_F) \) and \( (D_{FL}) \), respectively, are less than or equal to the optimal objective value \( v(P) \) of the primal problem \( (P) \). Henceforth we are going to investigate the relations between the optimal objective values of the three dual problems.

2.2.1 The general case

For the beginning we remain within the most general case, namely, without any special assumptions concerning the set \( X \) or the functions \( f, F, g \) and \( G \).

Proposition 2.1 The inequality \( v(D_{FL}) \leq v(D_L) \) holds.

Proof. Let \( p \in \mathbb{R}^n \), \( q \in \mathbb{R}^m \), \( q' \in \mathbb{R}^l \) and \( t \in \mathbb{R}^k_+ \) be fixed. By the definition of the conjugate function we have

\[-f^*(q) = - \sup_{y \in \mathbb{R}^m} \{ q^T y - f(y) \} = \inf_{y \in \mathbb{R}^m} \{ f(y) - q^T y \} \leq \inf_{x \in X} \{ f(F(x)) - q^T F(x) \}, \]

\[-(t^T g)^* (q') = - \sup_{z \in \mathbb{R}^l} \{ q'^T z - t^T g(z) \} = \inf_{z \in \mathbb{R}^l} \{ t^T g(z) - q'^T z \}\]

\[\leq \inf_{x \in X} \{ t^T g(G(x)) - q'^T G(x) \}, \]

\[-(q^T F)^*_X (p) = - \sup_{x \in X} \{ p^T x - q^T F(x) \} = \inf_{x \in X} \{ q^T F(x) - p^T x \} \]

and

\[-(q^T G)^*_X (-p) = - \sup_{x \in X} \{ -p^T x - q^T G(x) \} = \inf_{x \in X} \{ q^T G(x) + p^T x \}. \]

Adding the inequalities from above we obtain

\[-f^*(q) - (t^T g)^* (q') - (q^T F)^*_X (p) - (q^T G)^*_X (-p) \leq \]

\[\inf_{x \in X} \{ f(F(x)) + t^T g(G(x)) \}. \]
By taking now the supremum over \( p \in \mathbb{R}^n, \ q \in \mathbb{R}^m, \ q' \in \mathbb{R}^l \) and \( t \in \mathbb{R}^k_+ \), we have
\[
\sup_{p \in \mathbb{R}^n, \ q \in \mathbb{R}^m, \ q' \in \mathbb{R}^l, \ t \in \mathbb{R}^k_+} \left\{ -f^*(q) - (t^T g)^*(q') - (q^T F)^*_{X}(p) - (q' G)^*_{X}(-p) \right\}
\leq \sup_{t \in \mathbb{R}^k_+} \inf_{x \in X} \left\{ f(F(x)) + t^T g(G(x)) \right\}.
\]
This is nothing but \( v(D_{FL}) \leq v(D_L) \). \( \square \)

**Remark 2.3** We call the problems \( (D_{FL}) \) and \( (D_L) \) equivalent if the equality \( v(D_{FL}) = v(D_L) \) is just fulfilled.

**Proposition 2.2** The inequality \( v(D_{FL}) \leq v(D_F) \) holds.

**Proof.** Let \( p \in \mathbb{R}^n \) and \( q' \in \mathbb{R}^l \) be fixed. For each \( t \in \mathbb{R}^k_+ \) we have
\[
- (t^T g)^*(q') - (q^T G)^*_{X}(-p) = - \sup_{z \in \mathbb{R}^l} \left\{ q^T z - t^T g(z) \right\} - \sup_{x \in X} \left\{ -p^T x - q^T G(x) \right\}
\leq \inf_{x \in X} \left\{ t^T g(G(x)) - q^T G(x) \right\} + \inf_{x \in X} \left\{ q^T G(x) + p^T x \right\}
\leq \inf_{x \in A} \left\{ t^T g(G(x)) + p^T x \right\} \leq \inf_{x \in A} p^T x = -\delta^*_A(-p).
\] (2.14)
The last two inequalities in (2.14) hold because \( A \subseteq X \) and \( t^T g(G(x)) \leq 0 \) for all \( x \in A \). Additionally, let \( q \) be an arbitrary element of \( \mathbb{R}^m \). By adding first \( -f^*(q) - (q^T F)^*_{X}(p) \) to both sides of (2.14) and by taking then the supremum over \( p \in \mathbb{R}^n, \ q \in \mathbb{R}^m, \ q' \in \mathbb{R}^l \) and \( t \in \mathbb{R}^k_+ \), we obtain
\[
\sup_{p \in \mathbb{R}^n, \ q \in \mathbb{R}^m, \ q' \in \mathbb{R}^l, \ t \in \mathbb{R}^k_+} \left\{ -f^*(q) - (t^T g)^*(q') - (q^T F)^*_{X}(p) - (q' G)^*_{X}(-p) \right\}
\leq \sup_{p \in \mathbb{R}^n, \ q \in \mathbb{R}^m} \left\{ -f^*(q) - (q^T F)^*_{X}(p) - \delta^*_A(-p) \right\},
\]
which is nothing but \( v(D_{FL}) \leq v(D_F) \). \( \square \)

**Remark 2.4** Considering similar counterexamples like G. WANKA and R. I. BOT in [70], it can be shown that the inequalities in Proposition 2.1 and Proposition 2.2 can be also strict. Moreover, in general, an ordering between \( v(D_L) \) and \( v(D_F) \) cannot be established.

**Remark 2.5** We call the problems \( (D_{FL}) \) and \( (D_F) \) equivalent if the equality \( v(D_{FL}) = v(D_F) \) is just fulfilled.

In the following, we are going to study the equivalence of the dual problems \( (D_L), (D_F) \) and \( (D_{FL}) \). In order to do this, let us consider first some definitions and preliminary results.
2.2 RELATIONS BETWEEN THE DUALS’ OBJECTIVE VALUES

**Definition 2.5** The function \( f : \mathbb{R}^m \to \mathbb{R} \) is called componentwise increasing, if for \( x = (x_1, \ldots, x_m)^T, \ y = (y_1, \ldots, y_m)^T \in \mathbb{R}^m \) where \( x_i \leq y_i, \ i = 1, \ldots, m \), follows that \( f(x) \leq f(y) \).

**Proposition 2.3** If \( f : \mathbb{R}^m \to \mathbb{R} \) is a componentwise increasing function, then \( f^*(q) = +\infty \) for all \( q \in \mathbb{R}^m \setminus \mathbb{R}^m_+ \).

**Proof.** Let \( q \in \mathbb{R}^m \setminus \mathbb{R}^m_+ \). Then there exists at least one \( i \in \{1, \ldots, m\} \) such that \( q_i < 0 \). But

\[
\begin{align*}
\sup_{q \in \mathbb{R}^m} \{ q^T d - f(d) \} & \geq \sup_{d \in (0, \ldots, d_i, \ldots, 0)} \{ q^T d - f(d) \} \\
& = \sup_{d_i \in \mathbb{R}} \{ q_i d_i - f(0, \ldots, d_i, \ldots, 0) \} \geq \sup_{d_i < 0} \{ q_i d_i - f(0, \ldots, d_i, \ldots, 0) \} \\
& \geq \sup_{d_i < 0} \{ q_i d_i \} - f(0, \ldots, 0) = +\infty.
\end{align*}
\]

Therefore \( f^*(q) = +\infty, \ \forall \ q \in \mathbb{R}^m \setminus \mathbb{R}^m_+ \). \( \square \)

**Proposition 2.4** Assume that \( X \) is a nonempty convex subset of \( \mathbb{R}^n \), \( F_i : X \to \mathbb{R}, \ i = 1, \ldots, m \), are convex functions and \( f : \mathbb{R}^m \to \mathbb{R} \) is a convex and componentwise increasing function. Then \( f \circ F : \mathbb{R}^n \to \mathbb{R} \) is convex.

**Proof.** We have to prove that for all \( x, y \in \mathbb{R}^n \) and for all \( \lambda \in \mathbb{R} \), with \( 0 \leq \lambda \leq 1 \),

\[ (f \circ F)(\lambda x + (1 - \lambda)y) \leq \lambda (f \circ F)(x) + (1 - \lambda)(f \circ F)(y). \quad (2.15) \]

If \( x, \ y \in X \), then we have

\[
\begin{align*}
(f \circ F)(\lambda x + (1 - \lambda)y) &= f(F(\lambda x + (1 - \lambda)y)) \\
& \leq f(\lambda F(x) + (1 - \lambda)F(y)) \\
& \leq \lambda (f \circ F)(x) + (1 - \lambda)(f \circ F)(y) \\
& = \lambda (f \circ F)(x) + (1 - \lambda)(f \circ F)(y).
\end{align*}
\]

If either \( x \notin X \) or \( y \notin X \), or both, we have either \( (f \circ F)(x) = +\infty \) or \( (f \circ F)(y) = +\infty \), or both. So, the inequality (2.15) holds again. \( \square \)

**Proposition 2.5** Assume that \( X \) is a nonempty convex subset of \( \mathbb{R}^n \), \( G_j : X \to \mathbb{R}, \ j = 1, \ldots, l \), are convex functions and \( g_i : \mathbb{R}^l \to \mathbb{R}, \ i = 1, \ldots, k \), are convex and componentwise increasing functions. Then \( g_i \circ G : \mathbb{R}^n \to \mathbb{R}, \ i = 1, \ldots, k \), are convex.

**Proof.** The proof is analogous to the proof of Proposition 2.4. \( \square \)

In what follows, we give three known theorems which will play an important role in the sequel.
CHAPTER 2. DUALITY FOR THE SINGLE-OBJECTIVE PROBLEM

Theorem 2.2 (cf. Theorem 16.4 in [53]) Let \( f_1, \ldots, f_n : \mathbb{R}^m \rightarrow \mathbb{R} \) be proper convex functions. If the sets \( ri(dom(f_i)), i = 1, \ldots, n \), have a point in common, then

\[
\left( \sum_{i=1}^{n} f_i \right)^* (p) = \inf \left\{ \sum_{i=1}^{n} f_i^*(p_i) : \sum_{i=1}^{n} p_i = p \right\},
\]

where for each \( p \in \mathbb{R}^m \) the infimum is attained.

The next theorem was given by Zălinescu in [77] for locally convex spaces, in the following we particularize and formulate it for Euclidean spaces.

Theorem 2.3 (cf. Theorem 2.8.10 in [77]) Let \( F = (F_1, \ldots, F_m) \) with \( F_i : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}, i = 1, \ldots, m \), be convex functions and \( f : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\} \) be a convex and componentwise increasing function. If the image \( F(\cap_{i=1}^{m} dom(F_i)) \) of the effective domain \( \cap_{i=1}^{m} dom(F_i) \) contains an interior point of \( dom(f) \), then it holds

\[
(f \circ F)^* (p) = \inf_{\lambda \in \mathbb{R}_+^n} \left\{ f^*(\lambda) + (\lambda^T F)^* (p) \right\},
\]

where for each \( p \in \mathbb{R}^n \) the infimum is attained.

In what follows let \( X \) be a nonempty subset of \( \mathbb{R}^n \), \( g : X \rightarrow \mathbb{R}^k \) a function and \((CQ_a)\) the constraint qualification

\[
(CQ_a) \quad \exists \ x' \in \ ri(X) : \begin{cases} g_i(x') \leq 0, \quad i \in L_a, \\ g_i(x') < 0, \quad i \in N_a, \end{cases}
\]

where

\[
L_a := \left\{ i \in \{1, \ldots, k\} \mid g_i : X \rightarrow \mathbb{R} \text{ is the restriction to } X \right\}
\]

of an affine function \( H_i : \mathbb{R}^n \rightarrow \mathbb{R} \)

and

\[
N_a := \{1, \ldots, k\} \setminus L_a.
\]

Let us consider the optimization problem

\[
(P_a) \quad \inf_{x \in A_a} f(x),
\]

\[
A_a = \left\{ x \in X : g(x) \leq 0 \right\},
\]

and its well-known Lagrange dual

\[
(D_a) \quad \sup_{t \in \mathbb{R}_+^k} \inf_{x \in X} \left\{ f(x) + t^T g(x) \right\},
\]

where \( f : X \rightarrow \mathbb{R} \) and \( g : X \rightarrow \mathbb{R}^k \) are functions.

The next theorem gives us the strong Lagrange duality for the problems \((P_a)\) and \((D_a)\).
2.2 RELATIONS BETWEEN THE DUALS’ OBJECTIVE VALUES

Theorem 2.4 (cf. Theorem 5.7 in [15]) Assume that $X$ is a nonempty convex subset of $\mathbb{R}^n$ and $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}^k$ are convex functions. If $v(P_a) > -\infty$, and the constraint qualification (CQ$_a$) is fulfilled, then it holds

$$v(P_a) = v(D_a)$$

and the dual problem $(D_a)$ has a solution.

2.2.2 The equivalence of the dual problems $(D_L)$ and $(D_{FL})$

In this subsection we assume that $X$ is a convex subset, $F_i : X \to \mathbb{R}$, $i = 1, \ldots, m$, $G_j : X \to \mathbb{R}$, $j = 1, \ldots, l$, are convex functions and $f : \mathbb{R}^m \to \mathbb{R}$, $g_i : \mathbb{R}^l \to \mathbb{R}$, $i = 1, \ldots, k$, are convex and componentwise increasing functions. Under these hypotheses we prove that the optimal objective values of the Lagrange and the Fenchel-Lagrange dual problems are equal. According to Proposition 2.3 in this case the dual $(D_{FL})$ becomes (cf. (2. 12))

$$(D_{FL}) \sup_{p \in \mathbb{R}^m, q \in \mathbb{R}^n, t \in \mathbb{R}^k} \left\{-f^*(q) - (t^T g)^*(q') - (q^T F)_X^*(p) - (q^T G)_X^*(-p) \right\}.$$

Theorem 2.5 Assume that $X \subseteq \mathbb{R}^n$ is a nonempty convex subset, $F_i : X \to \mathbb{R}$, $i = 1, \ldots, m$, $G_j : X \to \mathbb{R}$, $j = 1, \ldots, l$, are convex functions and $f : \mathbb{R}^m \to \mathbb{R}$, $g_i : \mathbb{R}^l \to \mathbb{R}$, $i = 1, \ldots, k$, are convex and componentwise increasing functions. Then it holds

$$v(D_L) = v(D_{FL}).$$

Proof. Let $t \in \mathbb{R}^k$. By using the extended functions $\tilde{F}$ and $\tilde{G}$, introduced at the beginning of this section, the infimum in the expression of the Lagrange dual is rewrites as

$$\inf_{x \in X} \left\{f(F(x)) + t^T g(G(x)) \right\} = \inf_{x \in \mathbb{R}^n} \left\{f \left(\tilde{F}(x)\right) + t^T g \left(\tilde{G}(x)\right) \right\} =$$

$$\inf_{x \in \mathbb{R}^n} \left\{ \left(f \circ \tilde{F}\right)(x) + \left(t^T g \circ \tilde{G}\right)(x) \right\} = - \left(f \circ \tilde{F} + t^T g \circ \tilde{G}\right)^*(0).$$

Because $ri \left(\text{dom} \left(f \circ \tilde{F}\right)\right) \cap ri \left(\text{dom} \left(t^T g \circ \tilde{G}\right)\right) = ri(X) \neq \emptyset$ and $f \circ \tilde{F}$, $t^T g \circ \tilde{G}$ are convex functions (cf. Proposition 2.4 and Proposition 2.5), Theorem 2.2 implies the existence of an element $\tilde{p} \in \mathbb{R}^n$ such that

$$- \left(f \circ \tilde{F} + t^T g \circ \tilde{G}\right)^*(0) = - \inf_{p \in \mathbb{R}^n} \left\{ \left(f \circ \tilde{F}\right)^*(p) + \left(t^T g \circ \tilde{G}\right)^*(-p) \right\}$$

$$= - \left(f \circ \tilde{F}\right)^*(\tilde{p}) - \left(t^T g \circ \tilde{G}\right)^*(-\tilde{p}). \quad (2. 16)$$
Furthermore, since \( \tilde{F} \left( \bigcap_{i=1}^{m} \text{dom} \left( \tilde{F}_i \right) \right) \cap \text{int}(\text{dom}(f)) = F(X) \cap \mathbb{R}^m \neq \emptyset \) and \( \tilde{G} \left( \bigcap_{j=1}^{l} \text{dom} \left( \tilde{G}_j \right) \right) \cap \text{int}(\text{dom}(t^Tg)) = G(X) \cap \mathbb{R}^l \neq \emptyset \), by Theorem 2.3, there exist some elements \( \bar{q} \in \mathbb{R}^m_+ \) and \( \bar{q}' \in \mathbb{R}^l_+ \) such that
\[
\left( f \circ \tilde{F} \right)^* (\bar{p}) = \inf_{q \in \mathbb{R}^m_+} \left\{ f^*(q) + \left( q^T \tilde{F} \right)^* (\bar{p}) \right\} = \left( q^T \tilde{F} \right)^* (\bar{p}) \quad (2.17)
\]
and
\[
\left( t^Tg \circ \tilde{G} \right)^* (-\bar{p}) = \inf_{q' \in \mathbb{R}^l_+} \left\{ \left( t^T g \right)^* (q') + \left( q'^T \tilde{G} \right)^* (-\bar{p}) \right\} \\
= \left( t^T g \right)^* (q') + \left( q'^T \tilde{G} \right)^* (-\bar{p}). \quad (2.18)
\]
Finally, the relations (2.16), (2.17) and (2.18) give us
\[
\inf_{x \in X} \left\{ f(F(x)) + t^Tg(G(x)) \right\} = -f^*(\bar{q}) - \left( t^T g \right)^* (q') - \left( q^T F \right)^* (\bar{p}) - \left( q^T G \right)^* (\bar{p}),
\]
which implies that \( v(D_L) = v(D_{FL}) \).

**Remark 2.6** We denoted here by \( \text{ri}(\mathcal{M}) \) the relative interior of a set \( \mathcal{M} \) and by \( \text{dom}(h) = \{ x \in \mathbb{R}^n : h(x) < +\infty \} \) the effective domain of a function \( h : \mathbb{R}^n \to \mathbb{R} \).

### 2.2.3 The equivalence of the dual problems \((D_F)\) and \((D_{FL})\)

The aim of this section is to investigate some sufficient conditions in order to ensure the equality between the optimal objective values of the duals \((D_F)\) and \((D_{FL})\), i.e. their equivalence.

Therefore we consider a constraint qualification, but first, let us divide the index set \( \{1, \ldots, k\} \) into two subsets,
\[
L := \left\{ i \in \{1, \ldots, k\} \mid g_i \circ G : X \to \mathbb{R} \text{ is the restriction to } X \text{ of an affine function } H_i : \mathbb{R}^n \to \mathbb{R} \right\}
\]
and \( N := \{1, \ldots, k\} \setminus L \). The constraint qualification follows
\[
(CQ) \quad \exists x' \in \text{ri}(X) : \left\{ \begin{array}{l} g_i(G(x')) \leq 0, \quad i \in L, \\ g_i(G(x')) < 0, \quad i \in N. \end{array} \right.
\]

Next we assume that the constraint qualification \((CQ)\) is fulfilled and, moreover, that \( X \) is a convex set, \( G_j : X \to \mathbb{R}, \ j = 1, \ldots, l \), are convex functions and that \( g_i : \mathbb{R}^l \to \mathbb{R}, \ i = 1, \ldots, k \), are convex and componentwise increasing functions.
2.2 RELATIONS BETWEEN THE DUALS’ OBJECTIVE VALUES

These will imply the equality of the optimal objective values of \((D_F)\) and \((D_{FL})\). Let us mention that under these hypotheses \((D_{FL})\) becomes (cf. Proposition 2.3)

\[
(D_{FL}) \sup_{p \in \mathbb{R}^n, q \in \mathbb{R}^m, q' \in \mathbb{R}^l_+, t \in \mathbb{R}^k_+} \left\{ -f^*(q) - (t^T g)^*(q) - (q^T F)^*_X(p) - (q^T G)^*_X(-p) \right\}.
\]

**Theorem 2.6** Assume that \(X \subseteq \mathbb{R}^n\) is a nonempty convex subset, \(G_j : X \to \mathbb{R}, j = 1, \ldots, l\), are convex functions, \(g_i : \mathbb{R}^l \to \mathbb{R}, i = 1, \ldots, k\), are convex and componentwise increasing functions and the constraint qualification \((CQ)\) is fulfilled. Then it holds

\[
v(D_F) = v(D_{FL}).
\]

**Proof.** Let \(p \in \mathbb{R}^n\) be arbitrary. If \(\inf_{x \in A} p^T x = -\infty\), then the relation

\[
\inf_{x \in A} p^T x = \sup_{t \in \mathbb{R}^k_+} \inf_{x \in X} \{ p^T x + t^T g(G(x)) \}
\]

holds trivially (the right hand side is smaller than or equal to the left hand side). Else, we can apply Theorem 2.4 (with \(f(x) := p^T x\)) and the equality holds either.

Now, let \(\inf_{x \in A} p^T x\) be finite. By Theorem 2.4 there is

\[
\inf_{x \in A} p^T x = \sup_{t \in \mathbb{R}^k_+} \inf_{x \in X} \{ p^T x + t^T g(G(x)) \}
\]

and the supremum is attained. Applying again Theorem 2.2 it follows that

\[
\inf_{x \in X} \{ p^T x + (t^T g \circ G)(x) \} = \inf_{x \in \mathbb{R}^n} \{ p^T x + (t^T g \circ \tilde{G})(x) \} =
\]

\[
- (p, \cdot)^* + t^T g \circ \tilde{G}^*(-u) = - \inf_{u \in \mathbb{R}^n} \left\{ (p, \cdot)^*(u) + (t^T g \circ \tilde{G})^*(-u) \right\},
\]

where the infimum is attained. We use here the usual notation \((p, x) := p^T x\). On the other hand, Theorem 2.3 gives us

\[
(t^T g \circ \tilde{G})^*(-u) = \inf_{q' \in \mathbb{R}^l_+} \left\{ (t^T g)^*(q') + (q^T \tilde{G})^*(-u) \right\},
\]

where the infimum is attained, and so

\[
\inf_{x \in A} p^T x = \sup_{u \in \mathbb{R}^n, q' \in \mathbb{R}^l_+, t \in \mathbb{R}^k_+} \left\{ -(p, \cdot)^*(u) - (t^T g)^*(q') - (q^T \tilde{G})^*(-u) \right\}.
\]

Since

\[
(p, \cdot)^*(u) = \begin{cases} 0, & \text{if } u = p, \\ +\infty, & \text{otherwise}, \end{cases}
\]

it follows that

\[
\inf_{x \in A} p^T x = \inf_{u \in \mathbb{R}^n} \left\{ -(p, \cdot)^*(u) - (t^T g)^*(q') - (q^T \tilde{G})^*(-u) \right\}.
\]
and $-\delta_A^*(-p) = \inf_{x \in A} p^T x$ we have

$$-\delta_A^*(-p) = \sup_{q' \in \mathbb{R}^1_+, t \in \mathbb{R}^k_+} \left\{ -(t^T g)^*(q') - \left(q^T G\right)^*_X (-p) \right\}$$

$$= \sup_{q' \in \mathbb{R}^1_+, t \in \mathbb{R}^k_+} \left\{ -(t^T g)^*(q') - \left(q^T G\right)^*_X (-p) \right\}. \quad (2.19)$$

By adding $-f(q) - (q^T F)^*_X (p)$ to both sides of relation (2.19) and by taking the supremum over $p \in \mathbb{R}^n$ and $q \in \mathbb{R}^m$ we obtain

$$\sup_{p \in \mathbb{R}^n, q \in \mathbb{R}^m} \left\{ -f^*(q) - (q^T F)^*_X (p) - \delta_A^*(-p) \right\} =$$

$$\sup_{p \in \mathbb{R}^n, q \in \mathbb{R}^m, q' \in \mathbb{R}^1_+, t \in \mathbb{R}^k_+} \left\{ -f^*(q) - (t^T g)^*(q') - \left(q^T F\right)^*_X (p) - \left(q^T G\right)^*_X (-p) \right\},$$

which is nothing but $v(D_F) = v(D_{FL})$.

\[ \square \]

### 2.3 Strong duality and optimality conditions

#### 2.3.1 Strong duality for $(D_L)$, $(D_F)$ and $(D_{FL})$

In the previous subsections we have presented some conditions which ensure the equality of the optimal objective values between the Lagrange and the Fenchel-Lagrange and between the Fenchel and the Fenchel-Lagrange dual problems, respectively. Combining the hypotheses of Theorems 2.5 and Theorem 2.6 it follows the equality of the optimal objective values of these three duals. Under the same conditions it can be proved that the optimal objective values of the duals are also equal to $v(P)$. In case that $v(P)$ is finite, results the strong duality.

**Theorem 2.7** Assume that $X \subseteq \mathbb{R}^n$ is a nonempty convex subset, $F_i : X \to \mathbb{R}$, $i = 1, \ldots, m$, $G_j : X \to \mathbb{R}$, $j = 1, \ldots, l$, are convex functions, $f : \mathbb{R}^m \to \mathbb{R}$, $g_i : \mathbb{R}^l \to \mathbb{R}$, $i = 1, \ldots, k$, are convex, componentwise increasing functions and the constraint qualification (CQ) is fulfilled. Then it holds

$$v(P) = v(D_L) = v(D_F) = v(D_{FL}).$$

Provided $v(P) > -\infty$, the duals have optimal solutions.

**Proof.** By Theorem 2.5 and Theorem 2.6 we obtain

$$v(D_L) = v(D_F) = v(D_{FL}). \quad (2.20)$$
Because \( A = \{ x \in X : g(G(x)) \leq 0 \} \neq \emptyset \), it holds \( v(P) \in [-\infty, +\infty) \). If \( v(P) = -\infty \), then the weak duality together with (2.20) give us
\[
v(D_L) = v(D_F) = v(D_{FL}) = -\infty = v(P).
\]

Suppose now that \( -\infty < v(P) < +\infty \). Because the constraint qualification \((CQ)\) is fulfilled, Theorem 2.4 states the existence of a \( \tilde{t} \in \mathbb{R}_+^k \) such that the strong Lagrange duality holds, namely
\[
v(P) = \sup_{t \in \mathbb{R}_+^k} \inf_{x \in X} \{ f(F(x)) + t^T g(G(x)) \}
= \inf_{x \in X} \{ f(F(x)) + \tilde{t}^T g(G(x)) \} = v(D_L). \tag{2.21}
\]

Therefore,
\[
v(P) = v(D_L) = v(D_F) = v(D_{FL}), \tag{2.22}
\]
and \( \tilde{t} \in \mathbb{R}_+^k \) is an optimal solution to the Lagrange dual \((D_L)\).

