Realization of source conditions for linear ill-posed problems by conditional stability

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Abstract

We prove some sufficient conditions for obtaining convergence rates in regularization of linear ill-posed problems in a Hilbert space setting and show that these conditions are directly related with the conditional stability in several concrete inverse problems for partial differential equations.

\textit{Keywords:} linear ill-posed problems, regularization, source conditions, inverse PDE problems, conditional stability, operator monotonicity, Löwner's theorem

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1 Introduction

Let $X$ and $Y$ be infinite dimensional separable Hilbert spaces, where the symbol $\| \cdot \|$ denotes the generic norms in both spaces as well as associated operator norms. Moreover $(\cdot, \cdot)$ denotes the inner product. We are going to study ill-posed linear operator equations

$$Ax = y \quad (x \in X, \ y \in Y) \quad (1.1)$$

with injective and bounded linear operators $A : X \to Y$ having a non-closed range $\mathcal{R}(A)$. In this paper, we focus on the special case that the operator $A$ is compact. Then the solution $x_0 \in X$ of (1.1) is uniquely determined for elements $y \in \mathcal{R}(A)$. For nearly all applications the element $y$ itself is unknown, but noisy data $y^\delta \in Y$ of form

$$y^\delta = Ax_0 + \delta \xi, \quad \| \xi \| \leq 1,$$

with

$$\| y^\delta - y \| \leq \delta$$

and some noise level $\delta > 0$ are available. In this context $\xi$ can be considered to be a normalized random element, which is not known. In order to find the solution $x_0$ from data $y^\delta$ in a stable manner, regularization methods are required. We will focus here on general linear regularization methods generated by piecewise continuous functions

$$h_\gamma(t) \quad (0 < t \leq a := \| A^* A \|, \ 0 < \gamma \leq \overline{\gamma} \leq a).$$

We are going to distinguish regularized solutions

$$x_\gamma = h_\gamma(A^* A)^* y$$

in the case of noise-free data and

$$x_\delta^\gamma = h_\gamma(A^* A)^* y^\delta$$

in the case of noisy data. For fixed $A$ and $x_0$ the regularization error of the noise-free case as a function of the regularization parameter $\gamma > 0$ can be written as

$$\| x_\gamma - x_0 \| = \| (h_\gamma(A^* A)^* A - I)x_0 \| = \| r_{\gamma}(A^* A)x_0 \|, \quad (1.2)$$

where

$$r_{\gamma}(t) := 1 - t h_{\gamma}(t) \quad (0 < t \leq a)$$

is the residual function of the regularization method. As obvious in regularization theory (cf. [3], [5] and [9]) we pose the following assumption:

Assumption 1.1 There exist two constants $C_1, C_2 > 0$ such that for all $0 < t \leq a$

$$\begin{align*}
(i) \quad & \lim_{\gamma \to 0} r_{\gamma}(t) = 0, \\
(ii) \quad & |r_{\gamma}(t)| \leq C_1 \quad (0 < \gamma \leq \overline{\gamma}); \\
(iii) \quad & \sqrt{t} |h_{\gamma}(t)| \leq \frac{C_2}{\sqrt{\gamma}} \quad (0 < \gamma \leq \overline{\gamma}).
\end{align*}$$


Example 1.2 The most prominent regularization method is classical Tikhonov regularization with generator function 
\[ h_\gamma(t) = \frac{1}{t + \gamma} \] and residual function 
\[ r_\gamma(t) = \frac{\gamma}{t + \gamma} . \]
This method satisfies Assumption 1.1 with \( C_1 = 1 \) and \( C_2 = 1/2 \).

The requirements (i) and (ii) of Assumption 1.1 ensure based on the noise-free error formula (1.2) the convergence 
\[ \| x_\gamma - x_0 \| \to 0 \text{ as } \gamma \to 0 , \]
but this convergence depends on properties of \( x_0 \) and can be arbitrarily slow. Taking into account the noise level \( \delta > 0 \) the total error of regularization can be estimated by the triangle inequality as 
\[ \| x_\gamma^\delta - x_0 \| \leq \| x_\gamma - x_0 \| + \| x_\gamma^\delta - x_\gamma \| , \]
and by the requirement (iii) of Assumption 1.1 further as 
\[ \| x_\gamma^\delta - x_0 \| \leq \| x_\gamma - x_0 \| + \frac{C_2 \delta}{\sqrt{\gamma}} \quad (0 < \gamma \leq \gamma) . \]

2 General source conditions

Index functions (cf. [9]) play an important role in our theory.

Definition 2.1 We call \( \eta(t) \) \( (0 \leq t \leq \tilde{t}) \) an index function if \( \eta(0) = 0 \) and this function is continuous and strictly increasing.

