Farkas - type results for convex and non-convex inequality systems

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Report

As the title already suggests the aim of the present work is to present Farkas - type results for inequality systems involving convex and/or non - convex functions. To be able to give the desired results, we treat optimization problems which involve convex and composed convex functions or non - convex functions like DC functions or fractions.

To be able to use the fruitful Fenchel - Lagrange duality approach, to the primal problem we attach an equivalent problem which is a convex optimization problem. After giving a dual problem to the problem we initially treat, we provide weak necessary conditions which secure strong duality, i.e., the case when the optimal objective value of the primal problem coincides with the optimal objective value of the dual problem and, moreover, the dual problem has an optimal solution.

Further, two ideas are followed. Firstly, using the weak and strong duality between the primal problem and the dual problem, we are able to give necessary and sufficient optimality conditions for the optimal solutions of the primal problem. Secondly, provided that no duality gap lies between the primal problem and its Fenchel - Lagrange - type dual we are able to demonstrate some Farkas - type results and thus to underline once more the connections between the theorems of the alternative and the theory of duality. One statement of the above mentioned Farkas - type results is characterized using only epigraphs of functions.

We conclude our investigations by providing necessary and sufficient optimality conditions for a multiobjective programming problem involving composed convex functions. Using the well-known linear scalarization to the primal multiobjective program a family of scalar optimization problems is attached. Further to each of these scalar problems the Fenchel - Lagrange dual problem is determined. Making use of the weak and strong duality between the scalarized problem and its dual the desired optimality conditions are proved. Moreover, the way the dual problem of the scalarized problem looks like gives us an idea about how to construct a vector dual problem to the initial one. Further weak and strong vector duality assertions are provided.

Keywords

conjugate duality; conjugate functions; composed convex optimization problems; constraint qualifications; DC functions; DC optimization problems; duality in multiobjective optimization; Farkas - type results; Fenchel - Lagrange duality; fractional programming problems; theorems of the alternative; weak and strong duality; weak and strong vector duality; optimality conditions; weakly efficient solutions
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Chapter 1

Introduction

Duality is one of the most important and most used techniques in optimization. It consists in attaching to an initial optimization problem of a so-called dual problem whose optimal objective value is less than or equal to the optimal objective value of the initial problem.

The duality approach used in the present work is a recently introduced one, namely the Fenchel-Lagrange duality. As the name suggests, this duality approach is a "combination" of the classical Lagrange duality and Fenchel duality. This approach has proved to be very useful in giving a compact formula not only for the dual problem of some primal problems which involve merely convex functions, but also when the primal problems make use of composed convex functions, DC functions or fractional functions. Moreover, the dual problem deals only with the conjugates of the functions involved by the primal problem. As a last remark, let us mention that the Fenchel - Lagrange duality approach can be successfully used also for problems which involves more general concepts of convexity.

The aim of the present work is twofold. Firstly, scalar optimization problems which employ not only convex functions are treated. To each problem the Fenchel-Lagrange-type dual is attached and, provided that some assumptions are fulfilled, weak and strong duality assertions are proved. These duality statements are used in order to demonstrate some Farkas-type results. Secondly, some necessary and sufficient conditions for the weakly efficient solutions of a multiobjective optimization problem are established. Moreover, a multiobjective dual is attached to the initial vector minimization problem and then weak and strong duality assertions are proved.

1.1 Optimization problems: importance and applications

In many applications it is necessary to solve an optimization problem, i.e. to find the minimal or the maximal value a function can take provided that some conditions are fulfilled. Although it seems to be quite simple, such a problem cannot be usually solved without some additional assumptions.

Among the various types of optimization problems which have been treated during the last decades, some of the most encountered programming problems are the convex optimization problems (i.e. problems which have a convex function as objective function and inequalities involving convex functions as constraints). Fortunately many optimization problems which arise in different applications are convex optimization problems and, as a result, this type of problems has been extensively studied by many mathematicians during the last decades.
The optimization problems which involve composed convex functions are very important, too, as they are generated by fields like location and transports, economics and finances. Besides the practical applications, another advantage of the optimization problems with composed convex functions are their generality, since many types of convex optimization problems are actually special instances of them. We mention here only two special cases, namely the min - max programming problems and the problems which involve the composition between a convex function and a linear operator. The usefulness of the optimization problems with composed convex functions arises also when multiobjective programming problems are treated, as their efficient solutions are among the optimal solutions of some scalar problems obtained from the initial one. Since the scalar problems are not always obtained using linear functions (see [13] and the references therein), the objective function is not a sum of convex functions, but rather a composition of convex functions.

But not all the practical problems generate convex or composed convex optimization problems and one of the best examples in this sense are the location problems. Supposing that a lodging house must be raised within a given area, sometimes it is important to be as close as possible to certain facilities (like stores and bus stations) and in the same time far away regarding some other buildings (like airports and central stations). If we write the previous problem in a mathematical form, it is not hard to see that the objective function of the programming problem we obtain is a DC function (a function which can be written as the difference of two convex functions). Several other practical problems like production - transportation programming, bridge location problem and design centering problems have objective and constraints functions involving DC functions (more details can be found in [53] and [88] and the references therein).

Another class of non - convex optimization problems is formed by the fractional programming problems. Such problems with the objective function of fractional type have a broad applicability. Among the fields where such problems arise we mention here only the return on investment and dividend coverage, production planning and scheduling, data mining and entropy optimization.

Until now we have spoken only about scalar optimization problems. But in many practical fields it is necessary to optimize concomitantly two or more objectives which usually are conflicting (for example maximizing profit and minimizing cost of a product or minimizing weight and maximizing the strength of a particular component of a vehicle are usually situations of this type). This kind of problems are called multiojective (or multicriteria) optimization problems. They have a very wide range of applications in fields like product and process design, finance, oil and gas industry, aircraft and automobile design, and the list is far from being at end.

1.2 A description of the contents

In this section is given a description of the way the thesis is organized.

As the name suggests, the first chapter has an introductory role. At its very beginning one find some words regarding the importance of convex and non - convex optimization problems. Further a description of the present work is given, and afterwards some preliminary notions and results are recalled.

The problem we treat in the second chapter consists in minimizing the sum between a convex function and the composition of a convex and $K$ - increasing function with a $K$ - convex function ($K$ is a convex cone). To this problem a Fenchel - Lagrange - type dual is attached and it is proved that strong duality holds if a generalized interior point condition is valid. Further necessary and sufficient optimality conditions are proved using the duality assertions. The same assertions are committed so as to vindicate a main result of the chapter, namely a Farkas -
type result. Special cases of the initial problem are treated, too, and it is shown that some recently obtained results in the literature are generalized by the ones given within this chapter.

In the third chapter DC optimization problems are studied. First we consider an optimization problem with a DC function as objective function and finitely many DC functions as constraints. Using a result presented by Martínez-Legaz and Volle in [71] we are able to determine a dual problem to the initial one. As the optimal objective values of the primal and the dual are equal when some conditions are fulfilled, a Farkas-type result arises as an easy consequence of this equality. From the special instances of the problem we treat encountered in the literature we mention here only two, namely the optimization problem with a convex objective function and finitely many reversed convex constrains and the optimization problem with a DC objective function and finitely many convex constraints.

The fractional programming problems are a very important class of non-convex optimization problems from many points of view and are treated in the fourth chapter using an approach due to Dinkelbach (see [40]). More precisely, we consider an optimization problem with a fraction function as objective function and finitely many convex functions as constraints. The optimal objective value of the initial problem is greater than or equal to a real number λ if and only if the optimal objective value of an optimization problem is greater than or equal to 0 (the objective function of the new problem is DC for λ strictly negative and convex otherwise). The Fenchel-Lagrange-type dual problem of the new problem is established and a Farkas-type result is derived. Moreover, it is proved that some recently obtained statements are actually special instances of the ones presented within this chapter.

The last chapter of the work is dedicated to multiobjective optimization. After presenting the problem we work with and a family of scalar problems associated to it, to each of these scalar problems a Fenchel-Lagrange-type dual problem is given. Using the weak and strong duality assertions some necessary and sufficient conditions involving the weakly efficient solutions of the initial multiobjective problem are established. A multiobjective dual is given and weak and strong vector duality assertions are proved, too. At last some older results are rediscovered as special cases of the initial problem.

1.3 Preliminary notions and results

In the following we present some notations used throughout the entire work. The definitions of the notions that frequently appear within this work are given, too, together with some basic results used later.

For a positive integer n, by $\mathbb{R}^n$ is denoted the n-dimensional real space. As usual, $\mathbb{R} = \mathbb{R} \cup \{\pm \infty\}$ denotes the set of extended reals and $\mathbb{R}_+ = [0, +\infty)$ the non-negative reals. The vectors are considered as columns vectors and an upper index $^T$ is used in order to transpose a column vector to a row one (or viceversa). For two vectors $x = (x_1, \ldots, x_n)^T$ and $y = (y_1, \ldots, y_n)^T$ arbitrarily taken in the n-dimensional real space, $x^T y = \sum_{i=1}^n x_i y_i$ denotes their inner product.

**Definition 1.1** The non-empty set $K \subseteq \mathbb{R}^n$ is called cone if $\lambda x \in K$ for all $x \in K$ and $\lambda \in \mathbb{R}_+$.

From the definition we notice that 0 $\in K$.

**Definition 1.2** The dual cone of $K$ is defined as

$$K^* = \{x^* \in \mathbb{R}^n : x^*^T x \geq 0, \forall x \in K\}.$$
Over the finite dimensional real space $\mathbb{R}^n$ we consider the partial ordered $\leq_K$ defined as follows

$$x \leq_K y \iff y - x \in K, \ x, y \in \mathbb{R}^n.$$  

If $K$ is the positive orthant $\mathbb{R}^n_+$, then instead of $\leq_{\mathbb{R}^n_+}$ the simplified notation "$\leq$" is used. Moreover, if for all $i = 1, \ldots, n$, it holds $x_i < y_i$, then we write $x < y$.

To $\mathbb{R}^n$ we attach $\infty \mathbb{R}^n \not\subseteq \mathbb{R}^n$ a maximal element with respect to $K$ and we denote by $\mathbb{R}^n = \mathbb{R}^n \cup \{ \infty \mathbb{R}^n \}$. We have $y \leq_K \infty \mathbb{R}^n$ for all $y \in \mathbb{R}^n$, and the addition and the multiplication with a positive scalar are also natural extended to $\mathbb{R}^n$ if we require $\infty \mathbb{R}^n + y = y + \infty \mathbb{R}^n = \infty \mathbb{R}^n$ and $t \infty \mathbb{R}^n = \infty \mathbb{R}^n$ for any $y \in \mathbb{R}^n$ and $t \geq 0$.

**Definition 1.3** We say that the non-empty set $X \subseteq \mathbb{R}^n$ is convex if for all $x, y \in X$ and $\lambda \in [0, 1]$ we have $\lambda x + (1 - \lambda) y \in X$.

From now on we consider $X$ a non-empty subset of $\mathbb{R}^n$ and $K \subseteq \mathbb{R}^n$ a convex cone. By $\text{cl}(X)$, $\text{int}(X)$ and $\text{ri}(X)$ are denoted the closure, the interior and the relative interior of $X$. Furthermore, the convex hull of $X$ is denoted by $\text{co}(X)$.

**Definition 1.4** The cone and the convex cone generated by the set $X$ are defined as

$$\text{cone}(X) = \bigcup_{\lambda \geq 0} \lambda X \quad \text{and} \quad \text{cone}(X) = \bigcup_{\lambda \geq 0} \lambda \text{co}(X),$$

respectively.

For a function $f : \mathbb{R}^n \to \mathbb{R}$ we have the effective domain $\text{dom}(f) = \{ x \in \mathbb{R}^n : f(x) < +\infty \}$ and the epigraph $\text{epi}(f) = \{ (x, r) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq r \}$. We say that the function $f$ is proper if its effective domain is a non-empty set and $f(x) > -\infty$ for all $x \in \mathbb{R}^n$.

**Definition 1.5** The function $f : \mathbb{R}^n \to \mathbb{R}$ is called convex if for any $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$ one has

$$f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y),$$

whenever the sum in the right-hand side is defined (see relation (2. 1)). We call the function $f$ concave if $-f$ is convex.

**Definition 1.6** A function $f : \mathbb{R}^n \to \mathbb{R}$ is called $K$-increasing if for all $x, y \in \mathbb{R}^n$ fulfilling $x \leq_K y$ the inequality $f(x) \leq f(y)$ holds.

Without any fear of confusion we say that the function $h : \mathbb{R}^m \to \mathbb{R}^n$ is proper if its effective domain $\text{dom}(h) = \{ x \in \mathbb{R}^m : h(x) \in \mathbb{R}^n \}$ is a non-empty set. Moreover, for all $\beta \in K^*$ the function

$$(\beta^T h) : \mathbb{R}^m \to \mathbb{R}, \quad (\beta^T h)(x) = \left\{ \begin{array}{ll} \beta^T h(x), & x \in \text{dom}(h), \\ +\infty, & \text{otherwise,} \end{array} \right.$$ 

is well-defined. For any non-empty set $U \subseteq \mathbb{R}^n$, $h^{-1}(U) = \{ x \in \mathbb{R}^m : h(x) \in U \}$. Through the restriction of the function $h$ to the set $X$ we understand the function

$$h_X : \mathbb{R}^m \to \mathbb{R}^n, \quad h_X(x) = \left\{ \begin{array}{ll} h(x), & x \in X, \\ \infty \mathbb{R}^n, & \text{otherwise.} \end{array} \right.$$ (1. 1)

**Definition 1.7** The vector function $h : \mathbb{R}^m \to \mathbb{R}^n$ is called $K$-convex if for all $x, y \in \mathbb{R}^m$ and $\lambda \in [0, 1]$ it fulfills the property

$$h(\lambda x + (1 - \lambda) y) \leq_K \lambda h(x) + (1 - \lambda) h(y).$$
The next notion we introduce is that of conjugate function.

**Definition 1.8** For \( f : \mathbb{R}^n \to \mathbb{R} \) an arbitrary function, by the conjugate function of \( f \) regarding the set \( X \) we understand the function

\[
f_X^* : \mathbb{R}^n \to \mathbb{R}, \quad f_X^*(x^*) = \sup_{x \in X} \left\{ x^T x - f(x) \right\}.
\]

The previous definition obviously implies

\[
f_X^*(x^*) + f(x) \geq x^T x, \quad \forall x \in X, \forall x^* \in \mathbb{R}^n. \tag{1.2}
\]

When \( X = \mathbb{R}^n \) the conjugate regarding the set \( X \) turns out to be the classical (Legendre - Fenchel) conjugate function \( f^* \), denoted by \( f^* \), and the inequality (1.2) becomes the well-known Fenchel - Young inequality

\[
f^*(x^*) + f(x) \geq x^T x, \quad \forall x \in \mathbb{R}^n, \forall x^* \in \mathbb{R}^n. \tag{1.3}
\]

Within the present work a very important role play the indicator function of a set \( X \subseteq \mathbb{R}^n \)

\[
\delta_X : \mathbb{R}^n \to \mathbb{R}, \quad \delta_X(x) = \begin{cases} 0, & x \in X, \\ +\infty, & x \notin X, \end{cases}
\]

and the support function of \( X \)

\[
\sigma_X : \mathbb{R}^n \to \mathbb{R}, \quad \sigma_X(y) = \sup_{x \in X} y^T x.
\]

It is not hard to see that

\[
f_X^* = (f + \delta_X)^* \quad \text{and} \quad \sigma_X = \delta_X^*, \quad \tag{1.4}
\]

while for all \( \lambda > 0 \) and for all \( \beta \in K^* \) one has

\[
(\lambda f)^*(\lambda x^*) = \lambda f^*(x^*) \quad \text{and} \quad (\beta^T h_X)^* = (\beta^T h)^*_X. \tag{1.5}
\]

The next linear operator is defined so as to commute the entries of a tuple, i.e.

\[
T : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}^n, \quad T(x, r) = (r, x).
\]

**Definition 1.9** If \( A : \mathbb{R}^n \to \mathbb{R}^m \) is a linear operator, then \( A^* : \mathbb{R}^m \to \mathbb{R}^n \) defined such that

\[
x^T (A^* y^*) = (Ax)^T y^*, \quad \forall x \in \mathbb{R}^n, \forall y^* \in \mathbb{R}^m
\]

is called its adjoint.

We present now two well-known theorems, as they play a prominent part in proving some results we give later.

**Theorem 1.1** (cf. [80]) Let \( f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R} \) be proper convex functions. If the set \( \cap_{i=1}^m \text{ri}(\text{dom}(f_i)) \) is non-empty, then for each \( x^* \in \mathbb{R}^n \)

\[
\left( \sum_{i=1}^m f_i \right)^*(x^*) = \inf \left\{ \sum_{i=1}^m f_i^*(x_i^*) : x^* = \sum_{i=1}^m x_i^* \right\},
\]

and the infimum is attained.
Theorem 1.2 (cf. [80]) Let $C_i$ be a convex set for each $i \in \mathcal{I}$ (an index set). Suppose that the sets $\text{ri}(C_i)$ have at least one point in common. Then

$$\text{cl} \left( \bigcap_{i \in \mathcal{I}} C_i \right) = \bigcap_{i \in \mathcal{I}} \text{cl}(C_i).$$

If $\mathcal{I}$ is finite, then also

$$\text{ri} \left( \bigcap_{i \in \mathcal{I}} C_i \right) = \bigcap_{i \in \mathcal{I}} \text{ri}(C_i).$$

The proofs of the next results are omitted here, as they can be easily proved using the previous theorems and the properties of the conjugate function.

Theorem 1.3 (cf. [18]) Let $f_1, \ldots, f_m : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be proper convex functions. If the set $\bigcap_{i=1}^m \text{ri}(\text{dom}(f_i))$ is non-empty, then

$$\text{epi} \left( \left( \sum_{i=1}^m f_i \right)^* \right) = \sum_{i=1}^m \text{epi}(f_i^*).$$

Theorem 1.4 (cf. [18]) Let $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be a proper function and $\alpha > 0$ a real number. One has

$$\text{epi}(\alpha f)^* = \alpha \text{epi}(f^*).$$

Theorem 1.5 (cf. [20]) Let $K \subseteq \mathbb{R}^n$ be a non-empty convex cone and $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ a proper and $K$-increasing function. Then $f^*(x^*) = +\infty$ for all $x^* \notin K^*$.

Let us also note that everywhere within this work we write $\min$ (max) instead of $\inf$ ($\sup$) when the infimum (supremum) is attained and for an optimization problem $(P)$ we denote its optimal objective value by $v(P)$. As a last remark, we would like to mention that everywhere in this thesis we work in finite dimensional spaces, although many results we present can be given also in infinite dimensional spaces.

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Chapter 2

Farkas - type results with composed convex functions

We consider an optimization problem whose objective function is the sum of a convex function and the composition of a convex and $K$ - increasing function with a $K$ - convex one ($K$ is a convex cone), and no restrictions are made regarding the variable (see also [15, 32–34, 65, 102]). To this primal problem a dual is attached such that strong duality holds if some interior point conditions are fulfilled. The approach we use is similar to the one in [12, 18], i.e., to the initial problem we attach an equivalent one, but whose dual can be much easier established. To the new optimization problem the classical Lagrange dual problem is attached. As the inner infimum of the Lagrange problem can be viewed as a minimization programming problem, its Fenchel dual is also determined in order to get what we call a Fenchel - Lagrange - type dual problem. The construction of the dual is described in detail and a constraint qualification which ensures the strong duality between the primal problem and its dual is given, too. The weak and strong duality assertions are used in order to prove some necessary and sufficient optimality conditions, as well as a Farkas - type result. Moreover, using only the epigraphs of the involved functions, we give an equivalent formulation for a statement of the above mentioned Farkas - type result. Further special instances of the initial problem are considered and some recently obtained results are rediscovered as special cases.

2.1 Problem formulation and the Fenchel - Lagrange - type dual

In the following the addition and the multiplication with a positive scalar are extended from $\mathbb{R}$ onto $\overline{\mathbb{R}}$ according to the rules

\begin{align*}
a + (+\infty) &= +\infty \quad \forall a \in (-\infty, +\infty], \quad a + (-\infty) = -\infty \quad \forall a \in [-\infty, +\infty), \\
a(+\infty) &= +\infty \quad \text{and} \quad a(-\infty) = -\infty \quad \forall a \in (0, +\infty], \\
a(+\infty) &= -\infty \quad \text{and} \quad a(-\infty) = +\infty \quad \forall a \in [-\infty, 0), \\
0(+\infty) &= +\infty, \quad 0(-\infty) = 0.
\end{align*}

(2. 1)

Let us mention moreover that $(+\infty) + (-\infty)$ and $(-\infty) + (+\infty)$ are not defined. The way we extend the multiplication onto $\overline{\mathbb{R}}$ allows us to affirm that

\begin{align}
0f &= \delta_{\text{dom}(f)} \quad \text{and} \quad \operatorname{epi}(0f^*) = \operatorname{epi}(\sigma_{\text{dom}(f)}).
\end{align}

(2. 2)

Let $K \subseteq \mathbb{R}^k$ be a non - empty convex cone. Consider the functions $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, $g : \mathbb{R}^k \to \overline{\mathbb{R}}$ and $h : \mathbb{R}^n \to \overline{\mathbb{R}^k}$ such that $f$ is proper and convex, $g$ is proper, convex
and $K$ - increasing and $h$ is proper and $K$ - convex. The function $g$ is extended to the space $\mathbb{R}^k$ by taking $g(\infty_{\mathbb{R}^k}) = +\infty$. Moreover, throughout the entire chapter we assume that $\text{dom}(f) \cap \text{dom}(h) \cap h^{-1}(\text{dom}(g)) \neq \emptyset$.

The problem we work with is

\[(P) \quad \inf_{x \in \mathbb{R}^n} \left\{ f(x) + (g \circ h)(x) \right\}.\]

**Definition 2.1** An $\mathbf{\bar{x}} \in \mathbb{R}^n$ is called an optimal solution for $(P)$ if the optimal objective value of the problem is attained at $\mathbf{\bar{x}}$, i.e. $f(\mathbf{\bar{x}}) + (g \circ h)(\mathbf{\bar{x}}) = v(P)$.

It is not hard to see that the condition imposed above secure the existence of some $x \in \mathbb{R}^n$ such that $f(x) + (g \circ h)(x) < +\infty$. Thus $v(P) < +\infty$ and therefore the problem makes sense. Moreover, the function $g \circ h$ is a convex function, thus the problem $(P)$ is actually a convex optimization problem and can be treated as such (see, for example, [22, 28, 95]). Using this approach sooner or later we are forced to calculate the conjugate of the function $g \circ h$ and this can be done using the general formulae given in [51] and [52]. Nevertheless, the conditions the functions $g$ and $h$ must fulfill in order to use the formulae given in the above mentioned works are quite strong, and that is why we use a different approach.

Consider the additional problem

\[(\overline{P}) \quad \inf_{(x, y) \in \mathbb{R}^n \times \text{dom}(g), \ G(x, y) \leq K \emptyset} F(x, y)\]

where

$$F : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}, \quad F(x, y) = f(x) + g(y)$$

and

$$G : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k, \quad G(x, y) = h(x) - y.$$

The next result proves the equivalence of the problems $(P)$ and $(\overline{P})$.

**Theorem 2.1** The equality $v(P) = v(\overline{P})$ is always true.

**Proof.** Consider an arbitrary $x \in \mathbb{R}^n$. If $x \notin \text{dom}(f) \cap h^{-1}(\text{dom}(g))$, at least one of $f(x) = +\infty$ and $(g \circ h)(x) = +\infty$ holds, thus $f(x) + (g \circ h)(x) = +\infty \geq v(\overline{P})$. If $x \in \text{dom}(f) \cap h^{-1}(\text{dom}(g))$, take $y = h(x) \in \text{dom}(g)$. Then $G(x, y) = h(x) - y = 0 \in K$ and the pair $(x, y)$ is feasible to $(\overline{P})$. Even more, as $f(x) + (g \circ h)(x) = f(x) + g(y) = F(x, y)$, this equality is enough to secure $f(x) + (g \circ h)(x) \geq v(\overline{P})$. Taking into consideration the inequalities obtained in the two cases considered above, the inequality $v(P) \geq v(\overline{P})$ arises as an immediate consequence.

In order to prove the reverse inequality, let us consider an arbitrary pair $(x, y)$ feasible to $(\overline{P})$. Let us assume first that $h(x) = \infty_{\mathbb{R}^k}$. As $y \in \text{dom}(g) \subseteq \mathbb{R}^k$ the inequality $G(x, y) \leq K \emptyset$ cannot hold and thus $h(x) \in \mathbb{R}^k$. As $G(x, y) \leq K \emptyset$ we have that $g(h(x)) \leq g(y)$, so the inequality $f(x) + g(h(x)) \leq F(x, y)$ is also fulfilled. Even more, we get $v(P) \leq F(x, y)$ and, since this inequality is true for an arbitrary pair $(x, y)$ feasible to $(\overline{P})$, the inequality $v(P) \leq v(\overline{P})$ follows at hand. This completes the proof. \[\square\]

**Remark 2.1** Beyond any doubt, the fact that the function $g$ is $K$ - increasing plays a crucial role in the proof of the previous result. But in many practical cases the function $g$ is $K$ - increasing not over the whole space $\mathbb{R}^k$, but only over some non - empty subset of it, and under the circumstances we can ask ourselves if Theorem
2.1 can be proved also under these weaker assumption. Following the same idea as in the previous proof we can demonstrate that it holds $v(P) = v(D)$ even if the function $g$ is $K$ - increasing only over the set $h(\text{dom}(h)) + K$. Moreover, all the results we give further remain valid also in this case.

The problem $(P)$ is a convex optimization problem (this is a trivial consequence of the convexity of the functions $f$, $g$ and $h$) and a dual problem of it can be easily established. We associate first to $(P)$ its Lagrange dual problem with $\beta \in K^*$ as dual variable

$$\sup_{\beta \in K^*} \inf_{(x,y) \in \mathbb{R}^n \times \text{dom}(g)} \left\{ F(x,y) + \beta^T G(x,y) \right\}. \tag{D}$$

Using the definition of the conjugate regarding to a set, the inner infimum becomes

$$\inf_{(x,y) \in \mathbb{R}^n \times \text{dom}(g)} \left\{ F(x,y) + \beta^T G(x,y) \right\} = \inf_{x \in \mathbb{R}^n} \left\{ f(x) + g(y) + \beta^T (h(x) - y) \right\}$$

$$= \inf_{x \in \mathbb{R}^n} \left\{ f(x) + \beta^T h(x) \right\} + \inf_{y \in \text{dom}(g)} \left\{ g(y) - \beta^T y \right\} = -\sup_{x \in \mathbb{R}^n} \left\{ -f(x) - \beta^T h(x) \right\}$$

$$- \sup_{y \in \mathbb{R}^n} \left\{ \beta^T y - g(y) \right\} = -(f + \beta^T h)^*(0) - g^*(\beta).$$

As the inequality

$$(f + \beta^T h)^*(0) \leq \inf_{x^* \in \mathbb{R}^n} \left\{ f^*(x^*) + (\beta^T h)^*(-x^*) \right\}$$

is always true, it is not hard to notice that $v(D) \leq v(D)$, where

$$\sup_{x^* \in \mathbb{R}^n, \beta \in K^*} \left\{ -g^*(\beta) - f^*(x^*) - (\beta^T h)^*(-x^*) \right\} \tag{D}$$

is a so-called Fenchel - Lagrange - type dual of $(P)$. The following weak duality theorem arises as an easy consequence of the previous relation.

**Theorem 2.2** *(weak duality)* Between the primal problem $(P)$ and its dual problem $(D)$ weak duality always holds, i.e. $v(P) \geq v(D)$.

**Proof.** It is well-known that the optimal objective value of a convex programming problem is greater than or equal to the optimal objective value of its Lagrange dual problem. Thus $v(P) \geq v(D)$ and because of Theorem 2.1 we get the desired result. $\square$

If the duality gap between the primal problem $(P)$ and the Lagrange dual problem $(D)$ is strictly positive (for an example see [95]), the inequality $v(P) > v(D)$ arises as a direct consequence. Thus strong duality holds between $(P)$ and $(D)$ only in some circumstances. We consider the following constraint qualification

$$\exists x' \in \text{ri}(\text{dom}(h)) \cap \text{ri}(\text{dom}(f)) : h(x') \in \text{ri}(\text{dom}(g)) - \text{ri}(K), \tag{CQ}$$

and our next step is to prove that this condition assures strong duality between the primal problem $(P)$ and its Fenchel - Lagrange - type dual problem $(D)$.

**Theorem 2.3** *(strong duality)* If $(CQ)$ is fulfilled, then between $(P)$ and $(D)$ strong duality holds, i.e. $v(P) = v(D)$ and the dual problem has an optimal solution.
\textbf{Proof.} In order to prove that strong duality holds between \((P)\) and \((D)\) we prove first that strong duality holds between the problems \((\overline{P})\) and \((\overline{D})\). According to [12, 44] a sufficient condition for this is that

\[ 0 \in \text{ri} \left( G((\text{dom}(h) \cap \text{dom}(f)) \times \text{dom}(g)) + K \right) \quad (2.3) \]

is fulfilled. Let us prove now that (2.3) is implied by \((CQ)\).

A simple look at the condition \((CQ)\) allows us to affirm that there exists some \(y' \in \text{ri}(\text{dom}(g))\) such that \(h(x') = y' - \text{ri}(K)\). The last relation can be rewritten as

\[ G(x', y') = h(x') - y' \in - \text{ri}(K). \]

Moreover, since \(x' \in \text{ri}(\text{dom}(h)) \cap \text{ri}(\text{dom}(f))\), by Theorem 1.2 we get

\[ G(x', y') \in G \left( \text{ri} \left( (\text{dom}(h) \cap \text{dom}(f)) \times \text{dom}(g) \right) \right). \]

Obviously the relation

\[ 0 \in G \left( \text{ri} \left( (\text{dom}(h) \cap \text{dom}(f)) \times \text{dom}(g) \right) \right) + \text{ri}(K) \]

is a direct consequence of the previous ones. Using the results and considerations in [12] (by the proof of Proposition 1 the closedness assumption for \(K\) is there superfluous) one can prove that the last condition is equivalent to (2.3).

Thus there is strong duality between the problem \((\overline{P})\) and its Lagrange dual \((\overline{D})\), i.e. \(v(\overline{P}) = v(\overline{D})\) and the dual problem \((\overline{D})\) has an optimal solution \(\beta \in K^*\). Because of Theorem 2.1 we get

\[ v(\overline{P}) = \inf_{(x, y) \in \mathbb{R}^n \times \text{dom}(g)} \left\{ F(x, y) + \beta^T G(x, y) \right\}. \]

Since \(\text{dom}(F) = \text{dom}(f) \times \text{dom}(g)\) and \(\text{dom}(\overline{G}^T G) = \text{dom}(h) \times \mathbb{R}^k\) it is not hard to see that the assumptions of Theorem 31.1 in [80] are fulfilled, so that there exist \((\overline{x}, \overline{y}) \in \mathbb{R}^n \times \mathbb{R}^k\) such that \(v(\overline{P}) = -F^*(\overline{x}, \overline{y}) - (\overline{G}^T G)^*(-\overline{x}, -\overline{y})\). Using only the definition of the conjugate function one can prove that \(F^*(\overline{x}, \overline{y}) = f^*(\overline{x}) + g^*(\overline{y})\), while \((\overline{G}^T G)^*(-\overline{x}, -\overline{y}) = (\overline{h}^T h)^*(-\overline{x})\) if \(-\overline{y} = \overline{h}\) and \(+\infty\) otherwise. Therefore we have \(v(\overline{P}) = -g^*(\overline{h}) - f^*(\overline{x}) - (\overline{h}^T h)^*(-\overline{x}) = v(D)\). Noticing that \(v(D)\) is attained at \((\overline{h}, \overline{x})\), we are done. \(\square\)

Based on the above proved weak and strong duality properties we are able to point out necessary and sufficient conditions for the optimal solutions of problem \((P)\). The subsequent theorem is devoted to that matter.

\textbf{Theorem 2.4} (optimality conditions) \(a)\) Suppose that the condition \((CQ)\) is fulfilled and let \(\overline{\pi} \in \mathbb{R}^n\) be an optimal solution of the problem \((P)\). Then there exist \((\overline{\beta}, \overline{x}) \in K^* \times \mathbb{R}^n\) optimal solution for \((D)\) such that

\begin{enumerate}
  \item[(i)] \(g^*(\overline{\beta}) + (g \circ h)(\overline{\pi}) = \overline{h}^T h(\overline{\pi});\)
  \item[(ii)] \(f^*(\overline{x}) + f(\overline{\pi}) = \overline{x}^T \overline{\pi};\)
  \item[(iii)] \((\overline{h}^T h)^*(-\overline{x}) + \overline{h}^T h(\overline{\pi}) = -\overline{x}^T \overline{\pi}.\)
\end{enumerate}

\(b)\) If there exists \(\pi \in \mathbb{R}^n\) such that for some \(\beta \in K^*\) and \(x \in \mathbb{R}^n\) the assertions \(i) - (iii)\) are satisfied, then \(\pi\) is an optimal solution of \((P)\), \((\beta, x)\) is an optimal solution for \((D)\) and \(v(P) = v(D)\).
2.2. A FARKAS - TYPE RESULT VIA WEAK AND STRONG DUALITY

Proof. (a) The assumptions made in the beginning secures the inequality \( v(P) < +\infty \). Since \( \pi \) is an optimal solution of the problem \( (P) \) and the functions \( f \) and \( g \circ h \) are proper, it is binding to have \( v(P) > -\infty \), so that \( v(P) \) is a real number. Moreover, as the condition \((CQ)\) is fulfilled, Theorem 2.3 ensures the strong duality between \((P)\) and \((D)\), i.e. \( v(P) = v(D) \) and the dual problem has an optimal solution. Thus there exist \( \beta \in K^* \) and \( \overline{\pi} \in \mathbb{R}^n \) such that

\[
 f(\pi) + (g \circ h)(\pi) = -g^*(\overline{\beta}) - f^*(\overline{\pi}) = (\overline{\beta}^T h)^*(-\overline{\pi}).
\]

The last equality is nothing else than

\[
0 = f(\pi) + (g \circ h)(\pi) + g^*(\overline{\beta}) + f^*(\overline{\pi}) + (\overline{\beta}^T h)^*(-\overline{\pi}) = [g^*(\overline{\beta}) + (g \circ h)(\pi) - \overline{\beta}^T h(\pi)]
+ [f^*(\overline{\pi}) + f(\pi) - \overline{\pi}^T \overline{\pi}] = [g^*(\overline{\beta}) + (g \circ h)(\pi) - \overline{\beta}^T h(\pi)]
+ [f^*(\overline{\pi}) + f(\pi) - \overline{\pi}^T \overline{\pi}].
\]

As all the terms within the brackets are non-negative (see relation (1.3)), each term must be equal to 0 and the relations \((i) - (iii)\) follow.

(b) Summing up the statements \((i) - (iii)\) we acquire

\[
f(\pi) + (g \circ h)(\pi) = -g^*(\overline{\beta}) - f^*(\overline{\pi}) = (\overline{\beta}^T h)^*(-\overline{\pi})
\]

and the desired conclusion arises as a consequence of Theorem 2.2. \(\square\)

Before going further let us mention that for the sufficiency of the conditions \((i) - (iii)\), i.e. for \((b)\), the fulfillment of \((CQ)\) and also the convexity and monotonicity assumptions regarding \((P)\) are not necessary.

2.2 A Farkas - type result via weak and strong duality

By means of the weak and strong duality between a convex optimization problem and its Fenchel - Lagrange dual, Bot and Wanka have presented in [28, 29] some Farkas - type results for inequality systems involving finitely many convex functions. Since the Fenchel - Lagrange dual can be successfully employed also for optimization problems which involve the composition of two convex functions as objective function, the results in [18] naturally extend the ones from [28, 29] to inequality systems which involve also composed convex functions. Our aim is to give an analogous result for the problem we treat. As expected, the weak and strong duality assertions proved above are the backbone in the demonstration of the first main result of the present section.

Theorem 2.5 Suppose that \((CQ)\) holds. Then the following assertions are equivalent:

\[(i) \quad f(x) + (g \circ h)(x) \geq 0, \quad \forall x \in \mathbb{R}^n;\]
\[(ii) \quad \text{there exist } \beta \in K^* \text{ and } x^* \in \mathbb{R}^n \text{ such that}\]

\[
g^*(\beta) + f^*(x^*) + (\beta^T h)^*(-x^*) \leq 0. \tag{2.4}
\]

Proof. Necessity. The statement \((i)\) implies \( v(P) \geq 0 \) and, since the assumptions of Theorem 2.3 are fulfilled, \( v(D) = v(P) \geq 0 \) and the dual \((D)\) has an optimal solution \((\beta, x^*)\). Thus there exist \( \beta \in K^* \) and \( x^* \in \mathbb{R}^n \) fulfilling (2.4).

Sufficiency. As we can find some \( \beta \in K^* \) and \( x^* \in \mathbb{R}^n \) fulfilling (2.4), it follows right away that \( v(D) \geq -g^*(\beta) - f^*(x^*) - (\beta^T h)(-x^*) \geq 0 \). Weak duality between
\( (P) \) and \( (D) \) always holds and thus we obtain \( v(P) \geq 0 \), i.e. \( (i) \) is true.

The following theorem of the alternative is nothing else than a reformulation of the previous Farkas - type result.

**Theorem 2.6** Assume \( (CQ) \) fulfilled. Then either the inequality system

\[
(I) \quad x \in \mathbb{R}^n, f(x) + (g \circ h)(x) < 0
\]

has a solution or the system

\[
(II) \quad g^*(\beta) + f^*(x^*) + (\beta^T h)^*(-x^*) \leq 0, \quad \beta \in K^*, x^* \in \mathbb{R}^n
\]

has a solution, but never both.

The next theorem presents an equivalent assertion to the statement \( (ii) \) in Theorem 2.5 using only the epigraphs of the functions involved.

**Theorem 2.7** The statement \( (ii) \) in Theorem 2.5 is equivalent to

\[
(0, 0, 0) \in \{0\} \times \mathcal{T}(\text{epi}(g^*)) + \text{epi}(f^*) \times \{0\} + \bigcup_{\beta \in K^*} \text{epi}((\beta^T h)^*) \times \{-\beta\}. \tag{2.5}
\]

**Proof.** Necessity. By \( (ii) \), there exist \( \beta \in K^* \) and \( x^* \in \mathbb{R}^n \) such that (2.4) holds. As \( g^*(\beta) \) and \( (\beta^T h)^*(-x^*) \) have both finite real values, one has \( (\beta, g^*(\beta)) \in \text{epi}(g^*) \) and \( (-x^*, (\beta^T h)^*(-x^*)) \in \text{epi}((\beta^T h)^*) \). Thus \( (-x^*, (\beta^T h)^*(-x^*), -\beta) \in \text{epi}((\beta^T h)^*) \times \{-\beta\} \) and it holds

\[
(-x^*, (\beta^T h)^*(-x^*), -\beta) \in \bigcup_{\beta \in K^*} \text{epi}((\beta^T h)^*) \times \{-\beta\}.
\]

Taking into consideration the definition of the operator \( \mathcal{T} \) introduced in the first chapter of the work, the relation

\[
(0, g^*(\beta), \beta) \in \{0\} \times \mathcal{T}(\text{epi}(g^*))
\]

follows at once. On the other hand the inequality \( f^*(x^*) \leq g^*(\beta) - (\beta^T h)^*(-x^*) \) is also fulfilled, and, as the value in the right-hand side is finite, it holds \( (x^*, g^*(\beta) - (\beta^T h)^*(-x^*)) \in \text{epi}(f^*) \). This implies

\[
(x^*, -g^*(\beta) - (\beta^T h)^*(-x^*), 0) \in \text{epi}(f^*) \times \{0\}.
\]

Therefore we get

\[
(0, 0, 0) = (0, g^*(\beta), \beta) + (x^*, -g^*(\beta) - (\beta^T h)^*(-x^*), 0) + (-x^*, (\beta^T h)^*(-x^*), -\beta)
\]

\[
\in \{0\} \times \mathcal{T}(\text{epi}(g^*)) + \text{epi}(f^*) \times \{0\} + \bigcup_{\beta \in K^*} \text{epi}((\beta^T h)^*) \times \{-\beta\},
\]

and the first part of the proof is complete.

Sufficiency. Since (2.5) holds one can find some \( x^* \in \mathbb{R}^n \) and \( r \in \mathbb{R} \) such that

\[
(x^*, r, 0) \in \text{epi}(f^*) \times \{0\}
\]

and

\[
(-x^*, -r, 0) \in \{0\} \times \mathcal{T}(\text{epi}(g^*)) + \bigcup_{\beta \in K^*} \text{epi}((\beta^T h)^*) \times \{-\beta\}.
\]
Using the definition of the epigraph of a function from the first relation we acquire directly
\[ f^*(x^*) \leq r. \]
By the second relation there exists a \( \beta \in K^* \) such that
\[ (-x^*, -r, 0) \in \{0\} \times T(\text{epi}(g^*)) + \text{epi}((\beta^T h)^*)_X \times \{-\beta\}. \]
The definition of the operator \( T \) and the previous relation imply that there exist two real numbers \( r_1 \) and \( r_2 \) such that
\[-r = r_1 + r_2, \]
while the pairs \((\beta, r_1)\) and \((-x^*, r_2)\) are in \( \text{epi}(g^*) \) and \( \text{epi}((\beta^T h)^*) \), respectively. Thus
\[ g^*(\beta) + f^*(x^*) + (\beta^T h)^*(-x^*) \leq r_1 + r + r_2 = -r + r = 0 \]
and the desired conclusion is reached. \( \square \)

**Remark 2.2** In case the variable \( x \) covers not the whole space \( \mathbb{R}^n \), but only a non-empty and convex set \( X \subseteq \mathbb{R}^n \), the problem \((P)\) becomes the problem studied in [17]. If we replace the function \( h \) with \( h_X \) it is not hard to see that the results we derive from the assertions given before coincide with the ones in [17]. Moreover, as \( \text{dom}(h_X) = X \cap \text{dom}(h) \), the hypotheses we work with are the same as in the paper mentioned above.

### 2.3 Composition with a linear operator

In the following let us suppose that the function \( h \) is linear. More precisely, we consider the function
\[ h : \mathbb{R}^n \rightarrow \mathbb{R}^k, \quad h(x) = Ax, \]
where \( A \) is an \( k \times n \) real matrix. Our initial problem becomes in this special case
\[ (P^A) \]
\[ \inf_{x \in \mathbb{R}^n} \left\{ f(x) + g(Ax) \right\}. \]
If we take \( K = \{0\} \subseteq \mathbb{R}^k \), then obviously the functions \( g \) and \( h \) are \( K \)-increasing and \( K \)-convex, respectively. As for all \( \beta \in K^* = \mathbb{R}^k \) we have
\[ (\beta^T h)^*(-x^*) = \begin{cases} 0, & A^* \beta = -x^*, \\ +\infty, & \text{otherwise,} \end{cases} \]
it is not hard to see that the dual problem of \((P^A)\) becomes in this case
\[ (D^A) \]
\[ \sup_{\beta \in \mathbb{R}^k} \left\{ -g^*(\beta) - f^*(-A^* \beta) \right\}. \]
The proofs of the next results are omitted, as they are simple consequences of the ones given in the previous section.

**Theorem 2.8** (weak duality) Between the primal problem \((P^A)\) and the dual problem \((D^A)\) weak duality always holds, i.e. \( v(P^A) \geq v(D^A) \).

In this case we have \( \text{dom}(h) = \mathbb{R}^n \) and \( \text{ri}(K) = \{0\} \), so that the subsequent constraint qualification
\[ (CQ^A) \]
\[ \exists x' \in \text{ri(dom}(f)) : Ax' \in \text{ri(dom}(g)) \]
secures the strong duality between the primal problem \((P^A)\) and its Fenchel-Lagrange dual \((D^A)\). Moreover, the constraint qualification we \((CQ^A)\) coincides with the one given by Rockafellar in [80].
Theorem 2.9 (strong duality) If \((CQA)\) is fulfilled, then between \((PA)\) and \((DA)\) strong duality holds, i.e. \(v(\text{PA}) = v(\text{DA})\) and the dual problem has an optimal solution.