As in the proof of Theorem 2.5 we obtain easily that the infima in the relations (2.16), (2.17) and (2.18) are attained and so, there exist \( \tilde{p} \in \mathbb{R}^n, \tilde{q} \in \mathbb{R}_+^m \) and \( \tilde{q'} \in \mathbb{R}_+^l \) such that
\[
v(P) = \inf_{x \in X} \{ f(F(x)) + \tilde{t}^T g(G(x)) \}
= \sup_{p \in \mathbb{R}^n, q \in \mathbb{R}_+^m, q' \in \mathbb{R}_+^l} \{ -f^*(\tilde{q}) - (\tilde{t}^T g)^*(q') - (q'T F)^*_X (p) - (q'T G)^*_X (\tilde{p}) \}
= -f^*(\tilde{q}) - (\tilde{t}^T g)^*(q') - (q'T F)^*_X (\tilde{p}) - (q'T G)^*_X (\tilde{p}) = v(D_{FL}).
\]

Therefore \((\tilde{p}, \tilde{q}, \tilde{q'}, \tilde{t})\) is an optimal solution to \((D_{FL})\).
It remains to show that \((\tilde{p}, \tilde{q})\) is actually an optimal solution to the Fenchel dual \((D_F)\). The relations (2.14) and (2.22) imply that
\[
v(D_{FL}) = -f^*(\tilde{q}) - (\tilde{t}^T g)^*(q') - (q'T F)^*_X (\tilde{p}) - (q'T G)^*_X (\tilde{p})
\leq -f^*(\tilde{q}) - (q'T F)^*_X (\tilde{p}) - \delta_A(\tilde{p})
\leq \sup_{p \in \mathbb{R}^n, q \in \mathbb{R}_+^m} \{ -f^*(q) - (q'T F)^*_X (p) - \delta_A(-p) \} = v(D_F) \leq v(P),
\]
and so, because of \( v(P) = v(D_{FL}) = v(D_F), \) there is
\[
v(P) = -f^*(\tilde{q}) - (\tilde{t}^T g)^*(q') - (q'T F)^*_X (\tilde{p}) - (q'T G)^*_X (\tilde{p})
= -f^*(\tilde{q}) - (q'T F)^*_X (\tilde{p}) - \delta_A(-\tilde{p}) = v(D_F),
\]
which states that \((\tilde{p}, \tilde{q})\) is an optimal solution to \((D_F)\). □
2.3.2 Optimality conditions

In what follows we present for each of the three presented dual problems \((D_L), (D_F)\) and \((D_{FL})\) the necessary and sufficient optimality conditions for the primal and the dual problems. Let us begin with the optimality conditions that are based on the Lagrange dual.

**Theorem 2.8** (a) Let the assumptions of Theorem 2.7 be fulfilled and let \(x\) be an optimal solution to \((P)\). Then there exists an element \(\bar{t} \in \mathbb{R}^k_+\), optimal solution to \((D_L)\), such that the following optimality conditions are satisfied

\[
(i) \quad f(F(\bar{x})) = \inf_{x \in X} \{ f(F(x)) + \bar{t}^T g(G(x)) \},
(ii) \quad \bar{t}^T g(G(\bar{x})) = 0.
\]

(b) Let \(\bar{x}\) be admissible to \((P)\) and \(\bar{t}\) be admissible to \((D_L)\), satisfying (i) and (ii). Then \(\bar{x}\) is an optimal solution to \((P)\), \(\bar{t}\) is an optimal solution to \((D_L)\) and strong duality holds.

**Proof.**

(a) By Theorem 2.7, there exists an element \(\bar{t} \in \mathbb{R}^k_+\), optimal solution to \((D_L)\), such that

\[
f(F(\bar{x})) = v(P) = v(D_L) = \inf_{x \in X} \{ f(F(x)) + \bar{t}^T g(G(x)) \}. \tag{2.23}
\]

As one may see, the equality (2.23) is equivalent to the following one

\[
f(F(\bar{x})) + \bar{t}^T g(G(\bar{x})) - \inf_{x \in X} \{ f(F(x)) + \bar{t}^T g(G(x)) \} - \bar{t}^T g(G(\bar{x})) = 0. \tag{2.24}
\]

\(\bar{x}\) and \(\bar{t}\) being admissible to \((P)\) and \((D_L)\), respectively, it follows that \(\bar{t}^T g(G(\bar{x})) \leq 0\), and, because \(f(F(\bar{x})) + \bar{t}^T g(G(\bar{x})) - \inf_{x \in X} \{ f(F(x)) + \bar{t}^T g(G(x)) \} \geq 0\), equation (2.24) implies relations (i) and (ii).

(b) By (i) and (ii), we obtain that

\[
v(D_L) \geq \inf_{x \in X} \{ f(F(x)) + \bar{t}^T g(G(x)) \} = f(F(\bar{x})) \geq v(P),
\]

which together with Theorem 2.1 assures the strong duality between \((P)\) and \((D_L)\).

In the following theorem we formulate the optimality conditions based on the Fenchel dual problem.
2.3 STRONG DUALITY AND OPTIMALITY CONDITIONS

**Theorem 2.9** (a) Let the assumptions of Theorem 2.7 be fulfilled and let $\bar{x}$ be an optimal solution to $(P)$. Then there exists a tuple $(\bar{p}, \bar{q}) \in \mathbb{R}^n \times \mathbb{R}^m$, optimal solution to $(D_F)$, such that the following optimality conditions are satisfied

\[(i) \quad f(F(\bar{x})) + f^*(\bar{q}) = \bar{q}^T F(\bar{x}),\]
\[(ii) \quad \bar{q}^T F(\bar{x}) + (\bar{q}^T F)^*_X(\bar{p}) = \bar{p}^T \bar{x},\]
\[(iii) \quad \delta_A^*(-\bar{p}) = -\bar{p}^T \bar{x}.\]

(b) Let $x$ be admissible to $(P)$ and $(\bar{p}, \bar{q})$ be admissible to $(D_F)$, satisfying $(i)$, $(ii)$ and $(iii)$. Then $\bar{x}$ is an optimal solution to $(P)$, $(\bar{p}, \bar{q})$ is an optimal solution to $(D_F)$ and strong duality holds.

**Proof.**

(a) Analogously to the proof above, by Theorem 2.7, there exists a tuple $(\bar{p}, \bar{q}) \in \mathbb{R}^n \times \mathbb{R}^m$, optimal solution to $(D_F)$, such that

\[f(F(\bar{x})) = v(P) = v(D_F) = -f^*(\bar{q}) - (\bar{q}^T F)^*_X(\bar{p}) - \delta_A^*(-\bar{p}). \quad (2.25)\]

This equality is equivalent to

\[f(F(\bar{x})) + f^*(\bar{q}) - \bar{q}^T F(\bar{x}) + \bar{q}^T F(\bar{x}) + (\bar{q}^T F)^*_X(\bar{p}) - \bar{p}^T \bar{x} + \bar{p}^T \bar{x} + \delta_A^*(-\bar{p}) = 0. \quad (2.26)\]

Because of the Young-Fenchel inequality, which is expressing that for a function $h : \mathbb{R}^m \to \mathbb{R}$,

\[h(x) + h^*(x^*) \geq x^T x, \text{ for all } x \in \mathbb{R}^m, \quad (2.27)\]

and in case $h : X \to \mathbb{R}$, with $X \subseteq \mathbb{R}^m$,

\[h(x) + h^*_X(x^*) \geq x^T x, \text{ for all } x \in X, \quad (2.28)\]

we have

\[f(F(\bar{x})) + f^*(\bar{q}) - \bar{q}^T F(\bar{x}) \geq 0 \quad (2.29)\]

and

\[\bar{q}^T F(\bar{x}) + (\bar{q}^T F)^*_X(\bar{p}) - \bar{p}^T \bar{x} \geq 0. \quad (2.30)\]

Because $\bar{p}^T \bar{x} + \delta_A^*(-\bar{p}) \geq 0$, equality (2.26) together with relations (2.29) and (2.30) imply the optimality conditions $(i)$, $(ii)$ and $(iii)$.

(b) By $(i)$, $(ii)$ and $(iii)$ we obtain first equation (2.26) and then by means of Theorem 2.1 the equation (2.25) which proves the assertion. \qed

The last theorem of this subsection gives us the optimality conditions using the Fenchel-Lagrange dual problem.
Theorem 2.10 (a) Let the assumptions of Theorem 2.7 be fulfilled and let \( \bar{x} \) be an optimal solution to (P). Then there exists a tuple \((\bar{p}, \bar{q}, \bar{q}', \bar{t}) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \times \mathbb{R}^k \), solution to \((D_{FL})\), such that the following optimality conditions are satisfied

\begin{align*}
(i) \quad f(F(\bar{x})) + f^*(\bar{q}) &= \bar{q}^T F(\bar{x}), \\
(ii) \quad \bar{q}^T F(\bar{x}) + (\bar{q}^T F)^*_X(\bar{p}) &= \bar{p}^T \bar{x}, \\
(iii) \quad \bar{t}^T g(G(\bar{x})) + (\bar{t}^T g)^*(\bar{q}') &= \bar{q}^T G(\bar{x}), \\
(iv) \quad \bar{q}^T G(\bar{x}) + (\bar{q}^T G)^*_X(-\bar{p}) &= (-\bar{p})^T \bar{x}, \\
(v) \quad \bar{t}^T g(G(\bar{x})) &= 0.
\end{align*}

(b) Let \( \bar{x} \) be admissible to (P) and \((\bar{p}, \bar{q}, \bar{q}', \bar{t}) \) be admissible to \((D_{FL})\), satisfying (i), (ii), (iii), (iv) and (v). Then \( \bar{x} \) is an optimal solution to (P), \((\bar{p}, \bar{q}, \bar{q}', \bar{t}) \) is an optimal solution to \((D_{FL})\) and strong duality holds.

Proof.

(a) By Theorem 2.7, there exists a tuple \((\bar{p}, \bar{q}, \bar{q}', \bar{t}) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \times \mathbb{R}^k \), solution to \((D_{FL})\), such that

\[
f(F(\bar{x})) = v(P) = v(D_{FL}) = -f^*(\bar{q}) - (\bar{t}^T g)^*(\bar{q}') - (\bar{q}^T F)^*_X(\bar{p}) - (\bar{q}^T G)^*_X(-\bar{p}) \tag{2.31}
\]

Equality (2.31) is equivalent to

\[
\left\{ f(F(\bar{x})) + f^*(\bar{q}) - \bar{q}^T F(\bar{x}) \right\} + \left\{ \bar{t}^T g(G(\bar{x})) + (\bar{t}^T g)^*(\bar{q}') - \bar{q}^T G(\bar{x}) \right\} + \left\{ \bar{q}^T F(\bar{x}) + (\bar{q}^T F)^*_X(\bar{p}) - \bar{p}^T \bar{x} \right\} + \left\{ \bar{q}^T G(\bar{x}) + (\bar{q}^T G)^*_X(-\bar{p}) - (-\bar{p})^T \bar{x} \right\} + \left\{ -\bar{t}^T g(G(\bar{x})) \right\} = 0. \tag{2.32}
\]

According to the Young-Fenchel inequality

\[
f(F(\bar{x})) + f^*(\bar{q}) - \bar{q}^T F(\bar{x}) \geq 0,
\]

\[
\bar{t}^T g(G(\bar{x})) + (\bar{t}^T g)^*(\bar{q}') - \bar{q}^T G(\bar{x}) \geq 0,
\]

\[
\bar{q}^T F(\bar{x}) + (\bar{q}^T F)^*_X(\bar{p}) - \bar{p}^T \bar{x} \geq 0,
\]

\[
\bar{q}^T G(\bar{x}) + (\bar{q}^T G)^*_X(-\bar{p}) - (-\bar{p})^T \bar{x} \geq 0,
\]

and because \( \bar{t} \in \mathbb{R}_+^k \) and \( \bar{x} \in A \), it follows that \(-\bar{t}^T g(G(\bar{x})) \geq 0\), and so, equation (2.32) together with the inequalities from above implies relations (i), (ii), (iii), (iv), and (v).

(b) By (i), (ii), (iii), (iv) and (v), we obtain that

\[
v(D_{FL}) \geq -f^*(\bar{q}) - (\bar{t}^T g)^*(\bar{q}') - (\bar{q}^T F)^*_X(\bar{p}) - (\bar{q}^T G)^*_X(-\bar{p}) = f(F(\bar{x})) \geq v(P),
\]

which together with Theorem 2.1 assures the strong duality between (P) and \((D_{FL})\). \(\square\)
2.4 Special cases

In the last part of this chapter we intend to investigate some special cases of the original problem \((P)\) and its duals and show how the duality concepts introduced above generalize some results obtained in the past.

2.4.1 The classical optimization problem with inequality constraints and its dual problems

Let \(X \subseteq \mathbb{R}^n\) be a nonempty set and \(F : X \to \mathbb{R}\), \(G = (G_1, \ldots, G_k)^T\), \(G_i : X \to \mathbb{R}\), \(i = 1, \ldots, k\), be given functions. We consider the constrained optimization problem

\[
(P') \inf_{x \in A'} F(x),
\]

where

\[
A' = \left\{ x \in X : G(x) \leq 0 \right\}.
\]

One may observe that \((P')\) is a particular case of the original problem \((P)\), that means, it can be obtained from \((P)\) by taking the functions \(f : \mathbb{R} \to \mathbb{R}\), \(F : X \to \mathbb{R}\); \(G = (G_1, \ldots, G_k)^T : X \to \mathbb{R}^k\) and \(g = (g_1, \ldots, g_k)^T : \mathbb{R}^k \to \mathbb{R}^k\), such that \(f(x) = x\) for all \(x \in \mathbb{R}\) and \(g_i(y) = y_i\) for all \(y \in \mathbb{R}^k\) and \(i = 1, \ldots, k\). Let us notice that \(f\) and \(g_i\), \(i = 1, \ldots, k\), are convex and componentwise increasing functions. In what follows, by deriving from the duals introduced for \((P)\) corresponding dual problems for \((P')\), we present how the results obtained in the previous subsections can be applied in this case.

Because of

\[
f^*(q) = \sup_{x \in \mathbb{R}} \{q^T x - f(x)\} = \sup_{x \in \mathbb{R}} \{(q - 1)x\} = \begin{cases} 0, & \text{if } q = 1, \\ +\infty, & \text{otherwise,} \end{cases} \quad (2.33)
\]

\[
(t^T g)^* (q') = \sup_{y \in \mathbb{R}^k} \{q'^T y - t^T g(y)\} = \sup_{y \in \mathbb{R}^k} \{q'^T y - t^T y\} = \sup_{y \in \mathbb{R}^k} \{(q' - t)^T y\} = \begin{cases} 0, & \text{if } q' = t, \\ +\infty, & \text{otherwise}, \end{cases} \quad (2.34)
\]

and

\[
\left(q'^T G \right)^*_X (-p) = (t^T G)^*_X (-p) = \sup_{x \in X} \left\{-p^T x - t^T G(x)\right\} = - \inf_{x \in X} \left\{p^T x + t^T G(x)\right\}, \quad (2.35)
\]

the three dual problems turn out to be

\[
(D'_L) \sup_{t \in \mathbb{R}^k} \inf_{x \in X} \left\{F(x) + t^T G(x)\right\}, \quad (2.36)
\]
\( (D'_F) \sup_{p \in \mathbb{R}^n} \{ -F^*_X(p) - \delta^*_\mathcal{X}(-p) \}, \) \hfill (2.37)

and

\( (D'_{FL}) \sup_{p \in \mathbb{R}^n, t \in \mathbb{R}^k} \left\{ -F^*_X(p) + \inf_{x \in \mathcal{X}} \{ p^T x + t^T G(x) \} \right\}. \) \hfill (2.38)

We note that the constraint qualification \((CQ)\) becomes in this case

\[ (CQ') \exists \, x' \in r(X) : \begin{cases} G_i(x') \leq 0, \quad i \in L, \\ G_i(x') < 0, \quad i \in N, \end{cases} \]

where

\[ L := \left\{ i \in \{1, \ldots, k\} \mid G_i : X \to \mathbb{R} \text{ is the restriction to } X \text{ of an} \right. \]

\[ \left. \text{affine function } H_i : \mathbb{R}^n \to \mathbb{R} \right\} \]

and \( N := \{1, \ldots, k\} \setminus L. \)

The theorems 2.5, 2.6 and 2.7 turn out to be the following results.

**Theorem 2.11** Assume that \( X \subseteq \mathbb{R}^n \) is a nonempty convex subset and \( F : X \to \mathbb{R}, \ G_j : X \to \mathbb{R}, \ j = 1, \ldots, k, \) are convex functions. Then it holds

\[ v(D'_L) = v(D'_{FL}). \]

**Theorem 2.12** Assume that \( X \subseteq \mathbb{R}^n \) is a nonempty convex subset, \( G_j : X \to \mathbb{R}, \ j = 1, \ldots, k, \) are convex functions and the constraint qualification \((CQ')\) is fulfilled. Then it holds

\[ v(D'_F) = v(D'_{FL}). \]

**Theorem 2.13** Assume that \( X \subseteq \mathbb{R}^n \) is a nonempty convex subset, \( F : X \to \mathbb{R}, \ G_j : X \to \mathbb{R}, \ j = 1, \ldots, k, \) are convex functions and the constraint qualification \((CQ')\) is fulfilled. Then it holds

\[ v(P') = v(D'_L) = v(D'_F) = v(D'_{FL}). \]

Provided \( v(P') > -\infty, \) the duals have optimal solutions.

The following results, derived from the theorems 2.8, 2.9 and 2.10, respectively, provide us the necessary and sufficient optimality conditions for the primal and the corresponding dual problems. Let us start with the optimality conditions coming from the Lagrange dual \((D'_L).\)
Theorem 2.14 (a) Let the assumptions of Theorem 2.13 be fulfilled and let $\bar{x}$ be an optimal solution to $(P')$. Then there exists an element $\bar{t} \in \mathbb{R}^k_+$, optimal solution to $(D'_{L})$, such that the following optimality conditions are satisfied

(i) $F(\bar{x}) = \inf_{x \in X} \left\{ F(x) + \bar{t}^T G(x) \right\},$

(ii) $\bar{t}^T G(\bar{x}) = 0.$

(b) Let $\bar{x}$ be admissible to $(P')$ and $\bar{t}$ be admissible to $(D'_{L})$, satisfying (i) and (ii). Then $\bar{x}$ is an optimal solution to $(P')$, $\bar{t}$ is an optimal solution to $(D'_{L})$ and strong duality holds.

The next theorem gives us the optimality conditions to be based on the Fenchel dual $(D'_{F})$.

Theorem 2.15 (a) Let the assumptions of Theorem 2.13 be fulfilled and let $\bar{x}$ be an optimal solution to $(P')$. Then there exists an element $\bar{p} \in \mathbb{R}^n$, optimal solution to $(D'_{F})$, such that the following optimality conditions are satisfied

(i) $F(\bar{x}) + F_x^*(\bar{p}) = \bar{p}^T \bar{x},$

(ii) $\delta^*_A(-\bar{p}) = -\bar{p}^T \bar{x}.$

(b) Let $\bar{x}$ be admissible to $(P')$ and $\bar{p}$ be admissible to $(D'_{F})$, satisfying (i) and (ii). Then $\bar{x}$ is an optimal solution to $(P')$, $\bar{p}$ is an optimal solution to $(D'_{F})$ and strong duality holds.

Finally, let us formulate the optimality conditions using the Fenchel-Lagrange dual $(D'_{FL})$.

Theorem 2.16 (a) Let the assumptions of Theorem 2.13 be fulfilled and let $\bar{x}$ be an optimal solution to $(P')$. Then there exists a tuple $(\bar{p}, \bar{t}) \in \mathbb{R}^n \times \mathbb{R}^k_+$, optimal solution to $(D'_{FL})$, such that the following optimality conditions are satisfied

(i) $F(\bar{x}) + F_x^*(\bar{p}) = \bar{p}^T \bar{x},$

(ii) $\inf_{x \in X} \left\{ \bar{p}^T x + \bar{t}^T G(x) \right\} = \bar{p}^T \bar{x},$

(iii) $\bar{t}^T G(\bar{x}) = 0.$

(b) Let $\bar{x}$ be admissible to $(P')$ and $(\bar{p}, \bar{t})$ be admissible to $(D'_{FL})$, satisfying (i), (ii) and (iii). Then $\bar{x}$ is an optimal solution to $(P')$, $(\bar{p}, \bar{t})$ is an optimal solution to $(D'_{FL})$ and strong duality holds.

Remark 2.7 The statements from above turn out to coincide with the results obtained by G. WANKA and R. I. BOT in [70].

Remark 2.8 In [4] R. I. BOT, G. KASSAY and G. WANKA gave some relations between the optimal objective values of $(D'_{L})$, $(D'_{F})$ and $(D'_{FL})$ as well as strong duality results for a class of generalized convex programming problems.
2.4.2 The optimization problem without constraints

Let $X$ be a nonempty subset of $\mathbb{R}^n$ and $F = (F_1, ..., F_m)^T$, $F_i : X \to \mathbb{R}$, $i = 1, ..., m$, be given functions. As a second special case of our original problem $(P)$, let us consider the non-constrained optimization problem

$$ (P'') \quad \inf_{x \in X} f(F(x)). $$

This problem was already treated in detail by R. I. Bot and G. Wanka in [5] and by G. Wanka, R. I. Bot and E. Vargyas in [72]. Our intention hereby is to show how the results obtained by the authors in the mentioned papers can be derived from the composed problem $(P)$. Therefore, let us observe that $(P'')$ can be directly obtained from $(P)$, by taking in the original problem the functions

$$ F = (F_1, ..., F_m)^T, \quad F_i : X \to \mathbb{R}, \quad i = 1, ..., m; \quad G = (G_1, ..., G_l)^T, \quad G_j : X \to \mathbb{R}, \quad j = 1, ..., l; \quad f : \mathbb{R}^m \to \mathbb{R}, \quad i = 1, ..., k, $$

such that $g_i(y) = 0$, $i = 1, ..., k$, for all $y \in \mathbb{R}^l$.

In order to deduce the results obtained by the authors in [5] and [72] we examine only the Fenchel-Lagrange dual problem

$$ (D_{FL}) \quad \sup_{p \in \mathbb{R}^n, q \in \mathbb{R}^m, q' \in \mathbb{R}^l, t \in \mathbb{R}_+^k} \left\{ -f^*(q) - (t^T g)^* (q') - (q^T F)^* (p) - (q^T G)^* (-p) \right\}. $$

Because of

$$ (t^T g)^* (q') = (0)^* (q') = \sup_{y \in \mathbb{R}^l} \left\{ y^T q' \right\} = \begin{cases} 0, & \text{if } q' = 0, \\ +\infty, & \text{otherwise}, \end{cases} $$

and $0^*_X (-p) = - \inf_{x \in X} p^T x = \delta^*_X (-p)$, the Fenchel-Lagrange dual problem becomes

$$ (D''_{FL}) \quad \sup_{p \in \mathbb{R}^n, q \in \mathbb{R}^m} \left\{ -f^*(q) - (q^T F)^* (p) - \delta^*_X (-p) \right\}. $$

Let us give now the strong duality theorem and the optimality conditions for $(P'')$ and its Fenchel-Lagrange dual $(D''_{FL})$.

**Theorem 2.17** Assume that $X \subseteq \mathbb{R}^n$ is a nonempty convex subset, $f : \mathbb{R}^m \to \mathbb{R}$ is a convex and componentwise increasing function, $F = (F_1, ..., F_m)^T$, and $F_i : X \to \mathbb{R}$, $i = 1, ..., m$, are convex functions. Then it holds

$$ v(P'') = v(D''_{FL}). $$

Provided that $v(P'') > -\infty$, the strong duality holds, i.e. the optimal objective values of the primal and the dual problem coincide and the dual has an optimal solution.
Theorem 2.18  (a) Let the assumptions of Theorem 2.17 be fulfilled and let $\bar{x}$ be an optimal solution to $(P'')$. Then there exists a tuple $(\bar{p}, \bar{q}) \in \mathbb{R}^n \times \mathbb{R}^m$, optimal solution to $(D''_F)$, such that the following optimality conditions are satisfied

(i) \[ f(F(\bar{x})) + f^*(\bar{q}) = \bar{q}^T F(\bar{x}), \]

(ii) \[ \bar{q}^T F(\bar{x}) + (\bar{q}^T F)^*(\bar{p}) = \bar{p}^T \bar{x}, \]

(iii) \[ \delta_X^*(-\bar{p}) = -\bar{p}^T \bar{x}. \]

(b) Let $\bar{x}$ be admissible to $(P'')$ and $(\bar{p}, \bar{q})$ be admissible to $(D''_F)$, satisfying (i), (ii) and (iii). Then $\bar{x}$ is an optimal solution to $(P'')$, $(\bar{p}, \bar{q})$ is an optimal solution to $(D''_F)$ and strong duality holds.

Proof. The optimality conditions can be derived in this special case from Theorem 2.10. \qed
Chapter 3
Location problems

Location problems play an important role in a lot of fields of applications, as they appear in many areas such as transportation planning, industrial engineering, telecommunication, computer science, etc. The aims of these problems are to locate some items, to optimize transportation costs, to minimize covered distances and so on. A lot of research has been carried out in location analysis. Among the large number of papers and books dealing with them we mention [5], [11], [12], [17], [23], [24], [27], [41], [43], [45], [46], [50], [52], [56], [71] and [72].

The most common model is the classical single facility location problem which is concerned with finding of a point in a real normed space $X$ in order to minimize some function depending on the distances to a finite number of given points (existing facilities). When applying this model to real world problems, two primal questions arise:

1. What kind of distances should be used in the model?
2. Why do we have to consider points as existing facilities?

In general, to determine the distances different kinds of norms are used. For the existing facilities instead of points one could consider set of points, but in this case one cannot use anymore the natural distance induced by a norm. Therefore, a new decision has to be made before, namely, one measures the distances to the closest points in the sets. For this distance-interpretation one has to consider the concept of infimal distances to sets, so-called gauges. In the past most of the references concerning location problems have considered distances induced by norms, but recently there have been published some papers that consider the use of gauges. This approach has the advantage that it leads to more general models, for example to model situations where the symmetry property of a norm does not make sense. For an overview on the location of extensive facilities see [17], [45], [56] and [65].

Through this work we use the distance-interpretation from above, but we mention that in the literature there are also other ones. For instance, interpretations which take into account the average behavior, so that any point in the set
is visited according to a probability distribution. For a larger review of these see [11] and [51].