To obtain convergence rates for the regularization method \( h_\gamma \) general source conditions
\[ x_0 = \eta(A^*A)w \quad (w \in X) \tag{2.1} \]
with index functions \( \eta(t) \) \( (0 \leq t \leq a) \) have to be used. Based on (2.1) we then have from spectral theory
\[ \| x_\gamma - x_0 \| = \| r_\gamma(A^*A) \eta(A^*A)w \| \leq \left( \sup_{0 < t \leq a} |r_\gamma(t)| \eta(t) \right) \| w \| . \tag{2.2} \]

This can be estimated further from above if we follow the ideas of Mathé and Pereverzev (see [8] and [9]) to consider the qualification of a regularization method to be an index function.

Definition 2.2 An index function \( \eta(t) \) \( (0 < t \leq a) \) is called a qualification with constant \( 1 \leq C_3 < \infty \) of the regularization method \( h_\gamma \) if
\[ \sup_{0 < t \leq a} |r_\gamma(t)| \eta(t) \leq C_3 \eta(\gamma) \quad (0 < \gamma \leq \gamma) . \]

Then from formula (2.2) we immediately obtain the following proposition.
Proposition 2.3 Let $x_0$ satisfy the source condition (2.1) and let the index function $\eta$ be a qualification with constant $1 \leq C_3 < \infty$ of the regularization method $h_\gamma$. Then
\[
\|x_\gamma - x_0\| \leq C_3 \eta(\gamma) \|w\| \quad (0 < \gamma \leq \bar{\gamma})
\]
and hence
\[
\|x_\delta - x_0\| \leq C_3 \eta(\gamma) \|w\| + \frac{C_2 \delta}{\sqrt{\gamma}} \quad (0 < \gamma \leq \bar{\gamma}).
\]
As is well-known (see [9]) by balancing the two terms in the bound of (2.3) for sufficiently small $\delta > 0$ we find a constant $K > 0$ such that
\[
\|x_\delta - x_0\| \leq K \eta(\Theta^{-1}(\delta)) \quad (0 < \delta \leq \bar{\delta}),
\]
where with $\eta$ also
\[
\Theta(\gamma) := \sqrt{\gamma} \eta(\gamma) \quad (0 < \gamma \leq \bar{\gamma})
\]
is an index function and the regularization parameter is chosen a priori as $\gamma(\delta) := \Theta^{-1}(\delta)$. Under weak additional assumptions the function $\eta(\Theta^{-1}(\delta))$ characterizes an order optimal convergence rate.

In particular for the Tikhonov regularization from the literature (see, e.g., [1]) we get a variety of sufficient conditions that characterize qualifications and therefore ensure estimates (2.3) and (2.4).

Proposition 2.4 Let $\eta(t)$ ($0 \leq t \leq a$) be an index function. If (a) $\eta(t)/t$ is monotonically decreasing on $(0, a]$, or (b) $\eta(t)$ is concave on $[0, a]$, then $\eta(t)$ is a qualification with constant $C_3 = 1$ of Tikhonov regularization. If there exists a real number $\hat{t} \in (0, a]$ such that (c) $\eta(t)/t$ is monotonically decreasing on $(0, \hat{t}]$ or (d) $\eta(t)$ is concave on $[0, \hat{t}]$, then the same is true, but with the constant $C_3 = \eta(a)/\eta(\hat{t})$.

Note that any function $\eta(t) = t^{\nu}$ with exponent $0 < \nu \leq 2$ is concave and hence a qualification of Tikhonov’s method (see Example 1.2) with constant $C_3 = 1$.

3 Link conditions to solution smoothness

To get uniform convergence rates in regularization for a wider class of elements $x_0$ one can suppose that
\[
x_0 = Gw \quad (w \in X)
\]
for some given linear compact and self-adjoint operator $G : X \to X$, which we assume to be strictly positive, that is, all eigenvalues of $G$ are positive values. In principle, the operators $A$ and $G$ can be independent, but for the special case $G = \varphi((A^*A)^{\frac{1}{2}})$ with some index function $\varphi(t)$ ($0 \leq t \leq a$) we have a general source condition (2.1) with
\[
\eta(t) = \varphi(\sqrt{t}) \quad (0 \leq t \leq a)
\]
implying the corresponding convergence rates (2.3) and (2.4).

In order to discuss interrelations between $A$ and $G$ we formulate the following two link conditions:
Condition (A). There exists an index function $\mu(t)$ ($0 \leq t \leq \|G\|$) such that
\[ \|\mu(G)x\| \leq C_0 \|(A^*A)^{\frac{1}{2}}x\| \quad (x \in X). \]

We note Condition (A) is equivalent to the range inclusion
\[ \mathcal{R}(\mu(G)) \subset \mathcal{R}((A^*A)^{\frac{1}{2}}) \]
(see [1, Proposition 2.1(b)]).

Condition (B). There exists an index function $\varphi(t)$ ($0 \leq t \leq a$) such that
\[ \mathcal{R}(G) \subset \mathcal{R}(\varphi(A^*A)) \]

Remark 3.1

(1) For concrete inverse problems for partial differential equations, it is often easier to verify Condition (A). On the other hand, Condition (B) directly yields error estimates and convergence rates for regularization methods as outline above.