The next theorem presents some necessary and sufficient conditions for the optimal solutions of the problem \((PA)\).

Theorem 2.10 (optimality conditions) (a) Suppose that the condition \((CQA)\) is fulfilled and let \(x \in \mathbb{R}^n\) be an optimal solution of the problem \((PA)\). Then there exists \(\beta \in \mathbb{R}^k\) optimal solution for \((DA)\) such that

(i) \(g^*(\beta) + g(Ax) = \beta^T A x\);
(ii) \(f^*(-A^*\beta) + f(x) = -\beta^T A x\).

(b) If there exists \(x \in \mathbb{R}^n\) such that for some \(\beta \in \mathbb{R}^k\) the assertions (i) - (iii) are satisfied, then \(x\) is an optimal solution of \((PA)\), \(\beta\) is an optimal solution for \((DA)\) and \(v(\text{PA}) = v(\text{DA})\).

Proof. By Theorem 2.4 there exist \(x^* \in \mathbb{R}^n\) and \(\beta^* \in K^\ast\) such that the statements (i) - (iii) hold. Because of the special form of the function \((\beta^T h)^\ast\) the statements (iii) holds if and only if \(\beta^* = -A^* \beta\) and now it is easy to see that (i) and (ii) are equivalent to (i) and (ii), respectively. □

The subsequent theorem is a consequence of Theorem 2.5 and that is why we skip its proof (all we have to do is to take into consideration the way the function \((\beta^T h)^\ast\) looks like).

Theorem 2.11 Suppose that \((CQA)\) holds. Then the following assertions are equivalent:

(i) \(f(x) + g(Ax) \geq 0, \forall x \in \mathbb{R}^n;\)
(ii) there exist \(\beta \in \mathbb{R}^k\) such that

\[
g^*(\beta) + f^*(-A^*\beta) \leq 0. \tag{2. 6}\]

As in the previous section we rewrite the previous theorem as a theorem of the alternative and we present an equivalent assertion to the statement (ii) using the epigraphs of the functions involved.

Theorem 2.12 Assume \((CQA)\) fulfilled. Then either the inequality system

(I) \(x \in \mathbb{R}^n, f(x) + g(Ax) < 0\)

has a solution or the system

(II) \(g^*(\beta) + f^*(-A^*\beta) \leq 0, \beta \in \mathbb{R}^k\)

has a solution, but never both.

Theorem 2.13 The statement (ii) in Theorem 2.11 is equivalent to

\( (0, 0, 0) \in \{0\} \times T(\text{epi}(g^*)) + \bigcup_{\beta \in \mathbb{R}^k} \left( \text{epi}(f^*) + (A^*\beta, 0) \right) \times \{-\beta\}. \tag{2. 7} \)
2.4. THE ORDINARY OPTIMIZATION PROBLEM AS A SPECIAL CASE

Proof. By Theorem 2.7 the statement (ii) in Theorem 2.11 holds if and only if

\[(0, 0, 0) \in \{0\} \times T(\text{epi}(g^*)) + \text{epi}(f^*) \times \{0\} + \bigcup_{\beta \in K^*} \text{epi}((\beta^T h)^*) \times \{-\beta\}.
\]

Since \(K^* = \mathbb{R}^k\) in this case, the last relation can be equivalently rewritten as

\[(0, 0, 0) \in \{0\} \times T(\text{epi}(g^*)) + \bigcup_{\beta \in \mathbb{R}^k} \left(\text{epi}(f^*) + \text{epi}((\beta^T h)^*) \right) \times \{-\beta\}.
\]

Because of the way the function \((\beta^T h)^*\) looks like we have \(\text{epi}((\beta^T h)^*) = \{A^* \beta\} \times \mathbb{R}_+\). Using the properties of the epigraph of a function, it is not hard at all to prove that for all \(\beta \in \mathbb{R}^k\) we have \(\text{epi}(f^*) + \text{epi}((\beta^T h)^*) = \text{epi}(f^*) + (A^* \beta, 0)\). Replacing this in the formula from above we reach the desired conclusion. \(\square\)

Remark 2.3 Before going further we would like to mention that if the function \(g\) is \(K\)-increasing (\(K \subseteq \mathbb{R}^k\) is an arbitrary non-empty convex cone) we have \(\text{dom}(g^*) \subseteq K^*\) (see Theorem 1.5) and the condition \(\beta \in \mathbb{R}^k\) can be replaced by \(\beta \in K^*\).

Further we consider \(f, g : \mathbb{R}^n \to \mathbb{R}\) two proper and convex functions and \(A\) the identity operator over the space \(\mathbb{R}^n\), i.e., \(A : \mathbb{R}^n \to \mathbb{R}^n, Ax = x\). Then \(A^* = A\) and, moreover, it is easy to see that the condition \((CQ^*)\) holds if and only if the set \(\text{ri}(\text{dom}(f)) \cap \text{ri}(\text{dom}(g))\) is non-empty. The next theorem, known as Fenchel’s duality theorem, can be easily deduced from Theorem 2.9.

Theorem 2.14 Let \(f\) and \(g\) be such that \(\text{ri}(\text{dom}(f)) \cap \text{ri}(\text{dom}(g)) \neq \emptyset\). Then

\[\inf_{x \in \mathbb{R}^n} \left\{f(x) + g(x)\right\} = \sup_{\beta \in \mathbb{R}^n} \left\{-g^*(\beta) - f^*(\beta)\right\}\]

and the supremum is attained.

The next result is an obvious consequence of Theorem 2.11.

Theorem 2.15 Suppose that \(\text{ri}(\text{dom}(f)) \cap \text{ri}(\text{dom}(g)) \neq \emptyset\). Then the following assertions are equivalent:

(i) \(f(x) + g(x) \geq 0, \forall x \in \mathbb{R}^n;\)

(ii) there exist \(\beta \in \mathbb{R}^n\) such that

\[g^*(\beta) + f^*(-\beta) \leq 0.\]

Theorem 2.16 The statement (ii) in Theorem 2.15 is equivalent to

\[(0, 0) \in \text{epi}(g^*) + \text{epi}(f^*).\]

2.4 The ordinary optimization problem as a special case

Let \(X \subseteq \mathbb{R}^n\) be a non-empty convex set and \(K \subseteq \mathbb{R}^k\) a non-empty convex cone. Consider the functions \(f : \mathbb{R}^n \to \mathbb{R}\) and \(h : \mathbb{R}^n \to \mathbb{R}^k, h = (h_1, \ldots, h_k)^T\), such that \(f\) is proper and convex and \(h\) is proper and \(K\)-convex.
Take the problem

\[(P^O) \quad \inf_{x \in X, \ h(x) \leq_K 0} f(x) \]

and assume that \( X \cap \text{dom}(f) \cap h^{-1}(-(K)) \neq \emptyset \) (in other words we assume that the optimal objective value of the problem \((P^O)\) is less than \(+\infty\)). It is not hard to observe that for all \( x \in \mathbb{R}^n \) we have

\[ x \in X, h(x) \leq_K 0 \Leftrightarrow h_X(x) \leq_K 0 \Leftrightarrow \delta_{-K}(h_X(x)) = 0 \Leftrightarrow (\delta_{-K} \circ h_X)(x) = 0, \]

where the function \( h_X \) is given by (1. 1). Obviously we get

\[ v(P^O) = \inf_{x \in \mathbb{R}^n} \left\{ f(x) + (\delta_{-K} \circ h_X)(x) \right\} \]

and with an abuse of notation we can consider further the problem

\[(P^O) \quad \inf_{x \in \mathbb{R}^n} \left\{ f(x) + (\delta_{-K} \circ h_X)(x) \right\}. \]

Since the function \( \delta_{-K} \) is \( K \) - increasing, the results obtained in the first section can be used. Thus to the problem \((P^O)\) we can associate the following dual problem

\[(D^O) \quad \sup_{x^* \in \mathbb{R}^n, \ \beta \in K^*} \left\{ -\delta^*_{-K}(\beta) - f^*(x^*) - (\beta^T h_X)^*(x^*) \right\}. \]

Even more, since for all \( \beta \in K^* \) \((\beta^T h_X)^*(-x^*) = (\beta^T h)_X^*(-x^*) \) and \( \delta^*_{-K}(\beta) = \delta_{K^*}(\beta) = 0 \) the dual \((D^O)\) becomes

\[(D^O) \quad \sup_{x^* \in \mathbb{R}^n, \ \beta \in K^*} \left\{ -f^*(x^*) - (\beta^T h)_X^*(-x^*) \right\}. \]

The subsequent weak duality theorem arises as a corollary of Theorem 2.2.

**Theorem 2.17** (weak duality) Between the primal problem \((P^O)\) and the dual problem \((D^O)\) weak duality always holds, i.e. \( v(P^O) \geq v(D^O) \).

In order to get strong duality between the primal problem \((P^O)\) and its Fenchel - Lagrange dual problem \((D^O)\) the fulfilling of the following constraint qualification is sufficient

\[(CQ^O) \quad \exists x' \in \text{ri}(\text{dom}(h_X)) \cap \text{ri}(\text{dom}(f)) : h_X(x') \in \text{ri}(\text{dom}(\delta_{-K})) \cap \text{ri}(K). \]

But \( \text{dom}(h_X) = X \cap \text{dom}(h) \) and, as \( \text{ri}(\text{dom}(\delta_{-K})) \cap \text{ri}(K) = - \text{ri}(K) \), the constraint qualification becomes

\[(CQ^O) \quad \exists x' \in \text{ri}(X \cap \text{dom}(h)) \cap \text{ri}(\text{dom}(f)) : h(x') \in - \text{ri}(K). \]

The following outcome is an easy consequence of the result proved within the first section of the chapter.

**Theorem 2.18** (strong duality) If \((CQ^O)\) is fulfilled, then between \((P^O)\) and \((D^O)\) strong duality holds, i.e. \( v(P^O) = v(D^O) \) and the dual problem has an optimal solution.

Let us give now necessary and sufficient conditions for the optimal solutions of the problem \((P^O)\).
2.4 THE ORDINARY OPTIMIZATION PROBLEM

**Theorem 2.19** (optimality conditions) (a) Suppose that the condition \((CQ^O)\) is fulfilled and let \(x^* \in \mathbb{R}^n\) be an optimal solution of the problem \((P^O)\). Then there exist \((\beta, x^*) \in K^* \times \mathbb{R}^n\) optimal solution for \((D^O)\) such that

\[
(iO) \quad f^*(x^*) + f(x^*) = x^T \beta; \\
(iiO) \quad (\beta^T h)^*_X(-x^*) + \beta^T h(x^*) = -x^T \beta; \\
(iiiO) \quad \beta^T h(x^*) = 0.
\]

(b) If there exists \(x^* \in \mathbb{R}^n\) such that for some \(\beta \in K^*\) and \(x^* \in \mathbb{R}^n\) the assertions \((iO)-(iiiO)\) are satisfied, then \(x^*\) is an optimal solution of \((P^O)\), \((\beta, x^*)\) is an optimal solution for \((D^O)\) and \(v(P^O) = v(D^O)\).

**Proof.** According to Theorem 2.4 there exist \(\beta \in K^*\) and \(x^* \in \mathbb{R}^n\) such that the conditions \((i)-(iii)\) hold. Obviously the statements \((iO)\) and \((iiO)\) are equivalent within case to \((ii)\) and \((iii)\), respectively. It remains to prove that \((iiiO)\) is equivalent within case to \((i)\). As \(x^*\) is an optimal solution of the problem \((P^O)\), we have \(h(x^*) \in -K\). Thus \((\delta^*_K \circ h)(x^*) = 0\) and, as \(\delta^*_K(\beta) = 0\), the equivalence between the statements \((iiiO)\) and \((i)\) is obvious. \(\square\)

**Remark 2.4** The statement \((iiO)\) can be equivalently rewritten as

\[
\bar{x}^T \beta + \beta^T h(x^*) = \inf_{x \in K} \{ \bar{x}^T x + \beta^T h(x) \}.
\]

The proof of the next result is skipped, because all we have to do is to use Theorem 2.5 and the special form of the function \(\delta^*_K\).

**Theorem 2.20** Suppose that \((CQ^O)\) holds. Then the following assertions are equivalent:

\[
(i) \quad x \in \mathbb{R}^n, h(x) \leq_K 0 \Rightarrow f(x) \geq 0; \\
(ii) \quad \text{there exist } \beta \in K^* \text{ and } x^* \in \mathbb{R}^n \text{ such that}
\]

\[
f^*(x^*) + (\beta^T h)^*_X(-x^*) \leq 0. \tag{2.8}
\]

**Theorem 2.21** Assume \((CQ^O)\) fulfilled. Then either the inequality system

\[
(I) \quad x \in X, h(x) \leq_K 0, f(x) < 0
\]

has a solution or the system

\[
(II) \quad f^*(x^*) + (\beta^T h)^*_X(-x^*) \leq 0, \\
\beta \in K^*, x^* \in \mathbb{R}^n
\]

has a solution, but never both.

Our next aim is to present an equivalent formulation to assertion \((ii)\) in Theorem 2.20 using only epigraphs. The next theorem is devoted to that matter.

**Theorem 2.22** The statement \((ii)\) in Theorem 2.20 is equivalent to

\[
(0, 0) \in \text{epi}(f^*) + \bigcup_{\beta \in K^*} \text{epi}((\beta^T h)^*_X). \tag{2.9}
\]
 CHAPTER 2. FARKAS RESULTS WITH COMPOSED FUNCTIONS

Proof. By Theorem 2.7 we know that the statement (ii) in Theorem 2.20 is equivalent to

\[(0, 0, 0) \in \{0\} \times T(\text{epi}((\delta_{-K}^*)) + \text{epi}(f^*) \times \{0\} + \bigcup_{\beta \in K^*} \text{epi}((\beta^T h_X)^*) \times \{-\beta\}).\]

Since \(\text{epi}((\delta_{-K}^*)) = K^* \times \mathbb{R}_+\) and \(\text{epi}((\beta^T h_X)^*) = \text{epi}((\beta^T h_X)_{X^*})\) it is easy to see that the last relation can be equivalently written as

\[(0, 0, 0) \in \{0\} \times \mathbb{R}_+ \times K^* + \text{epi}(f^*) \times \{0\} + \bigcup_{\beta \in K^*} \text{epi}((\beta^T h_X)_{X^*}) \times \{-\beta\}).\]

This holds if and only if

\[(0, 0) \in \{0\} \times \mathbb{R}_+ + \text{epi}(f^*) + \bigcup_{\beta \in K^*} \text{epi}((\beta^T h_X)_{X^*}).\]

Using only the definition of the epigraph of a function it is easy to prove that \(\{0\} \times \mathbb{R}_+ + \text{epi}(f^*) = \text{epi}(f^*)\). Replacing above we get

\[(0, 0) \in \text{epi}(f^*) + \bigcup_{\beta \in K^*} \text{epi}((\beta^T h_X)_{X^*})\]

and the proof is complete. \(\square\)

Before going further let us remark that some similar results were given by Bot and Wanka in [28]. To show that their results are actually special instances of the ones presented within this section, we need the following lemma. We would like to mention that the idea behind the next result can be found in [28, 29], where an analogous result has been proved.

**Lemma 2.1** Let \(X \subseteq \mathbb{R}^n\) be a non-empty convex set and \(h : \mathbb{R}^n \to \mathbb{R}^k\), \(h = (h_1, \ldots, h_k)^T\) be a \(\mathbb{R}^k_+\) - convex function. Then

\[\bigcup_{\beta \in \mathbb{R}^k_+} \text{epi}((\beta^T h)^*_X) = \text{coneco} \left( \bigcup_{i=1}^k \text{epi}(h_i^*) \right) + \text{epi}(\sigma_X).\]

**Proof.** "\(\subseteq\)" Consider an arbitrary \(\beta \in \mathbb{R}^k_+\). Let us suppose first that \(\beta = 0\). It is easy to see that in this case for all \(x \in X\) we have \(\delta_X(x) = \beta^T h(x)\). This implies \(\delta_X^*(x^*) = \sigma_X(x^*) = (\beta^T h)^*_X(x^*)\) for all \(x^* \in \mathbb{R}^n\) (see relation (1. 4) for the first equality), and from here the relation \(\text{epi}((\beta^T h)^*_X) = \text{epi}(\sigma_X)\) follows at hand. As \((0, 0) \in \text{coneco} \left( \bigcup_{i=1}^k \text{epi}(h_i^*) \right)\), we acquire

\[\text{epi}((\beta^T h)^*_X) \subseteq \text{coneco} \left( \bigcup_{i=1}^k \text{epi}(h_i^*) \right) + \text{epi}(\sigma_X).\]

Now let us suppose that \(\beta = (\beta_1, \ldots, \beta_k)^T \in \mathbb{R}^k_+ \setminus \{0\}\). Then the set \(I_\beta = \{i \in \{1, \ldots, k\} : \beta_i > 0\}\) is non-empty and it holds

\[\beta^T h = \sum_{i \in I_\beta} \beta_i h_i.\]

By relation (1. 4) we get further

\[\text{epi}((\beta^T h)^*_X) = \text{epi}((\beta^T h + \delta_X^*)^*) = \text{epi} \left( \left( \sum_{i \in I_\beta} \beta_i h_i + \delta_X \right) \right)^*.\]
Moreover, as $\cap_{i=1}^k ri(dom(h_i) \cap ri(X)) = ri(X) \neq \emptyset$ (the set $X$ is non-empty and convex), the previous relation and Theorem 1.3 imply
\[
\text{epi}((\beta^T h)^*_X) = \sum_{i \in I_\beta} \text{epi}((\beta_i h_i)^*) + \text{epi}(\delta^*_X).
\]
As for all $i \in I_\beta$ the equality $\text{epi}((\beta_i h_i)^*) = \beta_i \text{epi}(h_i^*)$ holds (see Theorem 1.4), the relation
\[
\text{epi}((\beta^T h)^*_X) = \sum_{i \in I_\beta} \beta_i \text{epi}(h_i^*) + \text{epi}(\sigma_X)
\]
follows at hand. But
\[
\sum_{i \in I_\beta} \beta_i \text{epi}(h_i^*) = \left( \sum_{i \in I_\beta} \beta_i \right) \left( \sum_{i \in I_\beta} \frac{\beta_i}{\beta} \text{epi}(h_i^*) \right) \subseteq \text{coneco} \left( \bigcup_{i \in I_\beta} \text{epi}(h_i^*) \right)
\]
\[
\subseteq \text{coneco} \left( \bigcup_{i=1}^k \text{epi}(h_i^*) \right).
\]
and, combining this relation with the previous one we get
\[
\text{epi}((\beta^T h)^*_X) \subseteq \text{coneco} \left( \bigcup_{i=1}^k \text{epi}(h_i^*) \right) + \text{epi}(\sigma_X).
\]
"$\supseteq$" Take an arbitrary pair
\[
(x^*, r) \in \text{coneco} \left( \bigcup_{i=1}^k \text{epi}(h_i^*) \right) + \text{epi}(\sigma_X).
\]
Then there exist $(v^*, v) \in \text{coneco} \left( \bigcup_{i=1}^k \text{epi}(h_i^*) \right)$ and $(u^*, u) \in \text{epi}(\sigma_X)$ such that
\[
(x^*, r) = (v^*, v) + (u^*, u).
\]
Moreover, there exist $q \geq 0$, $t_i \geq 0$ and $(v_i^*, v_i) \in \text{epi}(h_i^*)$, $i = 1, \ldots, k$, such that
\[
\sum_{i=1}^k t_i = 1
\]
and
\[
(v^*, v) = q \sum_{i=1}^k t_i (v_i^*, v_i).
\]
As $\delta^*_X(u^*) = \sigma_X(u^*) \leq u$ and $h_i^*(v_i^*) \leq v_i$, $i = 1, \ldots, k$, and, since $qt_i \geq 0$ for all $i = 1, \ldots, k$, we get
\[
r = v + u = \sum_{i=1}^k qt_i v_i + u \geq \sum_{i=1}^k qt_i h_i^*(v_i^*) + \delta^*_X(u^*).
\]
Let us prove that for all $i = 1, \ldots, k$, we have $qt_i h_i^*(v_i^*) = (qt_i h_i)^*(qt_i v_i^*)$. If $qt_i > 0$ this is a well-known property of the conjugate function (see relation (1. 5)). If $qt_i = 0$ then, as $h_i^*(v_i^*) \in \mathbb{R}$, we have $qt_i h_i^*(v_i^*) = 0 = \delta^*_0(0) = (qt_i h_i)^*(qt_i v_i^*)$ (the last equality is a consequence of relation (2. 2)). Using the previous equalities and Theorem 1.1, we get
\[
r \geq \sum_{i=1}^k (qt_i h_i)^*(qt_i v_i^*) + \delta^*_X(u^*) \geq \left( \sum_{i=1}^k qt_i h_i + \delta^*_X \right)^*(\sum_{i=1}^k qt_i v_i^* + u^*).
\]
As $x^* = v^* + u^* = \sum_{i=1}^k q_i v_i^* + u^*$, for $\beta = (\beta_1, \ldots, \beta_k)^T \in \mathbb{R}_+^k$, where $\beta_i = q_i$, $i = 1, \ldots, k$, the inequality above becomes
\[ r \geq \left( \sum_{i=1}^k \beta_i h_i + \delta_X \right)^* (x^*). \]

Therefore $r \geq (\beta^T h + \delta_X)^* (x^*) = (\beta^T h)_X^*(x^*)$, which is nothing but $(x^*, r) \in \text{epi}((\beta^T h)_X^*)$, thus the lemma is proved. □

In order to rediscover the results given in [28] we take $K = \mathbb{R}_+^k$ and let $h : \mathbb{R}^n \to \mathbb{R}^k$, $h = (h_1, \ldots, h_k)^T$, be a $\mathbb{R}_+^k$ - convex function. The constraint qualification $(CQ_O)$ becomes in this case
\[ (CQ_O^Q) \quad \exists x' \in \text{ri}(X) \cap \text{ri(dom}(f)) : h(x') \in - \text{ri}(\mathbb{R}_+^k), \]
which is actually the Slater constraint qualification
\[ (CQ_O^S) \quad \exists x' \in \text{ri}(X) \cap \text{ri(dom}(f)) : h(x') < 0. \]

The next results are easy outcomes of Theorem 2.20 and Theorem 2.22 and the previous lemma, respectively.

**Theorem 2.23** Suppose that $(CQ_O^Q)$ holds. Then the following assertions are equivalent:

(i) $x \in X, h(x) \leq 0 \Rightarrow f(x) \geq 0$;
(ii) there exist $\beta \in \mathbb{R}_+^k$ and $x^* \in \mathbb{R}^n$ such that
\[ f^*(x^*) + (\beta^T h)_X^*(-x^*) \leq 0. \]

**Theorem 2.24** The statement (ii) in Theorem 2.23 is equivalent to
\[ (0, 0) \in \text{epi}(f^*) + \text{coneco} \left( \bigcup_{i=1}^k \text{epi}(h_i^*) \right) + \text{epi}(\sigma_X). \]

**Remark 2.5** We would like also to mention that the results remain valid if in the condition $(CQ_O^S)$ instead of $h(x') < 0$ we impose the weaker assumption (see [28])
\[ \left\{ \begin{array}{ll} h_i(x') \leq 0, & i \in L, \\ h_i(x') < 0, & i \in N, \end{array} \right. \]

where $L := \{i \in \{1, \ldots, k\} : h_i \text{ is an affine function}\}$ and $N := \{1, \ldots, k\} \setminus L$.

### 2.5 The problem with composed objective function and convex constraints

Within this section we consider an optimization problem having as objective function the composition of some convex functions and geometric and conical constraints, namely the problem
\[ (P^C) \quad \inf_{x \in X, \ |G(x)\| \leq q} (f \circ F)(x), \]
2.5 COMPOSED OBJECTIVE FUNCTION AND CONVEX CONSTRAINTS

where \( f : \mathbb{R}^k \to \mathbb{R} \) is proper, convex and \( K \) - increasing, \( F : \mathbb{R}^n \to \mathbb{R}^k \), \( F = (F_1, \ldots, F_k)^T \), is \( K \) - convex and \( G : \mathbb{R}^n \to \mathbb{R}^m \), \( G = (G_1, \ldots, G_m)^T \) is \( Q \) - convex. Here \( X \subseteq \mathbb{R}^n \) is a non - empty convex set and \( K \subseteq \mathbb{R}^k \) and \( Q \subseteq \mathbb{R}^m \) are non - empty convex cones. Moreover, we assume that \( X \cap G^{-1}(-Q) \cap F^{-1}(\text{dom}(f)) \neq \emptyset \) (otherwise \( v(P^C) = +\infty \)). The approach we use is similar to the one used in the first section of the chapter. Namely, to \( (P^C) \) we associate the convex optimization problem

\[
(P^C) \quad \inf_{(x,y) \in X \times \mathbb{R}^k} \tilde{f}(x,y),
\]

where \( \tilde{K} = K \times Q \) and \( \tilde{f} : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R} \) and \( \tilde{h} : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k \times \mathbb{R}^m \) are defined as

\[
\tilde{f}(x,y) = f(y) \quad \text{and} \quad \tilde{h}(x,y) = \left\{ \begin{array}{ll}
(F(x) - y, G(x)), & x \in \text{dom}(F) \cap \text{dom}(G), \\
\infty \times \mathbb{R}^m, & \text{otherwise}.
\end{array} \right.
\]

The problem \( (P^C) \) is obviously convex, as \( \tilde{K} \) is a convex cone and the functions \( \tilde{f} \) and \( \tilde{h} \) are convex and \( \tilde{K} \) - convex, respectively. Regarding the optimal objective values of the problems \( (P^C) \) and \( (\bar{P}^C) \) the following result can be established.

**Theorem 2.25** The equality \( v(P^C) = v(\bar{P}^C) \) is always fulfilled.

**Proof.** For an arbitrary \( x \) feasible to \( (P^C) \) take \( y = F(x) \). If \( F(x) \notin \text{dom}(f) \) or \( F(x) = \infty \times \mathbb{R}^m \) we obviously have \( f(F(x)) = +\infty \geq v(\bar{P}^C) \). If none of the previous situations holds, then \( F(x) - y = 0 \leq K \) and, since the feasibility of \( x \) involves \( G(x) \leq Q \), we can conclude that \( \tilde{h}(x,y) = (F(x) - y, G(x)) \leq R \). Thus \( (x,y) \) is feasible to \( (\bar{P}^C) \) and, moreover, it holds \( f(F(x)) = f(y) = \tilde{f}(x,y) \geq v(\bar{P}^C) \). Thus \( f(F(x)) \geq v(\bar{P}^C) \) for all \( x \) feasible to \( (P^C) \) and that implies \( v(P^C) \geq v(\bar{P}^C) \).

In order to prove the opposite inequality, let us consider \( (x,y) \) feasible to \( (P^C) \). If \( y \notin \text{dom}(f) \), then obviously \( v(P^C) \leq f \circ F)(x) \leq f(y) = \tilde{f}(x,y) = +\infty \). Otherwise, since \( \tilde{h}(x,y) \leq R \), the way we have defined the function \( \bar{h} \) allows us to affirm that \( F(x) - y \leq K \) and \( G(x) \leq Q \). Since \( x \in X \) we have \( x \) feasible to the problem \( (P^C) \). As the function \( f \) is \( K \) - increasing we get \( v(P^C) \leq f(F(x)) \leq f(y) = \tilde{f}(x,y) \).

Taking the infimum on the right-side regarding \( (x,y) \) feasible to \( (\bar{P}^C) \) we obtain \( v(P^C) \leq v(\bar{P}^C) \). \( \square \)

Using the results from the previous section, to the problem \( (\bar{P}^C) \) we attach the Fenchel - Lagrange dual problem

\[
(\bar{D}^C) \quad \sup_{(x^*, y^*) \in \mathbb{X}^* \times \mathbb{R}^m} \left\{ -\tilde{f}^*(x^*, y^*) - (\gamma^T \tilde{h})^*_X \times \mathbb{R}^k (-x^*, -y^*) \right\}.
\]

Our next step is to determine the values of the conjugate functions which appear in the objective function of the problem \( (\bar{D}^C) \). Through simple calculations we acquire

\[
\tilde{f}^*(x^*, y^*) = \sup_{x \in \mathbb{R}^n, \ y \in \mathbb{R}^k} \left\{ x^* T x + y^* T y - \tilde{f}(x,y) \right\} = \sup_{x \in \mathbb{R}^n, \ y \in \mathbb{R}^k} \left\{ x^* T x + y^* T y - f(y) \right\} = \sup_{x \in \mathbb{R}^n} \left\{ x^* T x \right\} + \sup_{y \in \mathbb{R}^k} \left\{ y^* T y - f(y) \right\} = \left\{ \begin{array}{ll}
\tilde{f}^*(y^*), & x^* = 0, \\
+\infty, & \text{otherwise}.
\end{array} \right.
\]

Thus the dual \( (\bar{D}^C) \) is equivalent to

\[
(\bar{D}^C) \quad \sup_{x \in \mathbb{R}^n} \left\{ x^* T x \right\} = \left\{ \begin{array}{ll}
\tilde{f}^*(y^*), & x^* = 0, \\
+\infty, & \text{otherwise}.
\end{array} \right.
\]
Since \( \tilde{K} = K \times Q \) we have \( \tilde{K}^* = K^* \times Q^* \) and that is why \( \gamma \in \tilde{K}^* \) if and only if there exist \( \beta \in K^* \) and \( \alpha \in Q^* \) such that \( \gamma = (\beta, \alpha) \). Thus for all \( \gamma = (\beta, \alpha) \in K^* \times Q^* \) one has

\[
(\gamma^T \tilde{h})^*_X(x, y) = \sup_{(x, y) \in X \times R^k} \left\{ -x^T x - y^T y - \gamma^T \tilde{h}(x, y) \right\}
\]

\[
= \sup_{x \in X, \ y \in R^k} \left\{ -x^T x - y^T y - \beta^T F(x) - y - \alpha^T G(x) \right\} = \sup_{x \in X} \left\{ -x^T x - \beta^T F(x) - \alpha^T G(x) \right\}
\]

\[-\alpha^T G(x) + \sup_{y \in R^k} \left\{ -y^T y + \beta^T y \right\} = \left\{ (\beta^T F + \alpha^T G)^*_X(-x^*), \ \beta = y^*, \ \text{otherwise}. \right\}
\]

Taking into consideration the previous relations the dual problem becomes

\[
(D^C) \quad \sup_{\beta \in K^*, \ \alpha \in Q^*} \left\{ -f^*(\beta) - (\beta^T F + \alpha^T G)^*_X(0) \right\}.
\]

Since for all \( x^* \in R^n \) we have \( (\beta^T F + \alpha^T G)^*_X(0) \leq (\beta^T F)(x^*) + (\alpha^T G)^*_X(-x^*) \) (see Theorem 16.4 in [80]), it is not hard to see that the optimal objective value of the problem

\[
(D^C) \quad \sup_{\beta \in K^*, \ \alpha \in Q^*} \left\{ -f^*(\beta) - (\beta^T F)^*(x^*) - (\alpha^T G)^*_X(-x^*) \right\}
\]

is less than or equal to \( v(D^C) \). The next result arises as a straightforward consequence of the previous inequality and Theorem 2.17.

**Theorem 2.26** (weak duality) Between the primal problem \((P^C)\) and the dual problem \((D^C)\) weak duality always holds, i.e. \( v(P^C) \geq v(D^C) \).

**Proof.** Theorem 2.17 implies weak duality between the problems \((\overline{P}^C)\) and \((\overline{D}^C)\). As \( v(\overline{P}^C) = v(P^C) \) and \( v(\overline{D}^C) \geq v(D^C) \) the desired inequality is immediate. \( \square \)

The next constraint qualification secures the strong duality between the primal problem \((P^C)\) and its Fenchel - Lagrange - type dual problem \((D^C)\)

\[
(CQ^C) \quad \exists x' \in \text{ri}(X \cap \text{dom}(G)) \cap \text{ri}(\text{dom}(F)) : \left\{ F(x') \in \text{ri}(\text{dom}(f)) - \text{ri}(K), \ G(x') \in -\text{ri}(Q). \right\}
\]

**Theorem 2.27** (strong duality) If \((CQ^C)\) is fulfilled, then between \((P^C)\) and \((D^C)\) strong duality holds, i.e. \( v(P^C) = v(D^C) \) and the dual problem has an optimal solution.

**Proof.** In order to prove that strong duality holds between \((P^C)\) and \((D^C)\), we prove first that strong duality holds between \((\overline{P}^C)\) and \((\overline{D}^C)\). Theorem 2.18 assures this provided that the condition

\[
\exists (x', y') \in \text{ri}(X \times R^k \cap \text{dom}(\tilde{h})) \cap \text{ri}(\text{dom}(\tilde{f})) : \tilde{h}(x', y') \in -\text{ri}(\tilde{K})
\]

holds, so that our first step is to prove that the previous condition is implied by the constraint qualification \((CQ^C)\).

The way we have defined the functions \( \tilde{f} \) and \( \tilde{h} \) imply \( \text{dom}(\tilde{f}) = R^n \times \text{dom}(f) \) and

\[
X \times R^k \cap \text{dom}(\tilde{h}) = X \times R^k \cap (\text{dom}(F) \cap \text{dom}(G)) \times R^k = (X \cap \text{dom}(F) \cap \text{dom}(G)) \times R^k.
\]
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Moreover, since \( \text{ri}(\tilde{K}) = \text{ri}(K \times Q) = \text{ri}(K) \times \text{ri}(Q) \), the condition which ensures that strong duality holds between \((P^C)\) and \((D^C)\) becomes

\[
\exists (x', y') \in \text{ri}(\text{dom}(F) \cap \text{dom}(G)) \times \text{ri}(\text{dom}(f)) : \begin{cases} F(x') - y' \in -\text{ri}(K), \\ G(x') \in -\text{ri}(Q). \end{cases}
\]

Since the constraint qualification \((CQC)\) if fulfilled, there exist \( y' \in \text{ri}(\text{dom}(f)) \) such that \( F(x') - y' \in -\text{ri}(K) \), and from here we get \( F(x') \in \text{ri}(\text{dom}(f)) - \text{ri}(K) \).

Moreover, as the sets \( \text{ri}(\text{dom}(F)) \cap \text{dom}(G) \) and \( \text{ri}(X \cap \text{dom}(G)) \cap \text{ri}(\text{dom}(f)) \) are equal (see Theorem 1.2) we can affirm that the condition which ensures strong duality between the problems \((P^C)\) and \((D^C)\) is fulfilled. Thus \( v(P^C) = v(D^C) \) and the dual problem \((D^C)\) has an optimal solution \((\tilde{\beta}, \tilde{\pi}) \in K^* \times Q^*\). By Theorem 2.25 we get

\[
v(P^C) = -f^*(\beta) - (\tilde{\beta}^T F + \tilde{\pi}^T G)\chi(0).
\]

As the hypotheses of Theorem 1.1 are fulfilled there exists \( \pi^* \in \mathbb{R}^n \) such that

\[
(\tilde{\beta}^T F + \tilde{\pi}^T G)\chi(0) = (\beta^T F + \pi^T G + \delta)\chi(0)
\]

= \( (\tilde{\beta}^T F)^*(\pi^*) + (\pi^T G + \delta)\chi(-\pi^*) = (\tilde{\beta}^T F)^*(\pi^*) + (\pi^T G)^*\chi(-\pi^*).\)

Combining the previous relations we reach the desired result and the proof is complete. \( \square \)

The subsequent theorem presents some necessary and sufficient conditions for the optimal solutions of \((P^C)\). Although its proof is very similar to the proof of Theorem 2.4, we prefer to give it for the sake of completeness.

**Theorem 2.28 (optimality conditions)** (a) Suppose that the condition \((CQC)\) is fulfilled and let \( \pi \in \mathbb{R}^n \) be an optimal solution of the problem \((P^C)\). Then there exist \((\tilde{\beta}, \tilde{\pi}, \tilde{\pi^*}) \in K^* \times Q^* \times \mathbb{R}^n \) optimal solution for \((D^C)\) such that

\[
(i\text{c}) \quad f^*(\beta) + (f \circ F)(\pi) = \tilde{\beta}^T F(\pi);
\]

\[
(ii\text{c}) \quad (\tilde{\beta}^T F)^*(\pi^*) + \tilde{\beta}^T F(\pi) = \tilde{\pi^T} \pi;
\]

\[
(iii\text{c}) \quad (\pi^T G)^*(\pi^*) + \tilde{\pi^T} G(\pi) = -\tilde{\pi^T} \pi;
\]

\[
(iv\text{c}) \quad \tilde{\pi^T} G(\pi) = 0.
\]

(b) If there exists \( \pi \in \mathbb{R}^n \) such that for some \( \tilde{\beta} \in K^*, \tilde{\pi} \in Q^* \) and \( \tilde{\pi^*} \in \mathbb{R}^n \) the assertions \((i\text{c}) - (iv\text{c})\) are satisfied, then \( \pi \) is an optimal solution of \((P^C)\), \((\tilde{\beta}, \tilde{\pi}, \tilde{\pi^*})\) is an optimal solution for \((D^C)\) and \( v(P^C) = v(D^C) \).

**Proof.** (a) By Theorem 2.27 strong duality between \((P^C)\) and \((D^C)\), i.e. \( v(P^C) = v(D^C) \) and the dual problem \((D^C)\) has an optimal solution \((\tilde{\beta}, \tilde{\pi}, \tilde{\pi^*})\). Thus there exist \( \tilde{\beta} \in K^*, \tilde{\pi} \in Q^* \) and \( \tilde{\pi^*} \in \mathbb{R}^n \) such that

\[
(f \circ F)(\pi) = -f^*(\beta) - (\tilde{\beta}^T F)^*(\pi^*) - (\tilde{\pi^T} G)^*(\pi^*).
\]

The last equality is nothing else than

\[
0 = (f \circ F)(\pi) + f^*(\beta) + (\tilde{\beta}^T F)^*(\pi^*) + (\tilde{\pi^T} G)^*(\pi^*) = |f^*(\beta) + (f \circ F)(\pi) - \tilde{\beta}^T F(\pi)| + [(\tilde{\beta}^T F)^*(\pi^*) + \tilde{\beta}^T F(\pi) - \tilde{\pi^T} \pi] + [\tilde{\pi^T} G)^*(\pi^*) + \tilde{\pi^T} G(\pi) - (\tilde{\pi^T})^T \pi].
\]
As $\pi$ is feasible to $(P_C)$ we have $G(\pi) \leq 0$. Thus $-\pi^T G(\pi) \geq 0$ and all the terms within the brackets are non-negative (see also relation (1.3)). Therefore each term must be equal to 0 and the relations $(i^C) - (iv^C)$ follow.

(b) Summing up the statements $(i^C) - (iv^C)$ we acquire

$$ (f \circ F)(\pi) = -f^*(\beta) - (\beta^T F)^*(x^*) - (\alpha^T G)_\lambda^*(-x^*) $$

and the desired conclusion arises as a consequence of Theorem 2.26.

The weak and strong duality assertions are used to prove the following Farkas-type result.

**Theorem 2.29** Suppose that $(CQ_C)$ holds. Then the following assertions are equivalent:

(i) $x \in X, G(x) \leq 0 \Rightarrow (f \circ F)(x) \geq 0$;

(ii) there exist $\beta \in K^*$, $\alpha \in Q^*$ and $x^* \in \mathbb{R}^n$ such that

$$ f^*(\beta) + (\beta^T F)^*(x^*) + (\alpha^T G)_\lambda^*(-x^*) \leq 0. \quad (2.10) $$

**Proof.** Necessity. Obviously the condition (i) is equivalent to $v(P_C) \geq 0$. As the condition $(CQ_C)$ is fulfilled, Theorem 2.27 implies $v(D_C) \geq 0$. Thus there exist $x^* \in \mathbb{R}^n$ and $\beta \in K^*$ and $\alpha \in Q^*$ such that (2.10) holds.

Sufficiency. As (2.10) is fulfilled for some $\beta \in K^*$, $\alpha \in Q^*$ and $x^* \in \mathbb{R}^n$, it follows right away that $v(D_C) \geq -f^*(\beta) - (\beta^T F)^*(x^*) - (\alpha^T G)_\lambda^*(-x^*) \geq 0$. Weak duality between $(P_C)$ and $(D_C)$ always holds and thus we obtain $v(P_C) \geq 0$, i.e. (i) is true.

The previous Farkas-type result can be reformulated as a theorem of the alternative.

**Theorem 2.30** Assume $(CQ_C)$ fulfilled. Then either the inequality system

(I) $x \in X, G(x) \leq 0, (f \circ F)(x) < 0$

has a solution or the system

(II) $f^*(\beta) + (\beta^T F)^*(x^*) + (\alpha^T G)_\lambda^*(-x^*) \leq 0,$

$\beta \in K^*, \alpha \in Q^*, x^* \in \mathbb{R}^n$,

has a solution, but never both.

The next result presents an equivalent reformulation of the statement (ii) in Theorem 2.29. Although its proof is very similar to the proof of Theorem 2.7, we give it here for the sake of completeness.

**Theorem 2.31** The statement (ii) in Theorem 2.29 is equivalent to

$$(0, 0, 0) \in \{0\} \times \mathcal{T}(\text{epi}(f^*)) + \bigcup_{\beta \in K^*} \text{epi}((\beta^T F)^*) \times \{-\beta\} + \bigcup_{\alpha \in Q^*} \text{epi}((\alpha^T G)_\lambda^*) \times \{0\}. \quad (2.11)$$

**Proof.** Necessity. Since the statement (ii) holds, there exist $\beta \in K^*$, $\alpha \in Q^*$ and $x^* \in \mathbb{R}^n$ such that (2.10) holds. As $f^*(\beta)$ and $(\beta^T F)^*(x^*)$ have both finite real values, by definition follows $(\beta, f^*(\beta)) \in \text{epi}(f^*)$ and $(x^*, (\beta^T F)^*(x^*)) \in \text{epi}((\beta^T F)^*)$. Taking into consideration the definition of the operator $\mathcal{T}$ we get immediately

$$(0, f^*(\beta), \beta) \in \{0\} \times \mathcal{T}(\text{epi}(f^*)).$$
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On the other hand \((x^*, (\beta T F)^*(x^*), -\beta) \in \text{epi}(\beta T F)^* \times \{-\beta\}\), so that
\[(x^*, (\beta T F)^*(x^*), -\beta) \in \bigcup_{\beta \in K^*} \text{epi}(\beta T F)^* \times \{-\beta\}.
\]
The inequality \((\alpha^T G)^*_{\chi}(-x^*) \leq -f^*(\beta) - (\beta T F)^*(x^*)\) is also fulfilled, and, as the value in the right-hand side is finite, it holds \((-x^*, -f^*(\beta) - (\beta T F)^*(x^*)) \in \text{epi}(\alpha^T G)^*_{\chi}).\) This implies
\[(-x^*, -f^*(\beta) - (\beta T F)^*(x^*), 0) \in \bigcup_{\alpha \in Q^*} \text{epi}(\alpha^T G)^*_{\chi} \times \{0\}.
\]
Therefore we get
\[\big\{0, 0, 0\big\} = (0, f^*(\beta), \beta) + (x^*, (\beta T F)^*(x^*), -\beta) + (-x^*, -f^*(\beta) - (\beta T F)^*(x^*), 0)
\]
\[\in \{0\} \times \mathcal{T}(\text{epi}(f^*)) + \bigcup_{\beta \in K^*} \text{epi}(\beta T F)^* \times \{-\beta\} + \bigcup_{\alpha \in Q^*} \text{epi}(\alpha^T G)^*_{\chi} \times \{-0\},
\]
and first part of the proof is complete.