3.1 Duality for location problems

3.1.1 Motivation

Although many papers on location problems have been published, there are only a few which treat these problems via duality, most of them being concerned with a geometrical characterization of the set of optimal solutions. Our purpose in this section is to show the usefulness of the conjugate duality in location theory. In order to do this we consider first a quite general location problem, where the distances are given by monotonic gauges. Using some results of the previous chapter, we construct a dual problem to it, prove the strong duality between them and give the optimality conditions.

As known, under certain conditions, the gauges turn out to be norms and because the problems where the distances are given by several norms play an important role in location analysis, we study also the problem with monotonic norms.

The last part of this chapter was actually inspired by a paper of Y. HINOJOSA and J. PUERTO [27], in which the authors introduced a location problem, where the distances were measured by gauges of closed (not necessarily bounded) convex sets. For this problem the authors obtained a geometrical characterization of the set of optimal solutions and gave some methods to solve it. Finding out that this problem can be embedded in our general location model, we solve it via duality. Finally, as applications of it, the Weber and minmax problems with gauges of closed convex sets are considered.

3.1.2 Notations and preliminaries

In this first section we provide some definitions and preliminary results that we shall use in the sequel.

**Definition 3.1** Let $C \subseteq \mathbb{R}^m$ be a closed convex set containing the origin. The function $\gamma_C$ defined by

$$\gamma_C(x) := \inf \left\{ \alpha > 0 : x \in \alpha C \right\}$$

is called the gauge of $C$ (or the Minkowski functional associated to $C$). The set $C$ is called the unit ball associated with $\gamma_C$. As usual, we set $\gamma_C(x) := +\infty$, if there is no $\alpha > 0$ such that $x \in \alpha C$.

Recall that $\gamma_C$ is a monotonic gauge on $\mathbb{R}^m$ (cf. [2]), if

$$\forall u, v \in \mathbb{R}^m, \text{ s. t. } |u_i| \leq |v_i|, \; i, \ldots, m, \Rightarrow \gamma_C(u) \leq \gamma_C(v).$$
Definition 3.2 Let $C \subseteq \mathbb{R}^m$ be a closed convex set containing the origin. The set given by
\[ C^0 = \{ y \in \mathbb{R}^m : x^T y \leq 1, \ \forall x \in C \} \]
is called the polar set of $C$.

Remark 3.1 $C^0$ is a closed convex set containing the origin.

Definition 3.3 Let $C \subseteq \mathbb{R}^m$ be a convex set. The function $\sigma_C$ given by
\[ \sigma_C(y) := \sup \{ x^T y : x \in C \} \]
is called the support function of $C$.

Proposition 3.1 ([28]) Let $C$ be a closed convex set containing the origin. Then
(i) its gauge $\gamma_C$ is a non-negative closed sublinear function,
(ii) $\{ x \in \mathbb{R}^m : \gamma_C(x) \leq r \} = rC$, for all $r > 0$.

Proposition 3.2 ([28]) Let $C$ be a closed convex set containing the origin. Its gauge $\gamma_C$ is the support function of the set $C^0$, namely
\[ \gamma_C(x) = \sigma_{C^0}(x) = \sup \{ x^T y : y \in C^0 \}. \]

Lemma 3.1 ([28]) Let $C$ be a closed convex set containing the origin. Its support function $\sigma_C$ is the gauge of $C^0$ and is denoted by $\gamma_{C^0}$, i.e.
\[ \sigma_C(y) = \gamma_{C^0}(y) = \inf \{ \alpha > 0 : y \in \alpha C^0 \}. \]

Proposition 3.3 The conjugate function $\gamma^*_C : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{ +\infty \}$ of $\gamma_C$ verifies
\[ \gamma^*_C(y) = \begin{cases} 0, & \text{if } y \in C^0, \\ +\infty, & \text{otherwise}, \end{cases} \]
where $C^0$ is the polar set of $C$.

Proof. By the definition of the conjugate function of $\gamma_C(x)$ we get
\[
\gamma^*_C(y) = \sup_{x \in \mathbb{R}^m} \left\{ y^T \left( \frac{x}{\alpha} \right) - \gamma_C \left( \frac{x}{\alpha} \right) \right\} = \sup_{x \in \mathbb{R}^m} \left\{ y^T \left( \frac{x}{\alpha} \right) - \inf \left\{ \alpha > 0 : x \in \alpha C \right\} \right\} \\
= \sup_{x \in \mathbb{R}^m} \left\{ y^T x + \sup_{\alpha > 0, x \in \alpha C} (-\alpha) \right\} = \sup_{\alpha > 0, x \in \alpha C} \left\{ y^T x - \alpha \right\} = \sup_{\alpha > 0, z \in C} \left\{ y^T (\alpha z) - \alpha \right\} \\
= \sup_{\alpha > 0} \left\{ \sup_{z \in C} \left\{ y^T z - 1 \right\} \right\} = \begin{cases} 0, & \text{if } y \in C^0, \\ +\infty, & \text{otherwise}. \end{cases}
\]

Remark 3.2 By Proposition 3.1 and Remark 3.1 the fact that $y \in C^0$ is equivalent to the inequality $\gamma_{C^0}(y) \leq 1$, so, one can write
\[ \gamma^*_C(y) = \begin{cases} 0, & \text{if } \gamma_{C^0}(y) \leq 1, \\ +\infty, & \text{otherwise}. \end{cases} \]
3.1.3 The composed problem with monotonic gauges

Let us consider the following location problem

$$ (P^c) \inf_{x \in X} \gamma_C^+(F(x)), $$

where $X$ is a nonempty subset of $\mathbb{R}^n$, $\gamma_C : \mathbb{R}^m \to \mathbb{R}$ is a monotonic gauge of a closed convex set $C$ containing the origin, $\gamma_C^+ : \mathbb{R}^m \to \mathbb{R}$, $\gamma_C(t) := \gamma_C(t^+)$, with $t^+ = (t_1^+, \ldots, t_m^+)^T$ and $t_i^+ = \max\{0, t_i\}$, $i = 1, \ldots, m$, and $F = (F_1, \ldots, F_m)^T : X \to \mathbb{R}^m$ is a vector-valued function. As one can see, this problem is a particular case of the composed optimization problem $(P^m)$, which we studied at the end of the previous section. Before we construct a dual problem to it, let us formulate some properties of the function $\gamma_C^+$ and of its conjugate $\gamma_C^+*$.

**Proposition 3.4** The function $\gamma_C^+ : \mathbb{R}^m \to \mathbb{R}$ is convex and componentwise increasing.

**Proof.** First, let us point out that the function $(\cdot)^+ : \mathbb{R}^m \to \mathbb{R}_+^m$, defined by $t^+ = (t_1^+, \ldots, t_m^+)^T$ for $t \in \mathbb{R}^m$, is convex. This means that, for $u, v \in \mathbb{R}^m$ and $\alpha \in [0, 1]$, it holds

$$ (\alpha u + (1 - \alpha)v)^+ \leq \alpha u^+ + (1 - \alpha)v^+. $$

Here, $\leq$ is the ordering induced on $\mathbb{R}^m$ by the cone of non-negative elements $\mathbb{R}_+^m$. By the positive sublinearity and monotonicity of the gauge $\gamma_C$, we have for $u, v \in \mathbb{R}^m$ and $\alpha \in [0, 1]$, that

$$ \gamma_C^+(\alpha u + (1 - \alpha)v) = \gamma_C((\alpha u + (1 - \alpha)v)^+) \leq \gamma_C(\alpha u^+ + (1 - \alpha)v^+) \leq \alpha \gamma_C(u^+) + (1 - \alpha)\gamma_C(v^+) = \alpha \gamma_C^+(u) + (1 - \alpha)\gamma_C^+(v), $$

which means that the function $\gamma_C^+$ is convex.

In order to prove that $\gamma_C^+$ is componentwise increasing, let $u, v \in \mathbb{R}^m$ be such that $u_i \leq v_i$, $i = 1, \ldots, m$. It follows $u_i^+ \leq v_i^+$, which implies that $|u_i^+| \leq |v_i^+|$, $i = 1, \ldots, m$. $\gamma_C$ being a monotonic gauge, we have $\gamma_C(u^+) \leq \gamma_C(v^+)$, where $u^+ = (u_1^+, \ldots, u_m^+)^T$, $v^+ = (v_1^+, \ldots, v_m^+)^T$ or, equivalently, $\gamma_C^+(u) \leq \gamma_C^+(v)$.

Hence the function $\gamma_C^+$ is componentwise increasing. \qed

By the approach described in Chapter 2, the Fenchel-Lagrange dual problem to $(P^c)$ is

$$ (D^c_{F*}) \sup_{p \in \mathbb{R}^n, q \in \mathbb{R}^m} \left\{ -\langle \gamma_C^+ \rangle^*(q) - \langle q^T F \rangle_X^*(p) - \delta_X^*( -p) \right\}. $$

**Proposition 3.5** The conjugate function $\gamma_C^+ : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ of $\gamma_C^+$ verifies

$$ (\gamma_C^+)^*(q) = \begin{cases} 0, & \text{if } q \in \mathbb{R}_+^m \text{ and } \gamma_C^0(q) \leq 1, \\ +\infty, & \text{otherwise}, \end{cases} $$

where $\gamma_C^0$ is the gauge of the polar set $C^0$. 


3.1 THE CASE OF MONOTONIC GAUGES

Proof. For \( q \in \mathbb{R}^m \setminus \mathbb{R}_{+}^m \) the assertion is a consequence of Proposition 2.3 and Proposition 3.4.

Let \( q \in \mathbb{R}^m_+ \). For \( t \in \mathbb{R}^m \), we have \( |t_i| \geq |t_i^+|, i = 1, ..., m \), which implies that \( \gamma_C(t) \geq \gamma_C(t^+) = \gamma_C^+(t) \) and

\[
\gamma_C^+(q) = \sup_{t \in \mathbb{R}^m} \{ q^T t - \gamma_C(t) \} \leq \sup_{t \in \mathbb{R}^m} \{ q^T t - \gamma_C^+(t) \} = (\gamma_C^+)^*(q).
\]

On the other hand, for the conjugate of the gauge \( \gamma_C \) we have the following formula (see Remark 3.2)

\[
\gamma_C^*(q) = \sup_{t \in \mathbb{R}^m} \{ q^T t - \gamma_C(t) \} = \begin{cases} 
0, & \text{if } \gamma_C^0(q) \leq 1, \\
+\infty, & \text{otherwise}.
\end{cases}
\]

If \( \gamma_C^0(q) > 1 \), we have that \( +\infty = \gamma_C^+(q) \leq (\gamma_C^+)^*(q) \). From here, \( (\gamma_C^+)^*(q) = +\infty \). Let be now \( \gamma_C^0(q) \leq 1 \). Because \( q \geq 0 \), it follows that \( q^T t \leq q^T t^+ \), for every \( t \in \mathbb{R}^m \). Furthermore, by Proposition 3.1, from \( \gamma_C^0(q) \leq 1 \) it follows that \( q \in C^0 \) and then by Proposition 3.2 we obtain that \( q^T t^+ \leq \gamma_C(t^+) \). By these inequalities together with Proposition 3.3 we obtain for the conjugate function of \( \gamma_C^+ \)

\[
0 \leq \gamma_C^+(q) \leq (\gamma_C^+)^*(q) = \sup_{t \in \mathbb{R}^m} \{ q^T t - \gamma_C(t^+) \} \leq \sup_{t \in \mathbb{R}^m} \{ q^T t^+ - \gamma_C(t^+) \} \leq 0.
\]

Consequently, there is \((\gamma_C^+)^*(q) = 0\) and the proposition is proved. \( \square \)

By the proposition from above, the dual of \((PC)^c\) has the following formulation

\[
(D_{FL}^{\gamma_C}) \sup_{p \in \mathbb{R}^n, q \in \mathbb{R}^m_+} \{ - (q^T F)^*_{X} (p) - \delta_X^*(-p) \}, \quad (3.1)
\]

which is nothing but the dual problem obtained by the authors in [5] and [72] as a theoretical framework for some locations problems.

The following theorems provide us the strong duality and the optimality conditions for \((PC)^c\) and its Fenchel-Lagrangian dual \((D_{FL}^{\gamma_C})\).

**Theorem 3.1** Assume that \( X \subseteq \mathbb{R}^n \) is a nonempty convex subset, \( \gamma_C : \mathbb{R}^m \rightarrow \mathbb{R} \) is a monotonic gauge of a closed convex set \( C \) and \( F = (F_1, ..., F_m)^T, F_i : X \rightarrow \mathbb{R} \), \( i = 1, ..., m \), are convex functions. Then it holds

\[
v(PC) = v(D_{FL}^{\gamma_C}).
\]

Provided \( v(PC) > -\infty \), the dual has an optimal solution.

**Theorem 3.2** (a) Let the assumptions of Theorem 3.1 be fulfilled and let \( \bar{x} \) be an optimal solution to \((PC)^c\). Then there exists a tuple \((\bar{p}, \bar{q}) \in \mathbb{R}^n \times \mathbb{R}^m_+\), with
\(\gamma_{C}^{0}(\bar{q}) \leq 1\), optimal solution to \((D_{FL}^{C})\), such that the following optimality conditions are satisfied

\[
\begin{align*}
(i) \quad & \gamma_{C}^{+}(F(\bar{x})) = \bar{q}^{T}F(\bar{x}), \\
(ii) \quad & \bar{q}^{T}F(\bar{x}) + (\bar{q}^{T}F)_{X}^{*}(\bar{p}) = \bar{p}^{T}\bar{x}, \\
(iii) \quad & \delta_{X}^{*}(-\bar{p}) = -\bar{p}^{T}\bar{x}.
\end{align*}
\]

(b) Let \(\bar{x}\) be admissible to \((P^{C})\) and \((\bar{p}, \bar{q})\) be admissible to \((D_{FL}^{C})\), satisfying (i), (ii) and (iii). Then \(\bar{x}\) is an optimal solution to \((P^{C})\), \((\bar{p}, \bar{q})\) is an optimal solution to \((D_{FL}^{C})\) and strong duality holds.

**Proof.** The optimality conditions from above can be derived from Theorem 2.18, by means of Proposition 3.5. \(\square\)

### 3.1.4 The case of monotonic norms

In what follows, we consider the optimization problem in which the objective function is a composition of a monotonic norm with a vector function.

Let \(X\) be a nonempty subset of \(\mathbb{R}^{n}\), \(F = (F_1, ..., F_m)^{T} : X \rightarrow \mathbb{R}^{m}\) be a vector-valued function and \(l : \mathbb{R}^{m} \rightarrow \mathbb{R}\) be a monotonic norm on \(\mathbb{R}^{m}\) in the sense that \(l(u) \leq l(v)\) whenever \(|u_i| \leq |v_i|, i = 1, ..., m\). The problem which we consider here is the following one

\[
(P^l) \quad \inf_{x \in X} l^{+}(F(x))
\]

where \(l^{+} : \mathbb{R}^{m} \rightarrow \mathbb{R}, l^{+}(t) := l(t^{+}),\) with \(t^{+} = (t_1^{+}, ..., t_m^{+})^{T}\) and \(t_i^{+} = \max\{0, t_i\}, i = 1, ..., m\).

Analogously to Proposition 3.4, it can be proved the following proposition.

**Proposition 3.6** The function \(l^{+} : \mathbb{R}^{m} \rightarrow \mathbb{R}\) is convex and componentwise increasing.

**Proof.** See the proof of Proposition 3.4. \(\square\)

One may observe, that the results obtained in Subsection 2.4.2 can be used also in this case, which lead us to the following Fenchel-Lagrange dual problem

\[
(D_{FL}^{l}) \sup_{p \in \mathbb{R}^{n}, q \in \mathbb{R}^{m}} \left\{-l^{+}(q) - (q^{T}F)^{*}_{X}(p) - \delta_{X}^{*}(-p)\right\}.
\]

**Proposition 3.7** The conjugate function \((l^{+})^{*} : \mathbb{R}^{m} \rightarrow \mathbb{R} \cup \{+\infty\}\) of \(l^{+}\) verifies

\[
(l^{+})^{*}(q) = \begin{cases} 
0, & \text{if } q \in \mathbb{R}^{m}_{+} \text{ and } l^{0}(q) \leq 1, \\
+\infty, & \text{otherwise},
\end{cases}
\]

where \(l^{0}\) is the dual norm of \(l\).
3.1 THE MODEL WITH UNBOUNDED UNIT BALLS

Proof. See the proof of Proposition 3.5. □

By Proposition 3.7, the Fenchel-Lagrange dual becomes

$$(D_{FL})^* \sup_{p \in \mathbb{R}^n, q \in \mathbb{R}^m, \|q\| \leq 1} \{ - (q^T F)^* (p) - \delta^*_X(-p) \}.$$  (3.2)

Similarly to theorems 3.1 and 3.2, we have:

Theorem 3.3 Assume that $X \subseteq \mathbb{R}^n$ is a nonempty convex subset, $l : \mathbb{R}^m \to \mathbb{R}$ is a monotonic norm on $\mathbb{R}^m$ and $F = (F_1, \ldots, F_m)^T$, $F_i : X \to \mathbb{R}$, $i = 1, \ldots, m$, are convex functions. Then it holds

$$v(P^l) = v(D_{FL}^l).$$

Provided that $v(P^l) > -\infty$, the strong duality holds, i.e. the optimal objective values of the primal and the dual problem coincide and the dual has an optimal solution.

Theorem 3.4 (a) Let the assumptions of Theorem 3.3 be fulfilled and let $\bar{x}$ be an optimal solution to $(P^l)$. Then there exists a tuple $(\bar{p}, \bar{q}) \in \mathbb{R}^m \times \mathbb{R}_+^m$, with $\|\bar{q}\| \leq 1$, optimal solution to $(D_{FL}^l)$, such that the following optimality conditions are satisfied

\[
\begin{align*}
(i) & \quad l^+(F(\bar{x})) = \bar{q}^T F(\bar{x}), \\
(ii) & \quad \bar{q}^T F(\bar{x}) + (q^T F)^* (\bar{p}) = \bar{p}^T \bar{x}, \\
(iii) & \quad \delta_X^*(-\bar{p}) = -\bar{p}^T \bar{x}.
\end{align*}
\]

(b) Let $\bar{x}$ be admissible to $(P^l)$ and $(\bar{p}, \bar{q})$ be admissible to $(D_{FL}^l)$, satisfying (i), (ii) and (iii). Then $\bar{x}$ is an optimal solution to $(P^l)$, $(\bar{p}, \bar{q})$ is an optimal solution to $(D_{FL}^l)$ and strong duality holds.

3.1.5 The location model with unbounded unit balls

In this section we consider the single facility problem, treated by Y. HINOJOSA and J. PUERTO in [27], where gauges of closed convex sets are used to model distances.

Throughout this chapter let $\mathcal{F} := \{a^1, \ldots, a^m\}$ be a subset of $\mathbb{R}^n$ which represents the set of existing facilities. Each facility $a^i \in \mathcal{F}$ has an associated gauge $\varphi_{a^i}$, whose unit ball is a closed convex set $C_{a^i}$ containing the origin. Let $w = \{w_{a^1}, \ldots, w_{a^m}\}$ be a set of positive weights and let $\gamma_C : \mathbb{R}^m \to \mathbb{R}$ be a monotonic gauge of a closed convex set $C$ containing the origin. The distance from an existing facility $a^i \in \mathcal{F}$ to a new facility $x \in \mathbb{R}^n$ is given by $\varphi_{a^i}(x - a^i)$. By $\varphi_{a^i}^0$, we denote the gauge of the polar set $C_{a^i}^0$. 
The location problem studied in [27] is

\[
(P_{\text{NC}}(\mathcal{F})) \quad \inf_{x \in \mathbb{R}^n} \gamma_C(w_{a_1} \varphi_{a_1}(x - a_1), \ldots, w_{a_m} \varphi_{a_m}(x - a_m)).
\]

Let \( F : \mathbb{R}^n \to \mathbb{R}^m \) be the vector function defined by \( F(x) := (F_1(x), \ldots, F_m(x))^T \), where \( F_i(x) = w_{a_i} \varphi_{a_i}(x - a_i) \) for all \( i = 1, \ldots, m \).

Because

\[
\gamma_C^+(F(x)) = \gamma_C(F^+(x)) = \gamma_C(F(x)), \quad \forall x \in \mathbb{R}^n,
\]

\((P_{\text{NC}}(\mathcal{F}))\) can be written in the equivalent form

\[
(P_{\text{NC}}(\mathcal{F})) \quad \inf_{x \in \mathbb{R}^n} \gamma_C^+(F(x)),
\]

which is a particular case of the problem \((P_{\text{NC}})\) studied in Subsection 3.1.3. We mention that instead of the set \( X \subseteq \mathbb{R}^n \) considered in the case of problem \((P_{\text{NC}})\), we take here analogously to [27] the whole space \( \mathbb{R}^n \). Because

\[
-\delta_{\mathbb{R}^n}^*(-p) = \inf_{x \in \mathbb{R}^n} p^T x = \begin{cases} 0, & \text{if } p = 0, \\ -\infty, & \text{otherwise}, \end{cases}
\]

the Fenchel-Lagrange dual problem to \((P_{\text{NC}}(\mathcal{F}))\) becomes (cf. (3. 1))

\[
(D_{\text{FL}}^{\text{NC}}(\mathcal{F})) \quad \sup_{q \in \mathbb{R}_+^m, \gamma_C(q) \leq 1} \left\{ - (q^T F)^*(0) \right\}.
\]

By Proposition 3.1, \( F_i(x) = w_{a_i} \varphi_{a_i}(x - a_i), \) \( i = 1, \ldots, m, \) are convex functions and because \( q \in \mathbb{R}_+^m \), by Theorem 2.2, we have

\[
(q^T F)^*(0) = \left( \sum_{i=1}^m q_i F_i \right)^*(0) = \inf \left\{ \sum_{i=1}^m (q_i F_i)^*(p_i) : \sum_{i=1}^m p_i = 0 \right\},
\]

which implies

\[
(D_{\text{FL}}^{\text{NC}}(\mathcal{F})) \quad \sup_{p^i \in \mathbb{R}^n, i = 1, \ldots, m, \sum_{i=1}^m p^i = 0, q \in \mathbb{R}_+^m, \gamma_C(q) \leq 1} \left\{ - \sum_{i=1}^m (q_i F_i)^*(p_i) \right\}.
\]

In the objective function of this dual we separate the terms for which \( q_i > 0 \) from those for which \( q_i = 0 \) and then the dual can be written as

\[
(D_{\text{FL}}^{\text{NC}}(\mathcal{F})) \quad \sup_{p^i \in \mathbb{R}^n, i = 1, \ldots, m, \sum_{i=1}^m p^i = 0, q \in \mathbb{R}_+^m, \gamma_C(q) \leq 1, I \subseteq \{1, \ldots, m\}, q_i > 0, i \in I, q_i = 0, i \notin I} \left\{ - \sum_{i \in I} (q_i F_i)^*(p_i) - \sum_{i \notin I} (0)^*(p_i) \right\}.
\]
For $i \notin I$, it holds
\[
(0)^*(p^i) = \sup_{x \in \mathbb{R}^n} \{(p^i)^T x - 0\} = \sup_{x \in \mathbb{R}^n} \{(p^i)^T x\} = \begin{cases} 0, & \text{if } p^i = 0, \\ +\infty, & \text{otherwise.} \end{cases}
\]

For $i \in I$ there is $(q_i F_i)^*(p^i) = q_i F_i^* \left( \frac{p^i}{q_i} \right)$ (cf. [14]). Redenoting $\frac{1}{q_i} p^i$ by $p_i$, $i \in I$, we obtain
\[
(D_{FL}^{FC}(\mathcal{F})) \sup_{(I, p, q) \in Y_{FC}(\mathcal{F})} \left\{- \sum_{i \in I} q_i F_i^*(p^i)\right\},
\]
with
\[
Y_{FC}(\mathcal{F}) = \left\{(I, p, q) : I \subseteq \{1, \ldots, m\}, p = (p^1, \ldots, p^m), p_i \in \mathbb{R}^n, i = 1, \ldots, m, q = (q_1, \ldots, q_m)^T \in \mathbb{R}^m, \gamma_{C^0}(q) \leq 1, q_i > 0, i \in I, q_i = 0, i \notin I, \sum_{i \in I} q_i p^i = 0 \right\}.
\]

In our case $F_i(x) = w_i v_{\alpha_i}(x - a_i^i)$, $i = 1, \ldots, m$, hence (cf. [14])
\[
F_i^*(p^i) = (w_i v_{\alpha_i}(-a_i^i))^*(p^i) = (w_i v_{\alpha_i})^*(p^i) + (p^i)^T a_i^i = w_i v_{\alpha_i}^* \left( \frac{p^i}{w_i} \right) + (p^i)^T a_i^i.
\]

By Remark 3.2, $v_{\alpha_i}^* \left( \frac{p^i}{w_i} \right) = \begin{cases} 0, & \text{if } \varphi_{\alpha_i}^0 \left( \frac{p^i}{w_i} \right) \leq 1, \\ +\infty, & \text{otherwise,} \end{cases}$ and redenoting $\frac{p^i}{w_i}$ by $p_i$, $i \in I$, the dual problem to $(P_{FC}(\mathcal{F}))$ becomes
\[
(D_{FL}^{FC}(\mathcal{F})) \sup_{(I, p, q) \in Y_{FC}(\mathcal{F})} \left\{- \sum_{i \in I} q_i w_{\alpha_i}^*(p^i)^T a_i^i\right\},
\]
with
\[
Y_{FC}(\mathcal{F}) = \left\{(I, p, q) : I \subseteq \{1, \ldots, m\}, p = (p^1, \ldots, p^m), p_i \in \mathbb{R}^n, i = 1, \ldots, m, \varphi_{\alpha_i}^0(p^i) \leq 1, i \in I, q = (q_1, \ldots, q_m)^T \in \mathbb{R}^m, \gamma_{C^0}(q) \leq 1, q_i > 0, i \in I, q_i = 0, i \notin I, \sum_{i \in I} q_i w_{\alpha_i} p^i = 0 \right\}.
\]

The next theorem gives us the strong duality for the problems $(P_{FC}(\mathcal{F}))$ and $(D_{FL}^{FC}(\mathcal{F}))$.

**Theorem 3.5** If $v(P_{FC}(\mathcal{F})) > -\infty$, then the dual problem $(D_{FL}^{FC}(\mathcal{F}))$ has an optimal solution and strong duality holds,
\[
v(P_{FC}(\mathcal{F})) = v(D_{FL}^{FC}(\mathcal{F})).
\]
Furthermore, we give the optimality conditions for the problem \((P^\gamma_c(F))\).