(2) Our main theorems give sufficient conditions for (B), so that our theorems provide practical criteria for the source condition, which are applicable to inverse problems for partial differential equations.

(3) The implication Condition (A) \(\implies\) Condition (B) is proved by the Heinz-Kato inequality in the case of $\mu(t) = t^\kappa$ with $\kappa \geq 1$. Thus our consideration is related with some generalization of the Heinz-Kato inequality (e.g., Furuta [4]). For a more detailed discussion of this context we refer to [1].

(4) For further remarks on the interplay of solution smoothness (3.1), general source conditions (2.1), range inclusions and convergence rates for regularization methods we also refer to [2], [5], [6] and [10].

4 Main results

The operators $S$ and $T$ under consideration now are both assumed to be compact self-adjoint non-degenerate operators in an infinite dimensional Hilbert space $X$. For such a compact self-adjoint operator $S$, there exist a sequence \(\{\lambda_j\}_{j=1}^\infty\) of decreasing real numbers and a sequence \(\{e_j\}_{j=1}^\infty \subset X\) of orthonormal elements where $\lim_{j \to \infty} \lambda_j = 0$ , $Se_j = \lambda_j e_j$ ($j = 1, 2, ...$) and
\[ Sx = \sum_{j=1}^\infty \lambda_j (x,e_j)e_j \quad (x \in X). \]
For an index function \( \varphi \) defined on the closed interval \([0, \|S\|]\) containing the spectrum \( \sigma(S) \) we can define
\[
\varphi(S)x = \sum_{j \in \mathbb{N}} \varphi(\lambda_j)(x, e_j)e_j \quad (x \in X).
\]
In particular, we see that \( \|\varphi(S)\| \leq \sup_{j=1,2,...} |\varphi(\lambda_j)| \). Moreover if \( \varphi \) is holomorphic in a neighbourhood in \( \mathbb{C} \) of \( \sigma(S) \), then \( \varphi(S) \) coincides with the one defined by the Dunford integral (e.g., Yosida [13, pp. 225-228]).

First we define an operator monotone function and present several properties. Let \( \sigma \) be a bounded self-adjoint operators in a Hilbert space \( X \).\( \|
\text{Let} \ \sigma \) be an open interval \( \subseteq \mathbb{C} \) and maps \( \Pi = \sigma(T), \sigma(T) \subseteq I \). The Löwner theorem asserts that \( f \) is operator monotone in an open interval \( I \) if and only if \( f \) is analytically extended to \( I_{+} = \{ z \in \mathbb{C} : \text{Im} z > 0 \} \) and maps \( I_{+} \) to \( I_{+} \) (e.g. [11], [12]). Henceforth \( \mu^{-1} \) denotes the inverse function to \( \mu \) which always exists if \( \mu \) is an index function, and we distinguish the product \( fg \) from the composition \( f \circ g \) of two functions \( f \) and \( g \):
\[
(fg)(S) = (f(S))(g(S)), \quad \text{that is,} \quad [(fg)(S)](x) = (f(S))(g(S)x) \quad (x \in X),
\]
and
\[
(f \circ g)(S) = f(g(S)).
\]

**Lemma 4.1**

(i) If \( 0 \leq T \leq S \), then \( \|S^{\frac{r}{2}}x\| \geq \|T^{\frac{r}{2}}x\| \ (x \in X) \).

(ii) Let \( 0 \leq T \leq S \). Then \( \|Tx\| \leq \|T^{\frac{r}{2}}\|\|S^{\frac{r}{2}}x\| \ (x \in X) \). In particular, \( \mathcal{R}(T) \subseteq \mathcal{R}(S^{\frac{r}{2}}) \).

(iii) Let \( S, T \geq 0 \) and \( \|Tx\| \leq \|Sx\| \) for all \( x \in X \). Then \( T \leq S \).

(iv) Let \( \sigma(S) \subseteq (0, \infty) \). Let \( f_0 = f_0(z) \) be holomorphic in a neighbourhood in \( \mathbb{C} \) of \( \sigma(S) \) and \( f_0(\sigma(S)) \subseteq (0, \infty) \). Set \( f(z) = z^\gamma f_0(z) \) for \( \text{Re} z > 0 \) with \( \gamma > 0 \). Then
\[
\|f(S)x\| \leq \|f_0(S)\|\|S^r x\| \quad (x \in X).
\]

(v) Let \( f = f(t) \) be defined in \( (0, \alpha) \) and \( |f(t)^2| \) be an operator monotone function in \( (0, \alpha^2) \). If \( S, T \geq 0 \), \( \|Tx\| \leq \|Sx\| \) \( (x \in X) \) and \( \sigma(S), \sigma(T) \subseteq (0, \alpha) \), then
\[
\|f(T)x\| \leq \|f(S)x\| \quad (x \in X).
\]

**Proof.**

(i) Since \( (Sx, x) \leq (Tx, x) \), we have \( (S^{\frac{r}{2}}S^{\frac{r}{2}}x, x) \geq (T^{\frac{r}{2}}T^{\frac{r}{2}}x, x) \) and so we see the conclusion.