**Sufficiency.** Since (2. 11) holds we can find some \(\beta \in K^*\) and \(\alpha \in Q^*\) such that
\[0, 0, 0 \in \{0\} \times \mathcal{T}(\text{epi}(f^*)) + \text{epi}((\beta T F)^*) \times \{-\beta\} + \text{epi}((\alpha^T G)^*_{\chi} \times \{0\}.
\]
Thus there exist \(x^* \in \mathbb{R}^n\) and \(r \in \mathbb{R}\) such that
\[(x^*, r, 0) \in \{0\} \times \mathcal{T}(\text{epi}(f^*)) + \text{epi}((\beta T F)^*) \times \{-\beta\}
\]
and
\[(-x^*, -r, 0) \in \text{epi}((\alpha^T G)^*_{\chi}) \times \{0\}.
\]
Using the definition of the epigraph of a function from the second relation we acquire directly
\[(\alpha^T G)^*_{\chi}(-x^*) \leq -r.
\]
The definition of the operator \(\mathcal{T}\) and the first relation imply that there exist two real numbers \(r_1\) and \(r_2\) such that \(r = r_1 + r_2\), while the pairs \((\beta, r_1)\) and \((x^*, r_2)\) are in \(\text{epi}(f^*)\) and \(\text{epi}((\beta T F)^*)\), respectively. Thus
\[f^*(\beta) + (\beta T F)^*(x^*) + (\alpha^T G)^*_{\chi}(-x^*) \leq r_1 + r_2 - r = r - r = 0
\]
and the desired conclusion is reached. \(\square\)

Let us suppose that the functions \(f : \mathbb{R}^k \rightarrow \overline{\mathbb{R}}\), \(F : \mathbb{R}^n \rightarrow \mathbb{R}^k\) and \(G : \mathbb{R}^n \rightarrow \mathbb{R}^m\) are such that \(f\) is proper, \(\mathbb{R}^k_+\) - increasing and convex, while \(F\) and \(G\) are \(\mathbb{R}^n_+\) - convex and \(\mathbb{R}^m_+\) - convex, respectively. The constraint qualification \((CQ_C^C)\) becomes in this case
\[(CQ^C_S) \quad \exists x' \in \text{ri}(X) : \begin{cases} F(x') \in \text{ri}(\text{dom}(f)) - \text{int}(\mathbb{R}^k_+), \\ G(x') < 0, \end{cases}
\]
and it is not hard to see that the next results, which were given in [18], are actually special instances of our more general results (see Lemma 2.1).

**Theorem 2.32** Suppose that \((CQ^C_S)\) holds. Then the following assertions are equivalent:

1. \(x \in X, G(x) \leq 0 \implies (f \circ F)(x) \geq 0;\)
(ii) there exist $\beta \in \mathbb{R}_+^k$, $\alpha \in \mathbb{R}_+^m$ and $x^* \in \mathbb{R}^n$ such that

$$f^*(\beta) + (\beta^T F)^*(x^*) + (\alpha^T G)^*(x^*) \leq 0.$$ 

**Theorem 2.33** The statement (ii) in Theorem 2.32 is equivalent to

$$(0, 0, 0) \in \{0\} \times T(\text{epi}(f^*)) + \bigcup_{\beta \in \mathbb{R}_+^k} \text{epi}((\beta^T F)^*) \times \{-\beta\} + \text{cone}(\bigcup_{i=1}^m \text{epi}(G_i^*)) \times \{0\} + \text{epi}(\sigma_X) \times \{0\}.$$

**Remark 2.6** The results in Theorem 2.32 and Theorem 2.33 remain valid if in the condition (CQC) instead of $G(x') < 0$ we impose the weaker assumption (see [18])

$$\begin{cases}
G_i(x') \leq 0, & i \in L, \\
G_i(x') < 0, & i \in N,
\end{cases}$$

where $L := \{i \in \{1, \ldots, m\} : G_i \text{ is an affine function}\}$ and $N := \{1, \ldots, m\} \setminus L$.

### 2.6 Special cases

In this section we present some special instances of the results given above.

#### 2.6.1 Optimization problems with quadratic functions

A widely used quadratic programming problem is the Markowitz mean - variance portfolio optimization problem, where the quadratic objective is the portfolio variance, and the linear constraints specify a lower bound for the portfolio return. Other special problems which can be formulated as quadratic programs are the smallest enclosing ball problem and the optimal separating hyperplane problem (see also [1, 4, 37, 43]). Our aim is here to attach a dual problem to an optimization problem which involves quadratic functions. Moreover, we prove that strong duality always holds between the primal and the dual problem. Further we characterize the optimal solutions of the primal problem and a Farkas - type result involving quadratic functions is given, too.

The problem we treat in this section is a special instance of the problem $(P)$. More precisely, in this case we consider the $\mathbb{R}_+^k$ - convex function

$$h : \mathbb{R}^n \to \mathbb{R}^k, \quad h(x) = \left( (x^T B_1 x)^{\frac{1}{2}}, \ldots, (x^T B_k x)^{\frac{1}{2}} \right)^T,$$

where $B_1, \ldots, B_k$ are symmetric and positive semidefinite $n \times n$ real matrices and the function

$$g : \mathbb{R}^k \to \mathbb{R}, \quad g(y) = \begin{cases}
y^T y, & y \in \mathbb{R}_+^k, \\
+\infty, & \text{otherwise},
\end{cases}$$

which is $\mathbb{R}_+^k$ - increasing over the set $h(\text{dom}(h)) + \mathbb{R}_+^k = \mathbb{R}_+^k$. The problem $(P)$ becomes in this case

$$(P^Q) \quad \inf_{x \in \mathbb{R}^n} \left\{ f(x) + \sum_{i=1}^k x^T B_i x \right\}.$$ 

According to Remark 2.1 the theory developed for the problem $(P)$ can be successfully used also in this special case. Thus, in order to provide a dual problem.
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to \((P^q)\) it is necessary to compute \(g^*(\beta)\), for \(\beta = (\beta_1, \ldots, \beta_k)^T \in \mathbb{R}_+^k\). Through calculations we acquire

\[
g^*(\beta) = \sup_{y \in \mathbb{R}^n} \left\{ \beta^T y - g(y) \right\} = \sup_{y \in \mathbb{R}_+^n} \left\{ \beta^T y - y^T y \right\} = \sup_{y = (y_1, \ldots, y_k)^T} \left\{ \sum_{i=1}^{k} \beta_i y_i - \sum_{i=1}^{k} y_i^2 \right\} \]

\[
- \sum_{i=1}^{k} y_i^2 \right\} = \sup_{y_i \geq 0, k} \left\{ \sum_{i=1}^{k} \beta_i y_i - \sum_{i=1}^{k} y_i^2 \right\} = \sum_{i=1}^{k} \sup_{y_i \geq 0} \left\{ \beta_i y_i - y_i^2 \right\} = \sum_{i=1}^{k} \beta_i^2.
\]

It remains to find \((\beta^T h)^*\) for \(\beta \in \mathbb{R}_+^k\) and for this we need the following auxilary results.

**Lemma 2.2** Let \(B\) be a real symmetric positive semidefinite \(n \times n\) matrix and consider the function

\[
b : \mathbb{R}^n \to \mathbb{R}, \quad b(x) = (x^T B x)^{\frac{1}{2}}.
\]

Then

\[
b^*(x^*) = \begin{cases} 0, & x^* \in \text{dom}(b^*), \\ +\infty, & \text{otherwise} \end{cases}
\]

and, moreover, \(x^* \in \text{dom}(b^*)\) if and only if there exists \(w \in \mathbb{R}^n\) such that \(x^* = Bw\) and \(w^T B w \leq 1\).

**Proof.** It is easy to remark that \(b\) is a convex function positive homogeneous of degree 1. Theorem 3.18 in [42] says that

\[
b^*(x^*) = \begin{cases} 0, & x^* \in \text{dom}(b^*), \\ +\infty, & \text{otherwise} \end{cases}
\]

and \(x^* \in \text{dom}(b^*)\) if and only if it fulfills

\[
x^T x - b(x) \leq 0, \forall x \in \mathbb{R}^n.
\]

It remains to prove that the last relation holds if and only if there exists a \(w \in \mathbb{R}^n\) such that \(x^* = Bw\) and \(w^T B w \leq 1\).

**Necessity.** Let us consider now an element \(x^* \in \text{dom}(b^*)\) arbitrarily chosen. As \(B\) is a linear and symmetric operator on \(\mathbb{R}^n\) we have \(\mathbb{R}^n = \text{Im} B \oplus \text{Ker} B\) (see [51], volume I, pp. 390 – 393). Due to this fact, there exist \(w, v \in \mathbb{R}^n\) such that \(x^* = Bw + v\) and \(Bv = 0\). Since \(x^T x - b(x) \leq 0\) for all \(x \in \mathbb{R}^n\), the inquality \(x^T v - b(v) = w^T B v + v^T v - (v^T B v)^{\frac{1}{2}} = v^T v \leq 0\) must be also fulfilled. This implies \(v = 0\) and it remains to prove that \(w^T B w \leq 1\). We use once more the previous inequality for \(x = w\). Since it holds \(0 \geq x^T w - b(w) = w^T B w - (w^T B w)^{\frac{1}{2}}\) which is nothing else but \(w^T B w \leq 1\), the first half of the proof is over.

**Sufficiency.** Suppose that \(x^* = Bw\), where \(w \in \mathbb{R}^n\) such that \(w^T B w \leq 1\). Since we have \(x^T x - b(x) = w^T B x - (x^T B x)^{\frac{1}{2}} \leq (w^T B w)^{\frac{1}{2}} (x^T B x)^{\frac{1}{2}} - (x^T B x)^{\frac{1}{2}} \leq 0\) for each \(x \in \mathbb{R}^n\), the proof is finished. \(\Box\)

The previous lemma and Theorem 1.1 allow us to prove the next result.

**Lemma 2.3** Let \(\beta = (\beta_1, \ldots, \beta_k)^T \in \mathbb{R}_+^k\) arbitrarily taken. Then

\[
(\beta^T h)^*(x^*) = \begin{cases} 0, & x^* \in \text{dom}((\beta^T h)^*), \\ +\infty, & \text{otherwise} \end{cases}
\]

and, moreover, \(x^* \in \text{dom}((\beta^T h)^*)\) if and only if there exist \(w_1, \ldots, w_k \in \mathbb{R}^n\), \(w_i^T B_i w_i \leq 1, i = 1, \ldots, k\), such that \(x^* = \beta_1 B_1 w_1 + \ldots + \beta_k B_k w_k\).
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Proof. Since $\beta = (\beta_1, \ldots, \beta_k)^T \in \mathbb{R}_+^k$, the function $\beta^T h$ turns out to be a convex and positively homogeneous function of degree 1 (it is a sum of positively homogeneous functions of degree 1). According to Theorem 3.18 in [42] we have

$$(\beta^T h)^*(x^*) = \begin{cases} 0, & x^* \in \text{dom}((\beta^T h)^*) \\ +\infty, & \text{otherwise} \end{cases}$$

We prove further that $(\beta^T h)^*(x^*) = 0$ if and only if there exist $w_1, \ldots, w_k$ which satisfy the conditions in the statement. Consider further the functions

$$b_i : \mathbb{R}^n \to \mathbb{R}, \quad b_i(x) = (x^T B_i x)^{\frac{1}{2}},$$

where $i = 1, \ldots, k$. As $h = \sum_{i=1}^k \beta_i b_i$ and the hypotheses of Theorem 1.1 are fulfilled, the equality $(\beta^T h)^*(x^*) = 0$ holds if and only if there exist some $x_1^*, \ldots, x_k^* \in \mathbb{R}^n$ such that

$$x^* = \sum_{i=1}^k x_i^* \quad \text{and} \quad 0 = (\beta^T h)^*(x^*) = \sum_{i=1}^k (\beta_i b_i)^*(x_i^*).$$

Let $i \in \{1, \ldots, k\}$ be arbitrarily taken. If $\beta_i = 0$, as $(\beta_i b_i)^*(x_i^*) = 0$ if and only if $x_i^* = 0$ (and $+\infty$ otherwise), for $w_i = 0 \in \mathbb{R}^n$ we get $x_i^* = \beta_i B_i w_i$ and the inequality $w_i^T B_i w_i \leq 1$ obviously holds. If $\beta_i > 0$ then by Lemma 2.2 we have $(\beta_i b_i)^*(x_i^*) = \beta_i^2 (\frac{1}{2} x_i^*) = 0$ if and only if there exists $w_i \in \mathbb{R}^n$ such that $\frac{1}{\beta_i} x_i^* = B_i w_i$ and $w_i^T B_i w_i \leq 1$. Thus there exist $w_1, \ldots, w_k \in \mathbb{R}^n$ such that for all $i = 1, \ldots, k$, we have $w_i^T B_i w_i \leq 1$ and $x_i^* = \beta_i B_i w_i$. Replacing into the previous relation we get $x^* = \beta_1 B_1 w_1 + \ldots + \beta_k B_k w_k$ and the desired conclusion is reached. \(\square\)

Taking into consideration the results from the previous sections of the chapter and the ones presented above, to the problem ($P^Q$) we associate the following dual problem

$$(D^Q) \sup_{w_i \in \mathbb{R}^n, \beta_i \geq 0} \left\{ - \sum_{i=1}^k \beta_i^2 - f^* \left( - \sum_{i=1}^k \beta_i B_i w_i \right) \right\}.$$

Since $\text{ri}(\text{dom}(g)) - \text{ri}(\mathbb{R}^k) = \text{ri}(\mathbb{R}^k) - \text{ri}(\mathbb{R}^k) = \mathbb{R}^k$, the constraint qualification (CQ) which secures strong duality is obviously fulfilled in this case. The next result is an easy consequences of Theorem 2.3, so that its proof is omitted.

Proposition 2.1 (strong duality) Between ($P^Q$) and ($D^Q$) strong duality holds, i.e. $v(P^Q) = v(D^Q)$ and the dual problem has an optimal solution.

The next proposition presents necessary and sufficient conditions for the optimal solutions of the problem ($P^Q$).

Proposition 2.2 (optimality conditions) (a) Let $\bar{x} \in \mathbb{R}^n$ be an optimal solution of the problem ($P^Q$). Then there exist $\bar{\beta} = (\bar{\beta}_1, \ldots, \bar{\beta}_k)^T = \bar{\beta}_i \beta_i T \bar{w}_i B_i \bar{w}_i \leq 1, i = 1, \ldots, k$, optimal solution for ($D^Q$) such that

1. $\bar{\alpha}^T B_i \bar{x} = \frac{\bar{x}}{2}, \ i = 1, \ldots, k$;
2. $f^* \left( - \sum_{i=1}^k \bar{\beta}_i B_i \bar{w}_i \right) + f(\bar{x}) = - \sum_{i=1}^k \bar{\beta}_i \bar{w}_i B_i \bar{x}$;
3. $\sum_{i=1}^k \bar{\beta}_i (\bar{x}^T B_i x)^{\frac{1}{2}} = \sum_{i=1}^k \bar{\beta}_i \bar{w}_i B_i \bar{x}$. 

}\]
(b) If there exists \( \bar{\beta} \in \mathbb{R}^n \) such that for some \( \beta_1, \bar{\beta}, \ldots, \bar{\beta}_k \in \mathbb{R}^n, \bar{\beta}_i^T B_i \bar{\beta}_i \leq 1, i = 1, \ldots, k \), the assertions (i\(Q\))–(iii\(Q\)) are satisfied, then \( \bar{x} \) is an optimal solution for (\( P^Q \)). By Theorem 2.4 there exist \( \beta_1, \bar{\beta}, \ldots, \bar{\beta}_k \in \mathbb{R}^n \) and \( \bar{x} \in \mathbb{R}^n \) such that the conditions (i)–(iii) are satisfied. A look at the function \( g^* \) allows us to rewrite the statement (i) as
\[
\sum_{i=1}^{k} \frac{\beta_i^2}{4} + \sum_{i=1}^{k} \bar{x}_i^T B_i \bar{x}_i - \sum_{i=1}^{k} \beta_i (\bar{x}_i^T B_i \bar{x}_i)^{1/2}.
\]
The last relation can be equivalently rewritten as
\[
0 = \sum_{i=1}^{k} \left( \frac{\beta_i^2}{4} + \bar{x}_i^T B_i \bar{x}_i - \beta_i (\bar{x}_i^T B_i \bar{x}_i)^{1/2} \right) = \sum_{i=1}^{k} \left( \frac{\beta_i^2}{4} - (\bar{x}_i^T B_i \bar{x}_i)^{1/2} \right)^2
\]
and now it is clear that the last relation holds if and only if (i\(Q\)) holds, too. Because of Lemma 2.3 there exist \( \bar{\beta}, \ldots, \bar{\beta}_k \in \mathbb{R}^n, \bar{\beta}_i^T B_i \bar{\beta}_i \leq 1, i = 1, \ldots, k \), such that \(-x^* = \beta_1 B_1 \bar{\beta} + \ldots + \beta_k B_k \bar{\beta}_k \) (otherwise the relation (iii) cannot hold) and the equivalences between (ii) and (ii\(Q\)) and between (iii) and (iii\(Q\)) are obvious. □

The proof of the following Farkas - type result is omitted, as it is a simple consequence of Theorem 2.5 and Lemma 2.3.

Proposition 2.3 The following assertions are equivalent:

(i) \( f(x) + \sum_{i=1}^{k} x_i^T B_i x_i \geq 0, \forall x \in \mathbb{R}^n \);

(ii) there exist \( \beta = (\beta_1, \ldots, \beta_k)^T \in \mathbb{R}^k, w_1, \ldots, w_k \in \mathbb{R}^n, w_i^T B_i w_i \leq 1, i = 1, \ldots, k \), such that
\[
\sum_{i=1}^{k} \beta_i^2 + f^* \left( -\sum_{i=1}^{k} \beta_i B_i w_i \right) \leq 0.
\]

The previous proposition can be reformulated as follows.

Proposition 2.4 Either the inequality system

\( (I) \quad x \in \mathbb{R}^n, f(x) + \sum_{i=1}^{k} x_i^T B_i x < 0 \)

has a solution or the system

\( (II) \quad \sum_{i=1}^{k} \beta_i^2 + f^* \left( -\sum_{i=1}^{k} \beta_i B_i w_i \right) \leq 0, \quad \beta = (\beta_1, \ldots, \beta_k)^T \in \mathbb{R}^k, w_1, \ldots, w_k \in \mathbb{R}^n, w_i B_i w_i \leq 1, i = 1, \ldots, k, \)

has a solution, but never both.

As in the literature the quadratic programming problems usually involve also affine constraints, we treat further the optimization problem

\( (P^Q) \quad \inf_{x \in \mathbb{R}^n} \left\{ a^T x + \sum_{i=1}^{k} x_i^T B_i x \right\}. \)
where $C$ is an $m \times n$ real matrix and $c \in \mathbb{R}^m$. If we consider the function

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^m, \quad f(x) = a^T x + \delta_{-\mathbb{R}_+^m}(Cx - c),$$

it is not hard to prove that the problem $(P^Q)$ can be equivalently rewritten as

$$(P^Q) \quad \inf_{x \in \mathbb{R}^n} \left\{ f(x) + \sum_{i=1}^k x^T B_i x \right\}.$$ 

Using the definition of the conjugate function and Theorem 2.2.1 from Chapter X of [51] (see also [12]) one can prove that for all $x^* \in \mathbb{R}^n$ we have

$$f^*(x^*) = \inf_{\alpha \in \mathbb{R}^m_+} \frac{\alpha^T c}{c^* \alpha - a^* x^*}$$

and, moreover, the infimum in the right-hand side is attained at some $\alpha \in \mathbb{R}^m_+$ if it is finite. Thus to the primal problem $(P^Q)$ we can attach the dual problem

$$(D^Q) \quad \sup_{w_i \in \mathbb{R}^n, \beta_i \geq 0, w_i^T B_i w_i \leq 1} \sup_{\alpha \in \mathbb{R}^m_+, \sum_{i=1}^k \beta_i w_i = a} \left\{ - \sum_{i=1}^k \frac{\beta_i^2}{1} - \alpha^T c \right\}.$$ 

Moreover, between the problem $(P^Q)$ and its Fenchel - Lagrange - type dual $(D^Q)$ strong duality holds. We give further a characterization of the optimal solutions of the problem $(P^Q)$.

**Proposition 2.5** (a) Let $\pi \in \mathbb{R}^n$ be an optimal solution of the problem $(P^Q)$. Then there exist $(\beta, \bar{\pi}, \bar{w}_1, \ldots, \bar{w}_k) \in \mathbb{R}^k_+ \times \mathbb{R}^m \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n$, $\bar{\beta} = (\bar{\beta}_1, \ldots, \bar{\beta}_k)^T \in \mathbb{R}^k_+$, $\bar{w}_i^T B_i \bar{w}_i \leq 1, \; i = 1, \ldots, k$, optimal solution for $(D^Q)$ such that

$$(i) \quad \bar{\beta}_i \pi = \frac{\pi}{2}, \; i = 1, \ldots, k;$$

$$(ii) \quad C^* \bar{\pi} = a - \sum_{i=1}^k \bar{\beta}_i B_i \bar{w}_i;$$

$$(iii) \quad \bar{\pi}^T c = c^T \pi - \sum_{i=1}^k \bar{\beta}_i (\bar{\pi}^T B_i \pi) \frac{1}{2}.$$ 

(b) If there exists $\pi \in \mathbb{R}^n$ such that for some $\bar{\beta} \in \mathbb{R}^k_+$, $\bar{\pi} \in \mathbb{R}^m$ and $\bar{w}_1, \ldots, \bar{w}_k \in \mathbb{R}^n$, $\bar{w}_i^T B_i \bar{w}_i \leq 1, \; i = 1, \ldots, k$, the assertions $(i) - (iii)$ are satisfied, then $\pi$ is an optimal solution for $(P^Q)$, $(\bar{\beta}, \bar{\pi}, \bar{w}_1, \ldots, \bar{w}_k)$ is an optimal solution for $(D^Q)$ and $v(P^Q) = v(D^Q)$.

### 2.6.2 Farkas - type results with non - differentiable functions

In the following we prove that some results due to Eisenberg (see [39]), Menhndiratta (see [74]) and Sinha (see [83,84]) are actually special instances of the results we give in Section 2.4 of the present chapter.

As in the previous section, we consider $B$ and $B_i$, $i = 1, \ldots, k$, some symmetric and positive semidefinite real $n \times n$ matrices. Moreover, let $C$ be a real $m \times n$ matrix, $a$ and $c$ two fixed vectors from $\mathbb{R}^n$ and, respectively, $\mathbb{R}^m$, and $M$ a real number. We give first a more general result than the ones given by the above mentioned authors.

**Proposition 2.6** The following assertions are equivalent:
Proposition 2.7 \( (\text{see [74]}). \)

Combining the above obtained results it holds that
\[ a^T x + \sum_{i=1}^k (x^T B_i x)^{\frac{1}{2}} \geq M; \]

\( (ii) \) \( \exists \beta \in \mathbb{R}^n_+ \) and \( w_1, \ldots, w_k \in \mathbb{R}^n \) such that \( M + \beta^T c \leq 0 \), \( C^T \beta + \sum_{i=1}^k B_i w_i + a = 0 \) and \( w_i^T B_i w_i \leq 1, i = 1, \ldots, k. \)

**Proof.** As in the proof of Lemma 2.3, for all \( i = 1, \ldots, k \), we define the function

\[ b_i : \mathbb{R}^n \to \mathbb{R}, \quad b_i(x) = (x^T B_i x)^{\frac{1}{2}}. \]

Moreover, we consider also the functions

\[ f : \mathbb{R}^n \to \mathbb{R}, \quad f(x) = a^T x + \sum_{i=1}^k b_i(x) - M \]

and

\[ h : \mathbb{R}^n \to \mathbb{R}, \quad h(x) = Cx - c. \]

As the hypotheses of Theorem 2.23 are obviously fulfilled (see also Remark 2.5) the assertions

\[ x \in \mathbb{R}^n, h(x) \leq 0 \Rightarrow f(x) \geq 0 \]

and

\[ \exists \beta \in \mathbb{R}^n_+, x^* \in \mathbb{R}^n \text{ such that } f^*(x^*) + (\beta^T h)^*(x^*) \leq 0 \]

are equivalent. It is easy to observe that the first assertion is actually the statement (i). We prove further that the second one is equivalent to the statement (ii).

Using the definition of the conjugate function we get

\[ f^*(x^*) = \left( \sum_{i=1}^k b_i \right)^*(x^* - a) + M = \inf \left\{ \sum_{i=1}^k b_i^*(x^*_i) : x^* - a = \sum_{i=1}^k x^*_i \right\} + M \]

and the infimum is attained (see Theorem 1.1), i.e. there exist \( x_1^*, \ldots, x_k^* \in \mathbb{R}^n \) such that \( x^* = \sum_{i=1}^k x_i^* + a \) and \( f^*(x^*) = \sum_{i=1}^k b_i^*(x_i^*) + M \). Since \( b_i^*(x_i^*) \) is either 0 or +\( \infty \) (see Lemma 2.2), we have \( f^*(x^*) = M \) if and only if there exist \( w_1, \ldots, w_k \in \mathbb{R}^n \) such that \( x^* = a + \sum_{i=1}^k B_i w_i \) and \( w_i^T B_i w_i \leq 1 \) for each \( i = 1, \ldots, k \), and \( f^*(x^*) = +\infty \) otherwise. Moreover, we have

\[ (\beta^T h)^*(-x^*) = \sup_{x \in \mathbb{R}^n} \left\{ -x^T x - \beta^T (Cx - c) \right\} = \sup_{x \in \mathbb{R}^n} \left\{ -x^T x - \beta^T Cx + \beta^T c \right\} \]

\[ = \sup_{x \in \mathbb{R}^n} \left\{ (-C^T \beta - x^*)^T x \right\} + \beta^T c = \begin{cases} \beta^T c, & C^T \beta + x^* = 0, \\ +\infty, & \text{otherwise}. \end{cases} \]

Combining the above obtained results it holds \( f^*(x^*) + (\beta^T h)^*(-x^*) \leq 0 \) if and only if \( M + \beta^T c \leq 0 \) and there exist \( w_1, \ldots, w_k \in \mathbb{R}^n \) such that \( C^T \beta + \sum_{i=1}^k B_i w_i + a = 0 \) and \( w_i^T B_i w_i \leq 1 \) for each \( i = 1, \ldots, k \).

The next outcomes are special cases of the previous result. For \( c = 0 \) and \( M = 0 \) we rediscover the following result due to Sinha (see [83]) and Mehndiratta (see [74]).

**Proposition 2.7** The following assertions are equivalent:

\( (i) \) \( x \in \mathbb{R}^n, Cx \leq 0 \Rightarrow a^T x + \sum_{i=1}^k (x^T B_i x)^{\frac{1}{2}} \geq 0; \)
(ii) \( \exists \beta \in \mathbb{R}^m_+ \) and \( w_1, \ldots, w_k \in \mathbb{R}^n \) such that \( C^T \beta + \sum_{i=1}^k B_i w_i + a = 0 \) and \( w_i^T B_i w_i \leq 1, \ i = 1, \ldots, k. \)

For \( k = 1 \) Eisenberg proved in [39] the next result.

**Proposition 2.8** The following assertions are equivalent:

(i) \( x \in \mathbb{R}^n, Cx \leq c \Rightarrow a^T x + (x^T Bx) \frac{1}{2} \geq M; \)

(ii) \( \exists \beta \in \mathbb{R}^m_+ \) and \( w \in \mathbb{R}^n \) such that \( M + \beta^T c \leq 0, C^T \beta + Bw + a = 0 \) and \( w^T Bw \leq 1. \)

Eisenberg in [39] proved that if \( c = 0, M = 0 \) and \( k = 1, \) we can find a vector \( w \) such that the next equivalence holds.

**Proposition 2.9** The following assertions are equivalent:

(i) \( x \in \mathbb{R}^n, Cx \leq 0 \Rightarrow a^T x + (x^T Bx) \frac{1}{2} \geq 0; \)

(ii) \( \exists \beta \in \mathbb{R}^m_+ \) and \( w \in \mathbb{R}^n \) such that \( C^T \beta = Bw + a \) and \( w^T Bw \leq 1. \)

The subsequent result is very similar to Proposition 2.6.

**Proposition 2.10** The following assertions are equivalent:

(i) \( x \in \mathbb{R}^n, Cx \leq c \Rightarrow a^T x + \sum_{i=1}^k (x^T B_i x) \frac{1}{2} \geq M; \)

(ii) \( \exists \beta \in \mathbb{R}^m_+ \) and \( w_1, \ldots, w_k \in \mathbb{R}^n \) such that \( M + \beta^T c \leq 0, C^T \beta + \sum_{i=1}^k B_i w_i + a \geq 0 \) and \( w_i^T B_i w_i \leq 1, \ i = 1, \ldots, k. \)

**Proof.** The proof of the proposition is similar to the proof of Proposition 2.6. We consider the same functions \( f \) and \( h \) as in the proof of Proposition 2.6. By Theorem 2.23 and Remark 2.5 the assertions

\[
x \in \mathbb{R}^n_+, h(x) \leq 0 \Rightarrow f(x) \geq 0
\]

and

\[
\exists \beta \in \mathbb{R}^m_+, x^* \in \mathbb{R}^n \text{ such that } f^*(x^*) + (\beta^T h)_{\mathbb{R}^n_+}(-x^*) \leq 0
\]

are equivalent. The first member of the equivalence is the statement (i) of the proposition and we will prove again that the second term of the equivalence holds if and only if the statement (ii) holds. Following the same idea presented in the proof of Proposition 2.6, we can find \( w_1, \ldots, w_k \in \mathbb{R}^n \) such that \( x^* = a + \sum_{i=1}^k B_i w_i \) and \( w_i^T B_i w_i \leq 1 \) for each \( i = 1, \ldots, k. \) Using Definition 1.8 we get

\[
(\beta^T h)_{\mathbb{R}^n_+}(-x^*) = \sup_{x \in \mathbb{R}^n_+} \left\{ -x^* T x - \beta^T (Cx - c) \right\} = \sup_{x \in \mathbb{R}^n_+} \left\{ -x^* T x - \beta^T Cx + \beta^T c \right\}
\]

\[
= \sup_{x \in \mathbb{R}^n_+} \left\{ (-x^* - C^T \beta) x \right\} + \beta^T c = \begin{cases} \beta^T c, & \ x^* + C^T \beta \geq 0, \\ +\infty, & \text{otherwise.} \end{cases}
\]

Using all the relations obtained so far, the conclusion is straightforward. \( \square \)

The next relations given by Sinha in [84] turns out to be special instances of the previous result.

**Proposition 2.11** The following assertions are equivalent:
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(i) \( x \in \mathbb{R}^n_+, Cx \leq 0 \Rightarrow a^T x + \sum_{i=1}^{k} (x^T B_i x)^2 \geq 0; \)

(ii) \( \exists \beta \in \mathbb{R}^m_+ \) and \( w_1, \ldots, w_k \in \mathbb{R}^n \) such that \( C^T \beta + \sum_{i=1}^{k} B_i w_i + a \geq 0 \) and \( w_i^T B_i w_i \leq 1, i = 1, \ldots, k. \)

Proposition 2.12 The following assertions are equivalent:

(i) \( x \in \mathbb{R}^n_+, Cx \leq c \Rightarrow a^T x + \sum_{i=1}^{k} (x^T B_i x)^2 \geq 0; \)

(ii) \( \exists \beta \in \mathbb{R}^m_+ \) and \( w_1, \ldots, w_k \in \mathbb{R}^n \) such that \( C^T \beta \leq 0, C^T \beta + \sum_{i=1}^{k} B_i w_i + a \geq 0 \) and \( w_i^T B_i w_i \leq 1, i = 1, \ldots, k. \)

Proposition 2.13 The following assertions are equivalent:

(i) \( x \in \mathbb{R}^n_+, Cx \leq 0 \Rightarrow a^T x + (x^T B x)^2 \geq 0; \)

(ii) \( \exists \beta \in \mathbb{R}^m_+ \) and \( w \in \mathbb{R}^n \) such that \( C^T \beta + B w + a \geq 0 \) and \( w^T B w \leq 1. \)

2.6.3 Min – max programming

During the last decades the min – max programming problem has generated considerable research interest, as many practical problems which arise in fields like resource allocation, production control, game theory and so on (see [6, 35, 100]). In [29] Bot and Wanka treat a min – max optimization problem and provided a Farkas - type result. Our aim is to derive their result from our more general one.

Let us consider the functions \( F_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, k, \) and \( G : \mathbb{R}^n \to \mathbb{R}^m \) such that each function \( F_i \) is proper and convex and \( G \) is a \( Q \) - convex function \( (Q \subseteq \mathbb{R}^n \) is a non - empty convex cone). We prove first that the problem

\[
(P_{MM}) \quad \inf_{x \in X, \ G(x) \leq 0} \left\{ \max_{i=1, \ldots, k} F_i(x) \right\},
\]

is actually a special instance of the problem \((P_C)\). Let us consider the function

\[ f : \mathbb{R}^k \to \mathbb{R}, \quad f(x) = \max\{x_1, \ldots, x_k\}, \quad x = (x_1, \ldots, x_k)^T \in \mathbb{R}^k \]

and

\[ F : \mathbb{R}^n \to \mathbb{R}^k, \quad F(x) = \left\{ \begin{array}{ll} (F_1(x), \ldots, F_k(x))^T, & x \in \bigcap_{i=1}^{k} \text{dom}(F_i), \\ \infty_{\mathbb{R}^k}, & \text{otherwise}. \end{array} \right. \]

It is not hard to see that the problem \((P_{MM})\) can be equivalently rewritten as

\[
(P_{MM}) \quad \inf_{x \in X, \ G(x) \leq 0} \left\{ (f \circ F)(x) \right\}.
\]

Moreover, as the functions \( f \) and \( F \) are convex and \( \mathbb{R}^k_+ \) - increasing and, respectively, \( \mathbb{R}^k_+ \) - convex, the problem \((P_{MM})\) turns out to be a special instance of the more general problem \((P_C)\). The conjugate of the function \( f \) is (see [51])

\[ f^* : \mathbb{R}^k \to \mathbb{R}, \quad f^*(\beta) = \left\{ \begin{array}{ll} 0, & \beta \in \mathbb{R}_+^k, \sum_{i=1}^{k} \beta_i = 1, \\ +\infty, & \text{otherwise}, \end{array} \right. \]
so that to the primal problem \( (P^{MM}) \) we can attach the dual problem

\[
(D^{MM}) \quad \sup_{x^* \in \mathbb{R}^n} \left\{ - (\beta^T F^*)(x^*) - (\alpha^T G^*)_X(x^*) \right\},
\]

where \( \Delta_k = \{ \beta = (\beta_1, \ldots, \beta_k)^T \in \mathbb{R}_k^k : \sum_{i=1}^k \beta_i = 1 \} \). Moreover, \( \text{dom}(f) = \mathbb{R}^k \) and one may notice that the constraint qualification which secures the strong duality between the primal problem \( (P^{MM}) \) and the dual problem \( (D^{MM}) \) becomes

\[
(CQ^{MM}) \quad \exists x' \in \text{ri}(X \cap \text{dom}(G)) \cap \bigcap_{i=1}^k \text{dom}(F_i) : G(x') \in - \text{ri}(Q).
\]

Using these results, the following results can be formulated as special cases of Theorem 2.26 and Theorem 2.27.

**Proposition 2.14** (weak duality) Between the primal problem \( (P^{MM}) \) and the dual problem \( (D^{MM}) \) weak duality always holds, i.e. \( v(P^{MM}) \geq v(D^{MM}) \).

**Proposition 2.15** (strong duality) If \( (CQ^{MM}) \) is fulfilled, then between \( (P^{MM}) \) and \( (D^{MM}) \) strong duality holds, i.e. \( v(P^{MM}) = v(D^{MM}) \) and the dual problem has an optimal solution.

To each \( \beta = (\beta_1, \ldots, \beta_k)^T \in \Delta_k \) we attach the non-empty set \( \mathcal{I}_\beta = \{ i \in \{1, \ldots, k\} : \beta_i > 0 \} \). The next theorem provides some necessary and sufficient conditions for the optimal solutions of the problem \( (P^{MM}) \).

**Proposition 2.16** (optimality conditions) (a) Suppose that the condition \( (CQ^{MM}) \) is fulfilled and let \( \pi \in \mathbb{R}^n \) be an optimal solution of the problem \( (P^{MM}) \). Then there exist \( (\overline{\beta}, \overline{\alpha}, \overline{\pi}) \in \Delta_k \times Q^* \times \mathbb{R}^n \), optimal solution to \( (D^{MM}) \) such that

\[
(i^{MM}) \quad F_j(\pi) = \max_{i=1, \ldots, k} F_i(\pi), \quad j \in \mathcal{I}_\pi;
\]

\[
(ii^{MM}) \quad (\overline{\beta}^T F^*)(\overline{\pi}) + \overline{\beta}^T F(\pi) = \overline{\pi}^T \overline{\pi};
\]

\[
(iii^{MM}) \quad (\overline{\alpha}^T G^*)_X(\overline{\pi}) + \overline{\alpha}^T G(\pi) = - \overline{\pi}^T \overline{\pi};
\]

\[
(iv^{MM}) \quad \overline{\pi}^T G(\pi) = 0.
\]

(b) If there exists \( \pi \in \mathbb{R}^n \) such that for some \( \overline{\beta} \in \mathbb{R}_k^k \), \( \overline{\pi} \in Q^* \) and \( \overline{\pi}^T \in \mathbb{R}^n \) the assertions \( (i^{MM}) - (iv^{MM}) \) are satisfied, then \( \pi \) is an optimal solution of \( (P^{MM}) \), \( (\overline{\beta}, \overline{\alpha}, \overline{\pi}) \) is an optimal solution for \( (D^{MM}) \) and \( v(P^{MM}) = v(D^{MM}) \).

**Proof.** By Theorem 2.28 there exist \( \overline{\beta} = (\overline{\beta}_1, \ldots, \overline{\beta}_k)^T \in \mathbb{R}_k^k \), \( \overline{\pi} \in Q^* \) and \( \overline{\pi}^T \in \mathbb{R}^n \) such that the conditions \( (i^C) - (iv^C) \) are satisfied. As \( (ii^{MM}) \), \( (iii^{MM}) \) and \( (iv^{MM}) \) are obviously equivalent to \( (ii^C) \), \( (iii^C) \) and \( (iv^C) \), respectively, it remains to prove the equivalence between \( (i^{MM}) \) and \( (i^C) \). Because of the special form of the function \( f^* \) it follows immediately \( \overline{\beta} \in \Delta_k \). Moreover, \( (i^C) \) can be equivalently rewritten as

\[
\sum_{j=1}^k \overline{\beta}_j F_j(\pi) = \max_{i=1, \ldots, k} \{ F_i(\pi) \}. \quad \text{The last relation holds if and only if}
\]

\[
0 = \sum_{j=1}^k \overline{\beta}_j F_j(\pi) - \max_{i=1, \ldots, k} F_i(\pi) = \sum_{j \in \mathcal{I}_\pi} \overline{\beta}_j F_j(\pi) - \sum_{j \in \mathcal{I}_\pi} \max_{i=1, \ldots, k} F_i(\pi)
\]

\[
= \sum_{j \in \mathcal{I}_\pi} \overline{\beta}_j \left( F_j(\pi) - \max_{i=1, \ldots, k} F_i(\pi) \right).
\]
and, as the previous relation is equivalent to relation \((i^{MM})\), the proof is over. □

The proofs of the next results can be surely skipped (we use Theorem 2.29 and the formula we gave for \(f^*\)).

**Proposition 2.17** Suppose that \((CQ^{MM})\) holds. Then the following assertions are equivalent:

(i) \(x \in X, G(x) \leq Q 0 \Rightarrow \max_{i=1, \ldots, k} F_i(x) \geq 0;\)

(ii) there exist \(\beta \in \Delta_k, \alpha \in Q^*\) and \(x^* \in \mathbb{R}^n\) such that

\[
(\beta^T F)^*(x^*) + (\alpha^T G)^*_X(-x^*) \leq 0.
\]

(2.12)

**Proposition 2.18** Assume \((CQ^{MM})\) fulfilled. Then either the inequality system

\[
(I) \quad x \in X, G(x) \leq Q 0, \max_{i=1, \ldots, k} F_i(x) < 0
\]

has a solution or the system

\[
(II) \quad (\beta^T F)^*(x^*) + (\alpha^T G)^*_X(-x^*) \leq 0,
\]

\(\beta \in \Delta_k, \alpha \in Q^*, x^* \in \mathbb{R}^n\)

has a solution, but never both.

The next result present an equivalent reformulation of (2.12) using only epigraphs.

**Proposition 2.19** The statement (ii) in Proposition 2.17 is equivalent to

\[
(0, 0) \in \bigcup_{\beta \in \Delta_k} \text{epi}((\beta^T F)^*) + \bigcup_{\alpha \in Q^*} \text{epi}((\alpha^T G)^*_X).
\]

(2.13)

**Proof.** Theorem 2.31 ensures that the statement (ii) in Proposition 2.17 is actually equivalent to

\[
(0, 0, 0) \in \{0\} \times T(\text{epi}(f^*)) + \bigcup_{\beta \in \mathbb{R}^*_+} \text{epi}((\beta^T F)^*) \times \{-\beta\} + \bigcup_{\alpha \in Q^*} \text{epi}((\alpha^T G)^*_X) \times \{0\}.
\]

As \(\text{epi}(f^*) = \Delta_k \times \mathbb{R}_+\), it is easy to notice that the previous relation holds if and only if there exists \(\beta \in \Delta_k\) such that

\[
(0, 0, 0) \in \{0\} \times [0, +\infty) \times \{\beta\} + \text{epi}((\beta^T F)^*) \times \{-\beta\} + \bigcup_{\alpha \in Q^*} \text{epi}((\alpha^T G)^*_X) \times \{0\}.
\]

Using the definition of the epigraph of a function it can be easily shown that \(\text{epi}((\beta^T F)^*) = \text{epi}((\beta^T F)^*) + \{0\} \times \mathbb{R}_+\). Thus the above relation is equivalent to

\[
(0, 0) \in \text{epi}((\beta^T F)^*) + \bigcup_{\alpha \in Q^*} \text{epi}((\alpha^T G)^*_X)
\]

and the desired conclusion holds. □

In the following we rediscover as special cases some results given by Bot and Wanka in [29]. With this aim in mind let us consider the functions \(F_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, k\), and \(G : \mathbb{R}^n \to \mathbb{R}^m, G = (G_1, \ldots, G_m)^T\), such that each function \(F_i\) is proper and convex and \(G\) is a \(\mathbb{R}^m_+\) - convex function. The problem \((P^{MM})\) becomes in this special case
To the primal problem \((P^{MM})\) we attach the following dual problem

\[(D^{MM}) \quad \sup_{x^* \in \mathbb{R}^n, \beta \in \Delta_k, \alpha \in \mathbb{R}^m_+} \left\{ - \sum_{i=1}^k (\beta_i F_i)(x^*_i) - (\alpha^T G)_X \left( - \sum_{i=1}^k x^*_i \right) \right\},\]

The next theorem deals with weak duality between the problem \((P^{MM})\) and its Fenchel - Lagrange - type dual problem \((D^{MM})\).

**Proposition 2.20** (weak duality) Between the primal problem \((P^{MM})\) and the dual problem \((D^{MM})\) weak duality always holds, i.e. \(v(P^{MM}) \geq v(D^{MM})\).

The strong duality is secured by the subsequent constraint qualification

\[(CQ^{MM}) \quad \exists x' \in \bigcap_{i=1}^k \text{ri}(\text{dom}(F_i)) \cap \text{ri}(X) : G(x') < 0.\]

**Proposition 2.21** (strong duality) Assume that \(v(P^{MM})\) is finite. If \((CQ^{MM})\) is fulfilled, then between \((P^{MM})\) and \((D^{MM})\) strong duality holds, i.e. \(v(P^{MM}) = v(D^{MM})\) and the dual problem has an optimal solution.

**Proof.** Since the condition \((CQ^{MM})\) stands one can prove that the hypotheses of Proposition 2.15 are fulfilled (see Theorem 1.2). Thus there exist \(\bar{\beta} = (\bar{\beta}_1, \ldots, \bar{\beta}_k)^T \in \Delta_k, \bar{\pi} \in \mathbb{R}^m_+\) and \(\bar{x} \in \mathbb{R}^n\) such that

\[v(P^{MM}) = -\left( \sum_{i=1}^k \bar{\beta}_i F_i \right)^* (\bar{x}^*) - (\bar{\pi}^T G)_X (-\bar{x}).\]

As Theorem 1.1 implies the existence of some \(x^*_1, \ldots, x^*_k\) such that \(\bar{x}^* = x^*_1 + \ldots + x^*_k\) and \(\sum_{i=1}^k (\bar{\beta}_i F_i)^*(\bar{x}^*_i) = \sum_{i=1}^k (\bar{\beta}_i F_i)^*(-\bar{x})\) we replace that in the previous relation and the conclusion is reached. \(\Box\)

The following theorem presents some necessary and sufficient conditions for the optimal solutions of the problem \((P^{MM})\) (the proof is omitted, as it is a simple consequence of Proposition 2.16 and Theorem 1.1).