**Theorem 3.6** (a) Let \(\bar{x}\) be an optimal solution to \((P^\gamma_c(F))\). Then there exists a tuple \((\bar{I}, \bar{p}, \bar{q}) \in Y^\gamma_c(F)\), optimal solution to \((D^\gamma_c_{FL}(F))\), such that the following optimality conditions are satisfied

\[
\begin{align*}
(i) & \quad \bar{I} \subseteq \{1, \ldots, m\}, \bar{q}_i > 0, i \in \bar{I}, \bar{q}_i = 0, i \notin \bar{I}, \\
(ii) & \quad \gamma^\gamma_c(\bar{q}) \leq 1, \varphi^0_{a_i}(\bar{p}^j) \leq 1, i \notin \bar{I}, \sum_{i \in \bar{I}} \bar{q}_i w^i_{a_i} \bar{p}^j = 0, \\
(iii) & \quad \gamma_C(w_{a_i} \varphi_{a_i}(\bar{x} - a^1), \ldots, w_{a_m} \varphi_{a_m}(\bar{x} - a^m)) = \sum_{i \in \bar{I}} \bar{q}_i w^i_{a_i} \varphi_{a_i}(\bar{x} - a^i), \\
(iv) & \quad \varphi_{a_i}(\bar{x} - a^i) = (\bar{p}^j)^T (\bar{x} - a^i), i \in \bar{I}.
\end{align*}
\]

(b) If \(\bar{x} \in \mathbb{R}^n\), \((\bar{I}, \bar{p}, \bar{q}) \in Y^\gamma_c\) and (i), (ii), (iii) and (iv) are fulfilled, then \(\bar{x}\) is an optimal solution to \((P^\gamma_c(F))\), \((\bar{I}, \bar{p}, \bar{q}) \in Y^\gamma_c(F)\) is an optimal solution to \((D^\gamma_c_{FL}(F))\) and strong duality holds

\[
\gamma_C(w_{a_i} \varphi_{a_i}(\bar{x} - a^1), \ldots, w_{a_m} \varphi_{a_m}(\bar{x} - a^m)) = - \sum_{i \in \bar{I}} \bar{q}_i w^i_{a_i} (\bar{p}^j)^T a^i.
\]

**Proof.**

(a) Because the functions \(F_i(x) = w_{a_i} \varphi_{a_i}(x - a^i), i = 1, \ldots, m\), are convex (cf. Proposition 3.1), by Theorem 3.1 it follows that there exists an optimal solution \((\bar{I}, \bar{p}, \bar{q}) \in Y^\gamma_c(F)\) to \((D^\gamma_c_{FL}(F))\) such that (i) and (ii) are fulfilled and

\[
\gamma_C(w_{a_i} \varphi_{a_i}(\bar{x} - a^1), \ldots, w_{a_m} \varphi_{a_m}(\bar{x} - a^m)) = - \sum_{i \in \bar{I}} \bar{q}_i w^i_{a_i} (\bar{p}^j)^T a^i. \tag{3. 5}
\]

Because \((\bar{I}, \bar{p}, \bar{q}) \in Y^\gamma_c(F)\), it follows that \(\gamma^\gamma_c(\bar{q}) \leq 1, \bar{q}^j > 0, i \in \bar{I}, \bar{q}_i = 0, i \notin \bar{I}, \varphi^0_{a_i}(\bar{p}^j) \leq 1, i \in \bar{I}\) and \(\sum_{i \in \bar{I}} \bar{q}_i w^i_{a_i} \bar{p}^j = 0\). Additionally, by Remark 3.2 \(\gamma^\gamma_c(\bar{q}) = 0\), and so the equation (3. 5) is equivalent to the following one

\[
\begin{align*}
\gamma_C & (w_{a_i} \varphi_{a_i}(\bar{x} - a^1), \ldots, w_{a_m} \varphi_{a_m}(\bar{x} - a^m)) + \sum_{i \in \bar{I}} \bar{q}_i w^i_{a_i} (\bar{p}^j)^T a^i - \\
& \bar{q}^T (w_{a_i} \varphi_{a_i}(\bar{x} - a^1), \ldots, w_{a_m} \varphi_{a_m}(\bar{x} - a^m)) - \sum_{i \in \bar{I}} \bar{q}_i w^i_{a_i} (\bar{p}^j)^T \bar{x} + \\
& \bar{q}^T (w_{a_i} \varphi_{a_i}(\bar{x} - a^1), \ldots, w_{a_m} \varphi_{a_m}(\bar{x} - a^m)) + \gamma^\gamma_c(\bar{q}) = 0. \tag{3. 6}
\end{align*}
\]

Because \(\varphi^0_{a_i}(\bar{p}^j) \leq 1, i \in \bar{I}\), by equation (3. 3) and Remark 3.2 it follows that

\[
(w_{a_i} \varphi_{a_i}(\cdot - a^i))^* (w_{a_i} \bar{p}^j) = w_{a_i} (\bar{p}^j)^T a^i, \quad \forall \, i \in \bar{I}. \tag{3. 7}
\]

Using equality (3. 7), relation (3. 6) becomes

\[
\gamma^\gamma_c(\bar{q}) + \gamma_C(w_{a_i} \varphi_{a_i}(\bar{x} - a^1), \ldots, w_{a_m} \varphi_{a_m}(\bar{x} - a^m)) - \sum_{i \in \bar{I}} \bar{q}_i w^i_{a_i} \varphi_{a_i}(\bar{x} - a^i) +
\]
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\[
\sum_{i \in I} \bar{q}_i (w_a^i \varphi_{a^i} (\cdot - a^i)) (w_a^i \bar{p}^i) + \sum_{i \in I} \bar{q}_i w_a^i \varphi_{a^i} (\bar{x} - a^i) - \sum_{i \in I} \bar{q}_i w_a^i (\bar{p}^i)^T \bar{x} = 0,
\]

which is equivalent to

\[
\gamma_C^*(\bar{q}) + \gamma_C (w_a^1 \varphi_{a^1} (\bar{x} - a^1), ..., w_a^m \varphi_{a^m} (\bar{x} - a^m)) - \sum_{i \in I} \bar{q}_i w_a^i \varphi_{a^i} (\bar{x} - a^i) + \sum_{i \in I} \bar{q}_i ((w_a^i \varphi_{a^i} (\cdot - a^i))^* (w_a^i \bar{p}^i) + w_a^i \varphi_{a^i} (\bar{x} - a^i) - w_a^i (\bar{p}^i)^T \bar{x}) = 0 \quad (3.8)
\]

According to Young’s inequality

\[
\gamma_C^*(\bar{q}) + \gamma_C (w_a^1 \varphi_{a^1} (\bar{x} - a^1), ..., w_a^m \varphi_{a^m} (\bar{x} - a^m)) - \sum_{i \in I} \bar{q}_i w_a^i \varphi_{a^i} (\bar{x} - a^i) \geq 0,
\]

\[
w_a^i \varphi_{a^i} (\cdot - a^i))^* (w_a^i \bar{p}^i) + w_a^i \varphi_{a^i} (\bar{x} - a^i) - w_a^i (\bar{p}^i)^T \bar{x} \geq 0, \forall i \in I,
\]

and so, equation (3.8) together with relation (3.7) implies that

\[
\gamma_C (w_a^1 \varphi_{a^1} (\bar{x} - a^1), ..., w_a^m \varphi_{a^m} (\bar{x} - a^m)) = \sum_{i \in I} \bar{q}_i w_a^i \varphi_{a^i} (\bar{x} - a^i)
\]

and

\[
\varphi_{a^i} (\bar{x} - a^i) = (\bar{p}^i)^T (\bar{x} - a^i).
\]

(b) All the calculations and transformations done within part (a) may be carried out in the inverse direction.

\[\square\]

Remark 3.3 The optimality conditions obtained for the optimization problem \((P^{wc}(F))\) are the same as the conditions obtained by Y. HINOJOSA and J. PUERTO in [27]. In the paper cited above the authors gave a geometrical description of the set of optimal solutions, but, as one can see, by means of duality one obtains the same characterization of this set.

In the next two sections of this chapter we present some particular cases of the problem \((P^{wc}(F))\), namely, the Weber problem and the minmax problem with gauges of closed convex sets.

3.1.6 The Weber problem with gauges of closed convex sets

The Weber problem with gauges of closed convex sets is

\[
(P^{w}(F)) \quad \inf_{x \in \mathbb{R}^n} \sum_{i=1}^m w_a^i \varphi_{a^i} (x - a^i),
\]

where \(\varphi_{a^i}, \ i = 1, ..., m,\) are gauges whose unit balls are the closed convex sets \(C_{a^i}, \ i = 1, ..., m,\) which contain the origin, and \(w = \{w_a^1, ..., w_a^m\}\) is a set of
positive weights. As one can see, the problem above is equivalent to the following one
\[
(P_w(F)) \quad \inf_{x \in \mathbb{R}^n} l_1(F(x)),
\]
where \(l_1 : \mathbb{R}^m \to \mathbb{R}, \ l_1(q) = \sum_{i=1}^{m} |q_i|\) and \(F : \mathbb{R}^n \to \mathbb{R}^m\) is the vector function defined by \(F := (F_1, ..., F_m)^T\), with \(F_i(x) = w_{a^i}(x - a^i)\) for all \(i = 1, ..., m\). One may observe that the function \(l_1\) is a monotonic gauge, actually, a monotonic norm.

If we take for \(C\) the set \(\{x \in \mathbb{R}^m : \sum_{i=1}^{m} |x_i| \leq 1\}\) (i.e. the so-called Minkowski unit ball), then \(\gamma_C\) reduces to \(l_1\). By the results obtained in the previous section, the Fenchel-Lagrange dual problem to \((P_w(F))\) becomes
\[
(D_{FL}^w(F)) \quad \sup_{(I, p, q) \in Y^w(F)} \left\{ -\sum_{i \in I} q_i w_{a^i}(p^i)^T a^i \right\},
\]
with
\[
Y^w(F) = \left\{ (I, p, q) : I \subseteq \{1, ..., m\}, p = (p^1, ..., p^m), p^i \in \mathbb{R}^n, \ i = 1, ..., m, \right. \]
\[
\varphi^0_{a^i}(p^i) \leq 1, i \in I, q = (q_1, ..., q_m)^T \in \mathbb{R}^m, l_1^0(q) \leq 1, q_i > 0, i \in I, \]
\[
q_i = 0, i \notin I, \sum_{i \in I} q_i w_{a^i} p^i = 0. \]

**Remark 3.4** In case that the gauge \(\gamma_C\) of a convex set \(C\) is a norm, the gauge of the polar set \(C^0\) actually becomes the dual norm. Because the dual norm of the \(l_1\)-norm is \(l_1^0(q) = \max_{i=1, ..., m} |q_i|\), we obtain the following formulation for the dual problem
\[
(D_{FL}^w(F)) \quad \sup_{(I, p, q) \in Y^w(F)} \left\{ -\sum_{i \in I} q_i w_{a^i}(p^i)^T a^i \right\}, \quad (3.9)
\]
with
\[
Y^w(F) = \left\{ (I, p, q) : I \subseteq \{1, ..., m\}, p = (p^1, ..., p_m), p^i \in \mathbb{R}^n, \ i = 1, ..., m, \right. \]
\[
\varphi^0_{a^i}(p^i) \leq 1, i \in I, q = (q_1, ..., q_m)^T \in \mathbb{R}^m, \max_{i \in I} q_i \leq 1, q_i > 0, i \in I, \]
\[
q_i = 0, i \notin I, \sum_{i \in I} q_i w_{a^i} p^i = 0. \]

Let us give now the strong duality theorem and the optimality conditions for \((P_w(F))\) and its dual \((D_{FL}^w(F))\).
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Theorem 3.7 If \( v(P^w(F)) > -\infty \), then the dual problem \( (D^w_{FL}(F)) \) has an optimal solution and strong duality holds, i.e.

\[
v(P^w(F)) = v(D^w_{FL}(F)).\]

Theorem 3.8 (a) Let \( \bar{x} \) be an optimal solution to \( (P^w(F)) \). Then there exists a tuple \((\bar{I}, \bar{p}, \bar{q}) \in Y^w(F)\), optimal solution to \( (D^w_{FL}(F)) \), such that the following optimality conditions are satisfied

1. If \( 1, \ldots, m \) \( q \bar{q}_i > 0, i \in \bar{I}, \bar{q}_i = 0, i \notin \bar{I} \),
2. \( \max_{i \in \bar{I}} \bar{q}_i \leq 1, \varphi^0_{a^i}(\bar{p}^i) \leq 1, i \in \bar{I}, \sum_{i \in \bar{I}} \bar{q}_i w_{a^i} \bar{p}^i = 0 \),
3. \( \sum_{i=1}^m w_{a^i} \varphi_{a^i}(\bar{x} - a^i) = \sum_{i \in \bar{I}} \bar{q}_i w_{a^i} \varphi_{a^i}(\bar{x} - a^i) \),
4. \( \varphi_{a^i}(\bar{x} - a^i) = (\bar{p}^i)^T (\bar{x} - a^i), i \in \bar{I} \).

(b) If \( \bar{x} \in \mathbb{R}^n, (\bar{I}, \bar{p}, \bar{q}) \in Y^w(F) \) and (i), (ii), (iii) and (iv) are fulfilled, then \( \bar{x} \) is an optimal solution to \( (P^w(F)) \), \((\bar{I}, \bar{p}, \bar{q}) \in Y^w(F) \) is an optimal solution to \( (D^w_{FL}(F)) \) and strong duality holds

\[
\sum_{i=1}^m w_{a^i} \varphi_{a^i}(\bar{x} - a^i) = - \sum_{i \in \bar{I}} \bar{q}_i w_{a^i} (\bar{p}^i)^T a^i .
\]

Proof. Theorem 3.8 is a direct consequence of Theorem 3.6. \( \square \)

3.1.7 The minmax problem with gauges of closed convex sets

The optimization problem studied in the last part of this chapter is the minmax problem with gauges of closed convex sets

\[
(P^m(F)) \quad \inf_{x \in \mathbb{R}^n} \max_{i=1, \ldots, m} w_{a^i} \varphi_{a^i}(x - a^i),
\]

where \( \varphi_{a^i}, i = 1, \ldots, m \), and \( w = \{w_{a^1}, \ldots, w_{a^m}\} \) are considered like in the previous section. One can see that this problem is equivalent to the following one

\[
(P^m(F)) \quad \inf_{x \in \mathbb{R}^n} l_\infty(F(x)),
\]

where \( l_\infty : \mathbb{R}^m \rightarrow \mathbb{R}, l_\infty(q) = \max_{i=1, \ldots, m} |q_i| \) and \( F : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is the vector function defined by \( F := (F_1, \ldots, F_m)^T \), with \( F_i(x) = w_{a^i} \varphi_{a^i}(x - a^i) \) for all \( i = 1, \ldots, m \). One may observe that the function \( l_\infty \) is also a monotonic norm.
Taking \( \gamma_C(q) := l_\infty(q) \) for all \( q \in \mathbb{R}^m \), the Fenchel-Lagrange dual problem to \((P^m(\mathcal{F}))\) becomes

\[
(D_{FL}^m(\mathcal{F})) \sup_{(I, p, q) \in Y^m(\mathcal{F})} \left\{ -\sum_{i \in I} q_i w_a(i)^T a_i \right\},
\]
with

\[
Y^m(\mathcal{F}) = \left\{ (I, p, q) : I \subseteq \{1, \ldots, m\}, p = (p^1, \ldots, p^m), p^i \in \mathbb{R}^n, i = 1, \ldots, m, \right. \]

\[
\phi_a^0(p^i) \leq 1, \ i \in I, \ q = (q_1, \ldots, q_m)^T \in \mathbb{R}^m, \ l_\infty^0(q) \leq 1, \ q_i > 0, \ i \in I, \]

\[
q_i = 0, \ i \notin I, \ \sum_{i \in I} q_i w_a(p^i) = 0. \]

**Remark 3.5** Because the dual norm of the \( l_\infty \)-norm is \( l_\infty^0(q) = \sum_{i=1}^m |q_i| \), we obtain the following formulation for the dual problem

\[
(D_{FL}^m(\mathcal{F})) \sup_{(I, p, q) \in Y^m(\mathcal{F})} \left\{ -\sum_{i \in I} q_i w_a(i)^T a_i \right\}, \quad (3.10)
\]
with

\[
Y^m(\mathcal{F}) = \left\{ (I, p, q) : I \subseteq \{1, \ldots, m\}, p = (p^1, \ldots, p^m), p^i \in \mathbb{R}^n, i = 1, \ldots, m, \right. \]

\[
\phi_a^0(p^i) \leq 1, \ i \in I, \ q = (q_1, \ldots, q_m)^T \in \mathbb{R}^m, \ \sum_{i=1}^m q_i \leq 1, \ q_i > 0, \ i \in I, \]

\[
q_i = 0, \ i \notin I, \ \sum_{i \in I} q_i w_a(p^i) = 0. \]

As in the previous section, we give now the strong duality theorem and the optimality conditions for \((P^m(\mathcal{F}))\) and its dual \((D_{FL}^m(\mathcal{F}))\).

**Theorem 3.9** If \( v(P^m(\mathcal{F})) > -\infty \), then the dual problem \((D_{FL}^m(\mathcal{F}))\) has an optimal solution and strong duality holds, i.e.

\[
v(P^m(\mathcal{F})) = v(D_{FL}^m(\mathcal{F})).
\]

**Theorem 3.10** (a) Let \( \bar{x} \) be an optimal solution to \((P^m(\mathcal{F}))\). Then there exists a tuple \((\bar{I}, \bar{p}, \bar{q}) \in Y^m(\mathcal{F})\), optimal solution to \((D_{FL}^m(\mathcal{F}))\), such that the following optimality conditions are satisfied

(i) \( \bar{I} \subseteq \{1, \ldots, m\}, \ \bar{q}_i > 0, \ i \in \bar{I}, \ \bar{q}_i = 0, \ i \notin \bar{I}, \)

(ii) \( \sum_{i \in \bar{I}} \bar{q}_i \leq 1, \ \phi_a^0(\bar{p}^i) \leq 1, \ i \in \bar{I}, \ \sum_{i \in \bar{I}} \bar{q}_i w_a(\bar{p}^i) = 0, \)

(iii) \( \max_{i=1,\ldots,m} w_a(i) \phi_a(\bar{x} - a^i) = \sum_{i \in \bar{I}} \bar{q}_i w_a(\phi_a(\bar{x} - a^i)), \)

(iv) \( \phi_a(\bar{x} - a^i) = (\bar{p}^i)^T (\bar{x} - a^i), \ i \in \bar{I}. \)
(b) If \( \bar{x} \in \mathbb{R}^n \), \( (\bar{I}, \bar{p}, \bar{q}) \in Y^m(\mathcal{F}) \) and \((i), (ii), (iii)\) and \((iv)\) are fulfilled, then \( \bar{x} \) is an optimal solution to \((P^m(\mathcal{F}))\), \( (\bar{I}, \bar{p}, \bar{q}) \in Y^m(\mathcal{F}) \) is an optimal solution to \((D^m_{PL}(\mathcal{F}))\) and strong duality holds

\[
\max_{i=1,\ldots,m} w_i \varphi_{a_i}(\bar{x} - a^i) = - \sum_{i \in I} \bar{q}_i w_{a_i}(\bar{p}^i)^T a^i.
\]

**Proof.** Theorem 3.10 is a direct consequence of Theorem 3.6. \( \square \)
Chapter 4

Multiobjective optimization problems

Most real life optimization problems require simultaneous optimization of more than one objective function. Problems with multiple objectives and criteria are generally known as multiobjective optimization or multiple criteria optimization problems. In general, these problems are concerned with the minimization of a vector of objectives $f = (f_1, ..., f_s)^T$, $X \subseteq \mathbb{R}^n$, $f_i : X \rightarrow \mathbb{R}$, $i = 1, ..., s$, that can be subject of a number of constraints defined by $g = (g_1, ..., g_k)^T$, $g_j : X \rightarrow \mathbb{R}$, $j = 1, ..., k$, i.e.

$$\min_{x \in \mathcal{A}} \left( \begin{array}{c} f_1(x) \\ \vdots \\ f_s(x) \end{array} \right),$$

where $\mathcal{A} = \left\{ x \in X : g(x) \leq 0 \right\}$. Note here that "v-min" stands for vector minimization.

Because $f$ is a vector-valued function, there is no longer a single optimal solution but rather a whole set of possible solutions. There are different solution concepts for vector optimization problems, e.g. so-called Pareto efficient, weakly efficient and properly efficient solutions. Throughout this work we use the Pareto efficient and properly efficient solution concepts.

The Pareto efficient solutions for a multiobjective optimization problem are those ones for which it is not possible to increase the satisfaction of any single objective without decreasing the satisfaction of one or more other objectives. As a consequence, a feasible point is defined as optimal if there does not exist a different feasible point with the same or smaller objective function values such that there is a strict decrease in at least one objective function value. In general there is no single solution point of a vector optimization problem but the solutions are represented by a set of points. For an overview of Pareto efficiency see [1],

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The classical and still widely used approach for generating the Pareto optimal set is to convert the original multiobjective optimization problem into a scalar one by forming a linear combination of the objectives
\[
\inf_{x \in \mathcal{A}} \lambda^T f(x),
\]
where \(\lambda = (\lambda_1, ..., \lambda_s)^T \in \mathbb{R}^s\), \(\lambda_i > 0\), are the so-called weights. This method generates Pareto optimal solutions which can be easily shown. Assume that a feasible element \(x\) minimizes \(\lambda^T f\) and it would not be Pareto optimal. Then there is an admissible element \(y\) that is dominating \(x\), i.e. \(f_i(y) \leq f_i(x)\), for \(i = 1, ..., s\), and \(f_j(y) < f_j(x)\) for at least one \(j \in \{1, ..., s\}\). Therefore \(\lambda^T f(y) < \lambda^T f(x)\) which is a contradiction to the assumption that \(\lambda^T f(x)\) is a minimum. Solving this scalarized problem with classical techniques for single-objective optimization yields a set of solutions. This method, known as scalarization, is described in many books and papers. For a detailed discussion of some scalarization techniques see [18], [19], [20], [21], [26], [33], [34], [58] and [74].

The properly efficient solutions are slightly restricted, they eliminate some trade-offs between the objectives. The concept of proper efficiency was introduced for the first time by KUHN and TUCKER in [42], but since then other well-known definitions have been given by GEOFFRION [20], BORWEIN [6], BENSON [3] and HENIG [25]. By the results presented by SAWARAGI, NAKAYAMA and TANINO in [57], for the optimization problem presented in this work, all these four concepts turn out to be equivalent. Because the properly efficient solutions are characterized by optimizing associated utility-related scalar optimization problems, they provide us a useful framework for finding the Pareto optimal solutions. Therefore, in what follows, we first scalarize the multiobjective problem and solve it by the conjugate duality method described in the second chapter. Using the results from there we generalize them and move from scalar towards vector optimization problems.

4.1 Duality in multiobjective optimization

4.1.1 Motivation

In vector optimization, duality theorems and Lagrangian functions have been known for a long time. In the literature one can find papers devoted to linear and nonlinear problems, papers dealing with duality under smooth and non-smooth assumptions for both the objective and constraints functions, etc.

Our propose in this chapter is to show how conjugate duality, presented in Chapter 2, can contribute to study duality for multiobjective optimization problems. For this sake we consider a vector optimization problem where the objective functions as well as the constraints are given by composed functions. First, we
3.1 THE CONJUGATE DUAL OF THE COMPOSED PROBLEM

transform this problem into a scalarized one, and then, based on dual information obtained for appropriately formulated single objective problems we establish a theoretical frame on conjugate duality in multiobjective optimization.

Because various mathematical optimization models can be reduced to composed programming, the suggested problem turns out to be quite general, and so, it provides a unified framework for studying different multiobjective optimization problems. Similar problems were studied via numerical, geometrical, etc. methods by J. V. BURKE, R. A. POLIQUIN [8], C. J. GOH, X. Q. YANG [22], J. JAHN, W. KRABS [35], V. JEYAKUMAR, X. Q. YANG [37], [38], [39], [75], [76] and S. K. MISHRA, R. N. MUKHERJEE [47]. In most of these papers, in order to obtain some duals, the authors made use of more additional assumptions concerning the objective functions and the constraints, such as differentiability, invexity, etc. In the approach presented below we solve this problem using convexity and monotonicity assumptions.

4.1.2 Problem formulation

Let us consider a nonempty subset $X \subseteq \mathbb{R}^n$ and the vector-valued functions $F = (F_1, \ldots, F_m)^T : X \rightarrow \mathbb{R}^m$, $G = (G_1, \ldots, G_l)^T : X \rightarrow \mathbb{R}^l$, $f = (f_1, \ldots, f_s)^T : \mathbb{R}^m \rightarrow \mathbb{R}^s$ and $g = (g_1, \ldots, g_k)^T : \mathbb{R}^l \rightarrow \mathbb{R}^k$. We assume that $F_i, i = 1, \ldots, m$, $G_j, j = 1, \ldots, l$, are convex functions and $f_i, i = 1, \ldots, s$, and $g_j, j = 1, \ldots, k$, are convex and componentwise increasing functions.

The optimization problem which we consider in this chapter is

$$(P_v) \ v\text{-min}_{x \in A} f(F(x)),$$

where

$$A = \left\{ x \in X : g(G(x)) \leq 0 \right\}.$$

Note here that "v-min" stands for vector minimization.

In what follows let us give the efficiency definitions which we shall use throughout this chapter.

**Definition 4.1** An element $\bar{x} \in A$ is said to be efficient (or Pareto efficient) with respect to $(P_v)$ if from $f(F(x)) \leq f(F(\bar{x}))$, for $x \in A$, it follows that $f(F(x)) = f(F(\bar{x}))$.

**Definition 4.2** An element $\bar{x} \in A$ is said to be properly efficient with respect to $(P_v)$ if there exists $\lambda = (\lambda_1, \ldots, \lambda_s)^T \in \text{int}(\mathbb{R}_+^s)$, (i.e. $\lambda_i > 0, i = 1, \ldots, s$), such that $\lambda^T f(F(\bar{x})) \leq \lambda^T f(F(x))$, for all $x \in A$.

**Remark 4.1** As we have seen in the introduction of this chapter, each properly efficient element is also efficient.
4.1.3 Duality for the scalarized problem

According to the Definition 4.2 we consider the following scalarized problem \((P^\lambda)\) to \((P_v)\)

\[
(P^\lambda) \quad \inf_{x \in \mathcal{A}} \lambda^T f(F(x)),
\]

where \(\lambda = (\lambda_1, \ldots, \lambda_s)^T\) is a fixed vector in \(\text{int}(\mathbb{R}_+^s)\).