(ii) We have
\[
\|Tx\| = \|T^{\frac{r}{2}}T^{\frac{r}{2}}x\| \leq \|T^{\frac{r}{2}}\|\|T^{\frac{r}{2}}x\| \leq \|T^{\frac{r}{2}}\|\|S^{\frac{r}{2}}x\|
\]
by (i). Hence Proposition 2.1 (b) in [1], yields \( \mathcal{R}(T) \subseteq \mathcal{R}(S^{\frac{r}{2}}) \).

(iii) This is a direct consequence of Theorem 4.12 (p.292) in [7].

(iv) By Theorem (p.226) in [13] for example, we have \( f(S)x = (f_0(S))(S^r x) \). Therefore
\[
\|f(S)x\| \leq \|f_0(S)(S^r x)\| \leq \|f_0(S)\|\|S^r x\|.
\]
(v) By the spectral mapping theorem, we have $\sigma(S^2), \sigma(T^2) \subseteq (0, \alpha^2)$. Moreover $\|Tx\| \leq \|Sx\|$ implies

$$(T^2x, x) = (Tx, Tx) \leq (Sx, Sx) = (S^2x, x) \quad (x \in X),$$

so that $T^2 \leq S^2$. By the operator monotonicity of $[f(t^2)]^2$, we have $f(T)^2 \leq f(S)^2$, that is, $(f(T)^2x, x) \leq (f(S)^2x, x) \quad (x \in X)$. Hence $\|f(T)x\|^2 \leq \|f(S)x\|^2 \quad (x \in X)$, because $f(S)$ and $f(T)$ are self-adjoint.

Now we state our results.

**Theorem 4.2**

Suppose Condition (A) and let $\mu$ be an index function defined on the interval $[0, \alpha]$ Let $0 < \|G\| \leq \alpha$ and $0 < \|A\|  \leq \alpha$ with $\alpha \geq 1$, and let us set $\beta = \mu(\alpha)$. 

(i) We assume

$$\mu^{-1}(t) = g(\psi(t)) \quad (0 \leq t \leq \beta),$$

where $\psi$ is operator monotone in $[\varepsilon, C_0\alpha + \varepsilon] \cup [\mu(\varepsilon), \mu(\alpha + \varepsilon)]$, $g(z) = z^\gamma g_0(z)$ with $\gamma > 0$, $g_0|_\mathbb{R}$ is monotone increasing, $g_0(0) \geq 0$ and $g_0$ is continuous on $[0, \infty)$ and holomorphic in a neighbourhood of $[\varepsilon, \infty)$ for any $\varepsilon > 0$. Setting

$$\varphi_1(t) = \begin{cases} (\psi(C_0^{t^{1/2}}))^{\gamma/2} & (0 < \gamma < 1), \\ (\psi(C_0^{t^{1/2}}))^{1/2} & (\gamma \geq 1), \end{cases}$$

we have Condition (B) with $\varphi = \varphi_1$. 

(ii) We assume

$$\mu^{-1}(t) = \psi(g(t)) \quad (0 \leq t \leq \beta).$$

Here $\psi$ is operator monotone in the following intervals in $t$:

$$g(\varepsilon) < t < g(\mu(\alpha) + \varepsilon),$$

$$\varepsilon < t < (C_0\alpha + \varepsilon)^{1/2}$$

and

$$\varepsilon g(\mu(\alpha) + \varepsilon) \min_{0 \leq \gamma \leq 1} \{\mu(\alpha + \varepsilon)^{-\gamma/2}, \mu(\alpha + \varepsilon)^{-1/2}\} < t$$

$$< (C_0\alpha + \varepsilon)g(\mu(\alpha) + \varepsilon) \max_{0 \leq \gamma \leq 1} \{\mu(\alpha + \varepsilon)^{-\gamma/2}, \mu(\alpha + \varepsilon)^{-1/2}\}$$

and $g(z) = z^\gamma g_0(z)$ with $\gamma > 0$, $g_0|_\mathbb{R}$ is monotone increasing, $g_0(0) \geq 0$, and $g_0$ is continuous on $[0, \infty)$ and holomorphic in a neighbourhood of $[\varepsilon, \infty)$ for any $\varepsilon > 0$. Setting

$$C(\gamma) = \begin{cases} g(\mu(\alpha))^{\mu(\alpha)^{-\gamma/2}} & (0 < \gamma < 1), \\ g(\mu(\alpha))^{\mu(\alpha)^{-1/2}} & (\gamma \geq 1) \end{cases}$$
and
\[ \varphi_2(t) = \begin{cases} \sqrt{\psi(C(\gamma)C_0^{1/2}t^{3/4})} & (0 < \gamma < 1), \\ \sqrt{\psi(C(\gamma)C_0^{1/2}t^{1/4})} & (\gamma \geq 1), \end{cases} \]
we have Condition (B) with \( \varphi = \varphi_2 \).