**Proposition 2.22** (optimality conditions) (a) Suppose that the condition \((CQ^{MM})\) is fulfilled and let \(\bar{x} \in \mathbb{R}^n\) be an optimal solution of the problem \((P^{MM})\). Then there exist \((\bar{\beta}, \bar{\alpha}, \bar{x}_1, \ldots, \bar{x}_k) \in \Delta_k \times \mathbb{R}^m \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n, \bar{\beta} = (\bar{\beta}_1, \ldots, \bar{\beta}_k)^T, \)\( \bar{\alpha} \) optimal solution to \((D^{MM})\) such that

\[(t^{MM}) \quad F_j(\bar{x}) = \max_{i=1, \ldots, k} F_i(\bar{x}), j \in I_{\bar{x}};\]

\[(ii^{MM}) \quad (\bar{\beta}_i F_i)^*(\bar{x}^*_i) + \bar{\beta}_i F_i(\bar{x}) = \bar{x}^*_i^T \bar{x}, i = 1, \ldots, k;\]

\[(iii^{MM}) \quad (\bar{\pi}^T G)_X \left( - \sum_{i=1}^k \bar{x}^*_i \right) + \bar{\pi}^T G(\bar{x}) = - \sum_{i=1}^k \bar{x}_i^T \bar{x}_i;\]

\[(iv^{MM}) \quad \bar{\alpha}^T G(\bar{x}) = 0.\]
2.6 SPECIAL CASES

(b) If there exists $\pi \in \mathbb{R}^n$ such that for some $\beta \in \Delta_k$, $\alpha \in \mathbb{R}^m_+$ and $x_1^*, \ldots, x_k^* \in \mathbb{R}^n$ the assertions $(\mathbb{M}) - (\mathbb{M})'$ are satisfied, then $\pi$ is an optimal solution of $\mathbb{P}$, $(\beta, \alpha, x_1^*, \ldots, x_k^*)$ is an optimal solution for $\mathbb{D}$ and $v(\mathbb{P}) = v(\mathbb{D})$.

The next Farkas-type result can be easily proved using the weak and strong duality assertions between the primal problem $\mathbb{P}$ and its dual $\mathbb{D}$.

**Proposition 2.23** Suppose that $(\mathbb{CQ}^{MM})$ holds. Then the following assertions are equivalent:

(i) $x \in X, G(x) \leq 0 \Rightarrow \max_{i=1,\ldots,k} F_i(x) \geq 0$;

(ii) there exist $\beta \in \Delta_k, \alpha \in \mathbb{R}^m_+$ and $x_i^* \in \mathbb{R}^n, i = 1, \ldots, k$, such that

$$\sum_{i=1}^k (\beta_i F_i(x_i^*) + (\alpha^T G_i)_X^+ \left(- \sum_{i=1}^k x_i^* \right) \leq 0. \quad (2.14)$$

**Proposition 2.24** Assume $(\mathbb{CQ}^{MM})$ fulfilled. Then either the inequality system

(I) $x \in X, G(x) \leq 0, \max_{i=1,\ldots,k} F_i(x) < 0$

has a solution or the system

(II) $\sum_{i=1}^k (\beta_i F_i(x_i^*) + (\alpha^T G_i)_X^+ \left(- \sum_{i=1}^k x_i^* \right) \leq 0, \beta \in \Delta_k, \alpha \in \mathbb{R}^m_+, x_i^* \in \mathbb{R}^n, i = 1, \ldots, k,$

has a solution, but never both.

If the functions $F_i$ and $G_j$ are everywhere finite (i.e., their domains is the whole space $\mathbb{R}^n$), then the following characterization with epigraphs can be proved.

**Proposition 2.25** The statement (ii) in Proposition 2.23 is equivalent to

$$ (0,0) \in \text{co} \left( \bigcup_{i=1}^k \text{epi}(F_i^+) \right) + \text{cone} \left( \bigcup_{j=1}^n \text{epi}(G_j^+) \right) + \text{epi}(\sigma_X). \quad (2.15) $$

**Remark 2.7** The previous results remain valid if in the condition $(\mathbb{CQ}^{MM})$ instead of $G(x') < 0$ we impose the weaker assumption

$$ \begin{cases} G_i(x') \leq 0, & i \in L, \\ G_i(x') < 0. & i \in N. \end{cases} $$

2.6.4 Theorems of the alternative with vector functions

During the last decades many theorems of the alternative involving vector valued functions have been proved under various assumptions (see, for instance, [103] and the references therein). Our aim is to give two such theorems of the alternative which, to the best of our knowledge, are entirely new.

Let us consider the functions $F : \mathbb{R}^n \to \mathbb{R}^m$ and $G : \mathbb{R}^n \to \mathbb{R}^m$ be such that $F$ is $K$-convex and $G$ is $Q$-convex, where $K \subset \mathbb{R}^k$ and $Q \subset \mathbb{R}^m$ are convex cones and $K$ has a non-empty interior. Moreover, let us consider the constraint qualification

$(\mathbb{CQ}') \quad \exists x' \in \text{ri}(X \cap \text{dom}(G)) \cap \text{ri}(\text{dom}(F)) : G(x') \in - \text{ri}(Q).$
Proposition 2.26 Suppose that the condition \((CQ^V)\) holds. Then either the inequality system

\[(I) \quad x \in X, G(x) \leq_Q 0, F(x) \in -\text{int}(K)\]

has a solution or

\[(II) \quad (0, 0) \in \bigcup_{\beta \in K^*} \text{epi}((\beta^T F)^*) + \bigcup_{\alpha \in Q^*} \text{epi}((\alpha^T G)^*_X),\]

but never both.

Proof. It is well-known that the system \((I)\) has a solution if and only if there exists \(\overline{\beta} \in K^* \setminus \{0\}\) such that the system

\[x \in X, G(x) \leq_Q 0, \overline{\beta}^T F(x) < 0\]

has a solution. Taking the function

\[f : \mathbb{R}^n \to \mathbb{R}, \quad f(y) = \overline{\beta}^T y,\]

it is not hard to remark that the hypotheses of Theorem 2.30 are fulfilled (the condition \((CQ^C)\) becomes \((CQ^V)\) and to prove that the function \(f\) is \(K\)-increasing is trivial). Thus either the system \((I)\) has a solution or there exist \(x^* \in \mathbb{R}^n, \beta \in K^*\) and \(\alpha \in Q^*\) such that

\[f^*(\beta) + (\beta^T F)^*(x^*) + (\alpha^T G)^*_X(-x^*) \leq 0.\]

As \(f^*(\beta) = 0\) for \(\beta = \overline{\beta}\) and \(f^*(\beta) = +\infty\) otherwise, it is easy to prove that the last inequality holds if and only if \((II)\) holds, too (see Theorem 2.31).

If we take the functions \(F = (F_1, \ldots, F_k)^T\) and \(G = (G_1, \ldots, G_m)^T\) such that \(\text{dom}(F) = \text{dom}(G) = \mathbb{R}^n\) the condition \((CQ^V)\) becomes

\[(CQ^V)\quad \exists x' \in \text{ri}(X) : G(x') < 0.\]

The subsequent proposition can be easily proved using the same idea as above. Nevertheless, we prefer to prove it using another approach.

Proposition 2.27 Suppose that the condition \((CQ^V)\) holds. Then either the inequality system

\[(I) \quad x \in X, G(x) \leq 0, F(x) < 0\]

has a solution or

\[(II) \quad (0, 0) \in \text{co} \left( \bigcup_{i=1}^k \text{epi}(F_i^*) \right) + \text{cone} \left( \bigcup_{j=1}^m \text{epi}(G_j^*) \right) + \text{epi}(\sigma_X),\]

but never both.

Proof. It is not hard to see that either the system \((I)\) has a solution or

\[x \in X, G(x) \leq 0 \Rightarrow \max_{i=1,\ldots,k} F_i(x) \geq 0,\]

but never both. The desired conclusion arises as a direct consequence of Proposition 2.23 and Proposition 2.25.
Remark 2.8 The previous results remain valid if in the condition \((CQ^S)\) instead of \(G(x') < 0\) we impose the weaker assumption

\[
\begin{align*}
G_i(x') &\leq 0, \quad i \in L, \\
G_i(x') &< 0, \quad i \in N.
\end{align*}
\]
CHAPTER 2. FARKAS RESULTS WITH COMPOSED FUNCTIONS
Chapter 3

Farkas - type results with DC functions

Since optimization techniques became more and more used in various fields of applications, an increasing number of problems that cannot be solved using the methods of linear or convex programming arise. Many of these problems are DC optimization problems, i.e. problems whose objective and/or constraint functions are functions which can be written as differences of convex functions. More and more papers treating DC programming problems have appeared recently, as many authors have enriched the existing literature regarding this type of optimization problems. We mention here only some of them, namely [10, 27, 45, 47, 49, 50, 60, 64, 65, 67, 68, 71, 77, 86, 90, 91] and the list is far from being complete. Many papers present techniques of solving such kinds of problems, yet it is not in our peculiar interest to give a list of references so that we mention here only two of them, namely [53] and [88].

The problem we consider in this chapter has been treated by Bot, Hodrea and Wanka in [19]. It consists in minimizing a DC function when its variable runs over a convex subset of $\mathbb{R}^n$ such that finitely many DC constraint functions are non-positive. To this problem we attach a Fenchel - Lagrange - type dual problem, whose construction is described here in detail. Using the weak and strong duality assertions a Farkas - type result is proved. Moreover, using only the epigraphs of the involved functions an equivalent formulation is given for a statement of the above mentioned Farkas - type result. Further special instances of the initial problem are considered and some recently obtained results are rediscovered as special cases.

3.1 The DC optimization problem and its Fenchel - Lagrange - type dual

Within the present chapter we change some conventions imposed in the second chapter of the work and impose some new ones. More precisely, in what follows we suppose that

\[ (+\infty) - (+\infty) = (-\infty) - (-\infty) = (+\infty) + (-\infty) = (-\infty) + (+\infty) = +\infty \]

and

\[ 0(+\infty) = 0(-\infty) = 0. \]

It is not hard to see that the last assumptions imply

\[ 0f = 0 \quad \text{and} \quad (0f)_X^* = \sigma_X. \quad (3. 1) \]
As we have already said in the introduction, the primal problem we work with is

\[
(P^{DC}) \quad \inf_{x \in X, \ g_i(x) - h_i(x) \leq 0, \ i = 1, \ldots, m} \left\{ g(x) - h(x) \right\},
\]

where \( X \subseteq \mathbb{R}^n \) is a non-empty and convex set, \( g, h : \mathbb{R}^n \to \mathbb{R} \) are two proper and convex functions and \( g_i, h_i : \mathbb{R}^n \to \mathbb{R}, \ i = 1, \ldots, m, \) are proper and convex functions such that the condition

\[
\bigcap_{i=1}^m \text{ri}(\text{dom}(g_i)) \bigcap \text{ri}(\text{dom}(g)) \bigcap \text{ri}(X) \neq \emptyset
\]

is satisfied (it is obviously satisfied if the functions \( g \) and \( g_i, \ i = 1, \ldots, m, \) are everywhere finite). Further we denote by \( F = \{ x \in X : g_i(x) - h_i(x) \leq 0, \ i = 1, \ldots, m \} \) the feasible set of \((P^{DC})\) and we assume that \( F \neq \emptyset. \) Moreover, we assume that the function \( h \) is lower semicontinuous on \( F \) and that the functions \( h_i, \ i = 1, \ldots, m, \) are subdifferentiable on \( F. \)

**Remark 3.1** A careful look at the problem \((P^{DC})\) is enough in order to realize that the function \( g - h \) is well-defined because of the conventions made in the beginning. Without them one of the additional assumptions \( X \subseteq \text{dom}(g) \) or \( X \subseteq \text{dom}(h) \) is necessary in order to avoid the case \( g(x) = h(x) = +\infty. \) Moreover, a similar reasoning allows us to affirm that the functions \( g_i - h_i, \ i = 1, \ldots, m, \) are well-defined, too.

Before going further we would like to mention that the same DC problem has been treated also by Martínez-Legaz and Volle in [71]. In order to attach a dual problem to \((P^{DC})\) they give first a decomposition of the feasible set. Inspired by their result we give a similar result, too.

**Lemma 3.1** It holds

\[
F = \bigcup_{y_i^* \in \partial h_i(x)} \left\{ x \in X : g_i(x) - y_i^* T x + h_i^*(y_i^*) \leq 0, i = 1, \ldots, m \right\}.
\]

**Proof.** "\( \subseteq \)" Let \( x \in F. \) As for all \( i = 1, \ldots, m, \) the function \( h_i \) is subdifferentiable at \( x \) we can find some \( y_i^* \in \partial h_i(x). \) Since \( y_i^* \in \partial h_i(x) \) if and only if \( h_i^*(y_i^*) = y_i^* T x, \) using the previous relation we acquire \( g_i(x) - y_i^* T x + h_i^*(y_i^*) = g_i(x) - h_i(x) \leq 0, \) and the inclusion is proved.

"\( \supseteq \)" For the opposite inclusion, let \( (y_1^*, \ldots, y_m^*) \in \partial h_1^* \times \ldots \times \partial h_m^* \) and \( x \in X \) such that \( g_i(x) - y_i^* T x + h_i^*(y_i^*) \leq 0 \) for all \( i = 1, \ldots, m. \) For \( i = 1, \ldots, m, \) the conventions we use ensure that \( h_i(x) > -\infty \) and \( g_i(x) < +\infty \) (if one of the previous situations occurs we have \( g_i(x) - h_i(x) = +\infty \). Using the Young-Fenchel inequality \((1.3)\) we get \( g_i(x) - h_i(x) \leq g_i(x) - y_i^* T x + h_i^*(y_i^*) \leq 0 \) and this inclusion is also proved.

For the sake of simplicity, we consider further the set \( \Pi = \text{dom}(h_1^*) \times \ldots \times \text{dom}(h_m^*) \) and by writing \( y^* \in \Pi, \) we understand that \( y^* \) is the following \( m \)-tuple \((y_1^*, \ldots, y_m^*)\) with \( y_i^* \in \text{dom}(h_i^*), \ i = 1, \ldots, m. \) We give now a characterization of the optimal objective value of the problem \((P^{DC})\).
3.1 THE DC PROBLEM AND ITS DUAL

**Theorem 3.1** Under the hypotheses imposed in the beginning of this section we have

\[ v(P_{DC}) = \inf_{\substack{x^* \in \text{dom}(h^*), \\ y^* \in \Pi}} \left\{ g(x) - x^*^T x + h^*(x^*) \right\}. \]

**Proof.** Since \( h \) is proper, convex and lower semicontinuous on \( \mathcal{F} \) it holds

\[ h(x) = h^{**}(x) = \sup_{x^* \in \text{dom}(h^*)} \left\{ x^*^T x - h^*(x^*) \right\}. \]

Thus

\[ v(P_{DC}) = \inf_{x \in \mathcal{F}} \left\{ g(x) - h(x) \right\} = \inf_{\substack{x^* \in \text{dom}(h^*)}} \inf_{x \in \mathcal{F}} \left\{ g(x) - x^*^T x + h^*(x^*) \right\}. \]

Using the decomposition of the set \( \mathcal{F} \) given by Lemma 3.1, the conclusion is straightforward. \( \square \)

Taking a careful look at the relation given in the previous theorem, one may notice that the inner infimum can be seen as a convex optimization problem. Thus for some \( x^* \in \text{dom}(h^*) \) and \( y^* \in \Pi \) fixed it is quite natural to consider the optimization problem

\[ (P_{x^*, y^*}) \quad \inf_{\substack{x \in X, \\ g_i(x) - y_i^T x + h_i^*(y_i^*) \leq 0, \\ i = 1, \ldots, m}} \left\{ g(x) - x^*^T x + h^*(x^*) \right\}, \]

and to deal with it by means of Fenchel - Lagrange duality. Considering the functions

\[ \tilde{g} : \mathbb{R}^n \to \mathbb{R}, \quad \tilde{g}(x) = g(x) - x^*^T x + h^*(x^*) \]

and

\[ \tilde{g}_i : \mathbb{R}^n \to \mathbb{R}, \quad \tilde{g}_i(x) = g_i(x) - y_i^T x + h_i^*(y_i^*), \]

\( i = 1, \ldots, m \), the problem \((P_{x^*, y^*})\) can be equivalently written as

\[ (P_{x^*, y^*}) \quad \inf_{\substack{x \in X, \\ \tilde{g}_i(x) \leq 0, \\ i = 1, \ldots, m}} \tilde{g}(x). \]

Our next step is to construct a dual problem for \((P_{x^*, y^*})\) and to give sufficient conditions such that strong duality holds, i.e. the optimal objective value of the primal coincides with the optimal objective value of the dual and the dual has an optimal solution. Because of the way the function \( \tilde{g}_i \) is defined, it is not difficult to show that, since the function \( g_i \) is proper and convex, the function \( \tilde{g}_i \) is proper and convex, too, and this is true for all \( i = 1, \ldots, m \). Since \( g \) is convex and proper, the definition of the function \( \tilde{g} \) implies that \( \tilde{g} \) is convex and proper, too. Thus the problem \((P_{x^*, y^*})\) turns out to be a convex optimization problem.

Next we consider the Lagrange dual problem to \((P_{x^*, y^*})\) with \( \beta = (\beta_1, \ldots, \beta_m)^T \in \mathbb{R}_+^m \) as dual variable

\[ (D_{x^*, y^*}) \quad \sup_{\beta \in \mathbb{R}_+^m} \inf_{x \in X} \left\{ \tilde{g}(x) + \sum_{i=1}^m \beta_i \tilde{g}_i(x) \right\}. \]
Regarding the infimum concerning \( x \in X \) we have

\[
\inf_{x \in X} \left\{ \tilde{g}(x) + \sum_{i=1}^{m} \beta_i \tilde{g}_i(x) \right\} = - \sup_{x \in X} \left\{ - \tilde{g}(x) - \sum_{i=1}^{m} \beta_i \tilde{g}_i(x) \right\} = - \left( \tilde{g} + \sum_{i=1}^{m} \beta_i \tilde{g}_i \right)^*_{X}(0).
\]

Since for all \( p^* \in \mathbb{R}^n \)

\[
\left( \tilde{g} + \sum_{i=1}^{m} \beta_i \tilde{g}_i \right)^*_{X}(0) \leq \tilde{g}^*(p^*) + \left( \sum_{i=1}^{m} \beta_i \tilde{g}_i \right)^*_{X}(-p^*),
\]

between the optimal objective value of the problem \((\mathcal{D}_{x^*,y^*})\) and the optimal objective value of the problem

\[
(D_{x^*,y^*}) \quad \sup_{p^* \in \mathbb{R}^n, \beta \in \mathbb{R}_+^m} \left\{ - \tilde{g}^*(p^*) - \left( \sum_{i=1}^{m} \beta_i \tilde{g}_i \right)^*_{X}(-p^*) \right\}.
\]

the relation \( v(\mathcal{D}_{x^*,y^*}) \geq v(D_{x^*,y^*}) \) always holds. Let us give an equivalent form for the objective function of the problem \((D_{x^*,y^*})\).

Taking into consideration the way the functions \( \tilde{g} \) and \( \tilde{g}_i, i = 1, \ldots, m \), are defined, through some simple calculations we get

\[
\tilde{g}^*(p^*) = \sup_{x \in \mathbb{R}^n} \left\{ p^T x - \tilde{g}(x) \right\} = \sup_{x \in \mathbb{R}^n} \left\{ p^T x - g(x) + x^T x - h^*(x^*) \right\}
\]

\[
= \sup_{x \in \mathbb{R}^n} \left\{ (p^* + x^*)^T x - g(x) \right\} - h^*(x^*) = g^*(p^* + x^*) - h(x^*)
\]

and

\[
\left( \sum_{i=1}^{m} \beta_i \tilde{g}_i \right)^*_{X}(-p^*) = \sup_{x \in X} \left\{ -p^T x - \sum_{i=1}^{m} \beta_i \tilde{g}_i(x) \right\} = \sup_{x \in X} \left\{ -p^T x \right\}
\]

\[
- \sum_{i=1}^{m} \beta_i \left( g_i(x) - y_i^T x + h_i^*(y_i^*) \right)
\]

\[
- \sum_{i=1}^{m} \beta_i h_i^*(y_i^*) = \sum_{i=1}^{m} \beta_i g_i(x) \left( -p^* + \sum_{i=1}^{m} \beta_i y_i^* \right) - \sum_{i=1}^{m} \beta_i h_i^*(y_i^*)
\]

This leads to the following formulation for the dual \((D_{x^*,y^*})\)

\[
(D_{x^*,y^*}) \quad \sup_{p^* \in \mathbb{R}^n, \beta \in \mathbb{R}_+^m} \left\{ - g^*(p^* + x^*) + h(x^*) - \left( \sum_{i=1}^{m} \beta_i g_i \right)^*_{X}(-p^* + \sum_{i=1}^{m} \beta_i y_i^*) + \sum_{i=1}^{m} \beta_i h_i^*(y_i^*) \right\}.
\]

It follows immediately that the dual \((D_{x^*,y^*})\) can be equivalently rewritten as

\[
(D_{x^*,y^*}) \quad \sup_{p^* \in \mathbb{R}^n, \beta \in \mathbb{R}_+^m} \left\{ h^*(x^*) + \sum_{i=1}^{m} \beta_i h_i^*(y_i^*) - g^*(p^*) - \left( \sum_{i=1}^{m} \beta_i g_i \right)^*_{X} \left( x^* + \sum_{i=1}^{m} \beta_i y_i^* - p^* \right) \right\}.
\]

The following theorem gives a weak duality statement between the primal problem \((P_{x^*,y^*})\) and the dual \((D_{x^*,y^*})\).

**Theorem 3.2 (weak duality)** Between the primal problem \((P_{x^*,y^*})\) and the dual problem \((D_{x^*,y^*})\) weak duality always holds, i.e. \( v(P_{x^*,y^*}) \geq v(D_{x^*,y^*}) \).

**Proof.** Since weak duality holds between a primal problem and its Lagrange dual, the inequality \( v(P_{x^*,y^*}) \geq v(D_{x^*,y^*}) \) is always true. As \( v(D_{x^*,y^*}) \geq v(D_{x^*,y^*}) \) the
Theorem 3.3 (strong duality) If \((CQ_{y^*})\) is fulfilled, then between \((P_{x^*,y^*})\) and \((D_{x^*,y^*})\) strong duality holds, i.e. \(v(P_{x^*,y^*}) = v(D_{x^*,y^*})\) and the dual problem has an optimal solution.

Proof. To the problem \((P_{x^*,y^*})\) we associate its Lagrange dual problem \((D_{x^*,y^*})\). Since the condition \((CQ_{y^*})\) is fulfilled and all the involved functions are convex, it is well-known from the literature (see Theorem 28.2 in [80]) that the optimal value of the primal problem \((P_{x^*,y^*})\) and its Lagrange dual \((D_{x^*,y^*})\) are equal and, moreover, there exists an optimal solution \(\bar{\beta} = (\bar{\beta}_1, \ldots, \bar{\beta}_m)^T \in \mathbb{R}_+^m\) such that

\[
v(P_{x^*,y^*}) = \sup_{\beta \in \mathbb{R}^m} \inf_{x \in X} \left\{ \bar{g}(x) + \sum_{i=1}^m \beta_i \bar{g}_i(x) \right\} = \inf_{x \in X} \left\{ \bar{g}(x) + \sum_{i=1}^m \beta_i \bar{g}_i(x) \right\}.
\]

Further we deal with the infimum in the last term of the equality from above. Supposing that the hypotheses of Theorem 1.1 are fulfilled it holds

\[
v(P_{x^*,y^*}) = \inf_{x \in X} \left\{ \bar{g}(x) + \sum_{i=1}^m \beta_i \bar{g}_i(x) \right\} = -\sup_{x \in \mathbb{R}^n} \left\{ -\bar{g}(x) - \sum_{i=1}^m \beta_i \bar{g}_i(x) - \delta_X(x) \right\}
\]

\[
= -\left( \bar{g} + \sum_{i=1}^m \beta_i \bar{g}_i + \delta_X \right)(0) = -\inf_{p^* \in \mathbb{R}^n} \left\{ \bar{g}^*(p^*) + \left( \sum_{i=1}^m \beta_i \bar{g}_i + \delta_X \right)^*(-p^*) \right\}
\]

and, moreover, there exists a \(p^* \in \mathbb{R}^n\) such that the last supremum is attained. Denoting \(\bar{p}^* = p^* + x^*\) we get

\[
v(P_{x^*,y^*}) = \bar{g}^*(p^*) - \left( \sum_{i=1}^m \beta_i \bar{g}_i \right)^*_X(-p^*)
\]

\[
= -g^*(x^* + p^*) + h^*(x^*) - \left( \sum_{i=1}^m \beta_i \bar{g}_i \right)^*_X \left( \sum_{i=1}^m \beta_i \bar{g}_i - p^* \right) + \sum_{i=1}^m \beta_i h_i^*(y_i^*)
\]

\[
= h^*(x^*) + \sum_{i=1}^m \beta_i h_i^*(y_i^*) - g^*(\bar{p}^*) - \left( \sum_{i=1}^m \beta_i \bar{g}_i \right)^*_X \left( x^* + \sum_{i=1}^m \beta_i y_i^* - \bar{p}^* \right),
\]

so that \((\bar{\beta}, \bar{p}^*)\) is an optimal solution for \((P_{x^*,y^*})\).

It remains to demonstrate that the hypotheses of Theorem 1.1 are indeed satisfied by the functions \(\bar{g}\) and \(\sum_{i=1}^m \beta_i \bar{g}_i + \delta_X\). First of all let us remark that \(\text{dom}(\bar{g}) = \text{dom}(g)\). Since the condition \((CQ_{y^*})\) holds, if we prove that

\[
\bigcap_{i=1}^m \text{ri}(\text{dom}(g_i)) \cap \text{ri}(X) \subseteq \text{ri} \left( \text{dom} \left( \sum_{i=1}^m \beta_i \bar{g}_i + \delta_X \right) \right),
\]

then obviously the relative interiors of the domains of the functions \(\bar{g}\) and \(\sum_{i=1}^m \beta_i \bar{g}_i + \delta_X\) have at least one point in common.
As \( \text{dom}(\bar{\beta}g_i) = \text{dom}(\tilde{g}_i) = \text{dom}(g_i) \) if \( \bar{\beta} \neq 0 \) and \( \text{dom}(\bar{\beta}g_i) = \mathbb{R}^n \) otherwise (see relation (3.1) for the last equality), we treat further two cases. If \( \bar{\beta} = 0 \) then obviously \( \text{dom} \left( \sum_{i=1}^{m} \bar{\beta}g_i + \delta \chi \right) = X \) and the previous inclusion holds. If \( \bar{\beta} \neq 0 \) then the set \( I_\bar{\beta} = \{ i \in \{ 1, \ldots, m \} : \bar{\beta}_i > 0 \} \) is non-empty and it is easy to prove that
\[
\text{dom} \left( \sum_{i=1}^{m} \bar{\beta}g_i + \delta \chi \right) = \bigcap_{i \in I_\bar{\beta}} \text{dom}(g_i) \bigcap X.
\]
By Theorem 1.2 we get further
\[
\text{ri} \left( \text{dom} \left( \sum_{i=1}^{m} \bar{\beta}g_i + \delta \chi \right) \right) = \text{ri} \left( \bigcap_{i \in I_\bar{\beta}} \text{dom}(g_i) \bigcap X \right) = \bigcap_{i \in I_\bar{\beta}} \text{ri}(\text{dom}(g_i)) \bigcap \text{ri}(X)
\]
and the proof of the theorem is complete. \(\square\)

Taking into consideration the results given by Theorem 3.2 and Theorem 3.3, it seems natural to introduce the following dual problem to \((P^{DC})\)
\[
(D^{DC}) \quad \inf_{x^* \in \text{dom}(h^*)} \sup_{y^* \in \Pi} \left\{ h^*(x^*) + \sum_{i=1}^{m} \bar{\beta}_i h^*_i(y^*_i) - g^*(p^*) \right\} - \left( \sum_{i=1}^{m} \bar{\beta}_i g_i \right) (x^* + \sum_{i=1}^{m} \beta_i y_i^* - p^*) \bigcap X.
\]
By the construction of the dual problem \((D^{DC})\) there is a weak duality statement for \((P^{DC})\) and \((D^{DC})\) as follows.

**Theorem 3.4** (weak duality) Between the primal problem \((P^{DC})\) and the dual problem \((D^{DC})\) weak duality always holds, i.e. \(v(P^{DC}) \geq v(D^{DC})\).

Concerning the strong duality between \((P^{DC})\) and \((D^{DC})\) the considerations done above lead to the following assertion.

**Theorem 3.5** (strong duality) If \((CQ_{\varphi^*})\) is fulfilled for all \(y^* \in \Pi\), then the optimal objective values of the problems \((P^{DC})\) and \((D^{DC})\) are equal, i.e. \(v(P^{DC}) = v(D^{DC})\).

Before going further, we would like to mention that although we call the previous result a "strong duality theorem", we do not have a real "strong duality", as the dual problem does not necessarily have an optimal solution. Nevertheless, the previous theorem allows us to give some necessary and sufficient optimality conditions for the optimal solutions of the problem \((P^{DC})\). The subsequent theorem is dedicated to that matter.

**Theorem 3.6** (optimality conditions) (a) Suppose that the condition \((CQ_{\varphi^*})\) is fulfilled for all \(y^* \in \Pi\). Let \( \bar{x} \in \mathbb{R}^n \) be an optimal solution of the problem \((P^{DC})\) and assume further that the function \( h \) is subdifferentiable at \( \bar{x} \). Then there exist \( x^* \in \mathbb{R}^n \), \( y^* \in \Pi \), and \( (\bar{\beta}, \bar{p}^*) \in \mathbb{R}^n \times \mathbb{R}^m \), \( \bar{\beta} = (\bar{\beta}_1, \ldots, \bar{\beta}_m)^T \) optimal solution for \((D_{\bar{x},\bar{y}^*,\bar{p}^*})\) such that
\[
(v^{DC}) \quad h^*(x^*) + h(\bar{x}) = x^T \bar{x};
\]
\[
(ii^{DC}) \quad h^*_i(y^*_i) + h_i(\bar{x}) = y^*_i T \bar{x}, \quad i = 1, \ldots, m;
\]
\[
(iii^{DC}) \quad g^*(\bar{p}^*) + g(\bar{x}) = \bar{p}^T \bar{x};
\]
\[ (iv_{DC}) \left( \sum_{i=1}^{m} \beta_i g_i(x) \right)^* \left( x^T + \sum_{i=1}^{m} \beta_i y_i^* - p^T \right) + \sum_{i=1}^{m} \beta_i g_i(x) = \left( x^T + \sum_{i=1}^{m} \beta_i y_i^* - p^T \right)^T \tau; \]

\[ (v_{DC}) \sum_{i=1}^{m} \beta_i (g_i(x) - h_i(x)) = 0. \]

(b) Let \( \tau \in \mathbb{R}^n \) be such that for all \( \overline{\tau} \in \mathbb{R}^n, \overline{y} \in \Pi \), there exist \( \overline{\beta} \in \mathbb{R}^n_+ \) and \( \overline{y} \in \mathbb{R}^n \) which fulfill the assertions \((i_{DC}) - (v_{DC})\). Then \( \tau \) is an optimal solution of \((P_{DC})\) and \( v(P_{DC}) = v(D_{DC}) \).

**Proof.** (a) Since the functions \( h \) and \( h_i \) are subdifferentiable at \( \tau \) there exists \( \overline{\tau} \in \mathbb{R}^n \) and \( \overline{y}_i \in \mathbb{R}^n, i = 1, \ldots, m \), such that \( \overline{\tau} \in \partial h(\tau) \) and, respectively, \( \overline{y}_i \in \partial h_i(\tau) \), \( i = 1, \ldots, m \). Using the well-known connection between the subdifferential and the conjugate function the statements \((i_{DC})\) and \((ii_{DC})\) follow immediately. Since

\[ \overline{\tau} \in \text{dom}(h^*) \quad \text{and} \quad \overline{y} = (\overline{y}_1, \ldots, \overline{y}_m) \in \text{dom}(h_1^*) \times \ldots \times \text{dom}(h_m^*) \]

and

\[ g_i(\tau) - g^*(\overline{\tau}^T \tau + h_i^*(\overline{y}_i^*)) = g_i(\tau) - h_i(\tau) \leq 0, \quad i = 1, \ldots, m, \]

it is not hard to see that \( \tau \) is feasible to the problem \((P_{\overline{\tau} \overline{y}})\). Moreover, the equality \( g(\tau) - g^*(\tau^T \tau + h^*(\overline{\tau}^T \tau)) = g(\tau) - h(\tau) \) holds. Theorem 3.1 allows us to affirm that \( \tau \) is actually an optimal solution of the problem \((P_{\overline{\tau} \overline{y}})\) and it holds \( v(P_{DC}) = v(P_{\overline{\tau} \overline{y}}) \). By Theorem 3.3 there exist \( \overline{\beta} = (\overline{\beta}_1, \ldots, \overline{\beta}_m)^T \in \mathbb{R}^n_+ \) and \( \overline{p}^T \) such that

\[ g(\tau) - h(\tau) = h^*(\overline{\tau}) + \sum_{i=1}^{m} \overline{\beta}_i h_i^*(\overline{y}_i^*) - g^*(\overline{p}^T) + \left( \sum_{i=1}^{m} \beta_i g_i \right)^* \left( \overline{\tau} + \sum_{i=1}^{m} \beta_i y_i^* - \overline{p} \right) \]

The previous relation implies

\[ 0 = g(\tau) - h(\tau) - h^*(\overline{\tau}) - \sum_{i=1}^{m} \overline{\beta}_i h_i^*(\overline{y}_i^*) + g^*(\overline{p}^T) + \left( \sum_{i=1}^{m} \beta_i g_i \right)^* \left( \overline{\tau} + \sum_{i=1}^{m} \beta_i y_i^* - \overline{p} \right) \]

\[ = [g^*(\overline{p}^T) + g(\tau) - \overline{p}^T \tau] + [-h(\tau) + h^*(\overline{\tau}) + \overline{\tau}^T \tau] + \sum_{i=1}^{m} [h_i^*(\overline{y}_i^*) - h_i(\tau) + \overline{y}_i^T \tau] \]

\[ + \left[ \left( \sum_{i=1}^{m} \beta_i g_i \right)^* \left( \overline{\tau} + \sum_{i=1}^{m} \beta_i y_i^* - \overline{p} \right) + \sum_{i=1}^{m} \beta_i g_i(\tau) - \left( \overline{\tau} + \sum_{i=1}^{m} \beta_i y_i^* - \overline{p} \right)^T \tau \right] \]

\[ + \left[ - \sum_{i=1}^{m} \beta_i (g_i(\tau) - h_i(\tau)) \right] \]

and, as all the terms within the brackets are non-negative, each term must be equal to 0 and the relations \((iii_{DC}) - (iv_{DC})\) follow.

(b) Let \( \overline{x} \in \mathbb{R}^n \) and \( \overline{y} \in \Pi \) be arbitrarily taken and \( \overline{\beta} \in \mathbb{R}^m_+ \) and \( \overline{p}^T \in \mathbb{R}^n \) be such that the assertions \((i_{DC}) - (v_{DC})\) are fulfilled. Then obviously \((i_{DC}), (ii_{DC})\) and \((v_{DC})\) implies

\[ -h^*(\overline{x}) - h(\tau) = -\overline{x}^T \tau, \]

\[ -\sum_{i=1}^{m} \beta_i h_i^*(\overline{y}_i^*) - \sum_{i=1}^{m} \beta_i h_i(\tau) = \left( \sum_{i=1}^{m} \beta_i \overline{y}_i^* \right)^T \tau \]

and

\[ -\sum_{i=1}^{m} \beta_i g_i(\tau) + \sum_{i=1}^{m} \beta_i h_i(\tau)) = 0, \]
respectively. Summing up the previous three relations and the assertions (iii$^{DC}$) and (iv$^{DC}$), after some minor calculations we get
\[-h^*(\overline{x}) - h(\overline{x}) - \sum_{i=1}^{m} \beta_i h_i^*(\overline{y}_i^*) + g^*(\overline{p}) + g(\overline{x}) + \left( \sum_{i=1}^{m} \beta_i g_i \right)^* \left( \overline{x} + \sum_{i=1}^{m} \beta_i y_i^* - \overline{p} \right) = 0.\]

The previous relation can be equivalently rewritten as
\[g(\overline{x}) - h(\overline{x}) = h^*(\overline{x}) + \sum_{i=1}^{m} \beta_i h_i^*(\overline{y}_i^*) - g^*(\overline{p}) - \left( \sum_{i=1}^{m} \beta_i g_i \right)^* \left( \overline{x} + \sum_{i=1}^{m} \beta_i y_i^* - \overline{p} \right)\]
and from here the inequality
\[g(\overline{x}) - h(\overline{x}) \leq v(D_{\overline{x},\overline{y}})\]
can be easily deduced. We get further $v(P^{DC}) \leq g(\overline{x}) - h(\overline{x}) \leq v(D_{\overline{x},\overline{y}})$ and, as $\overline{x}$ and $\overline{y}$ are arbitrarily taken, we acquire (see the definition of the dual problem $(D^{DC})$ for the equality and Theorem 3.4 for the last inequality)
\[v(P^{DC}) \leq g(\overline{x}) - h(\overline{x}) \leq \inf_{\overline{x} \in \mathbb{R}^n, \overline{y} \in \Pi} v(D_{\overline{x},\overline{y}}) = v(D^{DC}) \leq v(P^{DC}).\]

From the last relation the equalities $g(\overline{x}) - h(\overline{x}) = v(P^{DC}) = v(D^{DC})$ can be easily deduced and the proof is finished. \[\square\]

Before going further we would like to mention that for the second part of the previous theorem neither the conditions (CQ$_{y^*}$), $y^* \in \Pi$, nor the subdifferentiability of the function $h$ are necessary.

### 3.2 A Farkas - type result with DC functions

Using the weak and strong duality between a primal programming problem and its Fenchel - Lagrange - type dual problem we have proved in the previous chapter a Farkas - type result. Following the same idea we prove a Farkas - type result using the weak and strong duality assertions between the problems $(P^{DC})$ and $(D^{DC})$.

**Theorem 3.7** Suppose that (CQ$_{y^*}$) holds for all $y^* \in \Pi$. Then the following assertions are equivalent:

(i) $x \in X, g_i(x) - h_i(x) \leq 0, i = 1, \ldots, m \Rightarrow g(x) - h(x) \geq 0;$

(ii) $\forall x^* \in \text{dom}(h^*)$ and $\forall y^* \in \Pi$, there exist $\beta \in \mathbb{R}^m_+$ and $p^* \in \mathbb{R}^n$ such that
\[h^*(x^*) + \sum_{i=1}^{m} \beta_i h_i^*(y_i^*) - g^*(p^*) - \left( \sum_{i=1}^{m} \beta_i g_i \right)^* \left( x^* + \sum_{i=1}^{m} \beta_i y_i^* - p^* \right) \geq 0. \tag{3.3} \]

**Proof.** *Necessity.* Let us consider $x^* \in \text{dom}(h^*)$ and $y^* \in \Pi$. The statement (i) implies $v(P^{DC}) \geq 0$ and using Theorem 3.1 we acquire $v(P_{x^*,y^*}) \geq 0$. Since the assumptions of Theorem 3.3 are satisfied, strong duality holds between the problems $(P_{x^*,y^*})$ and $(D_{x^*,y^*})$, i.e. $v(D_{x^*,y^*}) = v(P_{x^*,y^*}) \geq 0$ and the dual $(D_{x^*,y^*})$ has an optimal solution. Thus there exist $\beta \in \mathbb{R}^m_+$ and $p^* \in \mathbb{R}^n$ such that relation (3.3) is true and this part of the proof is over.

*Sufficiency.* Consider $x^* \in \text{dom}(h^*)$ and $y^* \in \Pi$. Then there exist $\beta \in \mathbb{R}^m_+$ and $p^* \in \mathbb{R}^n$ such that (3.3) holds and this implies $v(D_{x^*,y^*}) \geq 0$. But $x^*$ and $y^*$ were arbitrarily chosen and it is easy to see that we have $v(D^{DC})$ also being
The second case is concerning $\beta \neq 0$. The previous statement can be formulated as a theorem of the alternative, too.

**Theorem 3.8** Assume $(CQ_{y^*})$ fulfilled for all $y^* \in \Pi$. Then either the inequality system

\[(I) \quad x \in X, g_i(x) - h_i(x) \leq 0, i = 1, \ldots, m, g(x) - h(x) < 0\]

has a solution or each of the following systems

\[(II_{x^*, y^*}) \quad h^*(x^*) + \sum_{i=1}^{m} \beta_i h_i^*(y_i^*) - g^*(p^*) - \left( \sum_{i=1}^{m} \beta_i g_i \right)_x (x^* + \sum_{i=1}^{m} \beta_i y_i^* - p^*) \geq 0, \quad \beta \in \mathbb{R}^m, p^* \in \mathbb{R}^n,\]

where $x^* \in \text{dom}(h^*)$ and $y_i^* \in \text{dom}(h_i^*), i = 1, \ldots, m$, has a solution, but never both.

Like in the previous chapter we give an equivalent assertion to the statement (ii) in Theorem 3.7 using the epigraphs of the involved functions.

**Theorem 3.9** The statement (ii) in Theorem 3.7 is equivalent to

\[\text{epi}(h^*) \subseteq \bigcap_{y^* \in \Pi} \left\{ \text{epi}(g^*) + \text{cone}(\bigcup_{i=1}^{m} (\text{epi}(g_i) - (y_i^*, h_i^*(y_i^*)))) + \text{epi}(\sigma_X) \right\}. \quad (3.4)\]

**Proof.** Necessity. Take the fixed $(x^*, r) \in \text{epi}(h^*)$ and an arbitrary $m$-tuple $y^* = (y_1^*, \ldots, y_m^*) \in \Pi$. Then $x^* \in \text{dom}(h^*)$ and Theorem 3.7 implies the existence of some $\beta \in \mathbb{R}^m$ and $p^* \in \mathbb{R}^n$ such that the relation (3.3) is true. Further we deal with two cases.