We may observe that \((P^\lambda)\) is a special case of the original problem \((P)\) (cf. Subsection 2.1.1). Therefore we solve it using some of the results obtained for \((P)\). In what follows we take into consideration only the Fenchel-Lagrange dual problem because it is the most comprehensive and at the same time leads to some former results obtained by G. WANKA, R. I. BOŢ and E. VARGYAS in [71]. Applying the results of Subsection 2.1.4, the Fenchel-Lagrange dual problem of \((P^\lambda)\) becomes

\[
(D_{FL}^{\lambda}) \quad \sup_{p \in \mathbb{R}^m, q \in \mathbb{R}^m, q' \in \mathbb{R}^s, t \in \mathbb{R}^s} \left\{ - (\lambda^T f)^* (q) - (t^T g)^* (q') - (q^T F)^* \chi(p) - (q^T G)^* \chi(-p) \right\}.
\]

**Proposition 4.1** Assume that \(\lambda \in \text{int}(\mathbb{R}_+^s),\ f = (f_1, \ldots, f_s)^T : \mathbb{R}^m \rightarrow \mathbb{R}^s\) and \(f_i,\ i = 1, \ldots, s,\) are componentwise increasing functions. Then \(\lambda^T f : \mathbb{R}^m \rightarrow \mathbb{R}\) is componentwise increasing.

**Proof.** Let \(x, y \in \mathbb{R}^m\) be such that \(x_i \leq y_i\) for all \(i = 1, \ldots, m.\) We have to prove that \(\lambda^T f(x) \leq \lambda^T f(y).\) Because \(f_i,\ i = 1, \ldots, s,\) are componentwise increasing functions and \(\lambda \in \text{int}(\mathbb{R}_+^s),\) it follows that

\[
\lambda^T f(x) = \lambda_1 f_1(x) + \ldots + \lambda_s f_s(x) \leq \lambda_1 f_1(y) + \ldots + \lambda_s f_s(y) = \lambda^T f(y),
\]

which implies that \(\lambda^T f\) is componentwise increasing. \(\square\)

By propositions 2.3 and 4.1 we can take \(q \in \mathbb{R}^m\) and therefore \((D_{FL}^{\lambda})\) becomes

\[
(D_{FL}^{\lambda}) \quad \sup_{p \in \mathbb{R}^m, q \in \mathbb{R}^m, q' \in \mathbb{R}^s, t \in \mathbb{R}^s} \left\{ - (\lambda^T f)^* (q) - (t^T g)^* (q') - (q^T F)^* \chi(p) - (q^T G)^* \chi(-p) \right\}.
\]

Because \(\bigcap_{i=1}^s ri(\text{dom}(f_i)) \neq \emptyset,\) \(\lambda_i > 0,\ i = 1, \ldots, s,\) and \(f_i\) are convex for all \(i = 1, \ldots, s,\) we have (cf. Theorem 2.2)

\[
(\lambda^T f)^*(q) = \left( \sum_{i=1}^s \lambda_i f_i \right)^*(q) = \inf \left\{ \sum_{i=1}^s (\lambda_i f_i)^*(r^i) : \sum_{i=1}^s r^i = q \right\}.
\]

According to Proposition 2.3, \(r^i,\ i = 1, \ldots, s,\) have to be positive, and so, the dual \((D_{FL}^{\lambda})\) becomes

\[
(D_{FL}^{\lambda}) \quad \sup_{(p, q, q', r, t) \in \mathbb{Y}^\lambda} \left\{ - \sum_{i=1}^s (\lambda_i f_i)^*(r^i) - (t^T g)^* (q') - (q^T F)^* \chi(p) - (q^T G)^* \chi(-p) \right\},
\]
4.1 DUALITY FOR THE SCALARIZED PROBLEM

with

\[ Y^\lambda = \left\{ (p, q, q', r, t) : p \in \mathbb{R}^n, q \in \mathbb{R}_+^m, q' \in \mathbb{R}_+^l, r = (r^1, \ldots, r^s), \right. \\
\left. r^i \in \mathbb{R}_+^m, i = 1, \ldots, s, \sum_{i=1}^s r^i = q, t \in \mathbb{R}_+^k \right\}. \]

Since \( \lambda_i > 0 \), it follows that \((\lambda_i f_i)^* (r^i) = \lambda_i f_i^* \left( \frac{r^i}{\lambda_i} \right)\), for all \( i = 1, \ldots, s \). Redenoting \( \frac{r^i}{\lambda_i} \) by \( r^i \) we obtain

\[
(D_{FL})^\lambda \sup_{(p, q, q', r, t) \in Y^\lambda} \left\{ -\sum_{i=1}^s \lambda_i f_i^* (r^i) - (t^T g)^*(q') - (q^T F)^* (p) - (q^T G)^* (-p) \right\},
\]

with

\[ Y^\lambda = \left\{ (p, q, q', r, t) : p \in \mathbb{R}^n, q \in \mathbb{R}_+^m, q' \in \mathbb{R}_+^l, r = (r^1, \ldots, r^s), \right. \\
\left. r^i \in \mathbb{R}_+^m, i = 1, \ldots, s, \sum_{i=1}^s \lambda_i r^i = q, t \in \mathbb{R}_+^k \right\}. \]

As we will see in the following subsection, this form of the dual problem help us to find a dual to the multiobjective problem \((P_v)\).

By means of the strong duality presented in Theorem 2.7, we can formulate the strong duality for the scalarized problem \((P^\lambda)\) and its Fenchel-Lagrange dual \((D_{FL}^\lambda)\). In order to do this, let us prove first the convexity of the objective function of the scalarized problem \((P^\lambda)\).

**Proposition 4.2** Let \( \lambda \in \mathbb{R}_+^s \) be fixed. The function \( \lambda^T (f \circ F) : X \to \mathbb{R} \) is convex.

**Proof.** The convexity of \( \lambda^T (f \circ F) = \sum_{i=1}^s \lambda_i f_i (F_1, \ldots, F_m) \) follows from the convexity and monotonicity of the functions \( f_i, i = 1, \ldots, s \), and the convexity of \( F_j, j = 1, \ldots, m \), as well as the fact that \( \lambda \in \mathbb{R}_+^s \). \( \square \)

**Theorem 4.1** Assume that \( X \subseteq \mathbb{R}^n \) is a nonempty convex subset and the constraint qualification \((CQ)\) is fulfilled. Then it holds

\[ v(P^\lambda) = v(D_{FL}^\lambda). \]

Provided that \( v(P^\lambda) > -\infty \), the strong duality holds, i.e. the optimal objective values of the primal and the dual problem coincide and the dual problem \((D_{FL}^\lambda)\) has an optimal solution.
Proof. Theorem 4.1 is a direct consequence of Theorem 2.7. \hfill \qed

Remark 4.2 The constraint qualification (CQ) from above is the same as in the case of problem \((P)\), which was defined in Subsection 2.2.3.

To investigate later the multiobjective duality for \((P_v)\) we need the optimality conditions regarding to the scalar problem \((P^\lambda)\) and its dual \((D^\lambda_{FL})\). These are formulated in the following theorem.

**Theorem 4.2** (a) Let the assumptions of Theorem 4.1 be fulfilled and let \(\bar{x}\) be an optimal solution to \((P^\lambda)\). Then there exists a tuple \((\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t})\) \(\in Y^\lambda\), optimal solution to \((D^\lambda_{FL})\), such that the following optimality conditions are satisfied

\[
\begin{align*}
(i) & \quad f_i(F(\bar{x})) + f_i^*(\bar{r}) = (\bar{r}^i)^T F(\bar{x}), & i = 1, \ldots, s, \\
(ii) & \quad \bar{q}^T F(\bar{x}) + (\bar{q}^T F)^*_X (\bar{p}) = \bar{p}^T \bar{x}, \\
(iii) & \quad \bar{t}^T g(G(\bar{x})) + (\bar{t}^T g)^* (\bar{q}') = \bar{q}^T G(\bar{x}), \\
(iv) & \quad \bar{q}^T G(\bar{x}) + (\bar{q}^T G)^*_X (-\bar{p}) = (-\bar{p})^T \bar{x}, \\
(v) & \quad \bar{t}^T g(G(\bar{x})) = 0.
\end{align*}
\]

(b) Let \(\bar{x}\) be admissible to \((P^\lambda)\) and \((\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t})\) be admissible to \((D^\lambda_{FL})\), satisfying (i), (ii), (iii), (iv) and (v). Then \(\bar{x}\) is an optimal solution to \((P^\lambda)\), \((\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t})\) is an optimal solution to \((D^\lambda_{FL})\) and strong duality holds.

**Proof.** By Theorem 4.1 there exists a tuple \((\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t})\) \(\in Y^\lambda\), optimal solution to \((D^\lambda_{FL})\), such that

\[
\lambda^T f(F(\bar{x})) = -\sum_{i=1}^s \lambda_i f_i^*(\bar{r}^i) - (\bar{t}^T g)^* (\bar{q}') - (\bar{q}^T F)^*_X (\bar{p}) - (\bar{q}^T G)^*_X (-\bar{p}).
\]

The equality from above implies, analogously to Theorem 2.10, the following optimality conditions:

\[
\begin{align*}
(i') & \quad \lambda^T f(F(\bar{x})) + \sum_{i=1}^s \lambda_i f_i^*(\bar{r}^i) = \bar{q}^T F(\bar{x}), \\
(ii') & \quad \bar{q}^T F(\bar{x}) + (\bar{q}^T F)^*_X (\bar{p}) = \bar{p}^T \bar{x}, \\
(iii') & \quad \bar{t}^T g(G(\bar{x})) + (\bar{t}^T g)^* (\bar{q}') = \bar{q}^T G(\bar{x}), \\
(iv') & \quad \bar{q}^T G(\bar{x}) + (\bar{q}^T G)^*_X (-\bar{p}) = (-\bar{p})^T \bar{x}, \\
(v') & \quad \bar{t}^T g(G(\bar{x})) = 0.
\end{align*}
\]

Because \((\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t})\) \(\in Y^\lambda\), it follows that \(\sum_{i=1}^s \lambda_i \bar{r}^i = \bar{q}\), and so, relation \((i')\) becomes

\[
\sum_{i=1}^s \lambda_i f_i(F(\bar{x})) + \sum_{i=1}^s \lambda_i f_i^*(\bar{r}^i) = \sum_{i=1}^s \lambda_i (\bar{r}^i)^T F(\bar{x}),
\]
which together with the fact that $\lambda \in \text{int}(\mathbb{R}^s_+)$ implies that

$$f_i(F(\bar{x})) + f_i^*(\bar{r}^i) = (\bar{r}^i)^T F(\bar{x}), \ i = 1, \ldots, s.$$ 

The rest of the proof is a direct consequence of Theorem 2.10. 

### 4.1.4 The multiobjective dual problem

After we have studied the scalarized problem, we formulate by its help a multiobjective dual ($D_v$) to the problem ($P_v$), which will be actually a vector maximum problem. We define the Pareto optimal solutions to ($D_v$) in the sense of maximum and prove the weak and strong duality theorems between ($P_v$) and its dual.

The dual multiobjective optimization problem ($D_v$) is introduced by

$$(D_v) \quad \text{v-max} \quad h(p, q, q', r, t, \lambda, u),$$

with

$$h(p, q, q', r, t, \lambda, u) = \begin{pmatrix} h_1(p, q, q', r, t, \lambda, u) \\ \vdots \\ h_s(p, q, q', r, t, \lambda, u) \end{pmatrix},$$

$$h_i(p, q, q', r, t, \lambda, u) = -f_i^*(r^i) - \frac{1}{s\lambda_i} \left( (t^T g)^*(q') + \left( q^T F \right)_\lambda^* (p) + \left( q^T G \right)_\lambda^* (-p) \right) + u_i,$$

for $i = 1, \ldots, s$, and the dual variables

$$p = (p_1, \ldots, p_n)^T \in \mathbb{R}^n, \quad q = (q_1, \ldots, q_m)^T \in \mathbb{R}^m, \quad q' = (q'_1, \ldots, q'_l)^T \in \mathbb{R}^l, \quad r = (r_1, \ldots, r_s) \in \mathbb{R}^m \times \ldots \times \mathbb{R}^m, \quad t = (t_1, \ldots, t_k)^T \in \mathbb{R}^k, \quad \lambda = (\lambda_1, \ldots, \lambda_s)^T \in \mathbb{R}^s, \quad u = (u_1, \ldots, u_s)^T \in \mathbb{R}^s,$$

and the set of constraints

$$\mathcal{B} = \left\{ (p, q, q', r, t, \lambda, u) : q \in \mathbb{R}^m_+, \ q' \in \mathbb{R}^l_+, \ r^i \in \mathbb{R}^m_+, \ i = 1, \ldots, s, \ t \in \mathbb{R}^k_+, \ \lambda \in \text{int}(\mathbb{R}^s_+), \ \sum_{i=1}^s \lambda_i r^i = q, \ \sum_{i=1}^s \lambda_i u_i = 0 \right\}.$$

**Definition 4.3** An element $(\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t}, \bar{\lambda}, \bar{u}) \in \mathcal{B}$ is said to be efficient (or Pareto efficient) with respect to the problem ($D_v$) if from $h(p, q, q', r, t, \lambda, u) \geq h(\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t}, \bar{\lambda}, \bar{u})$, for $(p, q, q', r, t, \lambda, u) \in \mathcal{B}$, it follows that $h(p, q, q', r, t, \lambda, u) = h(\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t}, \bar{\lambda}, \bar{u})$.

The following theorem provides the weak duality between the vector problems ($P_v$) and ($D_v$).
**Theorem 4.3** There is no \( x \in \mathcal{A} \) and no \((p, q, q', r, t, \lambda, u) \in \mathcal{B}\) fulfilling \( f(F(x)) \leq h(p, q, q', r, t, \lambda, u) \) and \( f(F(x)) \neq h(p, q, q', r, t, \lambda, u) \).

**Proof.** Let us assume that there exist \( x \in \mathcal{A} \) and \((p, q, q', r, t, \lambda, u) \in \mathcal{B}\) such that \( f_i(F(x)) \leq h_i(p, q, q', r, t, \lambda, u) \), for all \( i = 1, \ldots, s \), and \( f_j(F(x)) < h_j(p, q, q', r, t, \lambda, u) \), for at least one \( j \in \{1, \ldots, s\} \). This implies

\[
\lambda^T f(F(x)) = \sum_{i=1}^s \lambda_i f_i(F(x)) < \sum_{i=1}^s \lambda_i h_i(p, q, q', r, t, \lambda, u) = \lambda^T h(p, q, q', r, t, \lambda, u).
\]

(4.2)

But

\[
\sum_{i=1}^s \lambda_i h_i(p, q, q', r, t, \lambda, u) = -\sum_{i=1}^s \lambda_i f_i^*(r^i) - \sum_{i=1}^s \lambda_i \frac{1}{s} \lambda_i \left( (t^T g)^*(q') + (q^T F)^*_X(p) \right) + (q^T G)^*_X(-p) + \sum_{i=1}^s \lambda_i u_i
\]

\[
= -\sum_{i=1}^s \lambda_i f_i^*(r^i) - (t^T g)^*(q') + (q^T F)^*_X(p) + (q^T G)^*_X(-p),
\]

and applying then for \( f_i, \quad i = 1, \ldots, s, \) \( t^T g, \) \( q^T F \) and \( q^T G \) the Young’s inequalities (2, 27) and (2, 28) we have

\[
-f_i^*(r^i) \leq f_i(F(x)) - (r^i)^T F(x), \quad \forall \ i = 1, \ldots, s,
\]

\[
-(t^T g)^*(q') \leq t^T g(G(x)) - q^T G(x), \quad \forall x \in X,
\]

\[
-(q^T F)^*_X(p) \leq q^T F(x) - p^T x, \quad \forall x \in X,
\]

\[
-(q^T G)^*_X(-p) \leq q^T G(x) + p^T x, \quad \forall x \in X.
\]

Additionally, because of \( \sum_{i=1}^s \lambda_i r^i = q, \) \( t \in \mathbb{R}_+^k \) and \( x \in \mathcal{A} \), we obtain

\[
\sum_{i=1}^s \lambda_i h_i(p, q, q', r, t, \lambda, u) \leq \sum_{i=1}^s \lambda_i f_i(F(x)) - \sum_{i=1}^s \lambda_i (r^i)^T F(x) + t^T g(G(x))
\]

\[
- q^T G(x) + q^T F(x) - p^T x + q^T G(x) + p^T x
\]

\[
= \sum_{i=1}^s \lambda_i f_i(F(x)) + t^T g(G(x))
\]

\[
\leq \sum_{i=1}^s \lambda_i f_i(F(x)).
\]

The inequality \( \sum_{i=1}^s \lambda_i h_i(p, q, q', r, t, \lambda, u) \leq \sum_{i=1}^s \lambda_i f_i(F(x)) \) contradicts relation (4.2). Thus the weak duality between \((P_v)\) and \((D_v)\) holds. \( \square \)
Theorem 4.4 gives us the strong duality between the multiobjective problems \((P_v)\) and \((D_v)\).

**Theorem 4.4** Assume that the constraints qualification \((CQ)\) is fulfilled and let \(\bar{x}\) be a properly efficient element to \((P_v)\). Then there exists an efficient solution \((\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t}, \bar{\lambda}, \bar{u}) \in B\) to the dual \((D_v)\) and the strong duality \(f(F(\bar{x})) = h(\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t}, \bar{\lambda}, \bar{u})\) holds.

**Proof.** Let \(\bar{x}\) be a properly efficient element to \((P_v)\). By Definition 4.2, it follows that there exists a vector \(\bar{\lambda} = (\bar{\lambda}_1, ..., \bar{\lambda}_s)^T \in int(\mathbb{R}_+^s)\) such that \(\bar{x}\) solves the scalar problem

\[
(P_{\bar{\lambda}}) \quad \inf_{x \in A} \bar{\lambda}^T f(x).
\]

Because the constraint qualification \((CQ)\) is fulfilled, by Theorem 4.2, there exists an optimal solution \((\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t})\) to the dual problem \((D_{\bar{\lambda}})\) such that the optimality conditions \((i), (ii), (iii), (iv)\) and \((v)\) are satisfied.

By means of \(\bar{x}\) and \((\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t})\) we determine now an efficient solution \((\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t}, \bar{\lambda}, \bar{u})\) to \((D_v)\). In order to do this let \(\bar{\lambda} = (\bar{\lambda}_1, ..., \bar{\lambda}_s)^T\) be the vector given by the proper efficiency of \(\bar{x}\), \(\bar{p} = (\bar{p}_1, ..., \bar{p}_n)^T := (\bar{p}_1, ..., \bar{p}_n)^T = \bar{p}, \bar{q} = (\bar{q}_1, ..., \bar{q}_m)^T := (\bar{q}_1, ..., \bar{q}_m)^T = \bar{q}, \bar{q}' = (\bar{q}'_1, ..., \bar{q}'_m)^T := (\bar{q}'_1, ..., \bar{q}'_m)^T = \bar{q}', \bar{r} = (\bar{r}_1, ..., \bar{r}_s)^T := (\bar{r}_1, ..., \bar{r}_s)^T = \bar{r}\) and \(\bar{t} = (\bar{t}_1, ..., \bar{t}_k)^T := (\bar{t}_1, ..., \bar{t}_k)^T = \bar{t}\). It remains to define the vector \(\bar{u} = (\bar{u}_1, ..., \bar{u}_s)^T\). Therefore, let for \(i = 1, ..., s\), be

\[
\bar{u}_i := \frac{1}{s\bar{\lambda}_i} \left( (\bar{t}^T g)^*(\bar{q}') + \left(\bar{q}' F\right)^*_X(\bar{p}) + \left(\bar{q}' G\right)^*_X(-\bar{p}) \right) + \left(\bar{r}^i\right)^T F(\bar{x}). \tag{4.3}
\]

For \((\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t}, \bar{\lambda}, \bar{u})\) it holds \(\bar{q} \in \mathbb{R}_+^m, \bar{q}' \in \mathbb{R}_+^k, \bar{r}^i \in \mathbb{R}_+, i = 1, ..., s, \bar{t} \in \mathbb{R}_+, \bar{\lambda} \in int(\mathbb{R}_+^s)\) and

\[
\sum_{i=1}^s \bar{\lambda}_i \bar{u}_i = \sum_{i=1}^s \bar{\lambda}_i \frac{1}{s\bar{\lambda}_i} \left( \left(\bar{t}^T g\right)^*(\bar{q}') + \left(\bar{q}' F\right)^*_X(\bar{p}) + \left(\bar{q}' G\right)^*_X(-\bar{p}) \right) + \sum_{i=1}^s \bar{\lambda}_i (\bar{r}^i)^T F(\bar{x})
\]

\[
= \left(\bar{t}^T g\right)^*(\bar{q}') + \left(\bar{q}' F\right)^*_X(\bar{p}) + \left(\bar{q}' G\right)^*_X(-\bar{p}) + \sum_{i=1}^s \bar{\lambda}_i (\bar{r}^i)^T F(\bar{x}).
\]

Because \(\sum_{i=1}^s \bar{\lambda}_i \bar{r}^i = \bar{q}\), from the optimality conditions derived in Theorem 4.2 we obtain

\[
\sum_{i=1}^s \bar{\lambda}_i \bar{u}_i = \bar{q}' G(\bar{x}) - (\bar{t}^T g(G(\bar{x}))) + \bar{p}^T \bar{x} - \bar{q}^T F(\bar{x}) + (\bar{p})^T \bar{x} - \bar{q}^T G(\bar{x}) + \bar{q}'^T F(\bar{x}) = 0,
\]

which actually means that the element \((\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t}, \bar{\lambda}, \bar{u})\) is feasible to \((D_v)\).
Finally, we show that the values of the objective functions are equal, namely, \( f(F(\bar{x})) = h(\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t}, \bar{\lambda}, \bar{u}) \). In order to do this, we prove that \( f_i(F(\bar{x})) = h_i(\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t}, \bar{\lambda}, \bar{u}) \), for all \( i = 1, \ldots, s \). By Theorem 4.2, we have for all \( i = 1, \ldots, s \),

\[
\begin{align*}
\frac{1}{s\lambda_i} & \left( (\bar{t}^T g)^* (\bar{q}') + (\bar{q}^T F)^*_X (\bar{p}) + (\bar{q}^T G)^*_X (\bar{p}) \right) + \frac{1}{s\lambda_i} (\bar{r}^i)^T F(\bar{x}) = -f_i^*(\bar{r}^i) + (\bar{r}^i)^T F(\bar{x}) = f_i(F(\bar{x})).
\end{align*}
\]

The maximality of \((\bar{p}, \bar{q}, \bar{q}', \bar{r}, \bar{t}, \bar{\lambda}, \bar{u})\) is given by Theorem 4.3.

\[\square\]

### 4.1.5 Duality for the classical multiobjective optimization problem with inequality constraints

In this subsection we consider the multiobjective optimization problem

\[
(P'_e) \quad \inf_{x \in \mathcal{A}'} F(x),
\]

where

\[
\mathcal{A}' = \left\{ x \in X : G(x) \leq 0 \right\},
\]

\( X \subseteq \mathbb{R}^n, \quad F = (F_1, \ldots, F_s)^T : X \to \mathbb{R}^s \) and \( G = (G_1, \ldots, G_k)^T : X \to \mathbb{R}^k \). Additionally, let us assume that \( F_i, \; i = 1, \ldots, s \), and \( G_j, \; j = 1, \ldots, k \), are convex functions.

Let us give first the definitions of the efficient and properly efficient elements with respect to problem \((P'_e)\).

**Definition 4.4** An element \( \bar{x} \in \mathcal{A}' \) is said to be efficient (or Pareto efficient) with respect to \((P'_e)\) if from \( F(x) \leq F(\bar{x}) \), for \( x \in \mathcal{A}' \), it follows that \( F(x) = F(\bar{x}) \).

**Definition 4.5** An element \( \bar{x} \in \mathcal{A}' \) is said to be properly efficient with respect to \((P'_e)\) if there exists \( \lambda = (\lambda_1, \ldots, \lambda_s)^T \in \text{int} (\mathbb{R}^+_s) \), (i.e. \( \lambda_i > 0, \; i = 1, \ldots, s \)), such that \( \lambda^T F(\bar{x}) \leq \lambda^T F(x) \), for all \( x \in \mathcal{A}' \).

One may observe that \((P'_e)\) is a special case of the multiobjective problem studied in the previous subsection. Taking in problem \((P_e)\) the functions \( F = (F_1, \ldots, F_s)^T : X \to \mathbb{R}^s, \; G = (G_1, \ldots, G_k)^T : X \to \mathbb{R}^k, \; f = (f_1, \ldots, f_s)^T : \mathbb{R}^s \to \mathbb{R}^s \),
and \( g = (g_1, ..., g_k)^T : X \to \mathbb{R}^k \), such that \( f_i(y) = y_i \) for all \( y \in \mathbb{R}^s \) and \( i = 1, ..., s \), and \( g_j(z) = z_j \) for all \( z \in \mathbb{R}^k \) and \( j = 1, ..., k \), we actually obtain the multiobjective problem \((P'_\lambda)\). Defining \( f_i, i = 1, ..., s \), and \( g_j, j = 1, ..., k \), in this way, the functions \( f = (f_1, ..., f_s)^T \) and \( g = (g_1, ..., g_k)^T \) will be obviously convex and componentwise increasing.