**Theorem 4.3**

Suppose Condition (A) and let \( \mu \) be an index function defined on the interval \([0, \infty)\) such that \([\mu^{-1}(t^{1/2})]^2\) is an operator monotone function on \([0, \infty)\). Then we have Condition (B) with \( \varphi(t) = \mu^{-1}(C_0t^{1/2}) \). Here \( C_0 > 0 \) is the constant in Condition (A).

**Example 4.4** We recall the well-known heat equation problem backward in time formulated in an \( L^2 \)-setting in Example 3 of the paper [6]. It will be shown that in this example Condition (A) implies Condition (B). First, we have by formula (5.15) on p.816 in [6] the equality of ranges \( \mathcal{R}(\mu(G)) = \mathcal{R}(A^*) \), where \( \mu(t) = e^{-\frac{t}{2}} \) \((0 < t < T)\). By \( \mathcal{R}(A^*) = \mathcal{R}(A^*) \), we obtain \( \mathcal{R}(\mu(G)) = \mathcal{R}(A^*) \). Hence Condition (A) is valid for that index function \( \mu \). On the other hand,
\[ \mu^{-1}(t) = \frac{T}{\log \frac{1}{t}} \quad (0 < t < \frac{1}{2}), \]
and the function
\[ [\mu^{-1}(t^{1/2})]^2 = \frac{4T^2}{(\log t)^2} \]
is not operator monotone because it does not map the set \( \Pi_+ \) to itself. So Theorem 4.3 is not applicable. However \( \mu^{-1} \) satisfies (4.1) with \( g(t) = t \) so that Theorem 4.2 is applicable and yields the required implication.

**Example 4.5** In Theorem 4.2, for \( \varphi_1 \), we have to take the square root such that in some cases the choice of \( \varphi \) in Condition (B) is not the best possible, by noting that \( \mathcal{R}(\psi(A^*A)) \subset \mathcal{R}(\psi(A^*))^{1/2} \) for an index function \( \psi \). For example, let \( \mu(t) = t^\kappa \) with exponents \( 0 < \kappa < 1 \). Then \([\mu^{-1}(t^{1/2})]^2 = \mu^{-1}(t) = t^\kappa \) satisfies the conditions in Theorems 4.2 and 4.3. Theorem 4.2 gives \( \mathcal{R}(G) \subset \mathcal{R}(A^*) \), while Theorem 4.3 gives \( \mathcal{R}(G) \subset \mathcal{R}(A^*) \). The latter is the best possible choice of \( \varphi \) because Condition (A) means that \( \mathcal{R}(G^\frac{1}{2}) \subset \mathcal{R}(A^*) \). Consequently, this provides us with \( \mathcal{R}(G) \subset \mathcal{R}(A^*) \) owing to the Heinz-Kato inequality.

**Example 4.6** Now we close this series of examples by recalling Example 4 from [6] using the same notations here. Then \( \mathcal{R}(e^{-Ct^2}) \subset \mathcal{R}(A^*) \). The continuous function \( \mu(t) \) is defined by \( \mu(t) = e^{-\frac{Ct}{\log 1/t}} \). Then setting \( g(t) = t^2 \) and \( \psi(t) = \frac{C}{\log 1/t} \) we have
\[ \mu^{-1}(t) = \frac{C^2}{(\log t)^2} = g(\psi(t)). \]
Therefore condition (4.1) holds true and Theorem 4.2 is applicable, while Theorem 4.3 is again not applicable, since \([\mu^{-1}(t^{1/2})]^2 \) fails to be operator monotone.
5 Proof of theorems

Proof of Theorem 4.2.
Since \( \|Ax\| = \|(A^*A)^{\frac{1}{2}}x\| \), \( x \in X \), we see by Condition (A) that there exists a constant \( C_0 > 1 \) such that
\[
\|\mu(G)x\| \leq \|C_0(A^*A)^{\frac{1}{2}}x\| \quad (x \in X).
\] (5.1)
Therefore by Lemma 4.1(iii), we have
\[
\mu(G) + \varepsilon \leq C_0(A^*A)^{\frac{1}{2}} + \varepsilon
\] (5.2)
for any \( \varepsilon > 0 \). Since \( \sigma(G) + \varepsilon = \sigma(G + \varepsilon) \subset \{ z \in \mathbb{C}; |z| \leq \|G + \varepsilon\| \} \) and \( \sigma(G) \subset [0, \infty) \), we obtain
\[
\sigma(G) + \varepsilon \subset [\varepsilon, \|G\| + \varepsilon] \subset [\varepsilon, \alpha + \varepsilon].
\] (5.3)
Similarly we see
\[
\sigma(C_0(A^*A)^{\frac{1}{2}} + \varepsilon) \subset [\varepsilon, C_0\alpha + \varepsilon]
\] (5.4)
because \( \|(A^*A)^{\frac{1}{2}}\| = \|A\| \leq \alpha \). Hence
\[
\sigma(\mu(G + \varepsilon)), \sigma(C_0(A^*A)^{\frac{1}{2}} + \varepsilon) \subset [\varepsilon, C_0\alpha + \varepsilon] \cup [\mu(\varepsilon), \mu(\alpha + \varepsilon)].
\] (5.5)

