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Sergej V. Nepomnyaschikh

# Domain Decomposition and Multilevel Techniques for Preconditioning Operators

Computing Center
Siberian Branch of Russian Academy of Sciences
Novosibirsk, 63090
Russia
e-mail: svnep@comcen.nsk.su

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# 1 Introduction

In recent years, domain decomposition methods have been used extensively to efficiently solve boundary value problems for partial differential equations in complex–form domains [4, 13, 16]. On the other hand, multilevel techniques on hierarchical data structures also have developed into an effective tool for the construction and analysis of fast solvers [2, 5, 15, 17]. But direct realization of multilevel techniques on a parallel computer system for the global problem in the original domain involves difficult communication problems. I this paper, we present and analyze a combination of these two approaches: domain decomposition and multilevel decomposition on hierarchical structures to design optimal preconditioning operators.

Let $\Omega \subset R^2$ be a polygon. In the domain $\Omega$ we consider the boundary value problem

$$\left\{ \begin{array}{rcll} -\sum\limits_{i,j=1}^{2} \dfrac{\partial}{\partial x_i}\Big(a_{ij}(x)\dfrac{\partial u}{\partial x_j}\Big) + a_0(x)u & = & f(x), & x \in \Omega \\ u(x) & = & 0, & x \in \Gamma_0 \\ \dfrac{\partial u}{\partial N} + \sigma(x)u & = & 0, & x \in \Gamma_1. \end{array} \right. \tag{1.1}$$

where

$$\frac{\partial u}{\partial N} = \sum_{i,j=1}^{2} a_{ij}(x)\frac{\partial u}{\partial x_j}\cos(n, x_i)$$

is the conormal derivative, $n$ denotes the outward normal to $\Gamma$, and $\Gamma_0$ is a union of a finite number of curvilinear segments, $\Gamma = \Gamma_0 \cup \Gamma_1, \Gamma_0 = \bar{\Gamma}_0$. Here $\bar{\Gamma}_0$ denotes the closure of $\Gamma_0$.

By $H^1(\Omega, \Gamma_0)$ we denote the subspace of the Sobolev space $H^1(\Omega)$

$$H^1(\Omega, \Gamma_0) = \Big\{ v \in H^1(\Omega) \,|\, v(x) = 0, \quad x \in \Gamma_0 \Big\}.$$

We introduce the bilinear form $a(u, v)$ and the linear functional $l(v)$ :

$$a(u,v) = \int\limits_{\Omega} \Big( \sum_{i,j=1}^{2} a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + a_0(x)uv \Big)\, dx + \int\limits_{\Gamma_1} \sigma(x)uv\, dx$$

$$l(v) = \int\limits_{\Omega} f(x)v dx.$$

Let us suppose that the operator coefficients and the right-hand side of the problem (1.1) are such that the bilinear form $a(u, v)$ is symmetric, elliptic, and continuous on $H^1(\Omega, \Gamma_0) \times H^1(\Omega, \Gamma_0)$, i.e.

$$a(u, v) = a(v, u) \quad \forall u, v \in H^1(\Omega, \Gamma_0)$$

$$\alpha_0 \|u\|^2_{H^1(\Omega)} \le a(u, u) \le \alpha_1 \|u\|^2_{H^1(\Omega)} \quad \forall u \in H^1(\Omega, \Gamma_0)$$

and the linear functional $l(v)$ is continuous on $H^1(\Omega, \Gamma_0)$:

$$|l(u)| \le \alpha \|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega, \Gamma_0).$$

The generalized solution $u \in H^1(\Omega, \Gamma_0)$ of (1.1) is, by definition, a solution to the projection problem [1]

$$u \in H^1(\Omega, \Gamma_0) \,:\, a(u, v) = l(v) \quad \forall v \in H^1(\Omega, \Gamma_0). \tag{1.2}$$

We know that under these assumptions for $a(u,v)$ and $l(v)$ there exists a unique solution of (1.2).

Let $\Omega$ be a union of $n$ nonoverlapping subdomains $\Omega_i$,

$$\bar{\Omega} = \bigcup_{i=1}^{n} \bar{\Omega}_i, \quad \Omega_i \cap \Omega_j = \emptyset, \ i \neq j,$$

where $\Omega_i$ are polygons with diameters on the order of $H$. Let us consider a coarse grid triangulation of $\Omega$

$$\Omega_0^h = \bigcup_{i=1}^{n} \Omega_{i,0}^h, \quad \Omega_{i,0}^h = \bigcup_{l=1}^{M_i^{(0)}} \bar{\tau}_{i,l}^{(0)},$$

$$\text{diam } (\tau_{i,l}^{(0)}) = 0(H)$$

and we refine $\Omega_{i,0}^h$ several times. This results in a sequence of nested triangulations

$$\Omega_{i,0}^h, \Omega_{i,1}^h, \ldots, \Omega_{i,J}^h$$

such that

$$\bar{\Omega}_{i,k}^h = \bigcup_{l=1}^{M_i^{(k)}} \bar{\tau}_{i,l}^{(k)}, \qquad k = 0, 1, \ldots, J\,,$$

where the triangles $\tau_{i,l}^{(k+1)}$ are generated by subdividing triangles $\tau_{i,l}^{(k)}$ into four congruent subtriangles by connecting the midpoints of the edges.