Applying the results derived in the first part of this section, we determine a multiobjective dual to \((P'_\lambda)\) and then we verify the weak and strong duality. In order to do this, let us first consider the scalarized problem

\[
(P'\lambda) \quad \inf_{x \in \mathcal{A}'} \lambda^T F(x),
\]

where \( \lambda = (\lambda_1, ..., \lambda_s)^T \) is a fixed vector in \( \text{int}(\mathbb{R}_+^s) \). According to relation (4.1), the Fenchel-Lagrange dual of a scalarized problem is

\[
(D^\lambda_{FL}) \quad \sup_{(p, q, q', r, t) \in Y^\lambda} \left\{ -\sum_{i=1}^s \lambda_i f_i^*(r^i) - (r^T g)^*(q') - (q^T F)^*_X(p) - (q^T G)^*_X(-p) \right\},
\]

with

\[
Y^\lambda = \{(p, q, q', r, t) : p \in \mathbb{R}^n, q \in \mathbb{R}_+^s, q' \in \mathbb{R}_+^k, r = (r^1, ..., r^s),
\]

\[
r^i \in \mathbb{R}_+^s, i = 1, ..., s, \sum_{i=1}^s \lambda_i r^i = q, t \in \mathbb{R}_+^k \}. \]

Taking into consideration the definitions of the functions \( f_i, i = 1, ..., s \), and \( g_j, j = 1, ..., k \), respectively, we have for all \( i = 1, ..., s \),

\[
f_i^*(r^i) = \sup_{y \in \mathbb{R}^s} \{(r^i)^T y - f_i(y)\} = \sup_{y \in \mathbb{R}^s} \{(r^i)^T y - y_i\}
\]

\[
= \sup_{y \in \mathbb{R}^s} \{(r^i_1, ..., r^i_i - 1, ..., r^i_s)^T y\}
\]

\[
= \begin{cases} 0, & \text{if } r^i_i = 1 \text{ and } r^j_j = 0, j = 1, ..., s, j \neq i, \\ +\infty, & \text{otherwise,} \end{cases} \quad (4.4)
\]

\[
(t^T g)^*(q') = \sup_{y \in \mathbb{R}^k} \{q^T y - t^T g(y)\} = \sup_{y \in \mathbb{R}^k} \{q^T y - t^T y\}
\]

\[
= \sup_{y \in \mathbb{R}^k} \{(q' - t)^T y\} = \begin{cases} 0, & \text{if } q' = t, \\ +\infty, & \text{otherwise,} \end{cases} \quad (4.5)
\]

and

\[
(q^T F)^*_X(p) = (\left(\sum_{i=1}^s \lambda_i r^i\right)^T F)^*_X(p) = (\lambda^T F)^*_X(p), \quad \text{(by (4.4))}. \quad (4.6)
\]
Relations (4.1), (4.4), (4.5) and (4.6) imply that the dual looks like

\[
(D_{FL}^\lambda) \sup_{p \in \mathbb{R}^n, t \in \mathbb{R}^k_+} \left\{ - \left( \lambda^T F \right)_X^* (p) - (t^T G)_X^* (-p) \right\}.
\] (4.7)

According to Theorem 4.1 we can formulate the following strong duality theorem.

**Theorem 4.5** Assume that \( X \subseteq \mathbb{R}^n \) is a nonempty convex subset and the constraint qualification (CQ') (see Subsection 2.4.1) is fulfilled. Then it holds

\[
v(P^\lambda) = v(D_{FL}^\lambda).
\]

Provided that \( v(P^\lambda) > -\infty \), the strong duality holds, i.e. the optimal objective values of the primal and the dual problem coincide and the dual problem \((D_{FL}^\lambda)\) has an optimal solution.

**Proof.** Theorem 4.5 follows directly from Theorem 4.1. \(\Box\)

Let us give now the optimality conditions regarding the problems \((P^\lambda)\) and \((D_{FL}^\lambda)\).

**Theorem 4.6** (a) Let the assumptions of Theorem 4.5 be fulfilled and let \( \bar{x} \) be an optimal solution to \((P^\lambda)\). Then there exists a tuple \((\bar{p}, \bar{t}) \in \mathbb{R}^n \times \mathbb{R}^k_+\), optimal solution to \((D_{FL}^\lambda)\), such that the following optimality conditions are satisfied

\[
\begin{align*}
(i) \quad \lambda^T F(\bar{x}) + (\lambda^T F)_X^* (\bar{p}) & = \bar{p}^T \bar{x}, \\
(ii) \quad (t^T G)_X^* (-\bar{p}) & = -\bar{p}^T \bar{x}, \\
(iii) \quad \bar{t}^T G(\bar{x}) & = 0.
\end{align*}
\]

(b) Let \( \bar{x} \) be admissible to \((P^\lambda)\) and \((\bar{p}, \bar{t})\) be admissible to \((D_{FL}^\lambda)\), satisfying (i), (ii) and (iii). Then \( \bar{x} \) is an optimal solution to \((P^\lambda)\), \((\bar{p}, \bar{t})\) is an optimal solution to \((D_{FL}^\lambda)\) and strong duality holds.

**Proof.**

(a) By Theorem 4.5 there exists a tuple \((\bar{p}, \bar{t}) \in \mathbb{R}^n \times \mathbb{R}^k_+\), optimal solution to \((D_{FL}^\lambda)\), such that

\[
\lambda^T F(\bar{x}) = - (\lambda^T F)_X^* (\bar{p}) - (t^T G)_X^* (-\bar{p}),
\]

which implies that

\[
\left\{ \lambda^T F(\bar{x}) + (\lambda^T F)_X^* (\bar{p}) - \bar{p}^T \bar{x} \right\} + \left\{ t^T G(\bar{x}) + (t^T G)_X^* (-\bar{p}) + \bar{p}^T \bar{x} \right\} - \bar{t}^T G(\bar{x}) = 0.
\]

Using Young’s inequality we have

\[
\lambda^T F(\bar{x}) + (\lambda^T F)_X^* (\bar{p}) - \bar{p}^T \bar{x} \geq 0
\]
4.1 MULTICRITERIA PROBLEM WITH INEQUALITY CONSTRAINTS

\[ i^T G(\bar{x}) + (i^T G)_X^* (\bar{p}) + \bar{p}^T \bar{x} \geq 0. \]

Because \( \bar{x} \) is an optimal solution to \((P^\lambda)\) and \( i \in \mathbb{R}_+^k \), it follows that \( i^T G(\bar{x}) \leq 0 \), which actually implies relations (i), (ii) and (iii).

(b) All the calculations and transformations done within part (a) may be carried out in the inverse direction. \( \square \)

Having determined the optimality conditions for the scalarized problem, we are now able to construct a multiobjective dual problem to \((P'_v)\). Therefore, let us consider the following optimization problem

\[ (D'_v) \quad \text{v-max} \quad h'(p, t, \lambda, u), \]

with

\[ h'(p, t, \lambda, u) = \left( \begin{array}{c} h'_1(p, t, \lambda, u) \\ \vdots \\ h'_s(p, t, \lambda, u) \end{array} \right), \]

\[ h'_i(p, t, \lambda, u) = -\frac{1}{s\lambda_i} \left( (\lambda^T F)_X^* (p) + (i^T G)_X^* (\bar{p}) \right) + u_i, \quad i = 1, ..., s, \]

the dual variables

\[ p = (p_1, ..., p_n)^T \in \mathbb{R}^n, \ t = (t_1, ..., t_k)^T \in \mathbb{R}^k, \ \lambda = (\lambda_1, ..., \lambda_s)^T \in \mathbb{R}^s, \]

\[ u = (u_1, ..., u_s)^T \in \mathbb{R}^s, \]

and the set of constraints

\[ \mathcal{B}' = \left\{ (p, t, \lambda, u) : t \in \mathbb{R}_+^k, \ \lambda \in \text{int}(\mathbb{R}_+^s), \sum_{i=1}^s \lambda_i u_i = 0 \right\}. \]

The next two theorems yield the weak and strong duality for the multiobjective problems \((P'_v)\) and \((D'_v)\).

**Theorem 4.7** There is no \( x \in \mathcal{A}' \) and no \( (p, t, \lambda, u) \in \mathcal{B}' \) fulfilling \( F(x) \preceq_{\mathbb{R}_+^s} h'(p, t, \lambda, u) \) and \( F(x) \neq h'(p, t, \lambda, u) \).

**Proof.** Analogous to the proof of Theorem 4.3. \( \square \)

**Theorem 4.8** Assume that the constraints qualification \((CQ')\) is fulfilled and let \( \bar{x} \) be a properly efficient element to \((P'_v)\). Then there exists an efficient solution \((\bar{p}, \bar{t}, \bar{\lambda}, \bar{u}) \in \mathcal{B}' \) to the dual \((D'_v)\) and the strong duality \( F(\bar{x}) = h(\bar{p}, \bar{t}, \bar{\lambda}, \bar{u}) \) holds.
\textbf{Proof.} Let $\bar{x}$ be a properly efficient element to $(P'_v)$. By Definition 4.5, it follows that there exists a vector $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_s)^T \in \text{int}(\mathbb{R}^+_s)$ such that $\bar{x}$ solves the scalar problem

$$(P^{\bar{\lambda}}) \quad \inf_{x \in A} \bar{\lambda}^T F(x).$$

Because the constraint qualification (CQ') is fulfilled, by Theorem 4.6, there exists $(\bar{p}, \bar{t})$, an optimal solution to the dual problem $(D^\bar{p}_v)$, such that the optimality conditions (i), (ii) and (iii) are satisfied.

By means of $\bar{x}$ and $(\bar{p}, \bar{t})$ we determine now an efficient solution $(\bar{p}, \bar{t}, \bar{\lambda}, \bar{u})$ to $(D'_v)$. In order to do this let $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_s)^T$ be the vector given by the proper efficiency of $\bar{x}$, $\bar{p} = (\bar{p}_1, \ldots, \bar{p}_n)^T := (\bar{\lambda}_1, \ldots, \bar{\lambda}_s)^T = \bar{p}$, and $\bar{t} = (\bar{t}_1, \ldots, \bar{t}_k)^T := (\bar{t}_1, \ldots, \bar{t}_k)^T = \bar{t}$. It remains to define the vector $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_s)^T$. Therefore, let for $i = 1, \ldots, s$, be

$$\bar{u}_i := \frac{1}{s\bar{\lambda}_i} \left( (\bar{\lambda}^T F)_X^* (\bar{p}) + (\bar{t}^T G)_X^* (-\bar{p}) \right) + F_i(\bar{x}). \quad (4.8)$$

For $(\bar{p}, \bar{t}, \bar{\lambda}, \bar{u})$ it holds $\bar{t} \in \mathbb{R}^k, \bar{\lambda} \in \text{int}(\mathbb{R}^+_s)$ and

$$\sum_{i=1}^s \bar{\lambda}_i \bar{u}_i = \sum_{i=1}^s \bar{\lambda}_i \frac{1}{s\bar{\lambda}_i} \left( (\bar{\lambda}^T F)_X^* (\bar{p}) + (\bar{t}^T G)_X^* (-\bar{p}) \right) + \sum_{i=1}^s \bar{\lambda}_i F_i(\bar{x})$$

$$= \left( (\bar{\lambda}^T F)_X^* (\bar{p}) + (\bar{t}^T G)_X^* (-\bar{p}) \right) + \bar{\lambda}^T F(\bar{x}).$$

By the optimality conditions derived in Theorem 4.6 we have

$$\sum_{i=1}^s \bar{\lambda}_i \bar{u}_i = \bar{p}^T \bar{x} - \bar{\lambda}^T F(\bar{x}) - \bar{p}^T \bar{x} + \bar{\lambda}^T F(\bar{x}) = 0,$$

which actually means that the element $(\bar{p}, \bar{t}, \bar{\lambda}, \bar{u})$ is feasible to $(D'_v)$.

Finally, we show that the values of the objective functions are equal, namely, $F(\bar{x}) = h'_v(\bar{p}, \bar{t}, \bar{\lambda}, \bar{u})$. In order to do this, we prove that $F_i(\bar{x}) = h'_v(\bar{p}, \bar{t}, \bar{\lambda}, \bar{u})$, for all $i = 1, \ldots, s$. By Theorem 4.6, we have for all $i = 1, \ldots, s$,

$$h'_v(\bar{p}, \bar{t}, \bar{\lambda}, \bar{u}) = -\frac{1}{s\bar{\lambda}_i} \left( (\bar{\lambda}^T F)_X^* (\bar{p}) + (\bar{t}^T G)_X^* (-\bar{p}) \right) + \bar{u}_i$$

$$= -\frac{1}{s\bar{\lambda}_i} \left( (\bar{\lambda}^T F)_X^* (\bar{p}) + (\bar{t}^T G)_X^* (-\bar{p}) \right)$$

$$+ \frac{1}{s\bar{\lambda}_i} \left( (\bar{\lambda}^T F)_X^* (\bar{p}) + (\bar{t}^T G)_X^* (-\bar{p}) \right) + F_i(\bar{x}) = F_i(\bar{x}).$$

The maximality of $(\bar{p}, \bar{t}, \bar{\lambda}, \bar{u})$ is given by Theorem 4.7. \qed
4.1 MULTICRITERIA PROBLEM WITHOUT CONSTRAINTS

4.1.6 Duality for the multiobjective optimization problem without constraints

In what follows, let us consider the multiobjective optimization problem

\[(P_v^0) \quad \nu\text{-min } f(F(x)),\]

where \(X \subseteq \mathbb{R}^n, F = (F_1, \ldots, F_m)^T : X \to \mathbb{R}^m\) and \(f = (f_1, \ldots, f_s)^T : \mathbb{R}^m \to \mathbb{R}^s\). Assume that \(F_i, i = 1, \ldots, m\), are convex and \(f_j, j = 1, \ldots, s\), are convex and componentwise increasing functions.

Problem \((P_v^0)\) was already treated by G. WANKA, I. R. BOT and E. VARGYAS in [71], our purpose hereby is to show how these results can be obtained as special case from the general results formulated in subsections 4.1.3 and 4.1.4. Therefore, let us observe that problem \((P_v^0)\) can be obtained from problem \((P_v)\), by taking the functions \(G = (G_1, \ldots, G_l)^T : X \to \mathbb{R}^l,\ j = 1, \ldots, l,\) and \(g = (g_1, \ldots, g_k)^T : \mathbb{R}^l \to \mathbb{R}^k\), such that \(g_i(y) = 0\), for all \(i = 1, \ldots, k,\) and \(y \in \mathbb{R}^l\). Analogously to the previous section first we study the scalarized problem and then, by means of the scalarized dual, we determine a multiobjective dual to \((P_v^0)\). Finally, the weak and strong duality theorems are formulated.

Let us begin with the scalarized problem

\[(P_{\nu}^\lambda) \quad \inf_{x \in X} \lambda^T f(F(x)),\]

where \(\lambda = (\lambda_1, \ldots, \lambda_s)^T \in \text{int}(\mathbb{R}^*_+^s)\) is a fixed vector. By relation (4.1), the Fenchel-Lagrange dual of the scalarized problem is

\[
(D_{FL}^{\nu\lambda}) \quad \sup_{(p, q, q', r, t) \in Y_{\nu\lambda}} \left\{ -\sum_{i=1}^{s} \lambda_i f_i^*(r^i) - (t^T g)^*(q') - (q^T F)^* X(p) - (q'^T G)^* X(-p) \right\},
\]

with

\[
Y_{\nu\lambda} = \left\{ (p, q, q', r, t) : p \in \mathbb{R}^n, q \in \mathbb{R}^m_+, q' \in \mathbb{R}^l_+, r = (r^1, \ldots, r^s),
\right.
\]
\[
\left. r^i \in \mathbb{R}^+_+, i = 1, \ldots, s, \sum_{i=1}^{s} \lambda_i r^i = q, t \in \mathbb{R}^k_+ \right\}.
\]

Because in this case

\[
(t^T g)^*(q') = (0)^*(q') = \sup_{y \in \mathbb{R}^l} \{ y^T q' \} = \begin{cases} 0, & \text{if } q' = 0, \\ +\infty, & \text{otherwise}, \end{cases}
\]

and therefore

\[
(q'^T G)^*_X(-p) = 0_X^*(-p) = -\inf_{x \in X} p^Tx = \delta^*_X(-p),
\]
the Fenchel-Lagrange dual problem becomes
\[ (D_{FL}^{n\lambda}) \sup_{\begin{array}{c} p \in \mathbb{R}^n, q \in \mathbb{R}^m, r^i \in \mathbb{R}^m, i = 1, \ldots, s, \sum_{i=1}^{s} \lambda_i r^i = q \end{array}} \left\{ - \sum_{i=1}^{s} \lambda_i f^*_i(r^i) - (q^T F)^*_X(p) - \delta_X(-p) \right\}. \] (4.9)

According to theorems 4.1 and 4.2 we can formulate the strong duality theorem and give the optimality conditions for \((P_{00}^{n\lambda})\) and \((D_{FL}^{n\lambda})\).

**Theorem 4.9** Assume that \(X \subseteq \mathbb{R}^n\) is a nonempty convex subset. If \(v(P_{00}^{n\lambda}) > -\infty\), then its dual problem \((D_{FL}^{n\lambda})\) has an optimal solution and strong duality holds, i.e.
\[ v(P_{00}^{n\lambda}) = v(D_{FL}^{n\lambda}). \]

**Theorem 4.10** (a) Let the assumptions of Theorem 4.9 be fulfilled and let \(\bar{x}\) be an optimal solution to \((P_{00}^{n\lambda})\). Then there exists a tuple \((\bar{p}, \bar{q}, \bar{r})\), with \(\bar{r} = (\bar{r}^1, \ldots, \bar{r}^s)\), optimal solution to \((D_{FL}^{n\lambda})\), such that the following optimality conditions are satisfied
\[ \begin{align*}
(i) & \quad f^*_i(\bar{r}^i) + f_i(F(\bar{x})) = (\bar{r}^i)^T F(\bar{x}), \quad i = 1, \ldots, s, \\
(ii) & \quad (\bar{q}^T F)^*_X(\bar{p}) + \bar{q}^T F(\bar{x}) = \bar{p}^T \bar{x}, \\
(iii) & \quad \delta_X(-\bar{p}) = -\bar{p}^T \bar{x}.
\end{align*} \]

(b) Let \(\bar{x}\) be admissible to \((P_{00}^{n\lambda})\) and \((\bar{p}, \bar{t}, \bar{r})\), with \(\bar{r} = (\bar{r}^1, \ldots, \bar{r}^s)\), be admissible to \((D_{FL}^{n\lambda})\), satisfying (i), (ii) and (iii). Then \(\bar{x}\) is an optimal solution to \((P_{00}^{n\lambda})\), \((\bar{p}, \bar{t}, \bar{r})\) is an optimal solution to \((D_{FL}^{n\lambda})\) and strong duality holds.

In the following we construct a multiobjective dual to the problem \((P_{00}^{\nu})\).

Therefore, let us consider the optimization problem
\[ (D_{\nu}^{\nu}) \quad \text{v-max} \quad h''(p, q, r, \lambda, u), \]
with
\[ h''(p, q, r, \lambda, u) = \left( \begin{array}{c} h''_1(p, q, r, \lambda, u) \\ \vdots \\ h''_s(p, q, r, \lambda, u) \end{array} \right), \]
\[ h''_i(p, q, r, \lambda, u) = -f^*_i(r^i) - \frac{1}{s \lambda_i} \left( (q^T F)^*_X(p) + \delta_X(-p) \right) + u_i, \quad i = 1, \ldots, s, \]
the dual variables
\[ p = (p_1, \ldots, p_n)^T \in \mathbb{R}^n, \quad q = (q_1, \ldots, q_m)^T \in \mathbb{R}^m, \quad r = (r^1, \ldots, r^s), \quad r^i \in \mathbb{R}^m, \]
\[ i = 1, \ldots, s, \quad \lambda = (\lambda_1, \ldots, \lambda_s)^T \in \mathbb{R}^s, \quad u = (u_1, \ldots, u_s)^T \in \mathbb{R}^s, \]
4.1 MULTICRITERIA PROBLEM WITHOUT CONSTRAINTS

and the set of constraints

$$B'' = \left\{ (p, q, r, \lambda, u) : q \in \mathbb{R}_+^m, r^i \in \mathbb{R}_+^m, i = 1, \ldots, s, \lambda \in \text{int}(\mathbb{R}_+^s) \right\},$$

$$\sum_{i=1}^s \lambda_i r^i = q, \quad \sum_{i=1}^s \lambda_i u_i = 0.$$

The next two theorems provide us the weak and strong duality for the multi-objective problems \((P''_o)\) and \((D''_o)\).

**Theorem 4.11** There is no \(x \in X\) and no \((p, q, r, \lambda, u) \in B''\) fulfilling \(f(F(x)) \leq h''(p, q, r, \lambda, u)\) and \(f(F(x)) \neq h''(p, q, r, \lambda, u)\).

**Proof.** Analogous to the proof of Theorem 4.3. \(\square\)

**Theorem 4.12** Let \(\bar{x}\) be a properly efficient element to \((P''_o)\). Then there exists an efficient solution \(\bar{x} = (\bar{p}, \bar{q}, \bar{r}, \bar{\lambda}, \bar{u}) \in B''\) to the dual \((D''_o)\) and the strong duality \(f(F(\bar{x})) = h''(\bar{p}, \bar{q}, \bar{r}, \bar{\lambda}, \bar{u})\) holds.

**Proof.** Let \(\bar{x}\) be a properly efficient element to \((P''_o)\). By the definition of proper efficiency, it follows that there exists a vector \(\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_s)^T \in \text{int}(\mathbb{R}_+^s)\) such that \(\bar{x}\) solves the scalar problem

\[ (P''_{\bar{\lambda}}) \quad \inf_{x \in X} \bar{\lambda}^T f(F(x)). \]

By Theorem 4.10, there exists \((\bar{p}, \bar{q}, \bar{r})\), an optimal solution to the dual problem \((D''_{\bar{\lambda}o})\), such that the optimality conditions (i), (ii) and (iii) are satisfied.

By means of \(\bar{x}\) and \((\bar{p}, \bar{q}, \bar{r})\) we determine now an efficient solution \((\bar{p}, \bar{q}, \bar{r}, \bar{\lambda}, \bar{u})\) to \((D''_o)\). In order to do this let \(\bar{\lambda} = (\lambda_1, \ldots, \lambda_s)^T\) be the vector given by the proper efficiency of \(\bar{x}\), \(\bar{p} = (\bar{p}_1, \ldots, \bar{p}_n)^T := (\bar{p}_1, \ldots, \bar{p}_n)^T = \bar{p}, \bar{q} = (\bar{q}_1, \ldots, \bar{q}_m)^T := (\bar{q}_1, \ldots, \bar{q}_m)^T = \bar{q}, \) and \(\bar{r} = (\bar{r}^1, \ldots, \bar{r}^s)^T := (\bar{r}^1, \ldots, \bar{r}^s)^T = \bar{r}\). It remains to define the vector \(\bar{u} = (\bar{u}_1, \ldots, \bar{u}_s)^T\). Therefore, let for \(i = 1, \ldots, s\), be

\[ \bar{u}_i := \frac{1}{s \lambda_i} \left( (\bar{q}^T F)^*_X (\bar{p}) + \delta_X^* (-\bar{p}) \right) + (\bar{r}^i)^T F(\bar{x}). \quad (4.10) \]

For \((\bar{p}, \bar{q}, \bar{r}, \bar{\lambda}, \bar{u})\) it holds \(\bar{q} \in \mathbb{R}_+^m, \bar{r}^i \in \mathbb{R}_+^m, i = 1, \ldots, s, \bar{\lambda} \in \text{int}(\mathbb{R}_+^s)\) and

\[ \sum_{i=1}^s \bar{\lambda}_i \bar{u}_i = \sum_{i=1}^s \bar{\lambda}_i \frac{1}{s \lambda_i} \left( (\bar{q}^T F)^*_X (\bar{p}) + \delta_X^* (-\bar{p}) \right) + \sum_{i=1}^s \bar{\lambda}_i (\bar{r}^i)^T F(\bar{x}) \]

\[ = \left( (\bar{q}^T F)^*_X (\bar{p}) + \delta_X^* (-\bar{p}) \right) + \bar{q}^T F(\bar{x}). \]
By the optimality conditions derived in Theorem 4.10, we have

$$\sum_{i=1}^{s} \lambda_i u_i = \bar{p}^T \bar{x} - \bar{q}^T F(\bar{x}) - \bar{p}^T \bar{x} + \bar{q}^T F(\bar{x}) = 0,$$

which actually means that the element \((\bar{p}, \bar{q}, \bar{r}, \bar{\lambda}, \bar{u})\) is feasible to \((D''_v)\).

Finally, we show that the values of the objective functions are equal, namely, 

$$f(F(x)) = h''(\bar{p}, \bar{q}, \bar{r}, \bar{\lambda}, \bar{u}).$$

In order to do this we prove that 

$$f_i(F(x)) = h''_i(\bar{p}, \bar{q}, \bar{r}, \bar{\lambda}, \bar{u}),$$

for all \(i = 1, \ldots, s\). By Theorem 4.10, we have for all \(i = 1, \ldots, s\),

$$h''_i(\bar{p}, \bar{q}, \bar{r}, \bar{\lambda}, \bar{u}) = -f_i^*(\bar{r}^i) - \frac{1}{s\lambda_i} \left( (\bar{q}^T F)^*_X (\bar{p}) + \delta_X^*(\bar{p}) \right) + u_i$$

$$= -f_i^*(\bar{r}^i) - \frac{1}{s\lambda_i} \left( (\bar{q}^T F)^*_X (\bar{p}) + \delta_X^*(\bar{p}) \right)$$

$$+ \frac{1}{s\lambda_i} \left( (\bar{q}^T F)^*_X (\bar{p}) + \delta_X^*(\bar{p}) \right) + (\bar{r}^i)^T F(\bar{x}) = f_i(F(\bar{x})).$$

The efficiency of \((\bar{p}, \bar{q}, \bar{r}, \bar{\lambda}, \bar{u})\) is given by Theorem 4.11.

\[\square\]

### 4.2 Special cases

The last section of this work is motivated by a paper of S. NICKEL, J. PUERTO and A. M. RODRIGUEZ-CHIA [52], in which the authors studied a single-objective location problem with sets as existing facilities, giving a geometrical characterization of the set of optimal solutions. In [5], R. I. BOȚ and G. WANKA treated the same problem by means of conjugate duality. Our purpose here is to study, based on former results of this work, the duality for a multiobjective location problem involving sets as existing facilities.

In order to do this, first we consider a more general multiobjective problem in which the components of the objective function are composites of different monotonic norms with a vector-valued convex function. This problem turns out to be a special case of the nonconstrained problem \((P''_v)\) studied in Subsection 4.1.6. Applying the results obtained in the previous section we study the multiobjective problem from above, taking into consideration some properties of the monotonic norms. Using the results derived for monotonic norms we introduce the multiobjective dual problem and study the weak and strong duality for the multiobjective location model involving sets as existing facilities. Afterwards, as particular cases of this problem, the multiobjective Weber and minmax problems with infimal distances are studied. The last three location models were treated in detail by G. WANKA, R. I. BOȚ and E. VARGYAS in [71].
4.2 THE MULTICRITERIA MODEL WITH MONOTONIC NORMS

4.2.1 The case of monotonic norms

Let \( X \) be a nonempty subset of \( \mathbb{R}^n \) and \( F = (F_1, ..., F_m)^T : X \rightarrow \mathbb{R}^m \), \( l = (l_1, ..., l_s)^T : \mathbb{R}^m \rightarrow \mathbb{R}^s \) be vector-valued functions. Assume that \( F_i : X \rightarrow \mathbb{R}, i = 1, ..., m \), are convex functions on \( X \) and \( l_i : \mathbb{R}^m \rightarrow \mathbb{R}, i = 1, ..., s \), are monotonic norms on \( \mathbb{R}^m \). The optimization problem which we consider here is

\[
(P_{vl}) \quad \text{v-min} \quad l^+(F(x)),
\]

where \( l^+ = (l_1^+, ..., l_s^+)^T \) such that \( l_i^+(t) := l_i(t^+), i = 1, ..., s \), with \( t^+ = (t_1^+, ..., t_m^+)^T \) and \( t_j^+ = \max\{0, t_j\}, j = 1, ..., m \).