Case 1. We recall that \( \psi \circ \mu \) denotes the composition of two functions \( \psi \) and \( \mu \): \( (\psi \circ \mu)(z) = \psi(\mu(z)) \). Since \( \psi \) is operator monotone, in terms of (5.2) and (5.3) we obtain
\[
(\psi \circ \mu)(G + \varepsilon) \leq \psi(C_0(A^*A)^{\frac{1}{2}} + \varepsilon).
\] (5.6)
Here we note that \( (\psi \circ \mu)(G + \varepsilon)x = \psi(\mu(G + \varepsilon)x) \). By the spectral mapping theorem, by (5.3) and (5.4) we have
\[
\sigma((\psi \circ \mu)(G + \varepsilon)) = \psi(\sigma(\mu(G + \varepsilon))) \subset \psi[\mu(\varepsilon), \mu(\alpha + \varepsilon)]
\] (5.7)
and
\[
\sigma(\psi(C_0(A^*A)^{\frac{1}{2}} + \varepsilon)) = \psi(\sigma(C_0(A^*A)^{\frac{1}{2}} + \varepsilon)) \subset \psi[\varepsilon, C_0\alpha + \varepsilon].
\] (5.8)
It follows from (5.6) and Lemma 4.1(ii) that we have for all \( x \in X \)
\[
\|(\psi \circ \mu)(G + \varepsilon)x\| \leq \|(\psi \circ \mu)(G + \varepsilon)^{\frac{1}{2}}\|\|(\psi(C_0(A^*A)^{\frac{1}{2}} + \varepsilon))^{\frac{1}{2}}x\|
\leq |(\psi \circ \mu)(\alpha + \varepsilon)^{\frac{1}{2}}\|\|(\psi(C_0(A^*A)^{\frac{1}{2}} + \varepsilon))^{\frac{1}{2}}x\| \equiv C_1\|(\psi(C_0(A^*A)^{\frac{1}{2}} + \varepsilon))^{\frac{1}{2}}x\|.
\] (5.9)
Here we note that for any fixed \( \varepsilon_0 \) the constant \( C_1 = C_1(\varepsilon) > 0 \) can be taken uniformly in \( \varepsilon \in (0, \varepsilon_0] \). Henceforth \( C_j > 0 \) denote generic constants which are independent of \( \varepsilon \in (0, \varepsilon_0] \).
Since \( z \rightarrow z^\gamma \) is operator monotone by the Löwner-Heinz inequality (e.g., [4]) if \( 0 < \gamma \leq 1 \), we have
\[
((\psi \circ \mu)(G + \varepsilon))^\gamma \leq (\psi(C_0(A^*A)^{\frac{1}{2}} + \varepsilon))^\gamma,
\]
so that Lemma 4.1(ii) implies
\[
\|(\psi \circ \mu)(G + \varepsilon)\|^\gamma x \leq C_2\|(\psi(C_0(A^*A)^{\frac{1}{2}} + \varepsilon))^{\frac{1}{2}}x\| \quad (x \in X)
\] (5.10)
if \(0 < \gamma < 1\).

Since for \(\gamma \geq 1\), we have
\[
\|(\psi \circ \mu)(G + \varepsilon)\|^\gamma x \leq \|(\psi \circ \mu)(G + \varepsilon)\|^{\gamma-1}\|(\psi \circ \mu)(G + \varepsilon)x\| \leq C_2\|(\psi \circ \mu)(G + \varepsilon)x\|
\]
estimates (5.9) and (5.10) yield
\[
\|(\psi \circ \mu)(G + \varepsilon)\|^\gamma x \leq \begin{cases} C_2\|(\psi(C_0(A^*A)^{\frac{1}{2}} + \varepsilon))^{\frac{1}{2}}x\| & (x \in X), \text{if } 0 < \gamma < 1, \\ C_2\|(\psi(C_0(A^*A)^{\frac{1}{2}} + \varepsilon))^{\frac{1}{2}}x\| & (x \in X), \text{if } \gamma \geq 1. \end{cases}
\]
(5.11)

Hence by the assumption on \(g_0\), we can apply Lemma 4.1(iv) to have
\[
\|(g \circ \psi \circ \mu)(G + \varepsilon)x\| \leq C_3\|(\psi \circ \mu)(G + \varepsilon)\|^\gamma x\|
\]
Hence (5.11) implies
\[
\|(g \circ \psi \circ \mu)(G + \varepsilon)x\| \leq \begin{cases} C_2\|(\psi(C_0(A^*A)^{\frac{1}{2}} + \varepsilon))^{\frac{1}{2}}x\| & (x \in X), \text{if } 0 < \gamma < 1, \\ C_2\|(\psi(C_0(A^*A)^{\frac{1}{2}} + \varepsilon))^{\frac{1}{2}}x\| & (x \in X), \text{if } \gamma \geq 1. \end{cases}
\]
(5.12)