In the first case we suppose that $\beta = 0$. Because of (3.1) relation (3.3) becomes in this case

\[h^*(x^*) - g^*(p^*) - \delta_X(x^* - p^*) \geq 0.\]

Since $r \geq h^*(x^*)$ the previous relation implies $r - g^*(p^*) \geq \delta_X(x^* - p^*) = \sigma_X(x^* - p^*)$. Thus

\[(x^*, r) = (p^*, g^*(p^*)) + (x^* - p^*, r - g^*(p^*)) \in \text{epi}(g^*) + \text{epi}(\sigma_X)\]

\[\subseteq \text{epi}(g^*) + \text{cone}(\bigcup_{i=1}^{m} (\text{epi}(g_i) - (y_i^*, h_i^*(y_i^*)))) + \text{epi}(\sigma_X).\]

The second case is concerning $\beta \neq 0$. In this case the set $\mathcal{I}_\beta = \{i \in \{1, \ldots, m\} : \beta_i > 0\}$ is obviously non-empty and relation (3.3) can be equivalently rewritten as

\[h^*(x^*) + \sum_{i \in \mathcal{I}_\beta} \beta_i h_i^*(y_i^*) - g^*(p^*) - \left( \sum_{i \in \mathcal{I}_\beta} \beta_i g_i \right)_x (x^* + \sum_{i \in \mathcal{I}_\beta} \beta_i y_i^* - p^*) \geq 0.\]

As (3.2) holds, by Theorem 1.1 we have

\[
\left( \sum_{i \in \mathcal{I}_\beta} \beta_i g_i \right)_x (x^* + \sum_{i \in \mathcal{I}_\beta} \beta_i y_i^* - p^*)
= \inf \left\{ \sum_{i \in \mathcal{I}_\beta} (\beta_i g_i)^*(v_i^*) + \sigma_X(u^*) : x^* + \sum_{i \in \mathcal{I}_\beta} \beta_i y_i^* - p^* = \sum_{i \in \mathcal{I}_\beta} v_i^* + u^* \right\}.
\]
and this infimum is attained for some vectors \( u^* \in \mathbb{R}^n \) and \( v_i^* \in \mathbb{R}^n, i \in I_\beta \). Substituting this representation in the above inequality results in

\[
h^*(x^*) \geq g^*(p^*) + \sum_{i \in I_\beta} (\beta_i g_i)^*(v_i^*) - \sum_{i \in I_\beta} \beta_i h_i^*(y_i^*) + \sigma_X(u^*)
\]

and

\[
x^* + \sum_{i \in I_\beta} \beta_i y_i^* - p^* = \sum_{i \in I_\beta} v_i^* + u^*.
\]

Since \( \beta_i > 0 \) relation (1. 5) implies \((\beta_i g_i)^*(v_i^*) = \beta_i g_i^*\left(\frac{1}{\beta_i} v_i^*\right), i \in I_\beta\). Considering the vectors \( p_i^* \in \mathbb{R}^n, \beta_i = \frac{1}{\gamma^* v_i^*}, i \in I_\beta\), the relations obtained above imply

\[
x^* = p^* + \sum_{i \in I_\beta} \beta_i (p_i^* - y_i^*) + u^*
\]

and

\[
r \geq h^*(x^*) \geq g^*(p^*) + \sum_{i \in I_\beta} \beta_i (g_i^*(p_i^*) - h_i^*(y_i^*)) + \sigma_X(u^*).
\]

Therefore we get

\[
\left(p^*, r - \sum_{i \in I_\beta} \beta_i (g_i^*(p_i^*) - h_i^*(y_i^*)) - \sigma_X(u^*)\right) \in \text{epi}(g^*).
\]

Moreover, for all \( i \in I_\beta\)

\[
(p_i^* - y_i^*, g_i^*(p_i^*) - h_i^*(y_i^*)) \in \bigcup_{i=1}^{m} (\text{epi}(g_i^*) - (y_i^*, h_i^*(y_i^*))),
\]

so that

\[
\left(\sum_{i \in I_\beta} \beta_i(p_i^* - y_i^*), \sum_{i \in I_\beta} \beta_i(g_i^*(p_i^*) - h_i^*(y_i^*))\right) = \sum_{i \in I_\beta} \beta_i(p_i^* - y_i^*, g_i^*(p_i^*) - h_i^*(y_i^*))
\]

\[
\in \left(\sum_{i \in I_\beta} \beta_i\right)\left(\bigcup_{i=1}^{m} (\text{epi}(g_i^*) - (y_i^*, h_i^*(y_i^*))\right) \subseteq \text{cone co} \left(\bigcup_{i=1}^{m} (\text{epi}(g_i^*) - (y_i^*, h_i^*(y_i^*))\right).
\]

Combining all the results presented above it follows

\[
(x^*, r) = \left(p^*, r - \sum_{i \in I_\beta} \beta_i (g_i^*(p_i^*) - h_i^*(y_i^*)) - \sigma_X(u^*)\right) + \left(\sum_{i \in I_\beta} \beta_i(p_i^* - y_i^*), \sum_{i \in I_\beta} \beta_i(g_i^*(p_i^*) - h_i^*(y_i^*))\right) + (u^*, \sigma_X(u^*))
\]

\[
\in \text{epi}(g^*) + \text{cone co} \left(\bigcup_{i=1}^{m} (\text{epi}(g_i^*) - (y_i^*, h_i^*(y_i^*))\right) + \text{epi}(\sigma_X).
\]

Sufficiency. Let us consider \( x^* \in \text{dom}(h^*) \) and \( y^* \in \Pi \). As \((x^*, h^*(x^*)) \in \text{epi}(h^*)\) by relation (3. 4) there exist \((p^*, r) \in \text{epi}(g^*), (v^*, v) \in \text{cone co} \left(\bigcup_{i=1}^{m} (\text{epi}(g_i^*) - (y_i^*, h_i^*(y_i^*))\right)\) and \((u^*, u) \in \text{epi}(\sigma_X)\) such that

\[
(x^*, h^*(x^*)) = (p^*, r) + (v^*, v) + (u^*, u).
\]

The last relation obviously implies

\[
x^* = p^* + v^* + u^* \quad \text{and} \quad h^*(x^*) = r + v + u \geq g^*(p^*) + v + \delta_X(u^*).
\]
By Definition 1.4 there exist \( q \geq 0, t_i \geq 0 \) and \((v_i^*, v_i) \in \text{epi}(g_i^*)\), \( i = 1, \ldots, m \), such that \( \sum_{i=1}^m t_i = 1 \) and \((v^*, v) = q \sum_{i=1}^m t_i ((v_i^*, v_i) - (g_i^*, h_i^*(y_i^*))\). These imply further
\[
\sum_{i=1}^m q t_i v_i^* = v^* + \sum_{i=1}^m q t_i h_i^*(y_i^*) \quad \text{and} \quad v = \sum_{i=1}^m q t_i v_i - \sum_{i=1}^m q t_i h_i^*(y_i^*). 
\]
Moreover, for all \( i \in \{1, \ldots, m\} \) we have \( q t_i g_i^*(v_i^*) = (q t_i g_i^*)(q t_i v_i^*) \) (if \( q t_i = 0 \) the equality is a consequence of (3.1) and for \( q t_i > 0 \) see relation (1.5)). Combining all the results presented above we get
\[
q t_i \sum_{i=1}^m v_i \geq \sum_{i=1}^m q t_i g_i^*(v_i^*) = \sum_{i=1}^m (q t_i g_i^*)(q t_i v_i^*) \geq \left( \sum_{i=1}^m q t_i g_i \right)^* \left( \sum_{i=1}^m q t_i v_i \right)^* = \left( \sum_{i=1}^m q t_i g_i \right)^* \left( v^* + \sum_{i=1}^m q t_i y_i^* \right),
\]
where the second inequality is assured by Theorem 1.1. For \( \beta = (\beta_1, \ldots, \beta_m)^T \in \mathbb{R}_+^m, \beta_i = q t_i, i = 1, \ldots, m, \) we get
\[
v \geq - \sum_{i=1}^m \beta_i h_i^*(y_i^*) + \left( \sum_{i=1}^m \beta_i g_i \right)^* \left( v^* + \sum_{i=1}^m \beta_i y_i^* \right)
\]
and, taking into consideration some previous relations, we acquire further
\[
h^*(x^*) \geq g^*(p^*) - \sum_{i=1}^m \beta_i h_i^*(y_i^*) + \left( \sum_{i=1}^m \beta_i g_i \right)^* \left( v^* + \sum_{i=1}^m \beta_i y_i^* \right) + \delta_X^*(u^*)
\]
\[
= g^*(p^*) - \sum_{i=1}^m \beta_i h_i^*(y_i^*) + \left( \sum_{i=1}^m \beta_i g_i \right)^* \left( x^* - p^* - u^* + \sum_{i=1}^m \beta_i y_i^* \right) + \delta_X^*(u^*).
\]
Theorem 1.1 and relation (1.4) imply
\[
\left( \sum_{i=1}^m \beta_i g_i \right)^* \left( x^* - p^* - u^* + \sum_{i=1}^m \beta_i y_i^* \right) + \delta_X^*(u^*)
\]
\[
\geq \left( \sum_{i=1}^m \beta_i g_i + \delta_X \right)^* \left( x^* - p^* - u^* + \sum_{i=1}^m \beta_i y_i^* + u^* \right)
\]
\[
= \left( \sum_{i=1}^m \beta_i g_i \right)^* X \left( x^* + \sum_{i=1}^m \beta_i y_i^* - p^* \right)
\]
and the desired conclusion arises as a direct consequence of the previous inequalities.

\[\square\]

3.3 Special cases

Because of its general form many optimization problems turns out to be special instances of the problem \((P_{DC})\).

3.3.1 The case \( h = 0 \)

In this case the problem \((P_{DC})\) becomes an optimization problem with a convex objective function and finitely many DC constraint functions and the next results are easy consequences of Theorem 3.7 and Theorem 3.9.
Proposition 3.1 Suppose that \((CQ_{y^*})\) holds for all \(y^* \in \Pi\). Then the following assertions are equivalent:

(i) \(x \in X, g_i(x) - h_i(x) \leq 0, \; i = 1, \ldots, m \Rightarrow g(x) \geq 0;\)

(ii) \(\forall y^* \in \Pi, \) there exist \(\beta \in \mathbb{R}^m_+\) and \(p^* \in \mathbb{R}^n\) such that

\[
\sum_{i=1}^{m} \beta_i h_i^*(y_i^*) - g^*(p^*) - \left( \sum_{i=1}^{m} \beta_i g_i \right)^*_x \left( x^* + \sum_{i=1}^{m} \beta_i y_i^* - p^* \right) \geq 0.
\]

Proposition 3.2 The statement (ii) in Proposition 3.1 is equivalent to

\[(0,0) \in \bigcap_{y^* \in \Pi} \left\{ \text{epi}(g^*) + \text{coneco} \left( \bigcup_{i=1}^{m} (\text{epi}(g_i^*) - (y_i^*, h_i^*(y_i^*))) \right) + \text{epi}(\sigma_X) \right\}.\]

Proof. As \(\text{epi}(h_i^*) = \{0\} \times \mathbb{R}_+\), Theorem 3.9 ensures the equivalence between the statement (ii) of the previous proposition and the relation

\[
\{0\} \times \mathbb{R}_+ \subseteq \bigcap_{y^* \in \Pi} \left\{ \text{epi}(g^*) + \text{coneco} \left( \bigcup_{i=1}^{m} (\text{epi}(g_i^*) - (y_i^*, h_i^*(y_i^*))) \right) + \text{epi}(\sigma_X) \right\}.
\]

Using the properties of the epigraph it can be easily proved that this inclusion is equivalent to the relation given in the statement of the proposition. \(\square\)

3.3.2 The case \(h_i = 0, \; i = 1, \ldots, m\)

Within this case the problem \((P_{DC})\) turns out to be a programming problem with DC objective function and finitely many convex constraints. Moreover, as \(\text{dom}(h_i^*) = \{0\}\) and \(\text{epi}(h_i^*) = \{0\} \times \mathbb{R}_+\) for all \(i = 1, \ldots, m\), the condition \((CQ_{y^*})\) becomes

\[
(CQ_0) \quad \exists x^* \in \bigcap_{i=1}^{m} \text{ri}(\text{dom}(g_i)) \cap \text{ri}(\text{dom}(g)) \cap \text{ri}(X) : \begin{cases} g_i(x^*) \leq 0, \; i \in L, \\ g_i(x^*) < 0, \; i \in N \end{cases}
\]

and the next results can be easily deduced from Theorem 3.7 and Theorem 3.9, respectively.

Proposition 3.3 Suppose that \((CQ_0)\) holds. Then the following assertions are equivalent:

(i) \(x \in X, g_i(x) \leq 0, \; i = 1, \ldots, m \Rightarrow g(x) - h(x) \geq 0;\)

(ii) \(\forall x^* \in \text{dom}(h^*) \) there exist \(\beta \in \mathbb{R}^m_+\) and \(p^* \in \mathbb{R}^n\) such that

\[
h^*(x^*) - g^*(p^*) - \left( \sum_{i=1}^{m} \beta_i g_i \right)^*_x \left( x^* + \sum_{i=1}^{m} \beta_i y_i^* - p^* \right) \geq 0.
\]

Proposition 3.4 The statement (ii) in Proposition 3.3 is equivalent to

\[
\text{epi}(h^*) \subseteq \text{epi}(g^*) + \text{coneco} \left( \bigcup_{i=1}^{m} \text{epi}(g_i^*) \right) + \text{epi}(\sigma_X).
\]

Before going further let us mention that some similar results were given by Bot and Wanka in [28] and by Jeyakumar and Glover in [60].
3.3 SPECIAL CASES

3.3.3 The case $h = 0$ and $h_i = 0$, $i = 1, \ldots, m$

It is not hard to see that for $h = 0$ and $h_i = 0$, $i = 1, \ldots, m$, the problem ($P^{DC}$) becomes the convex optimization problem ($P^O$) and one can prove that Theorem 2.23 and Theorem 2.24 are actually special instances of Theorem 3.7 and Theorem 3.9, respectively. Moreover, the assumptions are similar, as the constraint qualification ($CQ_y$) becomes in this special case ($CQ_0$).

3.3.4 DC programming with quadratic functions

Let us consider the problem

$$(P^M) \quad \max_{x \in X, \ g_i(x) \leq 0, \ i = 1, \ldots, m} \left\{ h(x) - a^T x - \sum_{j=1}^k (x^T B_j x)^{\frac{1}{2}} \right\},$$

where $a \in \mathbb{R}^n$ is a given vector, $B_j$ is a positive semidefinite real $n \times n$ matrix, $j = 1, \ldots, k$, $h : \mathbb{R}^n \to \mathbb{R}$ is a convex function and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$, are proper and convex functions. We consider further the function

$$g : \mathbb{R}^n \to \mathbb{R}, \quad g(x) = a^T x + \sum_{j=1}^k (x^T B_j x)^{\frac{1}{2}}.$$  

Since

$$v(P^M) = -\inf_{x \in X, \ g_i(x) \leq 0, \ i = 1, \ldots, m} \left\{ g(x) - h(x) \right\}$$

and the right-hand term of the previous equality is actually a DC programming problem with finitely many convex constraints, to the problem ($P^M$) we attach in accordance with ($D^{DC}$) the dual problem

$$(D^M) \quad -\inf_{x^* \in \text{dom}(h^*)} \sup_{p^* \in \mathbb{R}^m_+} \left\{ h^*(x^*) - g^*(p^*) - \left( \sum_{i=1}^m \beta_i g_i \right)^* x^* - p^* \right\}.$$

Since $g^*(p^*) = 0$ if and only if there exist $w_1, \ldots, w_k \in \mathbb{R}^n$, $w_j^T B_j w_j \leq 1$, $j = 1, \ldots, k$, such that $p^* = a + \sum_{j=1}^m B_j w_j$ (see the proof of Proposition 2.6 and Proposition 2.10), the dual problem ($D^M$) becomes

$$(D^M) \quad \sup_{x^* \in \text{dom}(h^*)} \inf_{\beta \in \mathbb{R}^m_+, \ w_j^T B_j w_j \leq 1, \ j = 1, \ldots, k} \left\{ -h^*(x^*) + \left( \sum_{i=1}^m \beta_i g_i \right)^* x^* - a - \sum_{j=1}^m B_j w_j \right\}.$$  

The next theorems are consequences of the ones given above, so that their proof is omitted.

**Proposition 3.5** (weak duality) Between the primal problem ($P^M$) and the dual problem ($D^M$) weak duality always holds, i.e. $v(P^M) \leq v(D^M)$.

**Proposition 3.6** (strong duality) If ($CQ_0$) is fulfilled, then the optimal objective values of the problems ($P^M$) and ($D^M$) are equal, i.e. $v(P^M) = v(D^M)$.

**Proposition 3.7** (optimality conditions) (a) Suppose that the condition ($CQ_0$) is fulfilled. Let $\bar{x} \in \mathbb{R}^n$ be an optimal solution of the problem ($P^M$) and assume further that the function $h$ is subdifferentiable at $\bar{x}$. Then there exist $\bar{x}' \in \mathbb{R}^n$ and $(\bar{\beta}, \bar{w}_1, \ldots, \bar{w}_k) \in \mathbb{R}^n \times \ldots \times \mathbb{R}^n$, $\bar{\beta} = (\bar{\beta}_1, \ldots, \bar{\beta}_m)^T \in \mathbb{R}^m_+$, $\bar{w}_j^T B_j \bar{w}_j \leq 1$, $j = 1, \ldots, k$, such that
(iM) \( h^*(\overline{x}) + h(\overline{x}) = \overline{x}^T \overline{x}; \)

(iiM) \( \sum_{i=1}^{k} (\overline{x}^T B_i \overline{x})^{1/2} = \sum_{j=1}^{k} \overline{w}_j^T B_j \overline{x}; \)

(iiiM) \( \left( \sum_{i=1}^{m} \overline{\beta}_i g_i \right)^* (\overline{x}^T - a - \sum_{j=1}^{k} B_j \overline{w}_j) + \sum_{i=1}^{m} \overline{\beta}_i g_i(\overline{x}) = \left( \overline{x}^T - a - \sum_{j=1}^{k} B_j \overline{w}_j \right)^T \overline{x}; \)

(ivM) \( \sum_{i=1}^{m} \overline{\beta}_i g_i(\overline{x}) = 0. \)

(b) Let \( \overline{x} \in \mathbb{R}^n \) be such that for all \( \overline{x}_i \in \mathbb{R}^n \) there exist \( \overline{\beta} \in \mathbb{R}_+^m \) and \( \overline{w}_1, \ldots, \overline{w}_k \in \mathbb{R}^n \), \( \overline{w}_j^T B_j \overline{w}_j \leq 1, j = 1, \ldots, k, \) such that the assertions (iM)–(ivM) are satisfied. Then \( \overline{x} \) is an optimal solution of \( (P^M) \) and \( v(P^M) = v(D^M). \)

For \( a = 0 \) and \( k = 1 \) the problem \( (P^M) \) is treated by CHANDRA in [31].
Chapter 4

Farkas - type results for fractional programming

More and more papers treating optimization problems of fractional type have been published during the last decades. Although many papers are oriented more in the practical field, as they present techniques of solving such problems (see, for example, [40, 70, 98]), the theoretical side has not been neglected. In papers like [3, 7, 8, 36, 61, 81, 101] dual problems of various fractional programming problems are constructed and weak and strong duality assertions are also given.

The problem we work with consists in minimizing a fractional function when some geometrical and conical constraints are fulfilled. As our assumptions are more general as the assumptions imposed in [21], the results given in the above mentioned paper turns out to be special instances of the ones we give in this part of the work. The approach we use is the following. Considering $\lambda$ an arbitrary real number such that the optimal objective value of the initial fractional program is greater than or equal to $\lambda$, to the original problem we attach a new one, whose objective function is a convex function or the difference of two convex functions, while the constraints remain the ones of the initial problem. We would like to mention that the objective function of the new problem depends on the real parameter $\lambda$ (it is a convex function for $\lambda$ non-negative and a difference of convex functions for $\lambda$ strictly negative).

To the new problem we determine its Fenchel - Lagrange - type dual problem. The construction of the dual is described in detail and a constraint qualification which guarantees strong duality is presented. Using the relations between the optimal objective values of the attached problem and its dual, some necessary and sufficient optimality condition for the optimal solutions of the initial fractional programming problem are given. Moreover, a Farkas - type statement which generalizes some results recently given is proved.

4.1 Problem formulation

Before presenting some assumptions which we consider fulfilled throughout the entire chapter, we would like to mention that the conventions made at the beginning of the third chapter remain valid also in this part of the work. Further let $X$ be a non-empty convex subset of $\mathbb{R}^n$. The problem we work with is

\[(P_F) \quad \inf_{\substack{x \in X, \quad h(x) \leq \mu \quad \text{and} \quad g(x) \leq 0}} f(x),\]

where $K \subseteq \mathbb{R}^m$ is a non-empty convex cone, $f : \mathbb{R}^n \to \mathbb{R}$ is a proper and convex
function, $g : \mathbb{R}^n \to \mathbb{R}$ is a concave function such that $-g$ is proper and $h : \mathbb{R}^n \to [0,\infty]$ is a proper $K$-convex function such that $X \cap \text{dom}(f) \cap h^{-1}(-K) \neq \emptyset$. Moreover, we suppose that $g(x) > 0$ for all $x$ feasible to the problem $(P^F)$, i.e., for all $x \in X \cap h^{-1}(-K)$.

Before going further, we would like to underline some conclusions which can be easily extracted from the conditions already imposed. The first one concerns the objective value of the problem $(P^F)$. Namely, since the first of the previous relations is fulfilled, it is easy to see that $v(P^F) < +\infty$. The second inference we would like to mention regards the properness of the function $-g$. Although we suppose that $g(x) > 0$ over the feasible set and the later is considered non-empty, it is not hard to see that the fulfillment of this condition does not necessarily imply the properness of the function $-g$. Therefore the conditions imposed are not superfluous.

Lastly, for an arbitrary real number $\lambda$ let us consider the attached problem (for more details see [40])

$$(P^F) \quad \inf_{x \in X, \ h(x) \leq \mu, 0} \left\{ f(x) - \lambda g(x) \right\}.$$  

Then the following result, whose proof is skipped because of its simplicity, can also be proved.

**Lemma 4.1** The following equivalence holds

$$v(P^F) \geq \lambda \iff v(P^F_\lambda) \geq 0.$$  

Our next step is to construct a dual problem to $(P^F_\lambda)$ and to give sufficient conditions in order to achieve strong duality. Since the convexity of the objective function of the problem $(P^F_\lambda)$ depends on the sign of $\lambda$, we have to treat two different cases. First, we assume that $\lambda$ has a non-negative value. In this case the objective function of the problem $(P^F_\lambda)$ is convex and therefore the theory already developed for convex programming can be used. The second case occurs for $\lambda$ strictly negative. In this case the objective function of the problem $(P^F_\lambda)$ becomes the difference of two convex functions and therefore we have to use a slightly different approach inspired from DC programming.

### 4.2 The case $\lambda \geq 0$

A look at the objective function of the problem $(P^F_\lambda)$ shows us that the function $f - \lambda g$ is a convex function when $\lambda \geq 0$ and, using the methods of convex programming, a dual problem can be easily established. That is why to the problem $(P^F_\lambda)$ we associate first its Lagrange dual problem

$$(D^F_\lambda) \quad \sup_{\beta \in K^*} \inf_{x \in X} \left\{ (f + \lambda(-g))(x) + (\beta^T h)(x) \right\}.$$  

But for our aims it is important to point out the idea of reformulating the inner infimum of the Lagrange dual problem by using conjugate functions. Regarding this infimum concerning $x$, the definition of the conjugate relative to a set allows it to be rewritten as

$$\inf_{x \in X} \left\{ (f + \lambda(-g))(x) + (\beta^T h)(x) \right\} = \inf_{x \in \mathbb{R}^n} \left\{ f(x) + \lambda(-g)(x) + (\beta^T h)(x) + \delta_X(x) \right\}$$

$$= -\sup_{x \in \mathbb{R}^n} \left\{ -f(x) - \lambda(-g)(x) - (\beta^T h)(x) - \delta_X(x) \right\} = - (f + \lambda(-g) + (\beta^T h + \delta_X))^*(0).$$
4.2 THE CASE $\lambda \geq 0$

But Theorem 1.1 implies

$$\left(f + \lambda(-g) + (\beta^T h + \delta_X)\right)^*(0) \leq \inf_{u^*, v^* \in \mathbb{R}^n} \left\{ f^*(u^*) + \lambda(-g)^*(v^*) + (\beta^T h)^*_X(-u^* - v^*) \right\}$$

and, taking into consideration also the previous equality, it is easy to see that the optimal objective value of the dual problem $(D_\lambda^F)$ is greater than or equal to the optimal objective value of its Lagrange dual problem.

It is known that the optimal objective value of the problem $P$ is actually the Fenchel - Lagrange - type dual problem of $(D_\lambda^F)$. We consider $(D_\lambda^F)$ first for $\lambda > 0$. Using the definition of the conjugate function it can be easily proved that $\lambda(-g)^*(v^*) = \lambda(-g)^*(\frac{1}{\lambda} v^*)$. Introducing the new variables $x^* = u^*$ and $y^* = \frac{1}{\lambda} v^*$ one can write the dual $(D_\lambda^F)$ in the new form

$$(D_\lambda^F) \sup_{\beta \in \mathbb{R}^n, x^*, y^* \in \mathbb{R}^n} \left\{ -f^*(x^*) - \lambda(-g)^*(y^*) - (\beta^T h)^*_X(-x^* - \lambda y^*) \right\}.$$ 

Now we look at $(D_0^F)$ for $\lambda = 0$, which becomes in this case

$$(D_0^F) \sup_{\beta \in \mathbb{R}^n, x^*, y^* \in \mathbb{R}^n} \left\{ -f^*(x^*) - (0(-g))^*(v^*) - (\beta^T h)^*_X(-u^* - v^*) \right\}.$$ 

The second term in the objective function the definition of the conjugate function and relation (3.1) imply $(0(-g))^*(v^*) = 0$ if $v^* = 0$ and $+\infty$ otherwise. Therefore $(D_0^F)$ may be rewritten omitting $v^*$, namely

$$(D_0^F) \sup_{\beta \in \mathbb{R}^n, x^* \in \mathbb{R}^n} \left\{ -f^*(x^*) - (\beta^T h)^*_X(-u^*) \right\}.$$ 

But setting formally $\lambda = 0$ in the new form of $(D_0^F)$ we obtain

$$\sup_{x^*, y^* \in \mathbb{R}^n, \beta \in \mathbb{K}^*} \left\{ -f^*(x^*) - 0(-g)^*(y^*) - (\beta^T h)^*_X(-x^* - 0y^*) \right\} = \sup_{x^* \in \mathbb{R}^n, \beta \in \mathbb{K}^*} \left\{ -f^*(x^*) - (\beta^T h)^*_X(-u^*) \right\}$$

and this is indeed the above problem $(D_0^F)$. Thus we may write $(D_0^F)$ in the new form also for $\lambda = 0$.

Taking a closer look at the new form of the dual problem $(D_0^F)$, it is easy to see that it is actually the Fenchel - Lagrange - type dual problem of $(P_0^F)$. The next weak duality statement follows at hand.

**Theorem 4.1** (weak duality) Between the primal problem $(P_0^F)$ and the dual problem $(D_0^F)$ weak duality always holds, i.e. $v(P_0^F) \geq v(D_0^F)$.

**Proof.** It is known that the optimal objective value of the problem $(P_0^F)$ is always greater than or equal to the optimal objective value of its Lagrange dual problem $(D_0^F)$. Since the inequality $v(D_0^F) \geq v(D_0^F)$ has been proved above, the desired
result is follows immediately.

The following constraint qualification

\[(CQ^F) \quad \exists \alpha' \in \overline{\text{ri}(\text{dom}(f))} \cap \overline{\text{ri}(\text{dom}(-g))} \cap \overline{\text{ri}(\text{dom}(h) \cap X)} : h(\alpha') \in -\overline{\text{ri}(K)}\]

secures strong duality between the problems \((P^F_X)\) and \((D^F_X)\).

**Theorem 4.2 (strong duality)** If \((CQ^F)\) is fulfilled, then between \((P^F_X)\) and \((D^F_X)\) strong duality holds, i.e., \(v(P^F_X) = v(D^F_X)\) and the dual problem has an optimal solution.

**Proof.** To the problem \((P^F_X)\) we associate its Lagrange dual problem \((D^F_X)\). Since the condition \((CQ^F)\) is fulfilled, it is well-known from the existent literature (see also the Section 2.1 of the present work) that between \((P^F_X)\) and \((D^F_X)\) strong duality holds. This means nothing but the fact that the optimal objective values of \((P^F_X)\) and \((D^F_X)\) are equal and, moreover, there exists \(\beta \in K^*\) such that

\[
v(D^F_X) = \sup_{\beta \in K^*} \inf_{x \in X} \left\{ (f + \lambda(-g))(x) + (\beta^T h)(x) \right\} = \inf_{x \in X} \left\{ f(x) + \lambda(-g)(x) \right\}
\]

+(\beta^T h)(x)) = - \sup_{x \in \mathbb{R}^n} \left\{ - f(x) - \lambda(-g)(x) - (\beta^T h)(x) - \delta_X(x) \right\} = -(f + \lambda(-g) + (\beta^T h + \delta_X)^*(0).

As \(\overline{\text{dom}(\beta^T h + \delta_X)} = \overline{\text{dom}(h) \cap X}\) and \(\overline{\text{dom}(\lambda(-g))} = \overline{\text{dom}(-g)}\), the fulfillment of the condition \((CQ^F)\) implies

\[
\overline{\text{ri}(\text{dom}(f))} \cap \overline{\text{ri}(\text{dom}(-g))} \cap \overline{\text{ri}(\text{dom}(\beta^T h + \delta_X))} \neq \emptyset.
\]

By Theorem 1.1 we get further

\[
v(D^F_X) = - \inf_{u^*, v^* \in \mathbb{R}^n} \left\{ f^*(u^*) + (\lambda(-g))^*(v^*) + (\beta^T h + \delta_X)^*(-u^* - v^*) \right\},
\]

and there exist some \(\overline{u^*}, \overline{v^*} \in \mathbb{R}^n\) such that the infimum is attained, i.e.,

\[
v(D^F_X) = - f^*(\overline{u^*}) - (\lambda(-g))^*(\overline{v^*}) - (\beta^T h)^*_{\lambda^*}(-\overline{u^*} - \overline{v^*}).
\]

If we consider \(\overline{\tau} = \overline{u^*}\) and \(\overline{y^*} = \frac{1}{\lambda} \overline{v^*}\) for \(\lambda > 0\) and using for \(\lambda = 0\) the same arguments as above where we have derived the new formulation for \((D^F_X)\) we get

\[
v(D^F_X) = - f^*(\overline{\tau}) - (\lambda(-g))^*(\overline{y^*}) - (\beta^T h)^*_{\lambda^*}(-\overline{\tau} - \lambda\overline{y^*}).
\]

Indeed for \(\lambda = 0\) we have \(\overline{y^*} = 0\) in the optimum. Since \(v(P^F_X) = v(D^F_X)\) and \((\overline{\beta}, \overline{\tau}, \overline{y^*})\) is an optimal solution for \((D^F_X)\), the proof is complete.

Supposing that the optimal objective value of the problem \((P^F)\) is strictly positive, the next theorem provides necessary and sufficient conditions for the optimal solutions of the problem we study.

**Theorem 4.3 (optimality conditions)** (a) Suppose that the condition \((CQ^F)\) is fulfilled and \(v(P^F) > 0\). Let \(\overline{x} \in \mathbb{R}^n\) be an optimal solution of the problem \((P^F)\). Then there exist \(\lambda > 0\) and \((\overline{\beta}, \overline{x}, \overline{y^*}) \in K^* \times \mathbb{R}^n \times \mathbb{R}^n\) an optimal solution of \((D^F_X)\) such that
4.2 THE CASE $\lambda \geq 0$

\( (i^F) \) \( f^*(\overline{x}) + f(\overline{y}) = \overline{x}^T \overline{y} \);

\( (ii^F) \) \( g^*(\overline{y}) + g(\overline{y}) = \overline{y}^T \overline{y} \);

\( (iii^F) \) \( (\overline{\beta}^T h)\overline{X}(-\overline{x} - \overline{\lambda} \overline{y}) + \overline{\beta}^T h(\overline{y}) = -(\overline{x} - \overline{\lambda} \overline{y}) \overline{y}^T \overline{y}; \)

\( (iv^F) \) \( \overline{y}^T h(\overline{y}) = 0. \)

(b) If there exists $\overline{y} \in \mathbb{R}^n$ such that for some $\overline{\lambda} > 0$, $\overline{\beta} \in K^+$, and $\overline{x}^T \in \mathbb{R}^n$, $\overline{y}^T \in \mathbb{R}^n$ the assertions $(i^F) - (iv^F)$ are satisfied, then $\overline{y}$ is an optimal solution of $(P^F)$, $(\overline{\beta}, \overline{x}^T, \overline{y}^T)$ is an optimal solution of $(D^T)$ and, moreover, $v(P^F) = \overline{\lambda}.$

**Proof.** (a) Since $\overline{y}$ is an optimal solution of the problem $(P^F)$ it can be easily proved that for $\overline{\lambda} = \frac{f(\overline{x})}{g(\overline{y})} > 0$ the optimal objective value of the problem $(P^F)$ is equal to 0. As the condition $(CQ^F)$ is fulfilled, Theorem 4.2 ensures the strong duality between $(P^F)$ and $(D^F)$, i.e. $v(P^F) = v(D^F)$ and the dual problem has an optimal solution. Thus there exist $\overline{\beta} \in K^+$, $\overline{x}^T \in \mathbb{R}^n$ and $\overline{y}^T \in \mathbb{R}^n$ such that

\[ f(\overline{y}) + \overline{\lambda} g(\overline{y}) = -f^*(\overline{x}) - \overline{\lambda} g^*(\overline{y}) - (\overline{\beta}^T h)\overline{X}(-\overline{x} - \overline{\lambda} \overline{y}). \]

The last equality implies further

\[ 0 = [f(\overline{y}) + f^*(\overline{x})] + [\overline{\lambda} g(\overline{y})] + (\overline{\beta}^T h)\overline{X}(-\overline{x} - \overline{\lambda} \overline{y}) \]

\[ = [f(\overline{y}) + f^*(\overline{x}) - \overline{x}^T \overline{y}] + [\overline{\lambda} g(\overline{y})] + [-g^*(\overline{y}) - \overline{y}^T \overline{y}] \]

\[ + (\overline{\beta}^T h)\overline{X}(-\overline{x} - \overline{\lambda} \overline{y}) - (\overline{x}^T - \overline{\lambda} \overline{y}) \overline{y}^T \overline{y} + [-\overline{\beta}^T h(\overline{y})]. \]

According to (1.3) all the terms within the brackets are non-negative. Thus each term must be equal to 0 and the relations $(i^F) - (iv^F)$ follow immediately.

(b) Following the same transformations as above but in the reverse order, we obtain the desired conclusion. \( \square \)

Before going further, we would like to mention that for the second part of the previous theorem the fulfillment of the condition $(CQ^F)$ is not necessary.

**Remark 4.1** If the optimal objective value of the problem $(P^F)$ is equal to 0, then the optimality conditions we get are exactly the ones given in Theorem 2.19.

Further we give a Farkas-type result which involves also a fractional function.

**Theorem 4.4** Take $\lambda$ a non-negative real number and suppose that $(CQ^F)$ is fulfilled. Then the following assertions are equivalent:

\( (i) \) $x \in X, h(x) \leq K 0 \Rightarrow \frac{f(x)}{g(x)} \geq \lambda$;

\( (ii) \) there exist $\beta \in K^+$ and $x^*, y^* \in \mathbb{R}^n$ such that

\[ f^*(x^*) + \lambda(-g)^*(y^*) + (\beta^T h)^*(-x^* - \beta g^*) \leq 0. \] \( (4.1) \)

**Proof.** Since

\( (i) \Leftrightarrow v(P^F) \geq \lambda \Leftrightarrow v(P^F) \geq 0, \)

our aim is to prove that $v(P^F) \geq \lambda$ holds if and only if $(ii)$ holds, too.

**Necessity.** As the assumptions of Theorem 4.2 are valid, strong duality holds between $(P^F)$ and $(D^F)$, namely $v(P^F) = v(D^F)$ and the dual $(D^F)$ has an optimal
Theorem 4.5 Assume that \( \lambda \geq 0 \) is a real number and that \((CQ^F)\) is fulfilled. Then either the inequality system

\[
(D_1) \quad f^*(x^*) + \lambda (-g)^*(y^*) + (\beta^T h)_X^*(-x^* - \lambda y^*) \leq 0,
\]

\( \beta \in K^*, x^*, y^* \in \mathbb{R}^n, \)

has a solution, but never both.

The following result presents an equivalent assertion to the statement \((ii)\) in Theorem 4.4 using only the epigraphs of the functions involved.

Theorem 4.6 The statement \((ii)\) in Theorem 4.4 is equivalent to

\[
(0, 0) \in \text{epi}(f^*) + \lambda \text{epi}((-g)^*) + \bigcup_{\beta \in K^*} \text{epi}((\beta^T h)_X^*).
\]

(4.2)

Proof. Necessity. Since statement \((ii)\) in Theorem 4.4 is fulfilled, there exist some \( \beta \in K^* \) and \( x^*, y^* \in \mathbb{R}^n \) such that

\[
f^*(x^*) + \lambda (-g)^*(y^*) + (\beta^T h)_X^*(-x^* - \lambda y^*) \leq 0.
\]

Since \( f^*(x^*) \) and \((\beta^T h)_X^*(-x^* - \lambda y^*)\) are real values, by the definition of the epigraph we get

\[
(x^*, f^*(y^*)) \in \text{epi}(f^*)
\]

and

\[
(-x^* - \lambda y^*, (\beta^T h)_X^*(-x^* - \lambda y^*)) \in \text{epi}((\beta^T h)_X^*) \subseteq \bigcup_{\beta \in K^*} \text{epi}((\beta^T h)_X^*).
\]

Moreover, as \( \lambda (-g)^*(-y^*) \leq f^*(x^*) - (\beta^T h)_X^*(-x^* - \lambda y^*) \), we have

\[
\lambda (y^*, f^*(x^*) - (\beta^T h)_X^*(-x^* - \lambda y^*)) \in \lambda \text{epi}((-g)^*)
\]

even for \( \lambda = 0 \). Combining the previous relations we acquire

\[
(0, 0) = (x^*, f^*(x^*)) + \lambda (y^*, f^*(x^*) - (\beta^T h)_X^*(-x^* - \lambda y^*))
\]

\[
+ (-x^* - \lambda y^*, (\beta^T h)_X^*(-x^* - \lambda y^*)) \in \text{epi}(f^*) + \text{epi}((-g)^*) + \bigcup_{\beta \in K^*} \text{epi}((\beta^T h)_X^*)
\]
4.3. THE CASE $\lambda < 0$

If $\lambda$ is a strictly negative real number, it is not hard to see that the objective function of the problem $(P_\lambda^F)$, namely $f - \lambda g$, is not necessarily a convex function. Therefore, in order to determine a dual problem, the approach used in the previous section cannot be directly employed. Still, as the function $f + \lambda(-g)$ is actually the difference of the convex functions $f$ and $\lambda g$, it is well-known from the existent literature that for such kind of problem a dual can be established, provided that some sufficient assumptions are fulfilled. That is why, in addition to the conditions imposed at the beginning, we suppose further that the function $-g$ is lower semicontinuous over the feasible set of the problem $(P^F)$. The last condition allows us to affirm that for all $x$ feasible to the problem $(P^\lambda)$ we have

$$(-g)(x) = (-g)^*(x) = \sup_{y^* \in \text{dom}(-g)^*} \left\{ y^T x - (-g)^*(y^*) \right\}. \quad (4.3)$$

**Remark 4.2** As $g(x) > 0$ for all feasible $x$, we have that $X \cap h^{-1}(-K) \subseteq \text{dom}(-g)$. Since $-g$ is proper and convex it follows that $-g$ is continuous over $\text{ri}(\text{dom}(-g))$. Nevertheless, this is not sufficient, as relation (4.3) does not necessarily hold if the function $-g$ is not lower semicontinuous over the feasible set. Without this assumption the equality $g(x) = g^*(x)$ may not be fulfilled for all $x \in X \cap h^{-1}(-K)$. As an example, let us consider $m = n = 1$, $X = [0, +\infty)$ and the functions

$$g : \mathbb{R} \to \mathbb{R}, \quad g(x) = \begin{cases} -\infty, & x < 0, \\ 1, & x = 0, \\ 2, & x > 0, \end{cases}$$

and $h : \mathbb{R} \to \mathbb{R}$, $h(x) = -x$. The previous conditions are fulfilled, namely $-g$ is a proper and convex function such that $g(x) > 0$ for all feasible $x$ and $X \cap h^{-1}(-\mathbb{R}_+) \subseteq \text{dom}(-g)$. It is not hard to see that the function $(-g)^*$ takes the value 2 for $y^* \leq 0$ and $+\infty$ otherwise. Using this we get further

$$(-g)^*(0) = \sup_{y^* \in \text{dom}((-g)^*)} \left\{ y^T 0 - (-g)^*(y^*) \right\} = \sup_{y^* \leq 0} \left\{ y^T 0 - 2 \right\} = -2 < -g(0).$$

Regarding our case, there exist situations when the conditions imposed at the very beginning are enough to secure the lower semicontinuity of the function $-g$ over the feasible set of the problem $(P^F)$. As an example let us suppose that the feasible set is a subset of the relative interior of the domain of the function $-g$. Then the lower semicontinuity of the function $-g$ over the feasible set arises as a
consequence of its convexity and the fact that \(-g(x) < 0\) for all \(x\) feasible to \((P^F)\) (for details see [80]).

Making use of relation (4.3) the problem \((P^F)\) can be rewritten as

\[
(P^F) \quad \inf_{x \in X, \ h(x) \in K} \left\{ f(x) + \lambda \sup_{y^* \in \text{dom}(-g)^*} \left\{ y^T x - (-g)^*(y^*) \right\} \right\},
\]

and, after some minor calculations the following form is obtained

\[
(P^F) \quad \inf_{y^* \in \text{dom}(-g)^*} \inf_{x \in X, \ h(x) \leq 0} \left\{ f(x) + \lambda y^T x - \lambda(-g)^*(y^*) \right\}.
\]

Obviously, the inner infimum of this formulation is a convex optimization problem. Therefore for any \(y^* \in \text{dom}(-g)^*\) we consider the problem

\[
(P^F_{\lambda,y^*}) \quad \inf_{x \in X, \ h(x) \leq 0} \left\{ f(x) + (-\lambda)(-\tilde{g})(x) \right\}
\]

with

\[
\tilde{g} : \mathbb{R}^n \to \mathbb{R}, \quad \tilde{g}(x) = y^* T x - (-g)^*(y^*).
\]

Let us fix \(y^* \in \text{dom}(-g)^*\). Since the functions \(f\) and \(-\tilde{g}\) are convex and \(-\lambda > 0\), the results provided within the previous section allow us to affirm that the problem

\[
(D^F_{\lambda,y^*}) \quad \sup_{x^*,z^* \in \mathbb{R}^n} \left\{ -f^*(x^*) - (-\lambda)(-\tilde{g})^*(z^*) - (\beta^T h)^* X(-x^*) - (-\lambda) z^* \right\}
\]

is a Fenchel - Lagrange - type dual problem to \((P^F_{\lambda,y^*})\). Using only the definition of the conjugate of a function it is easy to calculate that

\[
(-\tilde{g})^*(z^*) = \left\{ \begin{array}{ll}
-(-g)^*(y^*), & z^* = -y^*, \\
+\infty, & \text{otherwise}.
\end{array} \right.
\]

Thus the dual \((D^F_{\lambda,y^*})\) becomes

\[
(D^F_{\lambda,y^*}) \quad \sup_{x^* \in \mathbb{R}^n, \beta \in K^*} \left\{ -f^*(x^*) - \lambda(-g)^*(y^*) - (\beta^T h)^* X(-x^*) - \lambda y^* \right\}
\]

As in the previous section, our aim is to give weak and strong duality assertions regarding the problem \((P^F_{\lambda,y^*})\) and its dual \((D^F_{\lambda,y^*})\). Therefore we impose the following constraint qualifications

\[
(CQ^F) \quad \exists x^* \in \text{ri}(\text{dom}(f)) \cap \text{ri}(\text{dom}(h) \cap X) : h(x^*) \in -\text{ri}(K).
\]

Since \(\text{dom}(-\tilde{g}) = \mathbb{R}^n\), the following two results can be easily proved using Theorem 4.1 and Theorem 4.2, respectively.

**Theorem 4.7** *(weak duality)* Between the primal problem \((P^F_{\lambda,y^*})\) and the dual problem \((D^F_{\lambda,y^*})\) weak duality always holds, i.e. \(v(P^F_{\lambda,y^*}) \geq v(D^F_{\lambda,y^*})\).

**Theorem 4.8** *(strong duality)* If \((CQ^F)\) is fulfilled, then between \((P^F_{\lambda,y^*})\) and \((D^F_{\lambda,y^*})\) strong duality holds, i.e. \(v(P^F_{\lambda,y^*}) = v(D^F_{\lambda,y^*})\) and the dual problem has an optimal solution.
4.3 THE CASE $\lambda < 0$

Taking into consideration the results presented in the last two theorems, it is natural to introduce the following dual problem to $(P^F_\lambda)$

$$(D^F_\lambda) \inf_{y^* \in \text{dom}((-g)^*)} \sup_{\beta \in K^*} \left\{ -f^*(x^*) - \lambda(-g)^*(y^*) - (\beta^T h)^X(-x^* - \lambda y^*) \right\}. $$

By the construction of the dual problem $(D^F_\lambda)$ there are weak and strong duality statements for $(P^F_\lambda)$ and $(D^F_\lambda)$ as follows.

**Theorem 4.9** (weak duality) Between the primal problem $(P^F_\lambda)$ and the dual problem $(D^F_\lambda)$ weak duality always holds, i.e. $v(P^F_\lambda) \geq v(D^F_\lambda)$.

**Theorem 4.10** (strong duality) If $(CQ^F)$ is fulfilled, then the optimal objective values of the problems $(P^F_\lambda)$ and $(D^F_\lambda)$ are equal, i.e. $v(P^F_\lambda) = v(D^F_\lambda)$.