Applying the results obtained in Subsection 4.1.6, we derive a multiobjective dual to \((P_{vl})\) and formulate the weak and strong duality theorems. Therefore, let us first consider the scalarized problem

\[
(P_{vl}) \quad \text{inf} \quad \lambda^T l^+(F(x)),
\]

where \( \lambda = (\lambda_1, ..., \lambda_s)^T \in \text{int}(\mathbb{R}_+^s) \) is a fixed vector. By the results obtained in Subsection 4.1.6, its Fenchel-Lagrange dual is (see relation (4. 9))

\[
(D_{FL}^{\lambda}) \quad \sup_{p \in \mathbb{R}^n, q \in \mathbb{R}_+^m, r^i \in \mathbb{R}_+^m, \sum_{i=1}^s \lambda_i r^i = q} \left\{ - \sum_{i=1}^s \lambda_i (l_i^+)^*(r^i) - (q^T F)^*_X (p) - \delta_X(-p) \right\}.
\]

Because \( l_i, i = 1, ..., s \), are monotoric norms, by Proposition 3.7 for all \( i = 1, ..., s \), we have

\[
(l_i^+)^*(r^i) = \begin{cases} 0, & \text{if } r^i \in \mathbb{R}_+^m \text{ and } l_i^0(r^i) \leq 1, \\ +\infty, & \text{otherwise,} \end{cases}
\]

where \( l_i^0 \) is the dual norm of \( l_i \), and so, the Fenchel-Lagrange dual becomes

\[
(D_{FL}^{\lambda}) \quad \sup_{p \in \mathbb{R}^n, q \in \mathbb{R}_+^m, r^i \in \mathbb{R}_+^m, \sum_{i=1}^s \lambda_i r^i = q} \left\{ - (q^T F)^*_X (p) - \delta_X(-p) \right\}. \quad (4. 11)
\]

Analogously to Theorem 4.9 and Theorem 4.10 we have:

**Theorem 4.13** Assume that \( X \subseteq \mathbb{R}^n \) is a nonempty convex subset. If \( v(P_{vl}) > -\infty \), then its dual problem \((D_{FL}^{\lambda})\) has an optimal solution and strong duality holds, i.e.

\[ v(P_{vl}) = v(D_{FL}^{\lambda}). \]
Theorem 4.14 (a) Let the assumptions of Theorem 4.13 be fulfilled and let $\bar{x}$ be an optimal solution to $(P^{\lambda})$. Then there exists a tuple $(\bar{p}, \bar{q}, \bar{r}^1, ..., \bar{r}^s)$, optimal solution to $(D^L_{FL})$, such that the following optimality conditions are satisfied

$$(i) \quad l^+(F(\bar{x})) = (\bar{r}^i)^TF(\bar{x}), \quad i = 1, ..., s,$$

$$(ii) \quad (\bar{q}^T F)_X^*(\bar{p}) + \bar{q}^T F(\bar{x}) = \bar{p}^T \bar{x},$$

$$(iii) \quad \delta_X^*(-\bar{p}) = -\bar{p}^T \bar{x}.$$

(b) Let $\bar{x}$ be admissible to $(P^{\lambda})$ and $(\bar{p}, \bar{q}, \bar{r}^1, ..., \bar{r}^s)$ be admissible to $(D^L_{FL})$, satisfying (i), (ii) and (iii). Then $\bar{x}$ is an optimal solution to $(P^{\lambda})$, $(\bar{p}, \bar{q}, \bar{r}^1, ..., \bar{r}^s)$ is an optimal solution to $(D^L_{FL})$ and strong duality holds.

Furthermore we construct a multiobjective dual problem to $(P^l_v)$,

$$(D^l_v) \quad \text{v-max} \quad h^l(p, q, r, \lambda, u),$$

with

$$h^l(p, q, r, \lambda, u) = \begin{pmatrix}
    h^l_1(p, q, r, \lambda, u) \\
    \vdots \\
    h^l_s(p, q, r, \lambda, u)
\end{pmatrix},$$

$$h^l_i(p, q, r, \lambda, u) = \frac{1}{s} \left( (q^TF)_X^*(p) + \delta_X^*(-p) \right) + u_i, \quad i = 1, ..., s,$$

the dual variables

$$p = (p_1, ..., p_n) \in \mathbb{R}^n, \quad q = (q_1, ..., q_m) \in \mathbb{R}^m, \quad r = (r^1, ..., r^s), \quad r^i \in \mathbb{R}^m, \quad i = 1, ..., s, \quad \lambda = (\lambda_1, ..., \lambda_s) \in \mathbb{R}^s, \quad u = (u_1, ..., u_s) \in \mathbb{R}^s,$$

and the set of constraints

$$\mathcal{B}^l = \left\{(p, q, r, \lambda, u) : q \in \mathbb{R}^m, \quad r^i \in \mathbb{R}^m, \quad l^0_i(r^i) \leq 1, \quad i = 1, ..., s, \right\}.$$

The next two theorems provide the weak and strong duality for the multiobjective problems $(P^l_v)$ and $(D^l_v)$.

Theorem 4.15 There is no $x \in X$ and no $(p, q, r, \lambda, u) \in \mathcal{B}^l$ fulfilling $l^+(F(x)) \leq h^l(p, q, r, \lambda, u)$ and $l^+(F(x)) \neq h^l(p, q, r, \lambda, u)$.

Proof. Analogous to the proof of Theorem 4.3. \qed

Theorem 4.16 Let $\bar{x}$ be a properly efficient element to $(P^l_v)$. Then there exists an efficient element $(\bar{p}, \bar{q}, \bar{r}, \bar{\lambda}, \bar{u}) \in \mathcal{B}^l$, solution to the dual $(D^l_v)$, and the strong duality $l^+(F(\bar{x})) = h^l(\bar{p}, \bar{q}, \bar{r}, \bar{\lambda}, \bar{u})$ holds.

Proof. Theorem 4.16 is a direct consequence of Theorem 4.12. \qed
4.2 THE MODEL INVOLVING SETS AS EXISTING FACILITIES

4.2.2 The multiobjective location model involving sets as existing facilities

Let $\mathcal{C} = \{C_1, ..., C_m\}$ be a family of convex sets in $\mathbb{R}^n$ such that $\bigcap_{i=1}^m \overline{C}_i = \emptyset$, where $\overline{C}_i$ denotes the closure of the set $C_i$, for all $i = 1, ..., m$. We consider the same vector function $d : \mathbb{R}^n \to \mathbb{R}^m$ as in [5], i.e.

$$d(x) := (d_1(x; C_1), ..., d_m(x; C_m))^T,$$

where

$$d_i(x; C_i) = \inf \{\gamma_i(x - y) : y \in C_i\}, \quad i = 1, ..., m,$$

and $\gamma_i$, $i = 1, ..., m$, are norms on $\mathbb{R}^n$.

**Remark 4.3** Because $C_i$ are convex sets and $\gamma_i$ are norms, $i = 1, ..., m$, it follows that the functions $d_i(x; C_i)$ are convex and continuous on $\mathbb{R}^n$, for all $i = 1, ..., m$.

The multiobjective location problem with sets as existing facilities is

$$(P^l(\mathcal{C})) \quad \text{v-min } l(d(x)), \quad x \in \mathbb{R}^n,$$

with $l = (l_1, ..., l_s)^T$ and $l_j : \mathbb{R}^m \to \mathbb{R}$, $j = 1, ..., s$, monotonic norms on $\mathbb{R}^m$. Because

$$l_j^+(d(x)) = l_j((d(x))^+) = l_j(d(x)), \quad \forall x \in \mathbb{R}^n, \quad j = 1, ..., s,$$

where $(d(x))^+ = ((d_1(x))^+, ..., (d_m(x))^+)$ with $(d_i(x))^+ = \max\{0, d_i(x)\}$, for $i = 1, ..., m$, we can write $(P^l(\mathcal{C}))$ in the equivalent form

$$(P^l(\mathcal{C})) \quad \text{v-min } l^+(d(x)).$$

As one can see, $(P^l(\mathcal{C}))$ is a particular case of the problem $(P^l_v)$. In order to study the duality for this problem, we study again at first the duality for the scalarized problem

$$(P^{l,\lambda}(\mathcal{C})) \quad \inf_{x \in \mathbb{R}^n} \lambda^T l^+(d(x)),$$

with $\lambda = (\lambda_1, ..., \lambda_s)^T \in \text{int}(\mathbb{R}_+^s)$ fixed.

According to relation (4.11), its Fenchel-Lagrange dual problem is

$$(D_{FL}^{l,\lambda}(\mathcal{C})) \quad \sup_{p \in \mathbb{R}^n, q \in \mathbb{R}^m, \sum_{j=1}^s \lambda_j r^j \leq q, r^j \leq 0} \left\{ -p^T d^*(p) - \delta_{\mathbb{R}_+^m}(-p) \right\}.$$

Taking into consideration that

$$-\delta_{\mathbb{R}_+}(-p) = \inf_{x \in \mathbb{R}^n} p^T x = \begin{cases} 0, & \text{if } p = 0, \\ -\infty, & \text{otherwise}, \end{cases}$$

we have

$$\inf_{x \in \mathbb{R}^n} p^T x = -\delta_{\mathbb{R}_+}(p).$$
and for $q \in \mathbb{R}_+^m$, by Theorem 2.2 and Remark 4.3,

$$(q^T d)^*(0) = \left( \sum_{i=1}^m q_id_i \right)^*(0) = \inf \left\{ \sum_{i=1}^m (q_id_i)^*(p^i) : \sum_{i=1}^m p^i = 0 \right\},$$

the dual problem becomes

$$(D_{FL}^{\lambda}(C)) \sup_{p^i \in \mathbb{R}^n, i=1,\ldots,m, \sum_{i=1}^m p^i = 0, q, r^j \in \mathbb{R}_+^m, \ell_j^0(r^j) \leq 1, j=1,\ldots,s, \sum_{j=1}^s \lambda_j r^j = q} \left\{ - \sum_{i=1}^m (q_id_i)^*(p^i) \right\}.$$  

In order to get the same results as the authors in [71], in the objective function of this dual we separate the terms for which $q_i > 0$ from those for which $q_i = 0$ and then the dual can be written as

$$(D_{FL}^{\lambda}(C)) \sup_{p^i \in \mathbb{R}^n, i=1,\ldots,m, \sum_{i=1}^m p^i = 0, q, r^j \in \mathbb{R}_+^m, \ell_j^0(r^j) \leq 1, j=1,\ldots,s, \sum_{j=1}^s \lambda_j r^j = q} \left\{ - \sum_{i \in I} (q_id_i)^*(p^i) - \sum_{i \notin I} (0)^*(p^i) \right\}.$$  

For $i \notin I$ we have

$$(0)^*(p^i) = \sup_{x \in \mathbb{R}^n} \{(p^i)^T x - 0\} = \sup_{x \in \mathbb{R}^n} \{(p^i)^T x\} = \begin{cases} 0, & \text{if } p^i = 0, \\ +\infty, & \text{otherwise}. \end{cases}$$

For $i \in I$, it holds $(q_id_i)^*(p^i) = q_id_i^*(p_i^i/\ell_i^0(r^j))$, (cf. [14]). Redenoting $\frac{1}{q_i} p^i$ by $p^i$, we obtain

$$(D_{FL}^{\lambda}(C)) \sup_{(I, p, q, r) \in Y(I)} \left\{ - \sum_{i \in I} q_id_i^*(p^i) \right\},\quad (4.12)$$

with

$$Y(I) = \left\{ (I, p, q, r) : I \subseteq \{1,\ldots,m\}, p = (p^1,\ldots,p^m), p^i \in \mathbb{R}^n, i = 1,\ldots,m, q = (q_1,\ldots,q_m)^T \in \mathbb{R}^m, q_i > 0, i \in I, q_i = 0, i \notin I, r = (r^1,\ldots,r^s), r^j \in \mathbb{R}_+^m, \ell_j^0(r^j) \leq 1, j = 1,\ldots,s, \sum_{i \in I} q_i p^i = 0, \sum_{j=1}^s \lambda_j r^j = q \right\}.$$

The next theorems present the strong duality and the optimality conditions for $(P_{FL}^{\lambda}(C))$ and $(D_{FL}^{\lambda}(C))$, respectively.
Theorem 4.17  If $v(P^{\lambda}(\mathcal{C})) > -\infty$, then the dual problem $(D^{\lambda}_{FL}(\mathcal{C}))$ has an optimal solution and strong duality holds,

$$v(P^{\lambda}(\mathcal{C})) = v(D^{\lambda}_{FL}(\mathcal{C})).$$

Theorem 4.18 (a) Let $\bar{x}$ be an optimal solution to $(P^{\lambda}(\mathcal{C}))$. Then there exists a tuple $(\bar{I}, \bar{p}, \bar{q}, \bar{r}) \in Y^l(\mathcal{C})$, optimal solution to $(D^{\lambda}_{FL}(\mathcal{C}))$, such that the following optimality conditions are satisfied

(i) $\bar{I} \subseteq \{1, ..., m\}$, $\bar{I} \neq \emptyset$, $\bar{q}_i > 0$, $i \in \bar{I}$, $\bar{q}_i = 0$, $i \notin \bar{I}$,
(ii) $\bar{r}^j \in \mathbb{R}^n_+$, $l^0_j(\bar{r}^j) = 1$, $j = 1, ..., s$, $\sum_{j=1}^{s} \lambda_j \bar{r}^j = \bar{q}$, $\sum_{i \in \bar{I}} \bar{q}_i \bar{p}^i = 0$,
(iii) $l_j(d(\bar{x})) = (\bar{r}^j)^T d(\bar{x})$, $j = 1, ..., s$,
(iv) $\bar{x} \in \partial d^*_i(\bar{p}^i)$, $i \in \bar{I}$.

(b) If $\bar{x} \in \mathbb{R}^n$, $(\bar{I}, \bar{p}, \bar{q}, \bar{r}) \in Y^l(\mathcal{C})$ and (i), (ii), (iii) and (iv) are fulfilled, then $\bar{x}$ is an optimal solution to $(P^{\lambda}(\mathcal{C}))$, $(\bar{I}, \bar{p}, \bar{q}, \bar{r}) \in Y^l(\mathcal{C})$ is an optimal solution to $(D^{\lambda}_{FL}(\mathcal{C}))$ and strong duality holds

$$X^Tl(d(\bar{x})) = -\sum_{i \in \bar{I}} \bar{q}_i d^*_i(\bar{p}^i).$$

Proof. Because $\bar{x}$ is an optimal solution to $(P^{\lambda}(\mathcal{C}))$, by Theorem 4.17 it follows that there exists $(\bar{I}, \bar{p}, \bar{q}, \bar{r}) \in Y^l(\mathcal{C})$, optimal solution to $(D^{\lambda}_{FL}(\mathcal{C}))$, such that

$$X^Tl(d(\bar{x})) = -\sum_{i \in \bar{I}} \bar{q}_i d^*_i(\bar{p}^i). \quad (4.13)$$

Because $(\bar{I}, \bar{p}, \bar{q}, \bar{r}) \in Y^l(\mathcal{C})$, it follows that $\bar{I} \subseteq \{1, ..., m\}$, $\bar{q}_i > 0$, $i \in \bar{I}$, $\bar{q}_i = 0$, $i \notin \bar{I}$, $\bar{r}^j \in \mathbb{R}^n_+$ such that $l^0_j(\bar{r}^j) \leq 1$, for all $j = 1, ..., s$, $\sum_{j=1}^{s} \lambda_j \bar{r}^j = \bar{q}$ and $\sum_{i \in \bar{I}} \bar{q}_i \bar{p}^i = 0$. Additionally, by Proposition 3.7 $l^0_j(\bar{r}^j) = 0$, $j = 1, ..., s$, and so, equation (4.13) becomes

$$\sum_{j=1}^{s} \lambda_j \left(l_j(d(\bar{x})) + l^*_j(\bar{r}^j) - (\bar{r}^j)^T d(\bar{x})\right) + \sum_{i \in \bar{I}} \bar{q}_i \left(d_i(\bar{x}, C_i) + d^*_i(\bar{p}^i) - (\bar{p}^i)^T \bar{x}\right) = 0,$$

which together with Young’s inequality implies that

$$l_j(d(\bar{x})) = (\bar{r}^j)^T d(\bar{x}), \quad j = 1, ..., s,$$

$$d_i(\bar{x}, C_i) + d^*_i(\bar{p}^i) = (\bar{p}^i)^T \bar{x}, \quad i \in \bar{I}.$$
If $\bar{I}$ would be empty, then it would follow that $\bar{q}_i = 0$, for all $i = 1, \ldots, m$, which together with $\sum_{j=1}^{s} \lambda_j \bar{r}^j = \bar{q}$ and $\bar{r}^j \geq 0$, $j = 1, \ldots, s$, imply that $\bar{r}^j = 0$, $j = 1, \ldots, s$. From (i') it holds then $l_j(d(\bar{x})) = 0$, which actually means that $d(\bar{x}) = 0$, i.e.

$$d_i(\bar{x}, C_i) = 0, \forall i = 1, \ldots, m.$$ 

But, this would imply that $\bar{x} \in \bigcap_{i=1}^{m} C_i$, which is a contradiction to the hypothesis $\bigcap_{i=1}^{m} C_i = \emptyset$. By this, the relation (i) is proved.

Now, let us show that $l_j^0(\bar{r}^j) = 1$, $j = 1, \ldots, s$. By the definition of the dual norm, we have

$$l_j^0(\bar{r}^j) = \sup_{l_j(v) \leq 1, v \in \mathbb{R}^m} \{(\bar{r}^j)^Tv\}, \quad j = 1, \ldots, s.$$ 

Because $\bigcap_{i=1}^{m} C_i = \emptyset$, it holds $l_j(d(\bar{x})) > 0$, for $j = 1, \ldots, s$. Let be $\bar{v}^j = \frac{1}{l_j(d(\bar{x}))}d(\bar{x}) \in \mathbb{R}^m$. We have $l_j(\bar{v}^j) = 1$, $j = 1, \ldots, s$, and then, by (i'),

$$l_j^0(\bar{r}^j) = l_j^0(\bar{r}^j)l_j(\bar{v}^j) \geq (\bar{r}^j)^Td(\bar{x}) = \frac{(\bar{r}^j)^Td(\bar{x})}{l_j(d(\bar{x}))} = 1.$$ 

In conclusion, $l_j^0(\bar{r}^j) = 1$, $j = 1, \ldots, s$.

For (iv), let us observe that (iiv) is equivalent to $\bar{p}^i \in \partial d_\ast(\bar{x}, C_i)$ for $i \in \bar{I}$ (cf. [14]). On the other hand, $d_\ast$ being a convex and continuous function, verifies (cf. [14])

$$\bar{p}^i \in \partial d_\ast(\bar{x}, C_i) \iff \bar{x} \in \partial d_\ast(\bar{p}^i), \quad i \in \bar{I},$$

which proves (iiv). \hfill \Box

**Remark 4.4** We denoted here by $\partial f(x)$ the subdifferential of the function $f$ at the point $x$.

As a multiobjective dual problem of the primal problem $(P^d(C))$ we can introduce

$$(D^d(C)) \quad \forall \max_{(I,p,q,r,\lambda,u) \in Y(C)} h^d(I,p,q,r,\lambda,u),$$

with

$$h^d(I,p,q,r,\lambda,u) = \begin{pmatrix} h^d_1(I,p,q,r,\lambda,u) \\ \vdots \\ h^d_s(I,p,q,r,\lambda,u) \end{pmatrix},$$

$$h^d_j(I,p,q,r,\lambda,u) = -\frac{1}{s\lambda_j} \left( \sum_{i \in I} q_i d^0_i(p^i) \right) + u_j, \quad j = 1, \ldots, s,$$
the dual variables
\[ I \subseteq \{1, \ldots, m\}, \quad p = (p^1, \ldots, p^m), \quad p^i \in \mathbb{R}^n, \quad r = (r^1, \ldots, r^s), \quad r^j \in \mathbb{R}^m, \quad j = 1, \ldots, s, \quad \lambda = (\lambda_1, \ldots, \lambda_s)^T \in \mathbb{R}^s, \quad u = (u_1, \ldots, u_s)^T \in \mathbb{R}^s, \]
and the set of constraints
\[ \mathcal{Y}^i(C) = \left\{(I, p, q, r, \lambda, u) : I \subseteq \{1, \ldots, m\}, q_i > 0, i \in I, q_i = 0, i \notin I, r^j \in \mathbb{R}^m, \right\} \]
\[ l_j(r^j) = 1, \quad j = 1, \ldots, s, \quad \lambda \in \text{int}(\mathbb{R}^s_+), \quad \sum_{i \in I} w_i p^i = 0, \quad \sum_{j=1}^s \lambda_j r^j = q, \quad \sum_{j=1}^s \lambda_j u_j = 0 \}. \]

The following theorems state the weak and strong duality assertions.

**Theorem 4.19** There is no \( x \in \mathbb{R}^n \) and no \((I, p, q, r, \lambda, u) \in \mathcal{Y}^i(C)\), such that \( l_j(d(x)) \leq h_j^d(I, p, q, r, \lambda, u), \quad j = 1, \ldots, s, \) and \( l_k(d(x)) < h_k^d(I, p, q, r, \lambda, u) \) for at least one \( k \in \{1, \ldots, s\} \).

**Theorem 4.20** Let \( \bar{x} \) be properly efficient element to \((P^1(C))\). Then there exists an efficient solution \((\bar{I}, \bar{p}, \bar{q}, \bar{r}, \bar{\lambda}, \bar{u}) \in \mathcal{Y}^i(C)\) to \((D^1(C))\) and strong duality
\[ l_j(d(\bar{x})) = -\frac{1}{s \lambda_j} \left( \sum_{i \in I} \bar{q}_i d_i^n(\bar{p}^i) \right) + \bar{u}_j, \quad j = 1, \ldots, s, \]
holds.

### 4.2.3 The biobjective Weber-minmax problem with infimal distances

In this subsection, for the same data set \( C = \{C_1, \ldots, C_m\} \) as in the previous one, we consider a multiobjective minimization problem with a two-dimensional objective function, its first component being given by the Weber location problem and the second one by the minmax location problem with infimal distances. Thus, the primal problem is
\[ (P^{W\text{M}}(C)) \]
\[ \text{v-min} \quad \left( \sum_{i=1}^m w_i d_i(x, C_i) \right) \]
where \( d_i(x, C_i) = \inf_{y_i \in C_i} \gamma_i(x - y_i), \quad i = 1, \ldots, m, \) and \( w_i > 0, \quad i = 1, \ldots, m, \) are positive weights. Let be, for \( i = 1, \ldots, m, \) the norms \( \gamma'_i : \mathbb{R}^n \to \mathbb{R}, \gamma'_i = w_i \gamma_i \) and the corresponding distance functions \( d'_i(x, C_i) : \mathbb{R}^n \to \mathbb{R}, \quad d'_i(x, C_i) = \inf_{y_i \in C_i} \gamma'_i(x - \)
The biobjective dual to the primal problem \((P^{WM}(C))\), as a special case of \((P^i(C))\), becomes

\[
(P^{WM}(C)) \quad \min_{x \in \mathbb{R}^n} \left( \frac{l_1(d'(x))}{l_\infty(d'(x))} \right),
\]

with \(d'(x) = (d'_1(x, C_1), \ldots, d'_m(x, C_m))\) and the norms \(l_1, l_\infty : \mathbb{R}^m \to \mathbb{R}\), \(l_1(z) = \sum_{i=1}^m |z_i|, l_\infty(z) = \max_{i=1, \ldots, m} |z_i|\), for \(z \in \mathbb{R}^m\). As for the dual norms, we recall that \(l_1^0(z) = l_\infty(z)\) and \(l_\infty^0(z) = l_1(z)\). Obviously, \(l_1\) and \(l_\infty\) are monotonic norms.

Taking into consideration the form of the dual problem \((D^i(C))\), observing that \(d^*_i(p^i) = (w_i d^*_i(p^i)) = w_i d^*_i(\frac{1}{w_i} p^i)\), and, redenoting \(\frac{1}{w_i} p^i\) by \(p^i\), we construct the biobjective dual to the primal problem \((P^{WM}(C))\). This becomes

\[
(D^{WM}(C)) \quad \max_{(I, p, q, r, \lambda, u) \in Y^{WM}(C)} \left( \frac{h_1(I, p, q, r, \lambda, u)}{h_2(I, p, q, r, \lambda, u)} \right),
\]

with

\[
\begin{align*}
    h_1(I, p, q, r, \lambda, u) &= -\frac{1}{2\lambda_1} \left( \sum_{i \in I} q_i w_i d^*_i(p^i) \right) + u_1, \\
    h_2(I, p, q, r, \lambda, u) &= -\frac{1}{2\lambda_2} \left( \sum_{i \in I} q_i w_i d^*_i(p^i) \right) + u_2,
\end{align*}
\]

the dual variables

\[
I \subseteq \{1, \ldots, m\}, p = (p^1, \ldots, p^m), p^i \in \mathbb{R}^n, i = 1, \ldots, m, q = (q_1, \ldots, q_m)^T \in \mathbb{R}^m, r = (r^1, r^2)^T, r^1, r^2 \in \mathbb{R}^m, \lambda = (\lambda_1, \lambda_2)^T \in \mathbb{R}^2, u = (u_1, u_2)^T \in \mathbb{R}^2,
\]

and the set of constraints

\[
Y^{WM}(C) = \left\{ (I, p, q, r, \lambda, u) : I \subseteq \{1, \ldots, m\}, q_i > 0, i \in I, q_i = 0, i \notin I, r^1, r^2 \in \mathbb{R}_+^m, \right\}
\]

\[
\max_{i=1, \ldots, m} r^1_i = 1, \sum_{i=1}^m r^2_i = 1, \lambda \in \text{int}(\mathbb{R}_+^2), \sum_{i \in I} q_i w_i p^i = 0, \sum_{j=1}^2 \lambda_j r^j = q, \sum_{j=1}^2 \lambda_j u_j = 0 \}
\]

Let us give also for these problems the weak and strong duality theorems.