By the Lebesgue convergence theorem for the series and the definition of \(f(G + \varepsilon)x\) and \(f(G)x\), we can verify that \(\lim_{\varepsilon \downarrow 0} \|f(G + \varepsilon)x\| = \|f(G)x\| \quad (x \in X)\) if \(f\) is continuous on \(\sigma(G)\). Hence, letting \(\varepsilon \downarrow 0\) in (5.12), we have
\[
\|(g \circ \psi \circ \mu)(G)x\| \leq \begin{cases} C_2\|(\psi(C_0(A^*A)^{\frac{1}{2}}))^{\frac{1}{2}}x\| & (x \in X), \text{if } 0 < \gamma < 1, \\ C_2\|(\psi(C_0(A^*A)^{\frac{1}{2}}))^{\frac{1}{2}}x\| & (x \in X), \text{if } \gamma \geq 1. \end{cases}
\]
Noting that \((g \circ \psi \circ \mu)(z) = z\) for \(0 \leq z \leq \alpha\), by (1), we have
\[
\|Gx\| \leq \begin{cases} C_2\|(\psi(C_0(A^*A)^{\frac{1}{2}}))^{\frac{1}{2}}x\| & (x \in X), \text{if } 0 < \gamma < 1, \\ C_2\|(\psi(C_0(A^*A)^{\frac{1}{2}}))^{\frac{1}{2}}x\| & (x \in X), \text{if } \gamma \geq 1. \end{cases}
\]

Since \(A\) is injective, it follows that the operators \((\psi(C_0(A^*A)^{\frac{1}{2}}))^{\frac{1}{2}}\) and \((\psi(C_0(A^*A)^{\frac{1}{2}}))^{\frac{1}{2}}\) are injective. Hence we see by e.g., Proposition 2.1 in [1] that Condition (B) follows.