Although we call the previous theorem a "strong duality theorem", we do not really have strong duality, since the dual problem does not necessarily have an optimal solution. Nevertheless, the previous result allows us to provide some necessary and sufficient condition for the optimal solutions of the problem $(P^F)$.

**Theorem 4.11** (optimality conditions) (a) Suppose that the condition $(CQ^F)$ is fulfilled and $v(P^F) < 0$. Let $\overline{x} \in \mathbb{R}^n$ be an optimal solution of the problem $(P^F)$ and assume further that the function $-g$ is subdifferentiable at $\overline{x}$. Then there exist $\overline{X} < 0$, $\overline{y} \in \mathbb{R}^n$ and $([\beta], \overline{x}) \in K^* \times \mathbb{R}^n$ an optimal solution of the problem $(D^F_{\overline{X}, \overline{y}})$ such that

$$(i^F) \quad f^*(\overline{x}) + f(\overline{x}) = \overline{x}^T \overline{x};$$

$$(ii^F) \quad (-g)^*(\overline{y}) - g(\overline{x}) = \overline{y}^T \overline{x};$$

$$(iii^F) \quad (\beta^T h)^X(-x^* - \lambda y^*) + \beta^T h(\overline{x}) = -(\overline{x}^* + \overline{x} \overline{y}^*)^T \overline{x};$$

$$(iv^F) \quad \overline{x}^T h(\overline{x}) = 0.$$  

(b) Let $\overline{x} \in \mathbb{R}^n$ and $\overline{X} < 0$ be such that for all $\overline{y} \in \mathbb{R}^n$ there exist $\overline{\beta} \in K^*$ and $\overline{x} \in \mathbb{R}^n$ which fulfill the assertions $((i^F)) -(i^F)$. Then $\overline{x}$ is an optimal solution of $(P^F)$ and, moreover, $v(P^F) = \overline{X}$.

**Proof.** (a) Since the function $-g$ is subdifferentiable at $\overline{x}$ there exists $\overline{y} \in \partial(-g)(\overline{x})$ and this implies further the statement $(ii^F)$. Even more, since $\overline{x}$ is an optimal solution of the problem $(P^F)$ it can be easily proved that for $\overline{X} = \frac{f(\overline{x})}{\partial(-g)(\overline{x}) \geq 0}$ we have $v(P^F_\overline{X}) = v(P^F_{\overline{X}, \overline{y}}) = 0$. By Theorem 4.8 strong duality between $(P^F_{\overline{X}, \overline{y}})$ and $(D^F_{\overline{X}, \overline{y}})$ holds, i.e. $v(P^F_{\overline{X}, \overline{y}}) = v(D^F_{\overline{X}, \overline{y}})$ and the dual problem has an optimal solution. Thus there exist $\overline{\beta} \in K^*$ and $\overline{x} \in \mathbb{R}^n$ such that

$$0 = f(\overline{x}) - \overline{X} g(\overline{x}) = -f^*(\overline{x}) - \overline{X}(-g)^*(\overline{y}) - (\overline{\beta}^T h)^X(-\overline{x} - \overline{X} \overline{y}).$$

Thus

$$0 = f(\overline{x}) - \overline{X} g(\overline{x}) + f^*(\overline{x}) + \overline{X}(-g)^*(\overline{y}) + (\overline{\beta}^T h)^X(-\overline{x} - \overline{X} \overline{y})$$

$$= [f(\overline{x}) + f^*(\overline{x}) - \overline{x}^T \overline{x}] + \overline{X}(-g)^*(\overline{y}) - (\overline{\beta}^T h)^X(-\overline{x} - \overline{X} \overline{y}) + [\overline{\beta}^T h(\overline{x}) + (\overline{\beta}^T h)^X(-\overline{x} - \overline{X} \overline{y}) - (-\overline{x} - \overline{X} \overline{y})^T \overline{x} + [-\overline{\beta}^T h(\overline{x})]$$

and, as all the terms within the brackets are non-negative, each of them must be equal to 0 and the relations $(i^F) - (iv^F)$ are immediate.
From the last relation the equality
\[ v(x) = f^*(\lambda) = \lambda, \]

Summing up the assertions \((ii)^\prime\), \((iii)^\prime\), \((iv)^\prime\) and the previous relation we get
\[ f^*(\lambda) + f^*(y) + \sum_{i} (\beta_i h_i) = 0. \]

Thus
\[ f^*(\lambda) = -f^*(y) = \sum_{i} (\beta_i h_i) \]

and the inequality
\[ f^*(\lambda) - \sum_{i} (\beta_i h_i) \leq 0 \]
can be easily deduced. As \(\bar{y}^*\) is arbitrarily chosen we get (see the way we define the dual problem \((D^\lambda_{xy})\) for the equality and Theorem 4.9 for the last inequality)
\[ v(P^F_X) \leq \sum_{i} (\beta_i h_i) \leq \inf_{\bar{y}^* \in \mathbb{R}^n} v(D^\lambda_{xy}) = v(D^\lambda_{xy}) \leq v(P^F_X). \]

From the last relation the equality \(v(P^F_X) = \lambda\) can be easily deduced and, as this implies further \(v(P^F_X) = \lambda\), the proof of the theorem is finished.
\[ \square \]

**Remark 4.3** Neither the subdifferentiability of the function \(-g\) nor the condition \((\tilde{CQ}^F)\) are necessary for the second part of this theorem.

As in the previous section we use further the weak and strong duality assertions presented in the previous theorems to prove the following Farkas-type result.

**Theorem 4.12** Take \(\lambda\) a strictly negative number and suppose that \((\tilde{CQ}^F)\) is fulfilled. Then the following assertions are equivalent:

(i) \(x \in X, h(x) \leq \lambda \Rightarrow \frac{f(x)}{g(x)} \geq \lambda;\)

(ii) for each \(y^* \in \text{dom}((-g)^*\)), there exist \(\beta \in K^*\) and \(x^* \in \mathbb{R}^n\) such that
\[ f^*(x^*) + \lambda(-g)^*(y^*) + (\beta^T h) \geq 0. \]

**Proof.** The proof is similar to the one of Theorem 4.4. Since the following equivalences
\[ \lambda \leq v(P^F_X) \Rightarrow \lambda \geq v(P^F_X) \]
hold, we prove that the last inequality is fulfilled if and only if \((ii)\) is fulfilled, too.

**Necessity.** Take \(y^* \in \text{dom}((-g)^*\)). As \(0 \leq v(P^F_X) = \inf_{y^* \in \text{dom}((-g)^*)} v(P^F_{xy})\) we get \(0 \leq v(P^F_{xy})\), too. By Theorem 4.8, whose hypotheses are fulfilled, strong duality holds between \((P^F_{xy})\) and \((D^\lambda_{xy})\), and this implies the existence of some \(\beta \in K^*\) and \(x^* \in \mathbb{R}^n\) which satisfy relation (4.4).

**Sufficiency.** Consider \(y^* \in \text{dom}((-g)^*\)). As we can find some \(\beta \in K^*\) and \(x^* \in \mathbb{R}^n\) such that relation (4.4) holds, it is obvious that \(v(D^\lambda_{xy}) \geq 0\). Since \(y^*\) was arbitrarily taken we get \(v(D^\lambda_{xy}) \geq 0\). As weak duality between \((P^F_X)\) and \((D^\lambda_{xy})\) always holds, we get \(v(P^F_X) \geq v(D^\lambda_{xy}) \geq 0\), too, and the proof is complete. \[ \square \]

The previous result can be reformulated as a theorem of the alternative in the following way.
4.3 THE CASE $\lambda < 0$

**Theorem 4.13** Assume that $\lambda < 0$ is a real number and that $(CQ^F)$ is fulfilled. Then either the inequality system

$$(I) \quad x \in X, h(x) \leq_k 0, \frac{f(x)}{g(x)} < \lambda$$

has a solution or each of the following systems

$$(II_y) \quad f^*(x^*) + \lambda(-g)^*(y^*) + (\beta^T h)^*(-x^* - \lambda y^*) \leq 0, \beta \in K^*, x^* \in \mathbb{R}^n,$$

where $y^* \in \text{dom}((-g)^*)$, has a solution, but never both.

As before, our next step is to provide an equivalent assertion to statement (ii) of Theorem 4.12 using only the epigraphs of the involved functions.

**Theorem 4.14** The statement (ii) in Theorem 4.12 is equivalent to

$$-\lambda \text{epi}((-g)^*) \subseteq \text{epi}(f^*) + \bigcup_{\beta \in K^*} \text{epi}((\beta^T h)^*). \quad (4.5)$$

**Proof.** **Necessity.** Take an arbitrary pair $(y^*, r) \in \text{epi}((-g)^*)$. Then $y^* \in \text{dom}((-g)^*)$ and assertion (ii) implies the existence of $\beta \in K^*$ and $x^* \in \mathbb{R}^n$ such that

$$f^*(x^*) + \lambda(-g)^*(y^*) + (\beta^T h)^*(x^*) \leq 0.$$

As the last inequality allows us to affirm that

$$-\lambda r + f^*(x^*) \geq (\beta^T h)^*(x^*) - \lambda y^*,$$

we finally get

$$-\lambda(y^*, r) = (x^*, f^*(x^*)) + (-x^* - \lambda y^*, -\lambda r - f^*(x^*)) \in \text{epi}(f^*) + \text{epi}((\beta^T h)^*) \subseteq \text{epi}(f^*) + \bigcup_{\beta \in K^*} \text{epi}((\beta^T h)^*)$$

and this part of the proof is finished.

**Sufficiency.** Take an arbitrary $y^* \in \text{dom}((-g)^*)$. As $(y^*, (-g)^*(y^*)) \in \text{epi}((-g)^*)$, by relation (4.5) we have

$$-\lambda(y^*, (-g)^*(y^*)) \in \text{epi}(f^*) + \bigcup_{\beta \in K^*} \text{epi}((\beta^T h)^*).$$

Thus there exist $\beta \in K^*$ and $(x^*, r) \in \text{epi}(f^*)$ and $(u^*, v^*) \in \text{epi}((\beta^T h)^*)$ such that

$$-\lambda(y^*, (-g)^*(y^*)) = (x^*, r) + (u^*, v^*).$$

Combining the inequalities $f^*(x^*) \leq r$ and $(\beta^T h)^*(u^*) \leq u$ with the equality from above we get

$$-\lambda y^* = x^* + u^* \quad \text{and} \quad -\lambda(-g)^*(y^*) \geq f^*(x^*) + (\beta^T h)^*(u^*),$$

and this completes the proof. $\square$
CHAPTER 4. FARAKAS - TYPE RESULTS WITH FRACTIONS

4.4 The case $K = \mathbb{R}^m_+$

Let us suppose now that $K = \mathbb{R}^m_+$ and $h : \mathbb{R}^n \to \mathbb{R}^m$, $h = (h_1, \ldots, h_m)^T$ is a $\mathbb{R}^m_+$-convex function. Following the same idea as in the proof of Theorem 3.9 it can be proved that

$$
\bigcup_{\beta \in \mathbb{R}^m_+} \text{epi}(\beta^T h)^*_X = \text{cone} \left( \bigcup_{i=1}^m \text{epi}(h^*_i) \right) + \text{epi}(\sigma_X).
$$

Moreover, the conditions $(CQ^F)$ and $(\overline{CQ}^F)$ become in this case

$$(CQ^F_S) \quad \exists x' \in \text{ri}(X) \cap \text{ri}(\text{dom}(f)) \cap \text{ri}(\text{dom}(-g)) : h(x') < 0
$$

and

$$(\overline{CQ}^F_S) \quad \exists x' \in \text{ri}(X) \cap \text{ri}(\text{dom}(f)) : h(x') < 0,
$$

respectively. Taking into consideration this information, it is not hard to see that the next pairs of theorems, given in [21], are actually special instances of our more general results. For $\lambda \geq 0$ one have the following.

**Theorem 4.15** Take $\lambda$ a non-negative real number and suppose that $(CQ^F_S)$ is fulfilled. Then the following assertions are equivalent:

(i) $x \in X, h(x) \leq 0 \Rightarrow \frac{f(x)}{\|x\|} \geq \lambda$;

(ii) there exist $\beta \in \mathbb{R}^m_+$ and $x^*, y^* \in \mathbb{R}^n$ such that

$$
f^*(x^*) + \lambda(-g)^*(y^*) + (\beta^T h)^*_X(-x^* - \lambda y^*) \leq 0.
$$

**Theorem 4.16** The statement (ii) in Theorem 4.15 is equivalent to

$$(0, 0) \in \text{epi}(f^*) + \lambda \text{epi}(-g)^* + \text{cone} \left( \bigcup_{i=1}^m \text{epi}(h^*_i) \right) + \text{epi}(\sigma_X).
$$

For $\lambda < 0$ we get the following statement.

**Theorem 4.17** Take $\lambda$ a strictly negative number and suppose that $(\overline{CQ}^F_S)$ is fulfilled. Then the following assertions are equivalent:

(i) $x \in X, h(x) \leq 0 \Rightarrow \frac{f(x)}{\|x\|} \geq \lambda$;

(ii) for each $y^* \in \text{dom}((-g)^*)$, there exist $\beta \in K^*$ and $x^* \in \mathbb{R}^n$ such that

$$
f^*(x^*) + \lambda(-g)^*(y^*) + (\beta^T h)^*_X(-x^* - \lambda y^*) \leq 0.
$$

**Theorem 4.18** The statement (ii) in Theorem 4.17 is equivalent to

$$
-\lambda \text{epi}(-g)^* \subseteq \text{epi}(f^*) + \text{cone} \left( \bigcup_{i=1}^m \text{epi}(h^*_i) \right) + \text{epi}(\sigma_X).
$$

**Remark 4.4** The results remain true if in the conditions $(CQ^F_S)$ and $(\overline{CQ}^F_S)$ instead of $h(x') < 0$ the weaker assumption

$$
\left\{ \begin{array}{ll}
    h_i(x') \leq 0, & i \in L, \\
    h_i(x') < 0, & i \in N,
\end{array} \right.
$$

where $L := \{i \in \{1, \ldots, m\} : h_i \text{ is an affine function}\}$ and $N := \{1, \ldots, m\} \setminus L$ is used (see [21]).
4.5 The ordinary convex optimization problem as a special case

Within this section we treat a special case of our general result. Two ideas are emphasized. First, within this special case our main statements “merge”, i.e. we give a pair of theorems whose conclusions do not depend on the sign of $\lambda$. On the other hand, the assertions within this section generalize some recently obtained results. Throughout this section $X$ and $f$ are considered as before, while $K = \mathbb{R}^m_+$, $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $h = (h_1, \ldots, h_m)^T$ and the function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is taken constant $g(x) = 1$. Using the definition it is not hard to prove that

$$(-g)^*(y^*) = \begin{cases} 1, & y^* = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Even more, in this special case the constraint qualifications ($CQ^F$) and ($\tilde{CQ}^F$) collapse into ($\tilde{CQ}_S^F$).

**Theorem 4.19** Suppose that ($\tilde{CQ}^F$) holds and let $\lambda$ be an arbitrary non-negative real number. Then the following assertions are equivalent:

(i) $x \in X, h(x) \leq 0 \Rightarrow f(x) \geq \lambda$;

(ii) there exist $\beta \in \mathbb{R}^m_+$ and $x^* \in \mathbb{R}^n$ such that

$$f^*(x^*) + (\beta^T h)^*_{\chi}(-x^*) \leq -\lambda.$$

**Proof.** By Theorem 4.4 we have (i) fulfilled if and only if there exist $\beta \in \mathbb{R}^m_+$ and $x^*, y^* \in \mathbb{R}^n$ such that

$$f^*(x^*) + \lambda (-g)^*(y^*) + (\beta^T h)^*_{\chi}(-x^* - \lambda y^*) \leq 0.$$

Since it is necessary to have $(-g)^*(y^*)$ different from $+\infty$, $y^*$ can take only the value 0 and the previous inequality becomes

$$f^*(x^*) + \lambda + (\beta^T h)^*_{\chi}(-x^*) \leq 0.$$

The equivalence between the previous relation and the one given in the statement is obvious. $\square$

**Theorem 4.20** The statement (ii) in Theorem 4.19 is equivalent to

$$(0, -\lambda) \in \text{epi}(f^*) + \text{coneco} \left( \bigcup_{i=1}^m \text{epi}(h_i^*) \right) + \text{epi}(\sigma X).$$

**Proof.** Theorem 4.6 ensures that the statement (ii) of Theorem 4.19 is equivalent to

$$(0, 0) \in \text{epi}(f^*) + \lambda \text{epi}((-g)^*) + \bigcup_{\beta \in \mathbb{R}^m_+} \text{epi}((\beta^T h)^*_{\chi}).$$

But

$$\lambda \text{epi}((-g)^*) = \lambda \left( \{0\} \times [1, +\infty) \right) = (0, \lambda) + \{0\} \times [0, +\infty),$$

and, since

$$\bigcup_{\beta \in \mathbb{R}^m_+} \text{epi}((\beta^T h)^*_{\chi}) = \text{coneco} \left( \bigcup_{i=1}^m \text{epi}(h_i^*) \right) + \text{epi}(\sigma X),$$
the previous relation becomes

\[(0, 0) \in \text{epi}(f^*) + (0, \lambda) + \{0\} \times [0, +\infty) + \text{coneco} \left( \bigcup_{i=1}^{m} \text{epi}(h_i^*) \right) + \text{epi}(\sigma_X).\]

Using the definition of the epigraph of a function it can be easily proved that \(\text{epi}(\sigma_X) + \{0\} \times [0, +\infty) = \text{epi}(\sigma_X)\) and the desired relation follows from the previous one. \(\Box\)

**Theorem 4.21** Suppose that \((\overline{CQ}_S^F)\) holds and let \(\lambda\) be a strictly negative real number. Then the following assertions are equivalent:

(i) \(x \in X, h(x) \leq 0 \Rightarrow f(x) \geq \lambda;\)

(ii) there exist \(\beta \in \mathbb{R}_+^m\) and \(x^* \in \mathbb{R}^n\) such that

\[f^*(x^*) + (\beta^T h)^*_X(-x^*) \leq -\lambda.\]

**Proof.** Theorem 4.12 assures that (i) is fulfilled if and only if there exist \(\beta \in \mathbb{R}_+^m\) and \(x^* \in \mathbb{R}^n\) such that

\[f^*(x^*) + \lambda + (\beta^T h)^*_X(-x^*) \leq 0.\]

Thus \(f^*(x^*) + (\beta^T h)^*_X(-x^*) \leq -\lambda\) and the proof is complete. \(\Box\)

**Theorem 4.22** The statement (ii) in Theorem 4.21 is equivalent with

\[(0, -\lambda) \in \text{epi}(f^*) + \text{coneco} \left( \bigcup_{i=1}^{m} \text{epi}(h_i^*) \right) + \text{epi}(\sigma_X).\]

**Proof.** By Theorem 4.14 we get (ii) is equivalent to

\[-\lambda \text{epi}((-g)^*) \subseteq \text{epi}(f^*) + \bigcup_{\beta \in \mathbb{R}_+^m} \text{epi}((\beta^T h)^*_X).\]

This can be equivalently written as

\[\{0\} \times [-\lambda, +\infty) \subseteq \text{epi}(f^*) + \text{coneco} \left( \bigcup_{i=1}^{m} \text{epi}(h_i^*) \right) + \text{epi}(\sigma_X).\]

As the properties of the epigraph assure that the previous inclusion holds if and only if

\[(0, -\lambda) \in \text{epi}(f^*) + \text{coneco} \left( \bigcup_{i=1}^{m} \text{epi}(h_i^*) \right) + \text{epi}(\sigma_X),\]

the proof is completed. \(\Box\)

The following statements unify the previous results and slightly extend Theorem 2.23 and Theorem 2.24, respectively.

**Theorem 4.23** Suppose that \((\overline{CQ}_S^F)\) holds and let \(\lambda\) be an arbitrary real number. Then the following assertions are equivalent:

(i) \(x \in X, h(x) \leq 0 \Rightarrow f(x) \geq \lambda;\)

(ii) there exist \(\beta \in \mathbb{R}_+^m\) and \(x^* \in \mathbb{R}^n\) such that

\[f^*(x^*) + (\beta^T h)^*_X(-x^*) \leq -\lambda.\]

**Theorem 4.24** The statement (ii) in Theorem 4.23 is equivalent to

\[(0, -\lambda) \in \text{epi}(f^*) + \text{coneco} \left( \bigcup_{i=1}^{m} \text{epi}(h_i^*) \right) + \text{epi}(\sigma_X).\]
4.6 Theorems of the alternative with fractions

In this section we consider $B$ a real $n \times n$ symmetric and positive semidefinite matrix, $a$ and $b > 0$ two vectors from $\mathbb{R}^n$, $C$ a real $m \times n$ matrix and $c$ an arbitrary vector from $\mathbb{R}^m$. The subsequent proposition can be easily proved using the results provided within the previous sections of the chapter.

**Proposition 4.1** Assume that $\lambda$ is an arbitrary real number. Then the following assertions are equivalent:

(i) $x \in \mathbb{R}^n_+ \setminus \{0\}, Cx \leq c \Rightarrow \frac{a^T x + (x^T B x)^{1/2}}{b^T x} \geq \lambda$;

(ii) there exists $\beta \in \mathbb{R}^m_+$ and $w \in \mathbb{R}^n, w^T Bw \leq 1$ such that $\beta^T c \leq 0$ and $a + \lambda b + Bw + C^T \beta \geq 0$.

**Proof.** We consider further the convex set $X = \mathbb{R}^n_+ \setminus \{0\}$, the convex cone $K = \{0\} \subseteq \mathbb{R}^m$ and the functions

$$f : \mathbb{R}^n \to \mathbb{R}, \quad f(x) = a^T x + (x^T B x)^{1/2},$$

$$g : \mathbb{R}^n \to \mathbb{R}, \quad g(x) = b^T x$$

and

$$h : \mathbb{R}^n \to \mathbb{R}^m, \quad h(x) = Cx - c.$$

Then the statement (i) is equivalent to

$$x \in X, h(x) \leq 0 \Rightarrow \frac{f(x)}{g(x)} \geq \lambda.$$

As $g^*(y^*) = 0$ if and only if $y^* = b$ (and $+\infty$ otherwise), Theorem 4.15 for $\lambda \geq 0$ and Theorem 4.17 for $\lambda < 0$ (see also Remark 4.4) allow us to affirm that the previous relation holds if and only if there exist $\beta \in \mathbb{R}^m_+$ and $x^* \in \mathbb{R}^n$ such that

$$f^*(x^*) + (\beta^T h)^*_X(z^*) = (\beta^T C x + \beta^T c) \leq 0.$$

By Lemma 2.2 it holds $f^*(x^*) = 0$ if and only if there exists $w \in \mathbb{R}^n, w^T Bw \leq 1$ such that $x^* - a = Bw$. The definition of the conjugate function relative to a set implies

$$(\beta^T h)^*_X(z^*) = \sup_{x \in X} \left\{ z^T x - \beta^T h(x) \right\} = \sup_{x \in \mathbb{R}^n_+ \setminus \{0\}} \left\{ z^T x - \beta^T C x + \beta^T c \right\}$$

$$= \sup_{x \in \mathbb{R}^n_+ \setminus \{0\}} \left\{ z^T x - \beta^T C x \right\} + \beta^T c = \sup_{x \in \mathbb{R}^n_+ \setminus \{0\}} \left\{ (z^* - C^T \beta)^T x \right\} + \beta^T c$$

$$= \left\{ \begin{array}{ll}
\beta^T c, & z^* - C^T \beta \leq 0, \\
+\infty, & \text{otherwise}.
\end{array} \right.$$
Chapter 5

Weak efficiency for vector optimization problems with composed functions

In multiobjective optimization we simultaneously optimize two or more objectives subject to certain constraints. Usually there should not be a single optimal solution that simultaneously minimizes each objective to its best and that is why we are looking for an optimal solution for which each objective has been optimized to the extent that if we try to optimize it any further, then the other objective(s) will suffer as a result. Finding such an efficient solution, and quantifying how much better this efficient solution is compared to other such efficient solutions (there will generally be quite many) is the goal when setting up and solving a multiobjective optimization problem. The most intuitive approach to solving the multiobjective problem is to combine all of the objective functions into a single functional form. A well-known (and widely used) combination is the weighted linear sum of the objectives. One specifies scalar weights for each objective to be optimized, and then combines them into a single function and the theory developed for single - criteria programming can be further used.

For vector optimization problems one can consider several types of efficient solutions, and among the most used of them there are the so-called weakly efficient solutions. Characterizations of the weakly efficient solutions were recently given under various assumptions in [2] and [59], while for applications of this type of efficient solutions in variational problems we refer to [5]. From the large amount of papers dealing with multiobjective programming problems we mention only [23–25, 27, 46, 48, 54–57, 62, 72, 79, 85–87, 89, 92–94].

Let us consider a vector valued function whose entries are the sum between a convex function and the compositions of some convex functions. Having a problem with an objective function of this kind and without constraints (i.e., the variable runs over the whole space $\mathbb{R}^n$), our aim is to provide necessary and sufficient conditions for its weakly efficient solutions, expressed by using the conjugates of the functions involved. To this end we associate to our initial problem a family of scalar optimization problems and to each scalar problem we provide a Fenchel - Lagrange - type dual. Regarding the construction of the Fenchel - Lagrange - type dual of the scalar problem, we would like to mention that the approach we use is similar to the one used in second chapter of the present work. The construction of the dual is described here in detail and a constraint qualification ensuring strong duality is introduced. Further, using only the strong duality between the scalar problem and its dual, we derive the necessary and sufficient conditions which characterize the
weakly efficient solutions of the primal vector problem. Moreover, a multiobjective
dual to the initial problem is given, and weak and strong duality assertions are
demonstrated. Finally, some special instances of the initial vector programming
problem are considered and some recent results are rediscovered as special cases.

5.1 The composed convex multiobjective problem

Before presenting the problem we work with let us mention that the rules involving
+∞ and −∞ are the same as in the second chapter of the work (see (2. 1)).

For each i = 1, . . . , k let Ki ⊆ Rni be a convex cone and fi : Rn → R, gi : Rni → R
and hi : Rn → Rni be such that fi is a proper convex function, gi is a proper, convex
and K_i - increasing function, while hi is a proper and K_i - convex one. The primal
vector optimization problem we treat within the present chapter is

\[(VP) \quad \text{v-min}_{x \in \mathbb{R}^n} \left( f_1(x) + (g_1 \circ h_1)(x), \ldots, f_k(x) + (g_k \circ h_k)(x) \right) \] T.

In the following we assume that the problem (VP) makes sense, i.e., there exists
at least one x ∈ Rn such that fi(x) + (g_i \circ h_i)(x) < +∞ for all i = 1, . . . , k (in
other words we suppose that the set \( \cap_{i=1}^{k} \text{dom}(f_i) \cap \text{dom}(h_i) \cap h_i^{-1}(\text{dom}(g_i)) \) is non
-empty).

**Definition 5.1** We call \( \pi \in \mathbb{R}^n \) a weakly efficient solution of the problem (VP) if
there exists no x ∈ Rn such that \( f_i(x) + (g_i \circ h_i)(x) < f_i(x) + (g_i \circ h_i)(\pi) \) for all
i = 1, . . . , k.

**Remark 5.1** Using only the definition one can prove that \( \pi \in \mathbb{R}^n \) is a weakly
efficient solution of the problem (VP) if and only if for all x ∈ Rn it holds fi(x) + 
(\( g_i \circ h_i \))(x) ≥ fi(x) + (\( g_i \circ h_i \))(\( \pi \)) for at least one i ∈ \{1, . . . , k\}.

In order to characterize its weakly efficient solutions, to (VP) we associate a fam-
ily of scalar optimization problems. Namely, for each \( \lambda = (\lambda_1, \ldots, \lambda_k)^T \in \mathbb{R}_+^k \setminus \{0\} \)
we consider the optimization problem

\[(VP_\lambda) \quad \inf_{x \in \mathbb{R}^n} \left\{ \sum_{i=1}^{k} \lambda_i (f_i(x) + (g_i \circ h_i)(x)) \right\}.
\]

The following well-known result gives a characterization of the weakly efficient sol-
lutions of a convex vector optimization problem via linear scalarization (see, for
instance, [56]).

**Theorem 5.1** A point \( \pi \in \mathbb{R}^n \) is a weakly efficient solution of the problem (VP)
if and only if there exists \( \lambda \in \mathbb{R}_+^k \setminus \{0\} \) such that \( \pi \) is an optimal solution of the
problem (VP_\lambda).

Further, our aim is to construct a Fenchel - Lagrange - type dual problem to
(VP_\lambda) and from the strong duality assertion to derive the optimality conditions
which characterize a weakly efficient solution for the problem (VP).

5.1.1 The dual problem of the scalarized problem

Let us consider an arbitrary \( \lambda = (\lambda_1, \ldots, \lambda_k)^T \in \mathbb{R}_+^k \setminus \{0\} \). To \( \lambda \) we associate the
set \( \mathcal{I}_\lambda = \{i \in \{1, \ldots, k\} : \lambda_i > 0\} \), which is obviously non - empty. The problem
(VP_\lambda) can be equivalently rewritten as
(V_P) \inf_{x \in \mathbb{R}^n} \left\{ \sum_{i \in I_\lambda} \lambda_i (f_i(x) + (g_i \circ h_i)(x)) \right\}

In order to find the Fenchel-Lagrange-type dual of the problem (V_P) the same idea as in the second chapter of the present work is used. Namely, to the problem (V_P) we associate the following optimization problem

(\overline{V_P}) \inf_{x \in \mathbb{R}^n, y_i \in \text{dom}(g_i), \ h_i(x) \leq k, \ i \in I_\lambda} \left\{ \sum_{i \in I_\lambda} \lambda_i (f_i(x) + g_i(y_i)) \right\}.

The next result allows us to affirm that any dual problem we associate to the problem (\overline{V_P}) becomes a dual problem of the problem (V_P), too.

**Theorem 5.2** It holds \(v(V_P) = v(\overline{V_P})\).

**Proof.** Consider an arbitrary \(x \in \mathbb{R}^n\) and let \(y_i = h_i(x)\) for all \(i \in I_\lambda\). If for at least one \(i \in I_\lambda\) we have \(h_i(x) \notin \text{dom}(g_i)\) or \(h_i(x) = \infty \mathbb{R}^n\), then obviously \(\sum_{i \in I_\lambda} \lambda_i (f_i(x) + g_i(y_i)) = \infty \geq v(\overline{V_P})\). If none of the previously mentioned situations occurs, then \(h_i(x) - y_i \leq K, \ i \in I_\lambda\), and the tuple formed by \(x\) and \(y_i, \ i \in I_\lambda,\) is a feasible solution for (\overline{V_P}). Thus \(\sum_{i \in I_\lambda} \lambda_i (f_i(x) + (g_i \circ h_i)(x)) = \sum_{i \in I_\lambda} \lambda_i (f_i(x) + g_i(y_i)) \geq v(\overline{V_P})\). Since \(x\) was arbitrarily taken in \(\mathbb{R}^n\) the inequality \(v(V_P) \geq v(\overline{V_P})\) is demonstrated.

In order to prove the opposite inequality, let \(x\) and \(y_i, \ i \in I_\lambda,\) be feasible to (\overline{V_P}). If \(x \notin \text{dom}(h_i)\) for some \(i \in I_\lambda\) then \(h_i(x) = \infty \mathbb{R}^n\), and, as \(y_i \in \text{dom}(g_i)\), the inequality \(h_i(x) - y_i \leq K\) cannot hold. Thus \(h_i(x) \in \mathbb{R}^n\) for all \(i \in I_\lambda\). As \(h_i(x) - y_i \leq K, \ i \in I_\lambda,\) by the hypothesis that \(g_i\) are \(K_i\) - increasing functions the inequalities \(g_i(h_i(x)) \leq g_i(y_i), \ i \in I_\lambda,\) are immediate. We acquire \(v(V_P) \leq \sum_{i \in I_\lambda} \lambda_i (f_i(x) + (g_i \circ h_i)(x)) \leq \sum_{i \in I_\lambda} \lambda_i (f_i(x) + g_i(y_i))\) and taking the infimum on the right-side regarding \(x\) and \(y_i, \ i \in I_\lambda,\) feasible to (\overline{V_P}) we obtain \(v(V_P) \leq v(\overline{V_P})\). \(\square\)

**Remark 5.2** We would like to mention that also in this case an observation similar to Remark 2.1 can be made. More precisely, following the same idea as in the proof from above we can demonstrate that \(v(V_P) = v(\overline{V_P})\) even if for some \(i \in \{1, \ldots, k\}\) the function \(g_i\) is \(K_i\) - increasing only over the set \(h_i(\text{dom}(h_i)) + K_i\). Moreover, all the results we give further remain valid also in this case.

In order to determine the Fenchel-Lagrange-type dual problem of the primal problem (\overline{V_P}) some preliminary work is necessary. Let us assume that \(I_\lambda = \{i_1, \ldots, i_l\}\) \((l \leq k)\) and take \(Y = \text{dom}(g_1) \times \cdots \times \text{dom}(g_l) \subseteq \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_l} = \mathbb{R}^N,\) where \(N = i_1 + \cdots + i_l.\) It is not hard to see that the problem (\overline{V_P}) can be equivalently written as

(\overline{V_P}) \inf_{(x,y) \in \mathbb{R}^n \times Y, \ \ G(x,y) \leq 0} F(x,y),

where \(K = K_{i_1} \times \cdots \times K_{i_l},\ y = (y_{i_1}, \ldots, y_{i_l}) \in \mathbb{R}^N\) and the functions \(F : \mathbb{R}^n \times \mathbb{R}^N \rightarrow \mathbb{R}\) and \(G : \mathbb{R}^n \times \mathbb{R}^N \rightarrow \mathbb{R}^N\) are given by

\[
F(x,y) = \lambda_i (f_i(x) + g_i(y_i)) + \cdots + \lambda_i (f_i(x) + g_i(y_i))
\]

and

\[
G(x, y) = \begin{cases} (h_i(x) - y_i, \ldots, h_i(x) - y_i)^T, & x \in \bigcap_{j=1}^l \text{dom}(h_{i_j}), \\ \infty \mathbb{R}^N, & \text{otherwise.} \end{cases}
\]
Making use of the assumptions made in the beginning of the chapter one can easily prove that $K$ is a convex cone, while the functions $F$ and $G$ are convex and $K$-convex, respectively. Thus the problem $(\overline{D}_\lambda)$ is actually a convex optimization problem. Moreover, between the problem $(\overline{D}_\lambda)$ and its Lagrange dual

$$(\overline{D}_\lambda) \quad \sup_{\beta \in \mathbb{K}^*} \inf_{(x,y) \in \mathbb{R}^n \times \mathbb{Y}} \left\{ F(x,y) + \beta^T G(x,y) \right\}$$

weak duality holds, i.e. $v(\overline{D}_\lambda) \leq v(\overline{D}_\lambda)$. Let $\beta \in \mathbb{K}^*$ be arbitrarily taken. Since $\mathbb{K}^* = K_1^* \times \ldots \times K_l^*$, there exist some vectors $\tilde{\beta}_i, \ldots, \tilde{\beta}_i \in K_i^*$ such that $\beta = (\tilde{\beta}_i, \ldots, \tilde{\beta}_i)$. Therefore

$$\inf_{(x,y) \in \mathbb{R}^n \times \mathbb{Y}} \left\{ F(x,y) + \beta^T G(x,y) \right\}$$

$$= \inf_{x \in \mathbb{R}^n, u \in \text{dom}(g_{i_j})} \left\{ \sum_{j=1}^l (\lambda_{i_j} f_{i_j}(x) + g_{i_j}(y_j)) + \sum_{j=1}^l \tilde{\beta}_{i_j} (h_{i_j}(x) - y_j) \right\}$$

$$= \inf_{x \in \mathbb{R}^n, u \in \text{dom}(g_{i_j})} \left\{ \sum_{j=1}^l (\lambda_{i_j} f_{i_j}(x) + \tilde{\beta}_{i_j} h_{i_j}(x)) + \sum_{j=1}^l (\lambda_{i_j} g_{i_j}(y_j) - \tilde{\beta}_{i_j} y_j) \right\}$$

$$= \inf_{x \in \mathbb{R}^n} \left\{ - \sum_{j=1}^l (\lambda_{i_j} f_{i_j}(x) - \tilde{\beta}_{i_j} h_{i_j}(x)) \right\} - \sup_{j=1}^l \inf_{u \in \text{dom}(g_{i_j})} \left\{ \lambda_{i_j} g_{i_j}(y_j) - \tilde{\beta}_{i_j} y_j \right\}$$

$$= -\left( \sum_{j=1}^l \lambda_{i_j} f_{i_j} + \tilde{\beta}_{i_j} h_{i_j} \right)^*(0) - \sum_{j=1}^l (\lambda_{i_j} g_{i_j})^*(\tilde{\beta}_{i_j}).$$

According to Theorem 1.1 for all $\tilde{x}_{i_j}, \tilde{p}_{i_j} \in \mathbb{R}^{n_{i_j}}, j = 1, \ldots, l$, such that $\sum_{j=1}^l \tilde{x}_{i_j} = \sum_{j=1}^l \tilde{p}_{i_j}$ it holds

$$\left( \sum_{j=1}^l \lambda_{i_j} f_{i_j} + \tilde{\beta}_{i_j} h_{i_j} \right)^*(0) \leq \sum_{j=1}^l (\lambda_{i_j} g_{i_j})^*(\tilde{\beta}_{i_j}) + \sum_{j=1}^l (\tilde{\beta}_{i_j} h_{i_j})^*(-\tilde{p}_{i_j}).$$

Therefore between the optimal objective value of the problem $(\overline{D}_\lambda)$ and the optimal objective value of the problem

$$(D_\lambda) \quad \sup_{\tilde{x}_{i_j}, \tilde{p}_{i_j} \in \mathbb{R}^{n_{i_j}}, \tilde{\beta}_{i_j} \in K_{i_j}^*, j = 1, \ldots, l, \sum_{j=1}^l \tilde{x}_{i_j} = \sum_{j=1}^l \tilde{p}_{i_j}} \left\{ - \sum_{j=1}^l (\lambda_{i_j} g_{i_j})^*(\tilde{\beta}_{i_j}) - \sum_{j=1}^l (\lambda_{i_j} f_{i_j})^*(\tilde{x}_{i_j}) - \sum_{j=1}^l (\tilde{\beta}_{i_j} h_{i_j})^*(-\tilde{p}_{i_j}) \right\}$$

the inequality $v(\overline{D}_\lambda) \geq v(D_\lambda)$ always holds. Our next step is to give an equivalent form for the problem $(D_\lambda)$ such that the constants $\lambda_{i_j}, j = 1, \ldots, l$, are factored out. As $\lambda_{i_j} > 0$ for all $j \in \{1, \ldots, l\}$ the equalities

$$(\lambda_{i_j} g_{i_j})^*(\tilde{\beta}_{i_j}) = \lambda_{i_j} g_{i_j}^*\left( \frac{1}{\lambda_{i_j}} \tilde{\beta}_{i_j} \right),$$

$$(\lambda_{i_j} f_{i_j})^*(\tilde{x}_{i_j}) = \lambda_{i_j} f_{i_j}^*\left( \frac{1}{\lambda_{i_j}} \tilde{x}_{i_j} \right),$$

$$(\tilde{\beta}_{i_j} h_{i_j})^*(-\tilde{p}_{i_j}) = \tilde{\beta}_{i_j} h_{i_j}^*\left( -\frac{\lambda_{i_j}}{\tilde{\beta}_{i_j}} \tilde{p}_{i_j} \right)$$
and
\[(\tilde{\beta}^T h_i)\ast(-\tilde{\beta}^T) = \lambda_i \left( \left( \frac{1}{\tilde{\beta}^T h_i} \right) T h_i \right)^\ast \left( - \frac{1}{\tilde{\beta}^T \tilde{\beta}^T} \right)\]
can be easily deduced from (1.5). Introducing the new variables \( \beta_i = \frac{1}{\lambda_i} \tilde{\beta}_i \),
\[x_i^* = \frac{1}{\lambda_i} \tilde{x}_i^* \text{ and } p_i^* = \frac{1}{\lambda_i} \tilde{p}_i^* \], \(j = 1, \ldots, l\), the dual problem \((VD_\lambda)\) becomes
\[(VD_\lambda) \sup_{x_i^*,p_i^* \in \mathbb{R}^{n_i}, \beta_i \in K_i^*} \left\{ - \sum_{j=1}^l \lambda_j g_i^* (\beta_i) - \sum_{j=1}^l \lambda_j f_i^* (x_i^*) \right\} \]
\[= \sum_{i=1}^l \lambda_i x_i^* = \sum_{j=1}^l \lambda_j p_i^* \]
\[- \sum_{j=1}^l \lambda_j (\beta_i^T h_i)^\ast(-p_i^*) \right\} \]
Finally, since \( I_\lambda = \{ i_j : j = 1, \ldots, l \} \), the dual \((VD_\lambda)\) can be rewritten as
\[(VD_\lambda) \sup \left\{ - \sum_{i \in I_\lambda} \lambda_i g_i^* (\beta_i) - \sum_{i \in I_\lambda} \lambda_i f_i^* (x_i^*) - \sum_{i \in I_\lambda} \lambda_i (\beta_i^T h_i)^\ast(-p_i^*) \right\} \]
and this final form is considered further. The next weak duality theorem is an obvious consequence of the previous calculations.

**Theorem 5.3** (weak duality) Between the primal problem \((VP_\lambda)\) and its dual problem \((VD_\lambda)\) weak duality always holds, i.e. \( v(P_\lambda) \geq v(D_\lambda) \).

In order to ensure the equality of the optimal objective values of the problems \((VP_\lambda)\) and \((VD_\lambda)\) we have to impose the constraint qualification
\[(VCQ) \exists x' \in \bigcap_{i=1}^k \text{ri}(\text{dom}(f_i)) \cap \text{ri}(\text{dom}(h_i)) : \{ h_i(x') \in \text{ri}(\text{dom}(g_i)) - \text{ri}(K_i), i = 1, \ldots, k. \}

The following assertion displays the strong duality between the optimization problems \((VP_\lambda)\) and \((VD_\lambda)\).

**Theorem 5.4** (strong duality) If \((VCQ)\) is fulfilled, then between \((VP_\lambda)\) and \((VD_\lambda)\) strong duality holds, i.e. \( v(P_\lambda) = v(D_\lambda) \) and the dual problem has an optimal solution.

**Proof.** If we demonstrate that strong duality holds between the problems \((VP_\lambda)\) and \((VD_\lambda)\) then the desired result arises as a consequence of Theorem 5.2.