**Theorem 4.21** There is no \(x \in \mathbb{R}^n\) and no \((I, p, q, r, \lambda, u) \in Y^{WM}(C)\) such that

\[
\sum_{i=1}^m w_i d_i(x, C_i) \leq h_1(I, p, q, r, \lambda, u), \quad \text{and} \quad \max_{i=1, \ldots, m} w_i d_i(x, C_i) \leq h_2(I, p, q, r, \lambda, u)
\]

and

\[
\sum_{i=1}^m w_i d_i(x, C_i) < h_1(I, p, q, r, \lambda, u) \quad \text{or} \quad \max_{i=1, \ldots, m} w_i d_i(x, C_i) < h_2(I, p, q, r, \lambda, u).
\]
4.2 MULTICRITERIA WEBER PROBLEM WITH INFIMAL DISTANCES

Theorem 4.22 Let $\bar{x}$ be properly efficient element to $(P^W(C))$. Then there exists an efficient solution $(\bar{I}, \bar{p}, \bar{q}, \bar{r}, \bar{\lambda}, \bar{u}) \in Y^W(C)$ to $(D^W(C))$ and the strong duality holds, i.e.

$$
\sum_{i=1}^{m} w_i d_i(\bar{x}, C_i) = -\frac{1}{2\lambda_1} \left( \sum_{i \in \bar{I}} \bar{q}_i w_i d_i^*(\bar{p}^i) \right) + \bar{u}_1
$$

and

$$
\max_{i=1,...,m} w_i d_i(\bar{x}, C_i) = -\frac{1}{2\lambda_2} \left( \sum_{i \in \bar{I}} \bar{q}_i w_i d_i^*(\bar{p}^i) \right) + \bar{u}_2.
$$

4.2.4 The multiobjective Weber problem with infimal distances

We consider as another application of the multiobjective duality results in Subsection 4.2.2 the multiobjective Weber problem with infimal distances for the data $C$

$$(P^W(C)) \ \ v\text{-min}_{x \in \mathbb{R}^n} \left( \sum_{i=1}^{m} w_i^1 d_i(x, C_i), ..., \sum_{i=1}^{m} w_i^s d_i(x, C_i) \right)^T,$$

where $d_i(x, C_i) = \inf_{y_i \in C_i} \gamma_i(x - y_i)$, $i = 1, ..., m$, $\gamma_i$, $i = 1, ..., m$, are norms defined on $\mathbb{R}^n$ and $w_i^j$, $i = 1, ..., m$, $j = 1, ..., s$, are positive weights. Considering the norms $l_j^W : \mathbb{R}^m \to \mathbb{R}$, $j = 1, ..., s$, defined by

$$l_j^W(z) := \sum_{i=1}^{m} w_i^j |z_i|,$$

we have

$$l_j^W(d(x)) = \sum_{i=1}^{m} w_i^j d_i(x, C_i).$$

We notice that $l_j^W$, $j = 1, ..., s$, are monotonic norms, with the dual norms $(l_j^W)^0(z) = \max_{i=1,...,m} \frac{|z_i|}{w_i^j}$. So, the primal problem $(P^W(C))$ becomes

$$(P^W(C)) \ \ v\text{-min}_{x \in \mathbb{R}^n} l^W(d(x)), $$

where $l^W = (l_1^W, ..., l_s^W)^T : \mathbb{R}^m \to \mathbb{R}^s$ and $d(x) = (d_1(x, C_1), ..., d_m(x, C_m))$. Due to Subsection 4.2.2, a multiobjective dual problem to $(P^W(C))$ is

$$(D^W(C)) \ \ v\text{-max}_{(I, p, q, r, \lambda, u) \in Y^W(C)} h^W(I, p, q, r, \lambda, u),$$
with \( h^W = (h^W_1, ..., h^W_s)^T \),

\[
h^W_j(I, p, q, r, \lambda, u) = -\frac{1}{s\lambda_j} \left( \sum_{i \in I} q_id^*_j(p^i) \right) + u_j, \quad j = 1, ..., s,
\]

the dual variables

\[
I \subseteq \{1, ..., m\}, \quad p = (p^1, ..., p^m), \quad p^i \in \mathbb{R}^n, \quad i = 1, ..., m, \quad q = (q_1, ..., q_m)^T \in \mathbb{R}^m, \quad r = (r^1, ..., r^s), \quad r^j \in \mathbb{R}^m, \quad j = 1, ..., s, \quad \lambda = (\lambda_1, ..., \lambda_s)^T \in \mathbb{R}^s, \quad u = (u_1, ..., u_s)^T \in \mathbb{R}^s,
\]

and the set of constraints

\[
Y^W(C) = \left\{ (I, p, q, r, \lambda, u) : I \subseteq \{1, ..., m\}, \quad q_i > 0, \quad i \in I, \quad q_i = 0, \quad i \notin I, \quad r^j \in \mathbb{R}^m, \quad \max_{i=1,...,m} \frac{r^j_i}{w^i} = 1, \quad j = 1, ..., s, \quad \lambda \in \text{int}(\mathbb{R}_+^s), \quad \sum_{i \in I} q_ip^i = 0, \quad \sum_{j=1}^s \lambda_j r^j = q, \quad \sum_{j=1}^s \lambda_j u_j = 0 \right\}.
\]

Using theorems 4.19 and 4.20 we can formulate the following duality results:

**Theorem 4.23** There is no \( x \in \mathbb{R}^n \) and no \((I, p, q, r, \lambda, u) \in Y^W(C)\) such that \( \sum_{i=1}^m w^i d_i(x, C_i) \leq h^W_j(I, p, q, r, \lambda, u), \quad i = 1, ..., s, \quad \text{and} \quad \sum_{i=1}^m w^i d_i(x, C_i) < h^W_k(I, p, q, r, \lambda, u) \) for at least one \( k \in \{1, ..., s\} \).

**Theorem 4.24** Let \( \bar{x} \) be properly efficient element to \((P^W(C))\). Then there exists an efficient solution \((\bar{I}, \bar{p}, \bar{q}, \bar{r}, \bar{\lambda}, \bar{u}) \in Y^W(C)\) to \((D^W(C))\) and strong duality, i.e.

\[
\sum_{i=1}^m w^i d_i(\bar{x}, C_i) = -\frac{1}{s\bar{\lambda}_j} \left( \sum_{i \in \bar{I}} \bar{q}_id^*_j(\bar{p^i}) \right) + \bar{u}_j, \quad j = 1, ..., s,
\]

holds.

### 4.2.5 The multiobjective minmax problem with infimal distances

The last optimization problem we are going to consider in this work is the multiobjective minmax location problem with infimal distances for the data \( C \)

\[
(P^M(C)) \quad \text{v-min}_{x \in \mathbb{R}^n} \left( \max_{i=1,...,m} w^i d_i(x, C_i), ..., \max_{i=1,...,m} w^i d_i(x, C_i) \right)^T,
\]

where \( d_i(x, C_i) = \inf_{y_i \in C_i} \gamma_i(x - y_i), \quad i = 1, ..., m, \quad \text{and} \quad w^i_j, \quad i = 1, ..., m, \quad j = 1, ..., s, \) are positive weights. Considering the norms \( l^M_j : \mathbb{R}^m \rightarrow \mathbb{R}, \quad j = 1, ..., s, \) defined by

\[
l^M_j(z) = \max_{i=1,...,m} w^i_j |z_i|,
\]
we have that
\[ l_j^M(d(x)) = \max_{i=1,...,m} w_i^j d_i(x, C_i). \]

We notice that \( l_j^M, j = 1, ..., s \), are monotonic norms, with the dual norm
\[ (l_j^M)^0(z) = \sum_{i=1}^m \frac{|z_i|}{w_i^j}. \] 
Thus, the primal problem \((P^M(C))\) becomes
\[ (P^M(C)) \quad \nu\text{-min}_{x \in \mathbb{R}^n} l^M(d(x)), \]
where \( l^M = (l_1^M, ..., l_s^M)^T : \mathbb{R}^m \rightarrow \mathbb{R}^s \). Due to Subsection 4.2.2, a multiobjective dual problem to \((P^M(C))\) is
\[ (D^M(C)) \quad \nu\text{-max}_{(I, p, q, r, \lambda, u) \in Y^M(C)} h^M(I, p, q, r, \lambda, u), \]
with \( h^M = (h_1^M, ..., h_s^M)^T \),
\[ h_j^M(I, p, q, r, \lambda, u) = -\frac{1}{s \lambda_j} \left( \sum_{i \in I} q_i d_i^r(p^i) \right) + u_j, \quad j = 1, ..., s, \]
the dual variables
\[ I \subseteq \{1, ..., m\}, \quad p = (p^1, ..., p^m), \quad p^i \in \mathbb{R}^n, \quad i = 1, ..., m, \quad q = (q_1, ..., q_m)^T \in \mathbb{R}^m, \]
\[ r = (r^1, ..., r^s), \quad r^j \in \mathbb{R}^m, \quad j = 1, ..., s, \quad \lambda = (\lambda_1, ..., \lambda_s)^T \in \mathbb{R}^s, \quad u = (u_1, ..., u_s)^T \in \mathbb{R}^s, \]
and the set of constraints
\[ Y^M(C) = \left\{ (I, p, q, r, \lambda, u) : I \subseteq \{1, ..., m\}, q_i > 0, i \in I, q_i = 0, i \notin I, r^j \in \mathbb{R}^m_+, \right. \]
\[ \sum_{i=1}^m \frac{r_i^j}{w_i^j} = 1, \quad j = 1, ..., s, \quad \lambda \in \text{int}(\mathbb{R}^s_+), \]
\[ \sum_{i \in I} q_i p^i = 0, \quad \sum_{j=1}^s \lambda_j r^j = q, \quad \sum_{j=1}^s \lambda_j u_j = 0. \]

\textbf{Remark 4.5} We emphasize the interesting observation that both dual problems \((D^W(C)) \) and \((D^M(C)) \) differ only in the constraints \( \max_{i=1,...,m} \frac{r_i^j}{w_i^j} = 1 \) and \( \sum_{i=1}^m \frac{r_i^j}{w_i^j} = 1 \), respectively.

The corresponding duality results for \((P^M(C)) \) and \((D^M(C)) \) are the following:

\textbf{Theorem 4.25} There is no \( x \in \mathbb{R}^n \) and no \( (I, p, q, r, \lambda, u) \in Y^M(C) \) such that \( \max_{i=1,...,m} w_i^j d_i(x, C_i) \leq h_j^M(I, p, q, r, \lambda, u), \quad j = 1, ..., l, \) and \( \max_{i=1,...,m} w_i^j d_i(x, C_i) < h_k^M(I, p, q, r, \lambda, u) \) for at least one \( k \in \{1, ..., s\} \).
Theorem 4.26  Let $\bar{x}$ be properly efficient element to $(P^M(\mathcal{C}))$. Then there exists an efficient solution $(\bar{I}, \bar{p}, \bar{q}, \bar{r}, \bar{\lambda}, \bar{u}) \in Y^M(\mathcal{C})$ to $(D^M(\mathcal{C}))$ and strong duality, i.e.

$$\max_{i=1,\ldots,m} w^j_i d_i(\bar{x}, C_i) = -\frac{1}{s\lambda_j} \left( \sum_{i \in I} \bar{q}_i d_i^*(\bar{p}^i) \right) + \bar{u}_j, \ j = 1, \ldots, s,$$

holds.
The main objective of this thesis is to establish a unified duality approach for both scalar and multiobjective convex composed programming problems. First, we study in the second chapter the single-valued composed optimization problem

\[ \inf_{x \in A} f(F(x)), \]

where

\[ A = \left\{ x \in X : g(G(x)) \leq 0_{\mathbb{R}_+^k} \right\}, \]

and

\[ X \subseteq \mathbb{R}^n, \quad F = (F_1, ..., F_m)^T : X \to \mathbb{R}^m, \quad G = (G_1, ..., G_l)^T : X \to \mathbb{R}^l, \]

\[ f : \mathbb{R}^m \to \mathbb{R} \quad \text{and} \quad g = (g_1, ..., g_k)^T : \mathbb{R}^l \to \mathbb{R}^k. \]

Using different perturbation functions we assign three dual supremum problems to the primal problem \((P)\), which we denote by \((D_L)\), \((D_F)\) and \((D_{FL})\). As one may observe, \((D_L)\) turns out to be the well-known Lagrange dual, \((D_F)\) the Fenchel dual and \((D_{FL})\) the so-called Fenchel-Lagrange dual problem. In what follows we analyze the relations between the optimal objective values of those three duals and then the relations between the optimal objective values of the primal and the dual problems, respectively. As a first result it can be stated that

\[ v(D_{FL}) \leq v(D_L) \quad \text{and} \quad v(D_{FL}) \leq v(D_F), \]

where \(v(D_L)\), \(v(D_F)\) and \(v(D_{FL})\) denote the optimal objective values of the corresponding duals. In fact, under some convexity assumptions and regularity conditions, they are even equal. The same convexity assumptions and regularity conditions will assure the strong duality between \((P)\) and \((D_L)\), \((D_F)\) and \((D_{FL})\), i.e. \(v(P) = v(D_L) = v(D_F) = v(D_{FL})\), where \(v(P)\) denotes the infimum of \((P)\). We mention that the weak duality between the primal and dual problems always holds, because of the construction of the duals. That means the suprema of the duals are less than or equal to the infimum of the primal problem \((P)\). Additionally, based on the verified strong duality, necessary and sufficient optimality conditions for each of these primal-dual pairs are derived.

As a first application of the general problem, the classical optimization
problem with inequality constraints

\[(P') \inf_{x \in A'} F(x),\]

\[A' = \{ x \in X : G(x) \leq 0 \} \]

is studied. Here \(X \subseteq \mathbb{R}^n\) is a nonempty set and \(F : X \to \mathbb{R}, G = (G_1, ..., G_k)^T, G_i : X \to \mathbb{R}, i = 1, ..., k,\) are vector-valued functions. Using the results obtained in the first part we construct three dual problems to \((P')\) and verify the strong duality for each of them. In conclusion, the optimality conditions are deduced. We mention that the results obtained by deriving them from the general problem coincide with those obtained by G. WANKA and R. I. BOT in [70].

3. Furthermore, the optimization problem without constraints

\[\inf_{x \in X} f(F(x))\]

is analyzed. In this case, \(X \subseteq \mathbb{R}^n\) and \(F = (F_1, ..., F_m)^T, F_i : X \to \mathbb{R}, i = 1, ..., m.\) This problem was already treated in detail by G. WANKA, R. I. BOT and E. VARGYAS in [71]. Our intention hereby is to show how the results obtained by these authors can be derived from the problem \((P)\). In order to do this we examine only the Fenchel-Lagrange dual problem. For this primal-dual pair we formulate a strong duality theorem and derive optimality conditions.

4. The third chapter of this work is devoted to location problems. First we consider a quite general problem

\[\inf_{x \in X} \gamma_C^+(F(x)),\]

where \(\gamma_C : \mathbb{R}^m \to \mathbb{R}\) is a monotonic gauge of a closed convex set \(C\) containing the origin, \(\gamma_C^+ : \mathbb{R}^m \to \mathbb{R}, \gamma_C^+(t) := \gamma_C(t^+),\) with \(t^+ = (t_1^+, ..., t_m^+)^T\) and \(t_i^+ = \max\{0, t_i\}, i = 1, ..., m,\) and \(F = (F_1, ..., F_m)^T : X \to \mathbb{R}^m\) is a vector-valued function. Embedding this problem into the general framework developed for the original problem \((P)\), we assign a Fenchel-Lagrange dual problem to it, prove the strong duality and derive the optimality conditions. The importance of this problem is that it provides a unified method for dealing with different location problems via conjugate duality.

5. As a first application of the previous problem we consider the model with monotonic norms

\[\inf_{x \in X} l^+(F(x))\]
where \( X \subseteq \mathbb{R}^n \), \( F = (F_1, ..., F_m)^T : X \to \mathbb{R}^m \) is a vector-valued function, \( l : \mathbb{R}^m \to \mathbb{R} \) is a monotonic norm on \( \mathbb{R}^m \) and \( l^+ : \mathbb{R}^m \to \mathbb{R} \), \( l^+(t) := l(t^+) \), with \( t^+ = (t_1^+, ..., t_m^+)^T \) and \( t_i^+ = \max\{0, t_i\} \), \( i = 1, ..., m \).

As further applications the location model with unbounded unit balls

\[
(P_{\infty}(\mathcal{F})) \quad \inf_{x \in \mathbb{R}^n} \gamma_C\left(w_{a_1} \varphi_{a_1}(x - a^1), ..., w_{a_m} \varphi_{a_m}(x - a^m)\right),
\]

the Weber problem with gauges of closed convex sets

\[
(P_w(\mathcal{F})) \quad \inf_{x \in \mathbb{R}^n} \sum_{i=1}^m w_{a_i} \varphi_{a_i}(x - a^i)
\]

and the minmax problem with gauges of closed convex sets

\[
(P_m(\mathcal{F})) \quad \inf_{x \in \mathbb{R}^n} \max_{i=1,...,m} w_{a_i} \varphi_{a_i}(x - a^i)
\]

are studied with respect to duality (see also G. WANKA, R. I. BOT and E. VARGYAS [72]). We mention that \( \mathcal{F} := \{a^1, ..., a^m\} \) is a set of \( m \) points of \( \mathbb{R}^n \) which represents the set of existing facilities, each facility \( a^i \in \mathcal{F} \) having an associated gauge \( \varphi_{a_i} \) whose unit ball is a closed convex set \( C_{a_i} \) containing the origin, \( w = \{w_{a_1}, ..., w_{a_m}\} \) is a set of positive weights and \( \gamma_C : \mathbb{R}^m \to \mathbb{R} \) is a monotonic gauge of a closed convex set \( C \) containing the origin. The last three problems were studied also by Y. HINOJOSA and J. PUERTO in [27]. There the authors gave a geometrical characterization of the set of optimal solutions.

6. The fourth chapter of this work is devoted to duality in multiobjective optimization. First, we study the composed multicriteria problem

\[
(P_v) \quad \vmin_{x \in A} f(F(x)),
\]

where \( X \subseteq \mathbb{R}^n \), \( F = (F_1, ..., F_m)^T : X \to \mathbb{R}^m \), \( G = (G_1, ..., G_l)^T : X \to \mathbb{R}^l \), \( f = (f_1, ..., f_s)^T : \mathbb{R}^m \to \mathbb{R}^s \) and \( g = (g_1, ..., g_k)^T : \mathbb{R}^l \to \mathbb{R}^k \). We assume that \( F_i, \ i = 1, ..., m \), \( G_j, \ j = 1, ..., l \), are convex functions and \( f_i, \ i = 1, ..., s \), and \( g_j, \ j = 1, ..., k \), are convex and componentwise increasing functions. In order to do this, first we examine the scalarized problem

\[
(P^\lambda) \quad \inf_{x \in A} \lambda^T f(F(x)),
\]

where \( \lambda = (\lambda_1, ..., \lambda_s)^T \) is a fixed vector in \( int(\mathbb{R}_+^s) \). Applying the results obtained for the single-valued problem \( (P) \), we determine its Fenchel-Lagrange
dual \((D_{FL}^\lambda)\). Analogously to the previous sections we prove the strong duality between \((P^\lambda)\) and \((D_{FL}^\lambda)\) and, in conclusion, we derive the optimality conditions. By means of the scalar dual, we construct the multiobjective dual problem \((D_v)\) to \((P_v)\). Finally, the weak and the strong duality between \((P_v)\) and \((D_v)\) are proved.

7. Closely related to \((P_v)\), two special problems are analyzed, first the classical multiobjective optimization problem with inequality constraints

\[
(P'_v) \quad \inf_{x \in A'} F(x),
\]

\[A' = \left\{ x \in X : G(x) \leq 0 \right\}, \]

where \(X \subseteq \mathbb{R}^n\), \(F = (F_1, ..., F_s)^T : X \to \mathbb{R}^s\) and \(G = (G_1, ..., G_k)^T : X \to \mathbb{R}^k\), and then the multiobjective optimization problem without constraints

\[
(P''_v) \quad v\text{-min}_{x \in X} f(F(x)),
\]

where \(X \subseteq \mathbb{R}^n\), \(F = (F_1, ..., F_m)^T : X \to \mathbb{R}^m\) and \(f = (f_1, ..., f_s)^T : \mathbb{R}^m \to \mathbb{R}^s\). We mention that the results obtained in this way for \((P'_v)\), \((P''_v)\) and their duals are identical to those obtained by using different approaches by G. WANKA and R. I. BOT in [69] and G. WANKA, R. I. BOT and E. VARGYAS in [71], respectively.

8. Similarly to the scalar case, some multiobjective location models are considered. The first one is the multicriteria problem with monotonic norms

\[
(P'_v) \quad \text{v-min}_{x \in X} l^+(F(x)),
\]

where \(X \subseteq \mathbb{R}^n\), \(F = (F_1, ..., F_m)^T : X \to \mathbb{R}^m\), \(l = (l_1, ..., l_s)^T : \mathbb{R}^m \to \mathbb{R}^s\), \(F_i : X \to \mathbb{R}\), \(i = 1, ..., m\), are convex functions on \(X\), \(l_i : \mathbb{R}^m \to \mathbb{R}\), \(i = 1, ..., s\), are monotonic norms on \(\mathbb{R}^m\) and \(l^+ = (l_1^+, ..., l_s^+)^T\) such that \(l_i^+(t) := l_i(t^+), i = 1, ..., s\), with \(t^+ = (t_1^+, ..., t_m^+)^T\) and \(t_j^+ = \max\{0, t_j\}, j = 1, ..., m\).

As a second application of the general theory we examine the multiobjective model involving sets as existing facilities

\[
(P''_v) \quad \text{v-min}_{x \in \mathbb{R}^n} l(d(x)),
\]

where \(C = \{C_1, ..., C_m\}\) is a family of convex sets in \(\mathbb{R}^n\) such that \(\bigcap_{i=1}^m C_i = \emptyset\), \(l = (l_1, ..., l_s)^T\) with \(l_j : \mathbb{R}^m \to \mathbb{R}\), \(j = 1, ..., s\), monotonic norms on \(\mathbb{R}^m\), \(d(x) := (d_1(x, C_1), ..., d_m(x, C_m))\) and \(d_i(x, C_i) = \inf\{\gamma_i(x - y_i) : y_i \in C_i\}, i = 1, ..., m\), where \(\gamma_i, i = 1, ..., m\), are norms on \(\mathbb{R}^n\). This model was
motivated by a paper of S. NICKEL, J. PUERTO and A. M. RODRIGUEZ-CHIA ([52]), where the authors give a geometrical characterization of the sets of optimal solutions. We establish duality results as well as necessary and sufficient optimality conditions.

9. Finally, closely related to the model involving sets as existing facilities, the biobjective Weber-minmax problem with infimal distances

\[
(P^{WM}(\mathcal{C})) \quad \text{v-min}_{x \in \mathbb{R}^n} \left( \sum_{i=1}^{m} w_i d_i(x, C_i) \right),
\]

the multiobjective Weber problem with infimal distances

\[
(P^W(\mathcal{C})) \quad \text{v-min}_{x \in \mathbb{R}^n} \left( \sum_{i=1}^{m} w_i^1 d_i(x, C_i), \ldots, \sum_{i=1}^{m} w_i^s d_i(x, C_i) \right)^T,
\]

and the multiobjective minmax problem with infimal distances

\[
(P^M(\mathcal{C})) \quad \text{v-min}_{x \in \mathbb{R}^n} \left( \max_{i=1,\ldots,m} w_i^1 d_i(x, C_i), \ldots, \max_{i=1,\ldots,m} w_i^s d_i(x, C_i) \right)^T,
\]

are studied. Also here we construct multiobjective dual problems and derive weak and strong duality assertions.
Index of notation

\( \mathbb{R} \) the set of real numbers
\( \overline{\mathbb{R}} \) the extended set of real numbers
\( \mathbb{R}^p \) the \( p \)-dimensional Euclidean space
\( \mathbb{R}^p_+ \) the non-negative orthant of \( \mathbb{R}^p \)
\( \text{int}(X) \) the interior of the set \( X \)
\( \text{ri}(X) \) the relative interior of the set \( X \)
\( \overline{X} \) the closure of the set \( X \)
\( \text{dom}(f) \) the domain of the function \( f \)
\( f^* \) the conjugate of the function \( f \)
\( f^*_X \) the conjugate of the function \( f \) relative to the set \( X \)
\( \delta_X \) the indicator function of the set \( X \)
\( \leq_{\mathbb{R}^p_+} \) the partial ordering induced by the non-negative orthant \( \mathbb{R}^p_+ \)
\( x^T y \) the inner product of the vectors \( x \) and \( y \)
\( \gamma_C^0 \) the dual of the gauge \( \gamma_C^0 \)
\( l^0 \) the dual norm of the norm \( l \)
\( \text{v-min} \) the notation for a multiobjective optimization problem in the sense of minimum
\( \text{v-max} \) the notation for a multiobjective optimization problem in the sense of maximum
\( v(P) \) the optimal objective value of a minimization problem \( (P) \)
\( v(D) \) the optimal objective value of a maximization problem \( (D) \)
Bibliography


Lebenslauf

**Persönliche Daten**
Name: Emese Tünde Vargyas
Adresse: Vettersstrasse 64/422 09126 Chemnitz
Geburtsdatum: 21.02.1975
Geburtsort: Reghin, Rumänien

**Schulausbildung**
09/1981 - 06/1989 Grundschule in Reghin, Rumänien
09/1989 - 06/1993 "Bolyai Farkas" Gymnasium in Târgu Mureș, Rumänien
Abschluß: Abitur

**Studium**
10/1993 - 06/1997 "Babeș-Bolyai" Universität Cluj-Napoca, Rumänien
Fakultät für Mathematik und Informatik
Fachbereich: Mathematik
Abschluß: Diplom in Mathematik
10/1997 - 06/1998 "Babeș-Bolyai" Universität Cluj-Napoca, Rumänien
Fakultät für Mathematik und Informatik
Masterstudium im Fachbereich "Konvexe Analysis und Approximationstheorie"
Abschluß: Masterdiplom

**Berufstätigkeit**
09/1998 - 02/2001 Mathematiklehrerin am Kollegium "George Coşbuc" in Cluj-Napoca, Rumänien
seit 03/2001 Wissenschaftliche Mitarbeiterin an der Technischen Universität Chemnitz, Fakultät für Mathematik
Erklärung gemäß §6 der Promotionsordnung

Hiermit erkläre ich an Eides Statt, dass ich die von mir eingereichte Arbeit ”Duality for convex composed programming problems” selbstständig und nur unter Benutzung der in der Arbeit angegebenen Hilfsmittel angefertigt habe.

Chemnitz, den 05.07.2004

Emese Tünde Vargyas