**Case 2.** By (5.2) and Lemma 4.1(ii) we have
\[
\|(\mu(G) + \varepsilon)x\| \leq |\mu(\alpha) + \varepsilon|\|(C_0(A^*A)^{\frac{1}{2}} + \varepsilon))^{\frac{1}{2}}x\| \quad (x \in X).
\]
(5.13)
Similarly to (5.7), we have
\[
\sigma(\mu(G) + \varepsilon) \subset [\varepsilon, \mu(\alpha) + \varepsilon].
\]
Therefore we can apply Lemma 4.1(iv) to have
\[
\|g(\mu(G) + \varepsilon)x\| \leq \|g_0(\mu(G) + \varepsilon)\| \|((\mu(G) + \varepsilon)^\gamma x\| \\
\leq |g_0(\mu(\alpha) + \varepsilon)\| \|((\mu(G) + \varepsilon)^\gamma x\| \quad (x \in X).
\]
(5.14)
Since \( z \rightarrow z^\gamma \) is operator monotone for \( 0 < \gamma \leq 1 \), relation (5.2) yields
\[
(\mu(G) + \varepsilon)^\gamma \leq (C_0(A^*A)^{\frac{1}{2}} + \varepsilon)^\gamma, \quad 0 < \gamma \leq 1.
\]
Therefore Lemma 4.1(ii) yields
\[
\|(\mu(G) + \varepsilon)^\gamma x\| \leq \|(\mu(G) + \varepsilon)^\gamma x\| \|(C_0(A^*A)^{\frac{1}{2}} + \varepsilon)^{\frac{\gamma}{2}} x\| \\
\leq |\mu(\alpha) + \varepsilon|^{\frac{\gamma}{2}} \|(C_0(A^*A)^{\frac{1}{2}} + \varepsilon)^{\frac{\gamma}{2}} x\| \quad (x \in X), \quad 0 < \gamma \leq 1.
\]
(5.15)
For \( 0 < \gamma \leq 1 \), we see from (5.14) and (5.15) that
\[
\|g(\mu(G) + \varepsilon)x\| \leq |g_0(\mu(\alpha) + \varepsilon)| |\mu(\alpha) + \varepsilon|^{\frac{\gamma}{2}} \|(C_0(A^*A)^{\frac{1}{2}} + \varepsilon)^{\frac{\gamma}{2}} x\| \quad (x \in X).
\]
Next let \( \gamma > 1 \). Since
\[
\|(\mu(G) + \varepsilon)^\gamma x\| = \|(\mu(G) + \varepsilon)^{\gamma-1}(\mu(G) + \varepsilon)x\| \leq |\mu(\alpha) + \varepsilon|^{\gamma-1}\|(\mu(G) + \varepsilon)x\|,
\]
we see from Lemma 4.1(iv), (5.14) and (5.13) that
\[
\|g(\mu(G) + \varepsilon)x\| \leq |g_0(\mu(\alpha) + \varepsilon)| |(\mu(\alpha) + \varepsilon)^{\gamma-\frac{1}{2}}\|(C_0(A^*A)^{\frac{1}{2}} + \varepsilon)^{\frac{\gamma}{2}} x\| \quad (x \in X).
\]
Lemma 4.1(iii) means that
\[
g(\mu(G) + \varepsilon) \leq \begin{cases} 
|g(\mu(\alpha) + \varepsilon)| |\mu(\alpha) + \varepsilon|^{\gamma-\frac{1}{2}}(C_0(A^*A)^{\frac{1}{2}} + \varepsilon)^{\frac{\gamma}{2}}, & 0 < \gamma \leq 1, \\
|g(\mu(\alpha) + \varepsilon)| |\mu(\alpha) + \varepsilon|^{\gamma-\frac{1}{2}}(C_0(A^*A)^{\frac{1}{2}} + \varepsilon)^{\frac{\gamma}{2}}, & \gamma \geq 1.
\end{cases}
\]
Set
\[
C(\varepsilon, \gamma) = \begin{cases} 
|g(\mu(\alpha) + \varepsilon)| |\mu(\alpha) + \varepsilon|^{\gamma-\frac{1}{2}}, & 0 < \gamma \leq 1, \\
|g(\mu(\alpha) + \varepsilon)| |\mu(\alpha) + \varepsilon|^{\gamma-\frac{1}{2}}, & \gamma \geq 1.
\end{cases}
\]
Similarly to (5.7) and (5.8), we can verify that
\[
\sigma(g(\mu(G) + \varepsilon)) \subset g[\varepsilon, \mu(\alpha) + \varepsilon]
\]
and
\[
\sigma((C_0(A^*A)^{\frac{1}{2}} + \varepsilon)^{\frac{\gamma}{2}}) \subset [\varepsilon^{\frac{\gamma}{2}}, (C_0A + \epsilon)^{\frac{\gamma}{2}}] \subset [\varepsilon, (C_0A + \epsilon)^{\frac{\gamma}{2}}]
\]
for \( 0 < \gamma \leq 1 \). Here we may assume that \( 0 < \varepsilon \leq 1 \) and recall that \( C_0, \alpha \geq 1 \), that is, \( C_0\alpha + \epsilon \geq 1 \). Consequently the operator monotonicity of \( \psi \) yields
\[
(\psi \circ g)(\mu(G) + \varepsilon) \leq \begin{cases} 
\psi(C(\varepsilon, \gamma)(C_0(A^*A)^{\frac{1}{2}} + \varepsilon)^{\frac{\gamma}{2}}), & 0 < \gamma \leq 1, \\
\psi(C(\varepsilon, \gamma)(C_0(A^*A)^{\frac{1}{2}} + \varepsilon)^{\frac{\gamma}{2}}), & \gamma \geq 1,
\end{cases}
\]
Lemma 4.1(ii) implies
\[ \|\mu^{-1}(\mu(G) + \varepsilon)x\| \leq \begin{cases} C_3\|\psi(C(\varepsilon, \gamma)(C_0(A^*A)^{\frac{1}{2}} + \varepsilon)^{\frac{1}{2}}) \|^{\frac{1}{2}}x\|, & 0 < \gamma \leq 1, \\ C_3\|\psi(C(\varepsilon, \gamma)(C_0(A^*A)^{\frac{1}{2}} + \varepsilon)^{\frac{1}{2}}) \|^{\frac{1}{2}}x\|, & \gamma \geq 1. \end{cases} \]

Letting \( \varepsilon \downarrow 0 \), we have
\[ \|Gx\| \leq \begin{cases} C_3\|\psi(C(\gamma)(C_0(A^*A)^{\frac{1}{2}})) \|^{\frac{1}{2}}x\|, & 0 < \gamma \leq 1, \\ C_3\|\psi(C(\gamma)(C_0(A^*A)^{\frac{1}{2}})) \|^{\frac{1}{2}}x\|, & \gamma \geq 1. \end{cases} \]

Thus by Proposition 2.1 in [1], we complete the proof of the theorem.

\[ \]

\textbf{Proof of Theorem 4.3.}
Condition (A) implies that \( \|\mu(G)x\| \leq \|C_0(A^*A)^{\frac{1}{2}}x\| \) for all \( x \in X \). By Lemma 4.1(v) we have
\[ \|(\mu^{-1}(\mu(G)))x\| \leq \|\mu^{-1}(C_0(A^*A)^{\frac{1}{2}})x\| \quad (x \in X), \]
that is,
\[ \|Gx\| \leq \|\varphi(A^*A)x\| \quad (x \in X). \]

Thus the proof of Theorem 4.3 is complete.

\[ \]

\textbf{References}