As the condition \((VCQ)\) is fulfilled, there exist \( y_i \in \text{ri}(\text{dom}(g_i)) \) such that
\[h_i(x') - y_i' \in - \text{ri}(K_i), j = 1, \ldots, l. \text{ For } y' = (y_1', \ldots, y_l') \text{ this can be equivalently rewritten as} \]
\[G(x', y') = (h_1(x') - y_1', \ldots, h_l(x') - y_l') \in - \text{ri}(K_1) \times \ldots \times \text{ri}(K_l) = - \text{ri}(K). \]

Theorem 1.2 allows us to affirm that \( x' \in \text{ri} \left( \bigcap_{j=1}^l \text{dom}(f_j) \cap \text{dom}(h_j) \right) = \bigcap_{j=1}^l \text{ri}(\text{dom}(f_j)) \cap \text{ri}(\text{dom}(h_j)) \) and, as \( y' \in \text{ri}(\text{dom}(g_1)) \times \ldots \times \text{ri}(\text{dom}(g_l)) = \text{ri}(Y) \), we acquire
\[0 \in G \left( \text{ri} \left( \bigcap_{j=1}^l \text{dom}(f_j) \cap \text{dom}(h_j) \right) \times \text{ri}(Y) \right) + \text{ri}(K). \]
Moreover, using a result given in [12] the previous condition can be rewritten as

$$0 \in \text{ri} \left( G \left( \bigcap_{j=1}^{l} \text{dom}(f_{i_j}) \cap \text{dom}(h_{i_j}) \right) \times Y \right) + K.$$ 

According to [44] the previous condition is enough to secure the strong duality between the primal problem \((\mathcal{VP}_\lambda)\) and its Lagrange dual \((\mathcal{VD}_\lambda)\). Therefore there exists \(\tilde{\beta} \in K^*\) such that

$$v(\mathcal{VP}_\lambda) = \inf_{(x,y) \in \mathbb{R}^n \times Y} \left\{ F(x,y) + \tilde{\beta}^T G(x,y) \right\}.$$ 

The next steps are identical to the ones made before, where we have deduced the final form of the dual problem \((\mathcal{VD}_\lambda)\). More precisely, using the definition of the conjugate function and the special form of the cone \(K^*\) one can easily prove that the previous equality holds if and only if there exists \(\tilde{\beta}_i \in K^*_i\), \(i = 1, \ldots, l\), such that

$$v(\mathcal{VP}_\lambda) = \left( \sum_{j=1}^{l} \lambda_{ij} g_{i_j} + \tilde{\beta}_{ij} h_{i_j} \right)^*(0) - \sum_{j=1}^{l} (\lambda_{ij} g_{i_j})^*(-\tilde{\beta}_{ij}).$$

Theorem 1.1 implies the existence of some vectors \(\tilde{x}_{i_j}, \tilde{p}_{i_j}, j = 1, \ldots, l\), such that

$$\sum_{i=1}^{l} \tilde{x}_{i_j} = \sum_{j=1}^{l} \tilde{p}_{i_j} \quad \text{and} \quad \left( \sum_{j=1}^{l} \lambda_{ij} f_{i_j} + \tilde{\beta}_{ij} h_{i_j} \right)^*(0) = \sum_{j=1}^{l} (\lambda_{ij} f_{i_j})^*(-\tilde{\beta}_{ij}) + \sum_{j=1}^{l} (\tilde{\beta}_{ij} h_{i_j})^*(-\tilde{p}_{i_j}).$$

Thus

$$v(\mathcal{VP}_\lambda) = - \sum_{j=1}^{l} (\lambda_{ij} g_{i_j})^*(-\tilde{\beta}_{ij}) + \sum_{j=1}^{l} (\lambda_{ij} f_{i_j})^*(-\tilde{x}_{i_j}) - \sum_{j=1}^{l} (\tilde{\beta}_{ij} h_{i_j})^*(-\tilde{p}_{i_j})$$

$$= - \sum_{j=1}^{l} (\lambda_{ij} g_{i_j})^*(\tilde{\beta}_{ij}) - \sum_{j=1}^{l} (\lambda_{ij} f_{i_j})^*(\tilde{x}_{i_j}) - \sum_{j=1}^{l} (\tilde{\beta}_{ij} h_{i_j})^*(-\tilde{p}_{i_j}),$$

where \(\tilde{\beta}_{ij} = \frac{1}{\lambda_{ij}} \tilde{\beta}_{ij}, \tilde{x}_{i_j} = \frac{1}{\lambda_{ij}} \tilde{x}_{i_j}\) and \(\tilde{p}_{i_j} = \frac{1}{\lambda_{ij}} \tilde{p}_{i_j}, j = 1, \ldots, l.\)

\[\square\]

**Remark 5.3** Although for the proof of the previous theorem we need just the weaker constraint qualification

\((VCQ)\) \(\exists x' \in \bigcap_{i \in I_\lambda} \text{ri}(\text{dom}(f_{i})) \cap \text{ri}(\text{dom}(h_{i})) : h_{i}(x') \in \text{ri}(\text{dom}(g_{i})) - \text{ri}(K_{i}), i \in I_\lambda,\)

we decided to consider \((VCQ)\) since this condition is independent from \(\lambda.\)

### 5.1.2 Optimality conditions for weakly efficient solutions

Based on the results provided above we point out necessary and sufficient optimality conditions for the weakly efficient solutions of the problem \((VP).\)

**Theorem 5.5** (optimality conditions) (a) Suppose that the condition \((VCQ)\) is fulfilled and let \(\bar{x} \in \mathbb{R}^n\) be a weakly efficient solution of the problem \((VP).\) Then there exist \(\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_k)^T \in \mathbb{R}^k \setminus \{0\}, \bar{\beta}_i \in K^*_i\) and \(\bar{x}_{i_j}, \bar{p}_{i_j} \in \mathbb{R}^n, i \in I_\bar{\lambda} = \{i \in \{1, \ldots, k\} : \bar{\lambda}_i > 0\}, such that
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(i) \( g_i^*(\beta_i) + (g_i \circ h_i)(\pi) = \beta_i^T h(\pi), \ i \in I_\pi; \)

(ii) \( f_i^*(\pi_i) + f_i(\pi) = \pi_i^T \pi, \ i \in I_\pi; \)

(iii) \( (\beta_i^T h_i)^*(-\pi_i^T) + \beta_i^T h_i(\pi) = -\beta_i^T \pi, \ i \in I_\pi; \)

(iv) \( \sum_{i \in I_\pi} \lambda_i x_i = \sum_{i \in I_\pi} \lambda_i \pi_i. \)

(b) If there exists \( \pi \in \mathbb{R}^n \) such that for some \( x \in \mathbb{R}_+^n \setminus \{0\}, \beta_i \in K_i^* \) and \( \pi_i, \pi_i^T \in \mathbb{R}^n, \ i \in I_\pi, \) the conditions (i) – (iv) are satisfied, then \( \pi \) is a weakly efficient solution of the multiobjective problem \((V P)\).

Proof. (a) Let \( \pi \in \mathbb{R}^n \) be a weakly efficient solution of the problem \((V P)\). Theorem 5.1 secures the existence of a vector \( \lambda = (\lambda_1, \ldots, \lambda_k)^T \in \mathbb{R}_+^k \setminus \{0\} \) such that \( \pi \) is an optimal solution of the problem \((V P_k)\). According to Theorem 5.4 strong duality holds between the primal problem \((V P_k)\) and its Fenchel - Lagrange dual problem \((V D_k)\), i.e. \( v(V P_k) = v(V D_k) \) and the dual problem \((V D_k)\) has an optimal solution. Thus there exist \( \pi_i, \pi_i^T \in \mathbb{R}^n \) and \( \beta_i \in K_i^* \), \( i \in I_\pi \), such that

\[
\sum_{i \in I_\pi} \lambda_i x_i = \sum_{i \in I_\pi} \lambda_i \pi_i^T
\]

and

\[
\sum_{i \in I_\pi} \lambda_i ((f_i(\pi) + (g_i \circ h_i)(\pi)) = - \sum_{i \in I_\pi} \lambda_i g_i^*(\beta_i) - \sum_{i \in I_\pi} \lambda_i f_i^*(\pi_i) - \sum_{i \in I_\pi} \lambda_i (\beta_i^T h_i)^*(-\pi_i^T).
\]

The last equality implies further

\[
0 = \sum_{i \in I_\pi} \lambda_i h_i(\pi) + (g_i \circ h_i)(\pi)) + \sum_{i \in I_\pi} \lambda_i g_i^*(\beta_i) + \sum_{i \in I_\pi} \lambda_i f_i^*(\pi_i) + \sum_{i \in I_\pi} \lambda_i (\beta_i^T h_i)^*(-\pi_i^T)
\]

\[
= \sum_{i \in I_\pi} \lambda_i g_i^*(\beta_i) + \sum_{i \in I_\pi} \lambda_i (g_i \circ h_i)(\pi)) - \sum_{i \in I_\pi} \lambda_i \beta_i^T h_i(\pi) + \sum_{i \in I_\pi} \lambda_i f_i^*(\pi_i)
\]

\[
+ \sum_{i \in I_\pi} \lambda_i f_i(\pi) - \sum_{i \in I_\pi} \lambda_i x_i^T \pi + \sum_{i \in I_\pi} \lambda_i (\beta_i^T h_i)^*(-\pi_i^T) + \sum_{i \in I_\pi} \lambda_i \beta_i^T h_i(\pi)
\]

\[
+ \sum_{i \in I_\pi} \lambda_i \pi_i^T \pi = \sum_{i \in I_\pi} \lambda_i ((g_i^*(\beta_i) + (g_i \circ h_i)(\pi)) - \beta_i^T h_i(\pi)) + \sum_{i \in I_\pi} \lambda_i (f_i^*(\pi_i)) + f_i(\pi)
\]

\[
- x_i^T \pi + \sum_{i \in I_\pi} \lambda_i ((\beta_i^T h_i)^*(-\pi_i^T) + \beta_i^T h_i(\pi)) - (-\pi_i^T \pi) + \sum_{i \in I_\pi} \lambda_i x_i^T \pi - \sum_{i \in I_\pi} \lambda_i \pi_i^T \pi.
\]

As all the terms inside the brackets of the previous sum are non- negative, each of them must be equal to 0 and the relations (i) – (iii) follow.

(b) Following the same steps as in (a), but in reverse order, the desired conclusion can be easily reached. \( \square \)

Remark 5.4 For the assertion (b) of the previous theorem, i.e. the sufficiency of the conditions (i) – (iv) for the weak efficiency of \( \pi \), the fulfillment of the condition \((\text{CQ})\) and also the convexity and monotonicity assumptions regarding \((V P)\) are not necessary.

5.1.3 The multiobjective dual problem

Inspired by the form of the dual problem \((V D_\lambda)\) (see also [11,24,25,27,92,93]) to the multiobjective problem \((V P)\) we attach the vector dual problem
\( (V D) \quad \text{v-max}_{\lambda, p, \beta, t \in \mathbb{B}} \left( H_1(\lambda, x^*, p^*, \beta, t), \ldots, H_k(\lambda, x^*, p^*, \beta, t) \right)^T, \)

where

\[
H_i(\lambda, x^*, p^*, \beta, t) = -g_i^*(\beta_i) - f_i^*(x_i^*) - (\beta_i^T h_i)^*(-p_i) + t_i,
\]

for all \( i = 1, \ldots, k, \) and the dual variables are \( \lambda = (\lambda_1, \ldots, \lambda_k)^T \in \mathbb{R}^k, \) \( x^* = (x_1^*, \ldots, x_k^*) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n, \) \( p^* = (p_1^*, \ldots, p_k^*) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n, \) \( \beta = (\beta_1, \ldots, \beta_k) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k} \) and \( t = (t_1, \ldots, t_k)^T \in \mathbb{R}^k. \) The feasible set of the problem \( (V D) \) is described by

\[
\mathcal{B} = \left\{ (\lambda, x^*, p^*, \beta, t) : \lambda \in \mathbb{R}^k_+ \setminus \{0\}, \sum_{i=1}^k \lambda_i x_i^* = \sum_{i=1}^k \lambda_i p_i^*, \sum_{i=1}^k \lambda_i t_i = 0, \beta_i \in K_i^*, i = 1, \ldots, k, \right\}.
\]

As for the primal problem \( (V P) \) we consider for the dual problem weakly efficient solutions, too.

**Definition 5.2** We call \( (\lambda^*, x^*, p^*, \beta^*, t^*) \) a called weakly efficient solution of the problem \( (V D) \) if there exists no \( (\lambda, x^*, p^*, \beta, t) \in \mathcal{B} \) such that \( H_i(\lambda, x^*, p^*, \beta, t) > H_i(\lambda^*, x^*, p^*, \beta^*, t^*) \) for all \( i = 1, \ldots, k. \)

Between the problems \( (V P) \) and \( (V D) \) the subsequent weak vector duality always holds.

**Theorem 5.6** (weak vector duality) There is no \( x \in \mathbb{R}^n \) and no \( (\lambda, x^*, p^*, \beta, t) \in \mathcal{B} \) such that \( f_i(x) + (g_i \circ h_i)(x) < H_i(\lambda, x^*, p^*, \beta, t) \) for all \( i = 1, \ldots, k. \)

**Proof.** Let us suppose that there exist \( x \in \mathbb{R}^n \) and \( (\lambda, x^*, p^*, \beta, t) \in \mathcal{B} \) such that \( f_i(x) + (g_i \circ h_i)(x) < H_i(\lambda, x^*, p^*, \beta, t) \) for all \( i = 1, \ldots, k. \) Since \( \lambda = (\lambda_1, \ldots, \lambda_k)^T \in \mathbb{R}^k_+ \setminus \{0\} , \) the relation

\[
\sum_{i=1}^k \lambda_i (f_i(x) + (g_i \circ h_i)(x)) < \sum_{i=1}^k \lambda_i H_i(\lambda, x^*, p^*, \beta, t)
\]

follows immediately. But

\[
\sum_{i=1}^k \lambda_i H_i(\lambda, x^*, p^*, \beta, t) = \sum_{i \in I_\lambda} \lambda_i H_i(\lambda, x^*, p^*, \beta, t)
\]

\[
= \sum_{i \in I_\lambda} \lambda_i \left( -g_i^*(\beta_i) - f_i^*(x_i^*) - (\beta_i^T h_i)^*(-p_i) + t_i \right)
\]

\[
= - \sum_{i \in I_\lambda} \lambda_i g_i^*(\beta_i) - \sum_{i \in I_\lambda} \lambda_i f_i^*(x_i^*) - \sum_{i \in I_\lambda} \lambda_i (\beta_i^T h_i)^*(-p_i),
\]

as \( \sum_{i \in I_\lambda} \lambda_i t_i = \sum_{i=1}^k \lambda_i t_i = 0. \) The inequalities

\[
- \sum_{i \in I_\lambda} \lambda_i g_i^*(\beta_i) \leq \sum_{i \in I_\lambda} \lambda_i (g_i \circ h_i)(x) - \sum_{i \in I_\lambda} \lambda_i \beta_i^T h_i(x)
\]

and

\[
- \sum_{i \in I_\lambda} \lambda_i f_i^*(x_i^*) \leq \sum_{i \in I_\lambda} \lambda_i f_i(x) - \sum_{i \in I_\lambda} \lambda_i x_i^T x
\]
are direct consequences of the Fenchel - Young inequality, as well as
\[- \sum_{i \in \mathcal{I}_\lambda} \lambda_i (\beta_i^T h_i)(x^*) \leq \sum_{i \in \mathcal{I}_\lambda} \lambda_i \beta_i^T h_i(x) + \sum_{i \in \mathcal{I}_\lambda} \lambda_i p_i^T x.\]
Summing up the previous three relations we get
\[
\sum_{i=1}^k \lambda_i H_i(\lambda, x^*, p^*, \beta, t) = - \sum_{i \in \mathcal{I}_\lambda} \lambda_i g_i^*(\beta_i) - \sum_{i \in \mathcal{I}_\lambda} \lambda_i h_i^*(x_i^*) - \sum_{i \in \mathcal{I}_\lambda} \lambda_i (\beta_i^T h_i)^*(-p_i^*)
\leq \sum_{i \in \mathcal{I}_\lambda} \lambda_i (f_i(x) + (g_i \circ h_i)(x)) = \sum_{i=1}^k \lambda_i (f_i(x) + (g_i \circ h_i)(x)).
\]
Taking into consideration all the relations given above we get
\[
\sum_{i=1}^k \lambda_i (f_i(x) + (g_i \circ h_i)(x)) < \sum_{i=1}^k \lambda_i H_i(\lambda, x^*, p^*, \beta, t) \leq \sum_{i=1}^k \lambda_i (f_i(x) + (g_i \circ h_i)(x)),
\]
which is impossible. Thus the initial assumption is false and the proof of the theorem is complete. \( \square \)

Between the vector problems (VP) and (VD) a strong duality assertion can be proved, too, and the next theorem is devoted to this matter.

**Theorem 5.7** (strong vector duality) Assume that (VCQ) is fulfilled and let \( \overline{\pi} \) be a weakly efficient solution of the primal problem (VP). Then there exists \( (\overline{\lambda}, \overline{\pi}, \overline{\beta}, \overline{t}) \in \mathcal{B} \) that is a weakly efficient solution to the dual problem (VD) and for all \( i = 1, \ldots, k \), one has
\[
f_i(x) + (g_i \circ h_i)(\overline{\pi}) = H_i(\overline{\lambda}, \overline{\pi}, \overline{\beta}, \overline{t}).
\]

**Proof.** Since \( \overline{\pi} \) is a weakly efficient solution of (VP) and the condition (VCQ) is fulfilled, by Theorem 5.5 there exist \( \lambda \in \mathbb{R}^k_+ \setminus \{0\} \), \( x_i^*, p_i^* \in \mathbb{R}^n \) and \( \beta_i \in K_i^* \), \( i \in \mathcal{I}_\lambda \), such that the conditions (i) – (iv) of the above mentioned theorem are fulfilled. Take an arbitrary \( i \in \{1, \ldots, k\} \setminus \mathcal{I}_\lambda \). Since the function \( g_i \) is proper and convex, the function \( g_i^* \) is proper and convex, too (for more details see Theorem 12.2 in [80]). Therefore there exists \( \overline{\beta}_i \in \mathbb{R}^n \) such that \( g_i^*(\overline{\beta}_i) \in \mathbb{R}^n \). Moreover, since the functions \( f_i \) and \( \beta_i^T h_i \) are proper and convex, we can find \( \overline{x}_i^*, \overline{p}_i^* \in \mathbb{R}^n \) such that \( f_i^*(\overline{x}_i^*) \in \mathbb{R} \) and \( (\beta_i^T h_i)^*(-\overline{p}_i^*) \in \mathbb{R} \). Choose
\[
\lambda_i := \lambda, \quad x_i^*: \overline{\pi}_i := \begin{cases} x_i^*, & i \in \mathcal{I}_\lambda; \\
\overline{x}_i^*, & i \notin \mathcal{I}_\lambda; \end{cases}, \quad \overline{p}_i^*: \overline{\beta}_i := \begin{cases} p_i^*, & i \in \mathcal{I}_\lambda; \\
\overline{\beta}_i, & i \notin \mathcal{I}_\lambda; \end{cases},
\]
and \( \overline{t}_i := \begin{cases} x_i^T \overline{\pi} - p_i^T \overline{\pi}, & i \in \mathcal{I}_\lambda; \\
f_i(\overline{\pi}) + (g_i \circ h_i)(\overline{\pi}) + g_i^*(\overline{\beta}_i) + f_i^*(\overline{x}_i^*) + (\beta_i^T h_i)^*(-\overline{p}_i^*), & i \notin \mathcal{I}_\lambda. \end{cases} \)

It is clear that for all \( i = 1, \ldots, k \), we have \( \overline{t}_i \in \mathbb{R} \) (all terms which occur in the definition of \( \overline{t}_i \) are finite), and that (see Theorem 5.5 (iv))
\[
\sum_{i=1}^k \lambda_i \overline{t}_i = \sum_{i \in \mathcal{I}_\lambda} \lambda_i \overline{t}_i = \sum_{i \in \mathcal{I}_\lambda} \lambda_i (x_i^T \overline{\pi} - p_i^T \overline{\pi}) = \left( \sum_{i \in \mathcal{I}_\lambda} \lambda_i x_i^* - \sum_{i \in \mathcal{I}_\lambda} \lambda_i p_i^* \right)^T \overline{\pi} = 0.
\]
It remains to prove that \( f_i(\overline{\pi}) + (g_i \circ h_i)(\overline{\pi}) = H_i(\overline{\lambda}, \overline{\pi}, \overline{\beta}, \overline{t}) \) for all \( i \in \mathcal{I}_\pi \) (for \( i \notin \mathcal{I}_\pi \) this is trivial as a consequence of the definition of \( \overline{t}_i \)). We have
\[
H_i(\overline{\lambda}, \overline{\pi}, \overline{\beta}, \overline{t}) = -g_i(\overline{\beta}_i) - f_i^*(\overline{x}_i^*) - (\overline{\pi}_i^T \overline{\beta}_i) \overline{p}_i^* + \overline{t}_i.
\]
For the last equalities we used Theorem 5.5 (i)–(iii). The fact that \((\bar{x}, \bar{x}^*, \bar{\beta}, \bar{t}, \bar{\lambda})\)
is a weakly efficient solution of the dual problem \((V_D)\) is a straightforward consequence of Theorem 5.6.

\[\square\]

5.2 Special case: composition with linear operators

In the following we suppose that for all \(i = 1, \ldots, k\), the function \(h_i\) is linear and continuous, i.e. \(h_i : \mathbb{R}^n \rightarrow \mathbb{R}^{n_i}, h_i(x) = A_i x,\) with \(A_i\) a \(n_i \times n\) real matrix. The problem \((V_P)\) becomes in this special case

\[(V_P^A) \quad \min_{x \in \mathbb{R}^n} f_1(x) + g_1(A_1x) + \ldots + f_k(x) + g_k(A_kx)\]

and it is easy to prove that for \(K_i^* = \{0\} \subseteq \mathbb{R}^{n_i}, i = 1, \ldots, k\), the problem \((V_P^A)\) is actually a special instance of the problem \((V_P)\). As \(\text{ri}(K_i) = \{0\}, i = 1, \ldots, k\), the constraint qualification \((V_CQ)\) turns out in this case

\[(V_CQ^A) \quad \exists x' \in \bigcap_{i=1}^k \text{ri}\left(\text{dom}(f_i)\right) : A_i x' \in \text{ri}\left(\text{dom}(g_i)\right), i = 1, \ldots, k.\]

Taking into consideration that

\[
(\beta^T h)^*(-p_i^*) = \begin{cases} 
0, & A_i^* \beta_i = -p_i^*, \\
+\infty, & \text{otherwise,}
\end{cases}
\]

the next result can be easily deduced from Theorem 5.5.

**Theorem 5.8** (optimality conditions) (a) Suppose that the condition \((V_CQ^A)\) is fulfilled and let \(\bar{x} \in \mathbb{R}^n\) be a weakly efficient solution of the problem \((V_P^A)\). Then there exist \(\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_k)^T \in \mathbb{R}^k \setminus \{0\}, \bar{\beta} \in K_i^*\) and \(\bar{x}_i^* \in \mathbb{R}^{n_i}, i \in I_\bar{x} = \{i \in \{1, \ldots, k\} : \bar{\lambda}_i > 0\}\), such that

\[(i^A) \quad g_i^*(\bar{\lambda}) + g_i(A_i\bar{x}) = \bar{\beta}_i^T A_i\bar{x}, \quad i \in I_\bar{x};\]

\[(ii^A) \quad f_i^*(\bar{x}_i^*) + f_i(\bar{x}) = \bar{x}_i^T \bar{\beta}_i, \quad i \in I_\bar{x};\]

\[(iii^A) \quad \sum_{i \in I_\bar{x}} \bar{x}_i^* = - \sum_{i \in I_\bar{x}} \bar{\lambda}_i A_i^* \bar{\beta}_i.\]

(b) If there exists \(\bar{x} \in \mathbb{R}^n\) such that for some \(\bar{\lambda} \in \mathbb{R}_-^k \setminus \{0\}, \bar{\beta}_i \in K_i^*\) and \(\bar{x}_i^* \in \mathbb{R}^{n_i}, i \in I_\bar{x},\) the conditions \((i^A) - (iii^A)\) are satisfied, then \(\bar{x}\) is a weakly efficient solution of the multiobjective problem \((V_P^A)\).

**Proof.** By Theorem 5.5 there exists \(\bar{x} = (\bar{x}_1, \ldots, \bar{x}_k)^T \in \mathbb{R}^k \setminus \{0\}, \bar{x}_i^*, \bar{\beta}_i \in \mathbb{R}^{n_i}\) and \(\bar{\lambda}_i \in K_i^*\), \(i \in I_\bar{x}\), such that the conditions (i) - (iv) hold. Because of the special form of the functions \((\beta^T h_i)^*\) relation \((iii)\) is fulfilled if and only if \(A_i^* \bar{\beta}_i = -p_i^*\) for all \(i \in I_\bar{x}\). Now it can be easily seen that the relations \((i^A), (ii^A)\) and \((iii^A)\) are equivalent now with (i), (ii) and (iv), respectively. \[\square\]

Taking into consideration the way the functions \((\beta^T h_i)^*, i = 1, \ldots, k,\) look like, to the problem \((V_D^A)\) we attach the vector dual problem

\[(V_D^A) \quad \max_{(\lambda, x^*, \beta, t) \in \mathbb{R}^A} \left( H_1(\lambda, x^*, \beta, t), \ldots, H_k(\lambda, x^*, \beta, t) \right)^T,\]
5.3 THE CLASSICAL MULTIOBJECTIVE PROBLEM

where

\[ H_i(\lambda, x^*, \beta, t) = -g_i^*(\beta_i) - f_i^*(x_i^*) + t_i, \]

for all \( i = 1, \ldots, k \), and the dual variables are \( \lambda = (\lambda_1, \ldots, \lambda_k)^T \in \mathbb{R}^k \), \( x^* = (x_1^*, \ldots, x_k^*) \in \mathbb{R}^n \times \ldots \times \mathbb{R}^n \), \( \beta = (\beta_1, \ldots, \beta_k) \in \mathbb{R}^n \times \ldots \times \mathbb{R}^n \) and \( t = (t_1, \ldots, t_k)^T \in \mathbb{R}^k \). The feasible set of the problem \((V D^A)\) turns out to be

\[
\mathcal{B}^A = \left\{ (\lambda, x^*, \beta, t) : \lambda \in \mathbb{R}^k \setminus \{0\}, \sum_{i=1}^k \lambda_i x_i^* = -\sum_{i=1}^k \lambda_i A_i^* \beta_i, \sum_{i=1}^k \lambda_i t_i = 0, \beta_i \in \mathbb{R}^n, i = 1, \ldots, k \right\}.
\]

The next weak and strong vector duality assertions are direct consequences of Theorem 5.6 and Theorem 5.7, respectively.

**Theorem 5.9** (weak vector duality) There is no \( x \in \mathbb{R}^n \) and no \((\lambda, x^*, \beta, t) \in \mathcal{B}^A\) such that \( f_i(x) + g_i(A_x) < H_i(\lambda, x^*, \beta, t)\) for all \( i = 1, \ldots, k \).

**Theorem 5.10** (strong vector duality) Assume that \((VCQ^A)\) is fulfilled and let \( \mathcal{X} \) be a weakly efficient solution of the primal problem \((VP^A)\). Then there exists \( (\lambda, x^*, \beta, t) \in \mathcal{B}^A\) that is a weakly efficient solution to the dual problem \((VD^A)\) and for all \( i = 1, \ldots, k \), one has

\[
f_i(x) + g_i(A, x^*) = H_i(\lambda, x^*, \beta, t).
\]

5.3 Special case: the classical convex multiobjective problem

Within the present subsection we treat the usual multiobjective program

\[
(V P^O) \quad \begin{array}{c}
\text{v-min} \\
_{x \in X, h(x) \leq K, 0}
\end{array} \begin{pmatrix} f_1(x), \ldots, f_k(x) \end{pmatrix}^T
\]

where \( K \subseteq \mathbb{R}^n \) is a non-empty convex cone, \( f_i : \mathbb{R}^n \to \mathbb{R} \), \( i = 1, \ldots, k \), are proper and convex functions and \( h : \mathbb{R}^n \to \mathbb{R}^m \) is a proper and \( K \)-convex function. We denote by \( A^O = \{ x \in X : h(x) \leq K \} \) the feasible set of the problem \((VP^O)\) and we assume that it is non-empty. Since for all \( x \in \mathbb{R}^n \) we have \( x \in X, h(x) \leq K 0 \Leftrightarrow (\delta_{-K} \circ h_{X})(x) = 0 \) (see relation (1.1) for the definition of the function \( h_{X} \)), it is apparent that a feasible \( x \in X \) is weakly efficient solution for the problem \((VP^O)\) if and only if it is a weakly efficient solution for the problem

\[
(V P^O) \quad \begin{array}{c}
\text{v-min} \\
_{x \in \mathbb{R}^n}
\end{array} \begin{pmatrix} f_1(x) + (\delta_{-K} \circ h_{X})(x), \ldots, f_k(x) + (\delta_{-K} \circ h_{X})(x) \end{pmatrix}^T.
\]

The last problem is obviously a special instance of \((VP)\) (for all \( i = 1, \ldots, k \), we consider \( g_i = \delta_{-K} \) and \( h_i = h_{X} \)), so that optimality conditions for the weakly efficient solution of the problem \((VP^O)\) and a vector dual of it can be derived from the results given before. As \( \text{ri(dom}(\delta_{-K})) = - \text{ri}(K) = \text{ri}(-K) = \text{ri}(K) = - \text{ri}(K) \) and \( \text{dom}(h_{X}) = X \cap \text{dom}(h) \), the constraint qualification \((VCQ)\) becomes

\[
(VCQ^O) \quad \exists x' \in \bigcap_{i=1}^k \text{ri(dom}(f_i)) \cap \text{ri}(X \cap \text{dom}(h)) : h(x') \in - \text{ri}(K).
\]

The next theorem deals with optimality conditions for the weakly efficient solutions of the problem \((VP^O)\).
Theorem 5.11 (optimality conditions) (a) Suppose that the condition (VCQO) is fulfilled and let $\bar{x} \in A^O$ be a weakly efficient solution of the problem (VPO). Then there exist $\lambda = (\lambda_1, \ldots, \lambda_k)^T \in \mathbb{R}^k \setminus \{0\}$, $\beta \in K^*$ and $\bar{x}_i^* \in \mathbb{R}^n$, $i \in I^*$ such that

\[(i^O) \quad f_i^*(\bar{x}_i^*) + f_i(\bar{x}) = \bar{x}_i^T \bar{x}, \quad i \in I^*; \]

\[(ii^O) \quad (\beta^T h)_X^k \left( - \sum_{i \in I^*} \lambda_i \bar{x}_i^T \right) + \beta^T h(\bar{x}) = - \sum_{i \in I^*} \lambda_i \bar{x}_i^T \bar{x}; \]

\[(iii^O) \quad \beta^T h(\bar{x}) = 0. \]

(b) If there exists $\bar{x} \in \mathbb{R}^n$ such that for some $\lambda \in \mathbb{R}_+^k \setminus \{0\}$, $\beta \in K^*$ and $\bar{x}_i^* \in \mathbb{R}^n$, $i \in I^*$, the conditions (iO) – (iiiO) are satisfied, then $\bar{x}$ is a weakly efficient solution of the multiobjective problem (VPO).

Proof. (a) By Theorem 5.5 there exist $\lambda = (\lambda_1, \ldots, \lambda_k)^T \in \mathbb{R}^k \setminus \{0\}$, $\bar{x}_i^*, \bar{p}_i^* \in \mathbb{R}^n$ and $\bar{x}_i \in K^*$, $i \in I^*$, such that the assertions (i) – (iv) are fulfilled. Obviously (iO) is equivalent to (ii). As $\bar{x} \in A^O$ implies $h(\bar{x}) < K$ it holds $(\delta_{-K} \circ h)(\bar{x}) = 0$. Even more, since $\delta_{-K} = \delta_{K^*}$, relation (i) implies $\beta_i^T h(\bar{x}) = 0$ for all $i \in I^*$ and now it is clear that for $\beta = \sum_{i \in I^*} \lambda_i \beta_i \in K^*$ relation (iiiO) is satisfied. It remains to prove that relation (iiO) holds, too. According to relation (iii) for all $i \in I^*$

$$\left( \beta_i^T h_X \right)^* (-\bar{p}_i^*) + \beta_i^T h_X(\bar{x}) = -\bar{p}_i^T \bar{x}.$$ 

As $h_X(\bar{x}) = h(\bar{x})$ and $(\beta_i^T h_X)^* (-\bar{p}_i^*) = (\bar{p}_i^T h)^* (\beta_i^T h_X^T (-\bar{p}_i^*))$ we can easily deduce that

$$\sum_{i \in I^*} \lambda_i (\beta_i^T h)^* (-\bar{p}_i^*) = - \sum_{i \in I^*} \lambda_i \bar{p}_i^T \bar{x} - \sum_{i \in I^*} \lambda_i \beta_i^T h(\bar{x}).$$

Taking into consideration the fact that $\lambda_i (\beta_i^T h)^* (-\bar{p}_i^*) = (\bar{p}_i \beta_i^T h)^* (-\lambda_i \bar{p}_i^*)$ and the way we define the vector $\bar{p}$ the previous relation imply

$$\sum_{i \in I^*} (\lambda_i \beta_i^T h)^* (-\lambda_i \bar{p}_i^*) = - \left( \sum_{i \in I^*} \lambda_i \bar{p}_i^T \right)^T \bar{x} - \beta^T h(\bar{x}).$$

According to Theorem 1.1

$$\sum_{i \in I^*} (\lambda_i \beta_i^T h)^* (-\lambda_i \bar{p}_i^*) \geq \left( \sum_{i \in I^*} \lambda_i \beta_i^T h \right)^* \left( - \sum_{i \in I^*} \lambda_i \bar{p}_i^T \right) = \left( \beta^T h \right)^* \left( - \sum_{i \in I^*} \lambda_i \bar{p}_i^T \right)$$

and combining the last two relations we get

$$\left( \beta^T h \right)^* \left( - \sum_{i \in I^*} \lambda_i \bar{p}_i^T \right) \leq - \left( \sum_{i \in I^*} \lambda_i \bar{p}_i^T \right)^T \bar{x} - \beta^T h(\bar{x}).$$
Since the reverse inequality is always true (it is a direct consequence of the definition of the conjugate function relative to a set) we acquire (see assertion (iv))

\[(\beta^T h)_X \left( - \sum_{i \in I_X} \lambda_i x_i^* \right) = \left( \sum_{i \in I_X} \lambda_i x_i^* \right)^T \bar{x} - \beta^T h(\bar{x}) \]

and from here (ii\(O\)) is automatically deduced.

(b) For

\[
\overline{p_i} = \frac{1}{\lambda_1 + \ldots + \lambda_k} \sum_{i \in I_X} \lambda_i x_i^* \quad \text{and} \quad \overline{\beta_i} = \frac{1}{\lambda_1 + \ldots + \lambda_k} \sum_{i \in I_X} \beta_i, i \in I_X
\]

the assertions (i) – (iv) can be easily deduced from (i\(O\)) – (iii\(O\)) (all we have to do is to use the properties of the conjugate function).

We introduce the following multiobjective dual problem to (\(VPO\))

\[\max_{(\lambda, x^*, p^*, \beta, t) \in B^O} \left( H_1(\lambda, x^*, p^*, \beta, t), \ldots, H_k(\lambda, x^*, p^*, \beta, t) \right)^T, \]

where

\[H_i(\lambda, x^*, p^*, \beta, t) = -f_i^*(x_i^*) - (\beta^T h_i)_X(-p_i^*) + t_i, \]

for all \(i = 1, \ldots, k\), and the dual variables are \(\lambda = (\lambda_1, \ldots, \lambda_k)^T \in \mathbb{R}^k\), \(x^* = (x_1^*, \ldots, x_k^*) \in \mathbb{R}^n \times \ldots \times \mathbb{R}^n\), \(p^* = (p_1^*, \ldots, p_k^*) \in \mathbb{R}^n \times \ldots \times \mathbb{R}^n\), \(\beta = (\beta_1, \ldots, \beta_k) \in \mathbb{R}^n \times \ldots \times \mathbb{R}^n\), and \(t = (t_1, \ldots, t_k)^T \in \mathbb{R}^k\). The feasible set of the problem (\(VDO\)) is described by

\[B^O = \left\{ (\lambda, x^*, p^*, \beta, t) : \lambda \in \mathbb{R}^k_+, \sum_{i=1}^k \lambda_i x_i^* = \sum_{i=1}^k \lambda_i p_i^*, \sum_{i=1}^k \lambda_i t_i = 0, \beta_i \in K_i^*, i = 1, \ldots, k \right\}. \]

The next weak and strong duality assertions can be easily deduced from Theorem 5.6 and Theorem 5.7, respectively.

**Theorem 5.12 (weak vector duality)** There is no \(x \in A^O\), and no \((\lambda, x^*, p^*, \beta, t) \in B^O\) such that \(f_i(x) < H_i(\lambda, x^*, p^*, \beta, t)\) for all \(i = 1, \ldots, k\).

**Theorem 5.13 (strong vector duality)** Assume that (\(VCQ^O\)) is fulfilled and let \(\bar{x} \in A^O\) be a weakly efficient solution of the primal problem (\(VPO\)). Then there exists \((\bar{\lambda}, \bar{x}, \bar{p}, \bar{\beta}, \bar{t}) \in B^O\) that is a weakly efficient solution to the dual problem (\(VDO\)) and for all \(i = 1, \ldots, k\), one has

\[f_i(\bar{x}) = H_i(\bar{\lambda}, \bar{x}, \bar{p}, \bar{\beta}, \bar{t}). \]

Before going further we would like to mention that some similar results were given in [20].

### 5.4 Special case: composed convex function and inequality constraints

Suppose that for all \(i = 1, \ldots, k\), the function \(f_i : \mathbb{R}^{n_i} \to \mathbb{R}\) is proper, convex and \(K_i\) - increasing, while the function \(F_i : \mathbb{R}^n \to \mathbb{R}^{n_i}\) is proper and \(K_i\) - convex. As
expected, $K_i \subseteq \mathbb{R}^{n_i}$ is a non-empty convex cone, $i = 1, \ldots, k$. Moreover, we consider $Q \subseteq \mathbb{R}^n$ a convex cone and $G : \mathbb{R}^n \to \mathbb{R}^m$ a proper and $Q$-convex function. In the following we treat the vector optimization problem

\[
(VP^C) \quad \text{v-min}_{x \in X, \ G(x) \leq_Q 0} \left( (f_1 \circ F_1)(x), \ldots, (f_k \circ F_k)(x) \right)^T.
\]

Even more, we suppose that the feasible set of the problem $(VP^C)$, i.e., the set $A^C = \{x \in X : G(x) \leq_Q 0\}$ is non-empty and, moreover, the intersection of the sets $A^C$ and

\[
\bigcap_{i=1}^k \text{dom}(f_i) \cap \text{dom}(h_i) \cap h_i^{-1}(\text{dom}(g_i))
\]

is non-empty. In order to apply the theory developed within the previous sections to the problem $(VP^C)$ we attach a new problem such that an arbitrary $\overline{x} \in A^C$ is a weakly efficient solution to $(VP^C)$ if and only if the tuple $(\overline{x}, F_1(\overline{x}), \ldots, F_k(\overline{x}))$ is weakly efficient solution to the new problem. To this aim we take $\overline{K} = Q \times K_1 \times \ldots \times K_k$ and $\overline{R} = \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_k}$. For all $i = 1, \ldots, k$ and $y = (y_1, \ldots, y_k) \in \overline{R}$ we take the functions $\overline{f}_i : \mathbb{R}^n \times \overline{R} \to \overline{R}$ and $\overline{h} : \mathbb{R}^n \times \overline{R} \to \mathbb{R}^{m_1} \times \overline{R}$ given by $\overline{f}_i(x, y) = f_i(y_i)$ and

\[
\overline{h}(x, y) = \begin{cases} (G(x), F_1(x) - y_1, \ldots, F_k(x) - y_k), & x \in \text{dom}(G) \cap \bigcap_{i=1}^k \text{dom}(F_i), \\ \infty, & \text{otherwise}. \end{cases}
\]

Consider the optimization problem

\[
(V^P^C) \quad \text{v-min}_{(x,y) \in X \times \overline{R}^N, \ \overline{h}(x,y) \leq_{\overline{K}} 0} \left( \overline{f}_1(x,y), \ldots, \overline{f}_k(x,y) \right)^T.
\]

The next result presents the relation between the weakly efficient solutions of the problems $(VP^C)$ and $(V^P^C)$.

**Theorem 5.14** A vector $\overline{x} \in A^C$ is a weakly efficient solution of the problem $(VP^C)$ if and only if for $\overline{y} = (F_1(\overline{x}), \ldots, F_k(\overline{x})) \in \overline{R}$ the tuple $(\overline{x}, \overline{y})$ is a weakly efficient solution of the problem $(V^P^C)$.

**Proof.** Necessity. Let us suppose that $\overline{x} \in A^C$ is a weakly efficient solution of the problem $(VP^C)$. Then $\overline{x} \in X$ and $G(\overline{x}) \leq_Q 0$ (by the definition of the set $A$) and, moreover, $\overline{y}_i = F_i(\overline{x}) \in \text{dom}(f_i)$ for all $i = 1, \ldots, k$ (see Definition 5.1). Moreover, taking a look at the definition of the function $\overline{h}$ we easily realize that for $\overline{y} = (\overline{y}_1, \ldots, \overline{y}_k) \in Y$ it holds $\overline{h}(\overline{x}, \overline{y}) \leq_{\overline{K}} 0$. Thus $(\overline{x}, \overline{y})$ is a feasible solution for the problem $(V^P^C)$.

Now let $(x, y) \in X \times \overline{R}^N$ be a feasible solution to $(V^P^C)$, $y = (y_1, \ldots, y_k)$. If for some $i \in \{1, \ldots, k\}$ we have $y_i \notin \text{dom}(f_i)$ then $+\infty = f_i(y_i) = \overline{f}_i(x, y)$ and this contradicts the fact that $y_i \in \text{dom}(f_i)$. Therefore it is binding to have $x \in \text{dom}(F_i)$ and, for $F_i(x) - y_i \leq_{K_i} 0$, we acquire $(f_i \circ F_i)(x) \leq f_i(y_i) = \overline{f}_i(x, y)$. Combining this relation with the previous one we obtain
\[ f_i(\tilde{x}, \tilde{y}) \geq \tilde{f}_i(x, y) \] and, as we do not impose additional conditions neither for \( x \), nor for \( y \), the conclusion is immediate.

**Sufficiency.** Take \( \tilde{x} \in X \) and \( \tilde{y} = (F_1(\tilde{x}), \ldots, F_k(\tilde{x})) \in \mathbb{R}^k \) such that the tuple \((\tilde{x}, \tilde{y})\) is a weakly efficient solution of the problem \((\nabla P_C)\). Then for all \( i = 1, \ldots, k \), we have \( f_i(\tilde{x}, \tilde{y}) = (f_i \circ F_i)(\tilde{x}) < +\infty \). To prove that \( \tilde{x} \) is weakly optimal solution to \((V P_C)\) we make use of Remark 5.1, i.e., we prove that for all \( x \in \mathbb{R}^n \) feasible to \((V P_C)\) we can find at least one indices \( i \in \{1, \ldots, k\} \) such that \( (f_i \circ F_i)(x) \geq (f_i \circ F_i)(\tilde{x}) \). Therefore let \( x \in A_C \) be an arbitrary feasible solution to the problem \((V P_C)\). If for some \( i \in \{1, \ldots, k\} \) we have either \( x \notin \text{dom}(F_i) \) or \( F_i(x) \notin \text{dom}(f_i) \), then obviously \( +\infty = (f_i \circ F_i)(x) \geq (f_i \circ F_i)(\tilde{x}) \). If the previously mentioned situations do not occur, then evidently the tuple \((x, y)\) with \( y = (F_1(x), \ldots, F_k(x)) \in \mathbb{R}^k \) is a feasible solution to the problem \((\nabla P_C)\). By Remark 5.1 there exists at least one \( i \in \{1, \ldots, k\} \) such that \( f_i(\tilde{x}, \tilde{y}) \geq f_i(\tilde{x}, \tilde{y}) \).

Because of the way we define the function \( f_i \) and the vectors \( y \) and \( \tilde{y} \) we get further \( (f_i \circ F_i)(x) \geq (f_i \circ F_i)(\tilde{x}) \) and the theorem is demonstrated. \( \square \)

Further we consider the following constraint qualification

\[ (VCQ_C) \exists x' \in \text{ri}(X \cap \text{dom}(G)) \cap \bigcap_{i=1}^k \text{ri}(\text{dom}(F_i)) : \begin{cases} F_i(x') \in \text{ri}(\text{dom}(f_i)) - \text{ri}(K_i), \\
 i = 1, \ldots, k, \\
 G(x') \in -\text{ri}(Q). \end{cases} \]

The subsequent result presents some necessary and sufficient conditions for the weakly efficient solutions of the problem \((V P_C)\).

**Theorem 5.15** (optimality conditions) \((a)\) Suppose that the condition \((VCQ_C)\) is fulfilled and let \( \tilde{x} \in A_C \) be a weak efficient solution of the problem \((V P_C)\). Then there exist \( \bar{x} \in \mathbb{R}_+^k \setminus \{0\}, \bar{\pi} \in Q^*, \bar{\alpha} \in K_+^i \) and \( \bar{x} \in \mathbb{R}^k \), \( i \in I_x = \{i \in \{1, \ldots, k\} : \bar{x}_i > 0\} \), such that

\[ (i^C) \ f_i^*(\bar{F}^\tau) + (f_i \circ F_i)(\tilde{x}) = \bar{x}^\tau F_i^*(\tilde{x}), \ i \in I_x; \]

\[ (ii^C) \ (\bar{x}^\tau F_i^*)^*(\bar{x}^\tau) + \bar{\alpha}^\tau F_i^*(\tilde{x}) = \bar{x}_i^\tau F_i^*(\tilde{x}), \ i \in I_x; \]

\[ (iii^C) \ (\bar{\pi}^\tau G)(\bar{\lambda}) + \sum_{i \in I_x} \bar{x}_i \bar{\pi}_i \bar{\lambda}_i^{\text{ri}} = \bar{\pi}^\tau G(\tilde{x}) \]

\[ (iv^C) \ \bar{\pi}^\tau G(\tilde{x}) = 0. \]

\((b)\) If there exists \( \tilde{x} \in \mathbb{R}^n \) such that for some \( \bar{x} \in \mathbb{R}_+^k \setminus \{0\}, \bar{\pi} \in Q^*, \bar{\alpha} \in K_+^i \) and \( \bar{x} \in \mathbb{R}^n \), \( i \in I_x \), the conditions \((i^C) - (iv^C)\) are satisfied, then \( \tilde{x} \) is a weakly efficient solution of the multiobjective problem \((V P_C)\).

**Proof.** \((a)\) Since \( \tilde{x} \) is a weak efficient solution of the problem \((V P_C)\), according to Theorem 5.14 the tuple \((\tilde{x}, \tilde{y})\), \( \tilde{y} = (F_1(\tilde{x}), \ldots, F_k(\tilde{x}))^T \), is a weak efficient solution of the problem \((\nabla P_C)\), too. Moreover, the problem \((\nabla P_C)\) can be treated as a special instance of the problem \((V P_O)\) and we can use Theorem 5.11 provided that the condition

\[ (V C Q^O) \exists (x', y') \in \bigcap_{i=1}^k \text{ri}(\text{dom}(f_i)) \cap \text{ri}(X \times \mathbb{R}^N) \cap \text{dom}(\tilde{K}) : \tilde{h}(x', y') \in -\text{ri}(\tilde{K}) \]

is fulfilled for some \( x' \in \mathbb{R}^n \) and \( y' = (y'_1, \ldots, y'_k) \in \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_k} \).

Since the condition \((VCQ_C)\) holds there exist \( x' \in \mathbb{R}^n \) and \( y'_i \in \text{ri}(\text{dom}(f_i)) \) such that \( G(x') \in -\text{ri}(Q) \) and \( F_i(x') \in y'_i - \text{ri}(K_i) \) for all \( i = 1, \ldots, k \). Let
Taking into consideration all the results given above one can easily see that for $f^*_i$ the problem

$$
(x', y') \in \bigcap_{i=1}^k \text{ri}(\text{dom}(\tilde{f}_i))
$$

and it remains to demonstrate that $(x', y') \in \text{ri}(X \times \mathbb{R}^N) \cap \text{dom}(\tilde{h})$. But

$$
\text{dom}(\tilde{h}) = \left( \bigcap_{i=1}^k \text{dom}(F_i) \right) \bigcap \text{dom}(G) \times \mathbb{R}^N
$$

and from here the relation

$$
(X \times \mathbb{R}^N) \cap \text{dom}(\tilde{h}) = \left( X \cap \text{dom}(G) \right) \bigcap \left( \bigcap_{i=1}^k \text{dom}(F_i) \right) \times \mathbb{R}^N
$$

can be easily deduced. Since $x' \in \text{ri}(X \cap \text{dom}(G)) \cap \text{ri}(\text{dom}(F_1)) \cap \ldots \cap \text{ri}(\text{dom}(F_k))$, by Theorem 1.2 we have $x' \in \text{ri}(X \cap \text{dom}(G) \cap \text{dom}(F_1) \cap \ldots \cap \text{dom}(F_k))$. Combining these results we acquire

$$
(x', y') \in \text{ri}(X \times \mathbb{R}^N) \cap \text{dom}(\tilde{h})
$$

and the fulfillment of the condition $(\tilde{V}C^O)$ is proved.

As the tuple $(\tilde{x}, \tilde{y}) = (F_1(\tilde{x}), \ldots, F_k(\tilde{x}))$, is a weakly efficient solution of the problem $(\tilde{V}P^O)$, according to Theorem 5.11 there exists $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_k)^T \in \mathbb{R}^k \setminus \{0\}$, $(\tilde{y}_1, \ldots, \tilde{y}_k) \in \mathbb{R}^n \times \mathbb{R}^N$, $i \in I_{\tilde{x}} = \{i \in \{1, \ldots, k\} : \tilde{x}_i > 0\}$, and $\tilde{\beta} \in \tilde{K}^*$ such that the conditions $(\tilde{x}^O) - (\tilde{ii}^O)$ are satisfied. Take an arbitrary $i \in I_{\tilde{x}}$. As $\tilde{y}_i \in \mathbb{R}^N$ if and only if $\tilde{y}_i = (\tilde{y}_{i,1}, \ldots, \tilde{y}_{i,k})$ with $\tilde{y}_{i,j} \in \mathbb{R}^n$, $j = 1, \ldots, k$, making use of the definition of the conjugate function we get

$$
\tilde{f}^*_i (\tilde{x}_i, \tilde{y}_i) = \sup_{(x,y) \in \mathbb{R}^n \times \mathbb{R}^N} \left\{ \tilde{x}_i^T x + \tilde{y}_i^T y - \tilde{f}_i(x,y) \right\}
$$

$$
= \sup_{x \in \mathbb{R}^n, y_j \in \mathbb{R}^n, j=1,\ldots,k} \left\{ \tilde{x}_i^T x + \sum_{j=1}^k \tilde{y}_{i,j}^T y_j - f_i(y_j) \right\}
$$

$$
= \sup_{x \in \mathbb{R}^n} \left\{ \tilde{x}_i^T x \right\} + \sup_{y_j \in \mathbb{R}^n, j=1,\ldots,k} \left\{ \tilde{y}_{i,j}^T y_j - f_i(y_j) \right\} + \sum_{j=1, j \neq i}^k \left\{ \tilde{y}_{i,j}^T y_j \right\}
$$

Therefore it holds

$$
\tilde{f}^*_i (\tilde{x}_i, \tilde{y}_i) = \left\{ \begin{array}{cl} f_i^*(\tilde{y}_{i,i}), & \tilde{x}_i = 0 \text{ and } \tilde{y}_{i,j} = 0, j = 1, \ldots, k, j \neq i, \\ +\infty, & \text{otherwise.} \end{array} \right.
$$

Moreover, since the function $f_i$ is $K_i$ - increasing, Theorem 1.5 implies $\tilde{y}_{i,i} \in K_i^*$. Taking into consideration all the results given above one can easily see that for $\beta_i = \tilde{y}_{i,i}$ the relation

$$
\tilde{f}^*_i (\tilde{x}_i, \tilde{y}_i) + \tilde{f}_i(\tilde{x}, \tilde{y}) = \tilde{x}_i^T \tilde{x} + \tilde{y}_{i,i}^T \tilde{y}
$$
5.4 THE COMPOSED MULTIOBJECTIVE PROBLEM

becomes

\[ f_i^*(\overline{\beta}) + (f_i \circ F_i)(\overline{\alpha}) = \overline{\beta}_T F_i(\overline{x}). \]

As \( i \in I_{\overline{\beta}} \) was arbitrarily taken the relation \((i^C)\) is proved.

Because of the special form of the tuple \((\overline{x}_i^*, \overline{\gamma}_i^*)\) we acquire immediately

\[ \sum_{i \in I_{\overline{\beta}}} \overline{x}_i (\overline{x}_i^*, \overline{\gamma}_i^*) = (0, \overline{\gamma}) \]

with \( \overline{\gamma} = (\overline{\gamma}_1, \ldots, \overline{\gamma}_k) \), \( \overline{\gamma}_i = \overline{x}_i \beta_i \) if \( i \in I_{\overline{\beta}} \) and \( \overline{\gamma}_i = 0 \) otherwise. This implies further

\[ \sum_{i \in I_{\overline{\beta}}} \overline{x}_i (\overline{x}_i^T x + \overline{\gamma}_i^T y) = \sum_{i \in I_{\overline{\beta}}} \overline{\gamma}_i \beta_i F_i(\overline{x}). \]

Moreover, as \( \hat{K}^* = Q^* \times K_1^* \times \ldots \times K_k^* \), there exists \( \overline{\alpha} \in Q^* \) and \( \beta_i \in K_i^*, i = 1, \ldots, k, \) such that \( \beta = (\overline{\alpha}, \beta_1, \ldots, \beta_k) \). Therefore

\[ (\beta T \hat{h})^*_{X \times R^n} \left( -\sum_{i \in I_{\overline{\beta}}} \overline{x}_i (\overline{x}_i^*, \overline{\gamma}_i^*) \right) = (\beta T \hat{h})^*_{X \times R^n} (0, \overline{\gamma}) = \sup_{(x,y) \in X \times R^n} \left\{ \overline{\gamma}^T y \right\} \]

\[ = (\beta T \hat{h})(x,y) \]

\[ = \sup_{x \in X, y_i \in R^n} \left\{ \sum_{i=1}^k \overline{\gamma}_i^T y_i - \overline{\alpha}^T G(x) - \sum_{i=1}^k \beta_i^T (F_i(x) - y_i) \right\} \]

\[ = \left( \sum_{i=1}^k \beta_i^T F_i \right)^* \left( 0 \right) + \sum_{i=1}^k \sup_{y_i \in R^n} \left\{ (\overline{x}_i \beta_i - \beta_i T y_i) \right\} \]

Since it is binding to have \( \beta_i = \overline{x}_i \beta_i \) for all \( i = 1, \ldots, k, \) we get

\[ (\beta T \hat{h})^*_{X \times R^n} \left( -\sum_{i \in I_{\overline{\beta}}} \overline{x}_i (\overline{x}_i^*, \overline{\gamma}_i^*) \right) = \left( \sum_{i \in I_{\overline{\beta}}} \overline{x}_i \beta_i^T F_i \right)^* \left( 0 \right) \]

and, moreover, \((iii^O)\) implies

\[ 0 = \beta T \hat{h}(\overline{x}, \overline{\gamma}) = \overline{\alpha}^T G(\overline{x}) + \sum_{i \in I_{\overline{\beta}}} \overline{x}_i \beta_i F_i(\overline{x}). \]

Combining all the previous results with \((ii^O)\) we acquire

\[ \left( \overline{\alpha}^T G + \sum_{i \in I_{\overline{\beta}}} \overline{x}_i \beta_i^T F_i \right)^* \left( 0 \right) = -\sum_{i \in I_{\overline{\beta}}} \overline{x}_i \beta_i^T F_i(\overline{x}). \]

By Theorem 1.1 there exist \( x_i^* \in \mathbb{R}^n, i \in I_{\overline{\beta}}, \) such that

\[ \left( \overline{\alpha}^T G + \sum_{i \in I_{\overline{\beta}}} \overline{x}_i \beta_i^T F_i \right)^* \left( 0 \right) = \sum_{i \in I_{\overline{\beta}}} \left( \overline{x}_i \beta_i F_i(x_i^*) \right)^* \left( x_i^* \right) + (\overline{\alpha}^T G)^* \left( -\sum_{i \in I_{\overline{\beta}}} x_i^* \right). \]

Since for \( \overline{x}_i^T = \frac{1}{\beta_i} x_i^*, i \in I_{\overline{\beta}}, \) we have \( (\overline{x}_i \beta_i^T F_i)^*(x_i^*) = \overline{x}_i (\beta_i^T F_i^*)^*(\overline{x}_i^*) \) the previous two relations imply

\[ 0 = \sum_{i \in I_{\overline{\beta}}} \overline{x}_i (\beta_i^T F_i)^*(\overline{x}_i^*) + (\overline{\alpha}^T G)^* \left( -\sum_{i \in I_{\overline{\beta}}} \overline{x}_i \overline{x}_i^T \right) + \sum_{i \in I_{\overline{\beta}}} \overline{x}_i \beta_i^T F_i(\overline{x}) - \sum_{i \in I_{\overline{\beta}}} \overline{x}_i \overline{x}_i^T \overline{x} + (\overline{\alpha}^T G)^* \left( -\sum_{i \in I_{\overline{\beta}}} \overline{x}_i \overline{x}_i^T \right). \]
\[-\left(\sum_{i \in I} \bar{\lambda}_i x_i^* \right)^T \pi = \sum_{i \in I} \bar{\lambda}_i \left(\beta_i^T F_i(x_i^*) + \beta_i^T F_i(\pi) - x_i^T \pi\right)\]
\[+ \left(\pi^T G\right)^T \left(\sum_{i \in I} \lambda_i x_i^* \right) + \pi^T G(\pi) - \left(\sum_{i \in I} \lambda_i x_i^* \right)^T \pi + -\pi^T G(\pi).\]

Since \(G(\pi) \in -Q\) (\(x\) is a feasible solution of the problem \((VP^C)\)) and \(\pi \in Q^*\) it holds \(-\pi^T G(\pi) \geq 0\). As all the other terms within brackets are greater than or equal to 0, too, each term of the previous sum must be equal to 0 and the assertions \((ii^C)\), \((iii^C)\) and \((iv^C)\) are demonstrated.

(b) Following the same steps, but in reversed order, the conclusion is reached.

\(\square\)

Before going further we would like to mention that for \(F_i\), \(i = 1, \ldots, k\), and \(G\) everywhere finite, the previous theorem has been proved in [20]. Moreover, the constraint qualification we need coincides with the one imposed in the mentioned paper.

To the problem \((VP^C)\) we attach the vector dual problem

\((VD^C)\)

\[
\vmax_{(\lambda, q, x^*, \beta, t) \in B^C} \left( H_1(\lambda, \alpha, x^*, \beta, t), \ldots, H_k(\lambda, \alpha, x^*, \beta, t) \right)^T,
\]

where

\[
H_i(\lambda, \alpha, x^*, \beta, t) = -f_i^*(\beta_i) - (\beta_i^T F_i)^*(x_i^*) - \frac{1}{|\lambda|}(\alpha^T G) \left(\sum_{i \in I} \lambda_i x_i^*\right) + t_i,
\]

for all \(i = 1, \ldots, k\), and the dual variables are \(\lambda = (\lambda_1, \ldots, \lambda_k)^T \in \mathbb{R}^k\), \(\alpha \in \mathbb{R}^n\), \(x^* = (x_1^*, \ldots, x_k^*) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n\), \(\beta = (\beta_1, \ldots, \beta_k) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}\) and \(t = (t_1, \ldots, t_k)^T \in \mathbb{R}^k\). We let \(|\lambda| = \sum_{i=1}^k \lambda_i\) and the feasible set of the problem \((VD^C)\) is described by

\[
B^C = \left\{ (\lambda, \alpha, x^*, \beta, t) : \lambda \in \mathbb{R}^k_+, \{0\}, \alpha \in Q^*, \beta_i \in K_i^*, i = 1, \ldots, k, \sum_{i=1}^k \lambda_i t_i = 0 \right\}.
\]

The proofs of the next results are similar to the proofs of Theorem 5.6 and Theorem 5.7, and that is why we skip them (see also [20]).

**Theorem 5.16** (weak vector duality) There is no \(x \in A^C\) and no \((\lambda, \alpha, x^*, \beta, t) \in B^C\) such that \((f_i \circ F_i)(x) < H_i(\lambda, \alpha, x^*, \beta, t)\) for all \(i = 1, \ldots, k\).

**Theorem 5.17** (strong vector duality) Assume that \((VCQ^C)\) is fulfilled. If \(\pi \in A^C\) is a weakly efficient solution of the primal problem \((VP^C)\), then there exists \((\lambda, \bar{\alpha}, \bar{x}^*, \bar{\beta}, \bar{t}) \in B^C\) that is a weakly efficient solution to the dual problem \((VD^C)\) and for all \(i = 1, \ldots, k\), one has

\[(f_i \circ F_i)(\pi) = H_i(\lambda, \bar{\alpha}, \bar{x}^*, \bar{\beta}, \bar{t}).\]

Before going further we would like to mention that a vector dual problem can be attached to the problem \((VP^C)\) using the connection between the problems \((VP^C)\) and \((VP^C)\). Nevertheless, as the vector dual acquired by this approach is too complicated, we prefer to use the vector dual given before.
5.5 Special case: weak efficiency for convex ratios

In the following we treat the multiobjective program (see also [11, 82, 94])

\[(V_{PC}^{CR}) \quad \inf_{x \in X, h(x) \leq 0} \left( \frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_k(x)}{g_k(x)} \right)^T.\]

We assume that the functions \(f_i : \mathbb{R}^n \rightarrow \mathbb{R}\) and \(g_i : \mathbb{R}^n \rightarrow \mathbb{R}\), \(i = 1, \ldots, k\), are convex and respectively, concave, while the function \(h : \mathbb{R}^n \rightarrow \mathbb{R}^m\) is Q - convex, \(Q \subseteq \mathbb{R}^m\) a convex cone. Moreover, for all \(x \in A^{CR} = \{ x \in X : h(x) \leq 0 \}\) and for all \(i = 1, \ldots, k\), let \(f_i(x) \geq 0\) and \(g_i(x) > 0\). We introduce the functions

\[f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f(a, b) = \begin{cases} \frac{a^2}{b}, & a \geq 0, b > 0, \\ +\infty, & \text{otherwise}, \end{cases}\]

and

\[F_i : \mathbb{R}^n \rightarrow \mathbb{R}^2, \quad F_i(x) = (f_i(x), g_i(x)), \quad i = 1, \ldots, k.\]

Evidently the problem \((V_{PC}^{CR})\) can be equivalently rewritten as

\[(V_{PC}^{CR}) \quad \inf_{x \in X, h(x) \leq 0} \left( (f \circ F_1)(x), \ldots, (f \circ F_k)(x) \right)^T.\]

Even more, for \(K = \mathbb{R}_+ \times (-\mathbb{R}_+)\) the function \(f\) is \(K\) - increasing and \(F_i\) is \(K\) - convex, \(i = 1, \ldots, k\). Thus the problem \((V_{PC}^{CR})\) turns out to be a special instance of the more general problem \((V_{PC})\). The constraint qualification \((VCQ^C)\) becomes in this case

\[(VCQ^{CR}) \quad \exists x' \in \text{ri}(X) : \begin{cases} F_i(x') \in \text{ri}(\text{dom}(f)) - \text{ri}(K), \quad i = 1, \ldots, k, \\ h(x') \in -\text{ri}(Q). \end{cases}\]

But \(\text{ri}(\text{dom}(f_i)) - \text{ri}(K) = (0, +\infty) \times (0, +\infty) - (0, +\infty) \times (-\infty, 0) = \mathbb{R} \times (0, +\infty)\) and we finally get

\[(VCQ^{CR}) \quad \exists x' \in \text{ri}(X) : \begin{cases} g_i(x') < 0, \quad i = 1, \ldots, k, \\ h(x') \in -\text{ri}(Q). \end{cases}\]

Taking into consideration the fact that for an arbitrary \((u, v) \in K^* = \mathbb{R}_+ \times (-\mathbb{R}_+)\) we have \(f^*(u, v) = 0\) if \(u^2 \leq 4v\) and \(f^*(u, v) = +\infty\) otherwise, we are able to prove the next necessary and sufficient conditions for the weakly efficient solutions of the problem \((V_{PC}^{CR})\).

**Theorem 5.18** (optimality conditions) (a) Suppose that the condition \((VCQ^{CR})\) is fulfilled and let \(\pi \in A^{CR}\) be a weak efficient solution of the problem \((V_{PC}^{CR})\). Then there exist \(\lambda \in \mathbb{R}_+^k \setminus \{0\}, \pi \in Q^*, \pi^T \in \mathbb{R}^n\) and \(\pi_i \geq 0, \pi_i \leq 0, \pi_i^2 \leq -4\pi_i, \quad i \in \mathcal{I}_\pi\), such that

\[(i^{CR}) \quad \frac{f_i^*(\mu)}{g_i^*(\mu)} = \overline{\mu}_i f_i(\pi) + \overline{\mu}_i g_i(\pi), \quad i \in \mathcal{I}_\pi;\]

\[(ii^{CR}) \quad \overline{\pi}_i f_i^*(\overline{\mu}_i) + \overline{\pi}_i g_i(\pi) = \overline{\pi}_i \overline{u}_i^T \pi, \quad i \in \mathcal{I}_\pi;\]

\[(iii^{CR}) \quad \overline{\pi}_i g_i^*(\overline{\mu}_i) + \overline{\pi}_i g_i(\pi) = \overline{\pi}_i \overline{v}_i^T \pi, \quad i \in \mathcal{I}_\pi;\]

\[(iv^{CR}) \quad (\pi^T h(\pi))_{\lambda}^X \left( - \sum_{i \in \mathcal{I}_\pi} X_i (\overline{\pi}_i \overline{u}_i^T + \overline{\pi}_i \overline{v}_i^T) \right) + \pi^T h(\pi) = - \sum_{i \in \mathcal{I}_\pi} X_i (\overline{\pi}_i \overline{u}_i^T \pi + \overline{\pi}_i \overline{v}_i^T \pi);\]

\[(v^{CR}) \quad \pi^T h(\pi) = 0.\]
(b) If there exists \( v \in \mathbb{R}^n \) such that for some \( x \in \mathbb{R}^k \setminus \{0\}, \pi \in \mathbb{Q}^+, \pi^T \in \mathbb{R}^n \) and \( \pi_i \geq 0, \pi_i \leq 0, \pi_i^2 \leq 4\pi_i, i \in I_\pi \), the conditions \((i^{CR}) - (iv^{CR})\) are satisfied, then \( \pi \) is a weakly efficient solution of the multiobjective problem \((VP^{CR})\).

**Proof.** (a) By Theorem 5.15 there exist \( x \in \mathbb{R}^k \setminus \{0\}, \pi \in \mathbb{Q}^+, \pi^T \in \mathbb{R}^n \) and \( \pi_i \in K^*, i \in I_\pi \), such that the conditions \((i^{C}) - (iv^{C})\) are fulfilled. The way we define the cone \( K \) implies \( \pi_i \in K^* \) if and only if \( \pi_i = (\pi_i, v_i) \) with \( \pi_i \geq 0 \) and \( \pi_i \leq 0 \), \( i = 1, \ldots, k \). Moreover, it cannot hold \( f^*(\pi_i) = \tilde{f}^*(\pi_i) = +\infty \) (the assertion \((i^{C})\) is not fulfilled in this case), so that \( \pi_i^2 \leq 4\pi_i \). Thus \( f^*(\pi_i, v_i) = 0 \) and, as \( (f \circ \tilde{f}^*) = \frac{f^*(\pi_i)}{\pi_i} \), assertion \((i^{CR})\) can be easily deduced from \((i^{C})\).

Take an arbitrary \( i \in I_\pi \). Since the hypotheses of Theorem 1.1 are fulfilled, there exist \( x_i, p_i \in \mathbb{R}^n \) such that \( x_i = x_i^* + p_i^* \) and \( (\pi_i f_i + \pi_i g_i)^*(\pi_i^T) = (\pi_i f_i)(x_i^*) + (\pi_i g_i)(p_i^*) \). Combining this with assertion \((iv^{C})\) we acquire

\[
(\pi_i f_i)^*(x_i^*) + (\pi_i g_i)^*(p_i^*) + \pi_i f_i(\pi) + \pi_i g_i(\pi) = x_i^T \pi + p_i^T \pi.
\]

This last relation can be equivalently rewritten as

\[
0 = [(\pi_i f_i)^*(x_i^*) + \pi_i f_i(\pi) - x_i^T \pi] + [(\pi_i g_i)^*(p_i^*) + \pi_i g_i(\pi) - p_i^T \pi].
\]

But both terms in the previous sum are non-negative so that it is binding to have

\[
(\pi_i f_i)^*(x_i^*) + \pi_i f_i(\pi) - x_i^T \pi = 0 \quad \text{and} \quad (\pi_i g_i)^*(p_i^*) + \pi_i g_i(\pi) - p_i^T \pi = 0.
\]

Our next step is to factor out the constant \( \pi_i \) and to this aim two cases are treated. If \( \pi_i = 0 \) then \( x_i^* = 0 \) (otherwise \( (\pi_i f_i)^*(x_i^*) = +\infty \)) and let \( \overline{u}_i = 0 \). Evidently

\[
(\pi_i f_i)^*(x_i^*) + \pi_i f_i(\pi) - x_i^T \pi = x_i^T \pi = 0.
\]

If \( \pi_i \neq 0 \) for \( \pi_i \) we have \( (\pi_i f_i)^*(x_i^*) = \pi_i f_i(\pi_i) = 0 \) and \( x_i^* = \overline{u}_i \pi_i \). If \( \pi_i \neq 0 \) for \( \pi_i \) we have \( (\pi_i f_i)^*(x_i^*) = \pi_i f_i(\pi_i) = 0 \) (see relation (1.5)). Taking into consideration these facts we get further

\[
(\pi_i f_i)^*(x_i^*) + \pi_i f_i(\pi) - x_i^T \pi = \pi_i f_i(\pi_i) + \pi_i f_i(\pi) - \pi_i \overline{u}_i^T \pi = 0.
\]

Following a similar reasoning we can find \( \overline{v}_i \in \mathbb{R}^n \) such that \( \overline{v}_i g_i(\overline{v}_i) + \pi_i g_i(\pi - \pi_i \overline{v}_i^T \pi = 0\) and, moreover, it holds \( p_i^* = \pi_i \overline{v}_i \). Since no assumption was made regarding \( i \in I_\pi \) the assertions \((ii^{CR})\) and \((iii^{CR})\) follow immediately. Even more, as

\[
\sum_{i \in I_\pi} \lambda_i x_i = \sum_{i \in I_\pi} \lambda_i (\pi_i \overline{u}_i^T + \pi_i \overline{v}_i^T)
\]

the assertion \((iv^{CR})\) can be easily deduced from \((iii^{C})\) and the theorem is demonstrated.

Further to the problem \((VP^{CR})\) we attach the vector dual problem

\[
(VD^{CR}) \quad \text{v-max}_{(\lambda, \alpha, u^*, v^*, u, v, t) \in \mathbb{R}^n} H_1(\lambda, \alpha, u^*, v^*, u, v, t), \ldots, H_k(\lambda, \alpha, u^*, v^*, u, v, t)
\]

where

\[
H_i(\lambda, \alpha, u^*, v^*, u, v, t) = -u_i f_i(u_i^*) + v_i g_i(v_i^*) - \frac{1}{|\lambda|} (\alpha^T h)(\sum_{i \in I_\lambda} \lambda_i (u_i u_i^* + v_i v_i^*)) + t_i
\]

for all \( i = 1, \ldots, k \), and the dual variables are \( \lambda = (\lambda_1, \ldots, \lambda_k)^T \in \mathbb{R}^k, \alpha \in \mathbb{R}^m, u^* = (u_1^*, \ldots, u_k^*) \in \mathbb{R}^n \times \ldots \times \mathbb{R}^n, v^* = (v_1^*, \ldots, v_k^*) \in \mathbb{R}^n \times \ldots \times \mathbb{R}^n, u = (u_1, \ldots, u_k) \in \mathbb{R}^k, u = (u_1, \ldots, u_k) \in \mathbb{R}^k, \)
5.5 CONVEX RATIOS

$v = (v_1, \ldots, v_k) \in \mathbb{R}^k$ and $t = (t_1, \ldots, t_k)^T \in \mathbb{R}^k$. The feasible set of the problem $(VD^{CR})$ is described by

$$B^{CR} = \left\{ (\lambda, \alpha, u^*, v^*, u, v, t) : \lambda \in \mathbb{R}^+_0 \setminus \{0\}, v \in Q^*, q \in Q^*, \sum_{i=1}^{k} \lambda_i t_i = 0, u_i^* v_i^* \in \mathbb{R}^n, u_i \geq 0, v_i \leq 0, u_i^2 \leq -4 v_i, i = 1, \ldots, k \right\}.$$

**Theorem 5.19** (weak vector duality) There is no $x \in A^{CR}$ and no $(\lambda, \alpha, u^*, v^*, u, v, t) \in B^{CR}$ such that $f(x) \leq H_i(\lambda, \alpha, u^*, v^*, u, v, t)$ for all $i = 1, \ldots, k$.

**Theorem 5.20** (strong vector duality) Assume that $(VCQ^{CR})$ is fulfilled. If $\pi \in A^{CR}$ is a weakly efficient solution of the primal problem $(VP^{CR})$, then there exists $(\lambda, \pi, x^0, v^0, \pi, \tilde{v}, t) \in B^{CR}$ that is a weakly efficient solution to the dual problem $(VD^{CR}$ and for all $i = 1, \ldots, k$, one has

$$\frac{f_i(x)}{g_i(\pi)} = H_i(\lambda, \pi, x^0, v^0, \pi, \tilde{v}, t).$$

As a special case of the problem $(VP^{CR})$ we consider the multiobjective quadratic - linear fractional programming problem

$$(VP^{QL}) \quad \min_{x \in X} \quad \frac{x^T B_i x}{a_i^T x + b_i},$$

where $B_i$ is a symmetric positive definite $n \times n$ matrix with real entries, $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$, $i = 1, \ldots, k$, and $C$ and $c$ are $m \times n$ real matrix and a $m$ - dimensional vector, respectively. Obviously the functions

$$f_i : \mathbb{R}^n \rightarrow \mathbb{R}, \quad f_i(x) = (x^T Q_i x)^{\frac{1}{2}}$$

and

$$g_i : \mathbb{R}^n \rightarrow \mathbb{R}, \quad g_i(x) = a_i^T x + b_i$$

are convex for each $i = 1, \ldots, k$. Moreover, for $b = (b_1, \ldots, b_k)^T$ we take

$$h : \mathbb{R}^n \rightarrow \mathbb{R}^m, \quad h(x) = C x - b$$

and we suppose that $g_i(x) > 0$ for each $x$ feasible to $(VP^{QL}), i = 1, \ldots, k$. Moreover, as in this case we deal with the cone $\mathbb{R}^m_+$, instead of the constraint qualification $(VCQ^{CR})$ we consider the weaker one

$$(VCQ^{QL}) \quad \exists x' \in ri(X) : g_i(x') < 0, i = 1, \ldots, k.$$ 

Taking into consideration Lemma 2.2 and the fact that

$$g_i^*(u_i^*) = \begin{cases} -b_i, & v_i^* = a_i, \\ +\infty, & \text{otherwise,} \end{cases} \quad \text{and} \quad (\alpha^T h)^*(x^*) = \begin{cases} \alpha^T c, & x^* = C^* \alpha, \\ +\infty, & \text{otherwise,} \end{cases}$$

the subsequent theorem can be easily deduced from Theorem 5.18.

**Theorem 5.21** (optimality conditions) (a) Suppose that the condition $(VCQ^{QL})$ is fulfilled and let $\pi \in \mathbb{R}^n$ be a weakly efficient solution of the problem $(VP^{QL})$. Then there exist $\lambda \in \mathbb{R}^+_0 \setminus \{0\}, \alpha \in \mathbb{R}^n, \bar{v}^0 \in \mathbb{R}^n, \bar{v}^0 T B_i^* \bar{v}^0 \leq 1$, and $\bar{\rho} \geq 0, \bar{\rho} \leq 0, \bar{\rho}^2 \leq -4 \bar{v}^0_i, i \in \mathbb{R}^n_0$, such that
(i) If there exists \( \bar{\lambda} \in \mathbb{R}^n \) such that for some \( \overline{\lambda} \in \mathbb{R}^n \setminus \{0\} \), \( \bar{\lambda} \in \mathbb{Q}^* \), \( \overline{\lambda} \in \mathbb{R}^n \), \( \overline{\lambda}^T B \bar{\lambda} \leq 1 \) and \( \overline{\lambda} \geq 0 \), \( \overline{\lambda}^2 \leq -4\overline{\lambda} \), \( \bar{\lambda} \in I_X \), the conditions \((i^{CR})\) - \((iv^{CR})\) are satisfied, then \( \bar{\lambda} \) is a weakly efficient solution of the primal problem \((V P^{QL})\).

Further to the problem \((V P^{QL})\) we attach the vector dual problem

\[(V D^{QL})\] \begin{align*} \text{v-max}_{(\lambda, x, w, u, v, t) \in B^{QL}} & \left( H_1(\lambda, x, w, u, v, t), \ldots, H_k(\lambda, x, w, u, v, t) \right)^T, \end{align*}

where

\[ H_i(\lambda, x, w, u, v, t) = -v_i b_i - \frac{1}{|\lambda|} \alpha^T c + t_i, \]

for all \( i = 1, \ldots, k \), and the dual variables are \( \lambda = (\lambda_1, \ldots, \lambda_k)^T \in \mathbb{R}^k \), \( \alpha \in \mathbb{R}^m \), \( w = (w_1, \ldots, w_k) \in \mathbb{R}^n \times \ldots \times \mathbb{R}^n \), \( u = (u_1, \ldots, u_k) \in \mathbb{R}^k \), \( v = (v_1, \ldots, v_k) \in \mathbb{R}^k \) and \( t = (t_1, \ldots, t_k)^T \in \mathbb{R}^k \). The feasible set of the problem \((V D^{QL})\) is described by

\[ B^{QL} = \left\{ (\lambda, x, w, u, v, t) : \lambda \in \mathbb{R}^k_+ \setminus \{0\}, \alpha \in \mathbb{R}^m_+, C^* \alpha = -\sum_{i=1}^k \lambda_i (u_i B_i w_i + v_i a_i), \right\} \]

\[ \sum_{i=1}^k \lambda_i t_i = 0, w_i \in \mathbb{R}^n, u_i \geq 0, v_i \leq 0, u_i^2 \leq -4v_i, i = 1, \ldots, k \}

Theorem 5.22 (weak vector duality) There is no \( x \in A^{QL} \) and no \( (\lambda, x, w, u, v, t) \in B^{QL} \) such that

\[ \frac{\bar{x}^T B \bar{x}}{a^T \bar{x} + b_i} < H_i(\lambda, x, w, u, v, t) \]

for all \( i = 1, \ldots, k \).

Theorem 5.23 (strong vector duality) Assume that \((V C^{QL})\) is fulfilled. If \( \bar{x} \in \mathbb{A}^{QL} \) is a weakly efficient solution of the primal problem \((V P^{QL})\), then there exists \( (\bar{\lambda}, \bar{\pi}, \bar{\alpha}, \bar{\nu}, \bar{\tau}, \bar{\beta}) \in B^{QL} \) that is a weakly efficient solution to the dual problem \((V D^{QL})\) and for all \( i = 1, \ldots, k \), one has

\[ \frac{\bar{x}^T B \bar{x}}{a^T \bar{x} + b_i} = H_i(\bar{\lambda}, \bar{\pi}, \bar{\alpha}, \bar{\nu}, \bar{\tau}, \bar{\beta}). \]
Theses

(1) We consider the general optimization problem

\[(P) \quad \inf_{x \in \mathbb{R}^n} \left\{ f(x) + (g \circ h)(x) \right\}, \]

where \( K \subseteq \mathbb{R}^k \) is a convex cone, the functions \( f : \mathbb{R}^n \to \mathbb{R}, g : \mathbb{R}^k \to \mathbb{R} \) and \( h : \mathbb{R}^n \to \mathbb{R}^k \) are taken such that \( f \) is proper and convex, \( g \) is proper, convex and \( K \)-increasing and \( h \) is \( K \)-convex. Moreover, we impose the feasibility condition \( \text{dom}(f) \cap \text{dom}(h) \cap h^{-1}(\text{dom}(g)) \neq \emptyset \). To the problem \((P)\) we attach the Fenchel - Lagrange - type dual problem

\[(D) \quad \sup_{x^* \in \mathbb{R}^n, \beta \in \mathbb{K}^*} \left\{ -g^*(\beta) - f^*(x^*) - (\beta^T h)^*(-x^*) \right\} \]

and, after giving a sufficient condition for strong duality, we deliver optimality conditions. A Farkas - type result is proved, too, and one of its statements is characterized using the epigraphs of the functions involved (see [17]).

(2) The problems

\[(P^A) \quad \inf_{x \in \mathbb{R}^n} \left\{ f(x) + g(Ax) \right\}, \]

and

\[(P^D) \quad \inf_{x \in X, h(x) \leq K_0} f(x) \]

are treated as special instances of the general problem \((P)\). Thus for each of these two problems a dual problem is derived and, moreover, a condition which secures strong duality and then necessary and sufficient optimality conditions are provided. Further a Farkas - type result is given and one of its statements is characterized by using epigraphs of the functions involved.

(3) Using an approach similar to the one used for the problem \((P)\) we are able to provide the Fenchel - Lagrange dual problem for the problem

\[(P^C) \quad \inf_{x \in X, G(x) \leq Q_0} (f \circ F)(x). \]

Moreover, we prove that strong duality holds provided that a sufficient condition is fulfilled. Further we deliver optimality conditions and a Farkas - type
result. Finally, a statement of the previous mentioned Farkas - type result is characterized using only the epigraphs of the functions involved.

(4) We deal with some special instances of the problems \((P)\) and \((P^C)\). Firstly an optimization problem with quadratic functions is treated as a special instance of \((P)\). Moreover, some existent results involving quadratic functions are generalized by two theorems we give. Secondly, we show that the min - max optimization problems is actually a special instance of the problem \((P^C)\). Finally, some theorems of the alternative with vector valued functions are provided.

(5) We treat the DC programming problem

\[
(P_{DC}) \inf_{x \in X} \left\{ g(x) - h(x) \right\},
\]

where \(X \subseteq \mathbb{R}^n\) is a non - empty and convex set, \(g, h, g_i, h_i : \mathbb{R}^n \to \mathbb{R}\) are proper and convex functions, \(i = 1, \ldots, m\). Using an approach based on an idea of MARTÍNEZ-LEGAZ AND VOLLE (see [71]), to the problem \((P_{DC})\) we attach the following dual problem

\[
(D_{DC}) \inf_{x^* \in \text{dom}(h^*)} \sup_{y^* \in \Pi} \left\{ h^*(x^*) + \sum_{i=1}^{m} \beta_i h_i^*(y_i^*) - g^*(p) \right\},
\]

where \(\Pi\) is a K - convex function.

After presenting a constraint qualification which secures strong duality between the problems \((P_{DC})\) and \((D_{DC})\) we deliver necessary and sufficient optimality conditions. Moreover, a Farkas - type results involving DC functions is proved, as well as a characterization with epigraphs (see [19]). Three special instances of the problem \((P_{DC})\) are also considered, namely the problem with DC objective function and finitely many convex constraints, the problem with convex objective function and finitely many DC constraints and the ordinary convex optimization problem.

(6) The next non - convex programming problem we work with is of fractional type. More precisely, we examine the problem

\[
(P^F) \inf_{x \in X, h(x) \leq \kappa 0} \left\{ \frac{f(x)}{g(x)} \right\},
\]

with \(K \subseteq \mathbb{R}^m\) non - empty convex cone, \(f : \mathbb{R}^n \to \mathbb{R}\) proper and convex function, \(g : \mathbb{R}^n \to \mathbb{R}\) concave function such that \(-g\) is proper and \(g(x) > 0\) for all \(x\) feasible to the problem \((P^F)\) and \(h : \mathbb{R}^n \to \mathbb{R}^m\) a K - convex function. Using an approach due to DINKELBACH (see [40]) to the problem \((P^F)\) we attach the problem

\[
(P_{\lambda}^F) \inf_{x \in X, h(x) \leq \kappa 0} \left\{ f(x) - \lambda g(x) \right\}.
\]

Since the objective function of the problem is a convex function for \(\lambda \geq 0\) and a DC function for \(\lambda < 0\), two cases are considered. In both cases the Fenchel
- Lagrange dual problem of \((P^C_{\lambda})\) is determined and weak and strong duality
assertions are delivered. These assertions are used in order to prove optimality
conditions and also to give a Farkas-type result which involves fractions.
Moreover, a characterization with epigraphs is given, too (see also [21]).

(7) For \(i = 1, \ldots, k\), let the functions \(f_i : \mathbb{R}^n \to \mathbb{R}\), \(g_i : \mathbb{R}^{n_i} \to \mathbb{R}\)
and \(h_i : \mathbb{R}^n \to \mathbb{R}^{n_i}\) be such that \(f_i\) is a convex function, \(g_i\) is a proper, convex and \(K_i\)-increasing
function, while \(h_i\) is a proper and \(K_i\)-convex one. We treat the multiobjective
programming problem

\[
(VP) \quad \text{v-min}_{x \in \mathbb{R}^n} \left( f_1(x) + (g_1 \circ h_1)(x), \ldots, f_k(x) + (g_k \circ h_k)(x) \right)^T.
\]

To the problem \((VP)\) we attach the scalarized problem

\[
(VP_{\lambda}) \quad \text{inf}_{x \in \mathbb{R}^n} \left\{ \sum_{i=1}^k \lambda_i (f_i(x) + (g_i \circ h_i)(x)) \right\}
\]

and for all \(\lambda = (\lambda_1, \ldots, \lambda_k)^T \in \mathbb{R}_+^k \setminus \{0\}\) the Fenchel-Lagrange dual problem
\((VD_{\lambda})\) of \((VP_{\lambda})\) is determined. After establishing weak and strong duality
assertions between the problems \((VP_{\lambda})\) and \((VD_{\lambda})\) necessary and sufficient
conditions for the weakly efficient solutions of the problem \((VP_{\lambda})\) are given.
Moreover, inspired by the form of the dual problem \((VD_{\lambda})\), to \((VP)\) we at-
tach a vector dual problem and weak and strong vector duality assertions are
proved, too. Some special instances of the problem \((VP)\) are considered, like
the classical vector programming problem.

(8) The multiobjective programming problem

\[
(VP^C) \quad \text{v-min}_{x \in X, \ G(x) \leq Q^0} \left( (f_1 \circ F_1)(x), \ldots, (f_k \circ F_k)(x) \right)^T
\]

is considered, too. Also in this case we give necessary and sufficient conditions
for the weakly efficient solutions. After attaching a vector dual problem to
\((VP^C)\) weak and strong vector duality assertions are proved.

(9) Finally, the problem

\[
(VP^{CR}_C) \quad \text{inf}_{x \in X, \ h(x) \leq Q^0} \left( \frac{f_1^2(x)}{g_1(x)}, \ldots, \frac{f_k^2(x)}{g_k(x)} \right)^T
\]

is considered (one can prove that this problem is a special instance of the
problem \((VP^C)\)). After we characterize its weakly efficient solutions we attach
to it a vector dual and weak and strong vector duality assertions are given,
too.
Index of notation

\( \mathbb{R} \) the set of real numbers
\( \mathbb{R}^m \) the extended set of real numbers
\( \mathbb{R}^m_+ \) the non-negative orthant of \( \mathbb{R}^m \)
\( \leq \) the partial ordering introduced on \( \mathbb{R}^m \) by \( \mathbb{R}^m_+ \)
\( K^* \) the dual cone of the cone \( K \)
\( \leq_K \) the partial ordering induced on \( \mathbb{R}^m \) by \( K \)
\( \infty_{\mathbb{R}^m} \) a maximal element with respect to \( K \)
\( \mathbb{R}^m_{\infty} \) the set \( \mathbb{R}^m \cup \infty_{\mathbb{R}^m} \)
\( \text{int}(X) \) the interior of the set \( X \)
\( \text{ri}(X) \) the relative interior of the set \( X \)
\( x^T y \) the inner product of the vectors \( x \) and \( y \)
\( \text{dom}(f) \) the domain of the function \( f \)
\( \text{epi}(f) \) the epigraph of the function \( f \)
\( f^* \) the conjugate of the function \( f \)
\( f^*_X \) the conjugate of the function \( f \) regarding the set \( X \)
\( h_X \) the restriction of the function \( h \) to the set \( X \)
\( \partial f \) the subdifferential of the function \( f \)
\( A^* \) the adjoint of the linear transformation \( A \)
\( \delta_X \) the indicator function of the set \( X \)
\( \sigma_X \) the support function of the set \( X \)
\( v(P) \) the optimal objective value of the optimization problem \( (P) \)
\( v\text{-min} \) the notation for a multiobjective optimization problem in the sense of minimum
\( v\text{-max} \) the notation for a multiobjective optimization problem in the sense of maximum
Bibliography


BIBLIOGRAPHY


Lebenslauf

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Erklärung gemäß §6 der Promotionsordnung

Hiermit erkläre ich an Eides Statt, dass ich die von mir eingereichte Arbeit “Farkas-type results for convex and non-convex inequality systems” selbstständig und nur unter Benutzung der in der Arbeit angegebenen Quellen und Hilfsmittel angefertigt habe.

Chemnitz, den 01.10.2007

Ioan Bogdan Hodrea