Introduce the spaces

$$\begin{aligned} W_{i,0} \subset W_{i,1} \subset \ldots \subset W_{i,J} &= H_h(\Omega_i), \\ V_{i,0} \subset V_{i,1} \subset \ldots \subset V_{i,J} &= H_h(\Gamma_i), \\ \Gamma_i = \partial\Omega_i, \quad i = 1, 2, \ldots, n. & \end{aligned} \tag{1.3}$$

Here the space $W_{i,k}$ consists of real-valued functions which are continuous on $\Omega$ and linear on the triangles in $\Omega_{i,k}^h$. The space $V_{i,k}$ is the space of traces on $\Gamma_i$ of functions from $W_{i,k}$:

$$V_{i,k} = \left\{ \varphi^h \mid \varphi^h = u^h|_{\Gamma_i}, \text{ with } u^h \in W_{i,k} \right\}.$$

We define the space $H_h(\Omega)$ of real continuous functions which are linear on each triangle of $\Omega^h$ and vanish at $\Gamma_0$.

Let us consider the projection problem

$$u^h \in H_h(\Omega): \ a(u^h, v^h) = l(v^h) \quad \forall v^h \in H_h(\Omega) \tag{1.4}$$

which is an approximation of the problem (1.2).

Each function $u^h \in H_h(\Omega)$ is put in correspondence with a real column vector $u \in R^N$ whose components are values of the function $u^h$ at the corresponding nodes of the triangulation $\Omega^h$. Then (1.4) is equivalent to the system of mesh equations

$$\begin{aligned} Au &= f, \\ (Au, v) &= a(u^h, v^h) \quad \forall u^h, v^h \in H_h(\Omega), \\ (f, v) &= l(v^h) \quad \forall v^h \in H_h(\Omega), \end{aligned} \tag{1.5}$$

where $u^h$ and $v^h$ are the respective interpolations of vectors $u$ and $v$; $(f, v)$ is the Euclidean scalar product in $R^N$.

The goal of this work is to construct a symmetric positive definite preconditioning operator $B$ for (1.5) so as to satisfy the inequalities

$$c_1(Bu, u) \leq (Au, u) \leq c_2(Bu, u) \tag{1.6}$$

where the positive constants $c_1$ and $c_2$ are independent of $h$ and $H$, the multiplication of a vector by $B^{-1}$ should be easy to implement.

Using a combination of Additive Schwarz and Fictitious Space Methods, optimal preconditioning operators have been constructed in [11, 12, 13] for the case of arbitrary (unstructured) grids. However, that construction involves explicit extension operators whose implementation for three dimensional problems is optimal from the arithmetic cost and the condition number points of view but difficult for practice realization. The main goal of this work is to construct, using the hierarchical structure (1.3), a robust optimal preconditioning operator. One of the crucial points in [11, 12, 13] and this paper is using of non–exact solvers in subdomains and explicit extension operators. It means, to construct optimal preconditioning operators, we can design norm preserving operators of functions given at $\Gamma_i$ into $\Omega_i$ with the optimal arithmetic cost (a number of arithmetic operations should be proportional to a number degrees of freedom) and then, instead of exact solvers in subdomains, we can use any spectrally equivalent preconditioning operators. Optimal extension operators have been presented in [8, 9, 11] for unstructured grids and robust explicit extension operators on hierarchical data structures in [5, 14].

The paper is organized as follows. In the Section 2, using Additive Schwarz Method, we describe general construction of a preconditioning operator with local multilevel preconditioning operators. In the Section 3, we present an optimal multilevel extension of grid functions from boundaries subdomains into inside subdomains. In the Section 4, we propose an optimal interface preconditioning operator at the boundaries of the subdomains which involves a multilevel decomposition and corresponding explicit extension operators at interfaces.

## 2 Domain decomposition – additive Schwarz-Method

To design the preconditioning operator for the system (1.5), we use the additive Schwarz–Method [7] and realize the main idea of the construction of preconditioners from [13] for the hierarchical grids. Denote by $\overset{\circ}{H}_h(\Omega_i)$ the subspace of $H_h(\Omega_i)$

$$\overset{\circ}{H}_h(\Omega_i) = \left\{ u^h \in H_h(\Omega_i) \mid u^h(x) = 0, \quad x \in \Gamma_i \right\}$$

and define the local preconditioning operators $B_i$ such that

$$B_i : \overset{\circ}{H}_h(\Omega_i) \rightarrow \overset{\circ}{H}_h(\Omega_i),$$

$$c_3 \|u^h\|^2_{H^1(\Omega_i)} \leq (B_i u, u) \leq c_4 \|u^h\|^2_{H^1(\Omega_i)} \quad \forall u^h \in \overset{\circ}{H}_h(\Omega_i),$$

where $c_3, c_4$ are independent of $h$ and $H$. We hereafter use the same designation for an operator and its matrix representation. For instance, to define $B_i$, we can use the so-called BPX–preconditioners [3]. To do it, denote by $\{f_l^{(k)}\}$ nodal basis functions from the k–th level and define

$$B_i^{-1} u^h = \sum_{k=0}^{J} \sum_{f_l^{(k)} \in \overset{\circ}{H}_h(\Omega_i)} (u^h, f_l^{(k)})_{L_2(\Omega_i)} \, f_l^{(k)}. \tag{2.1}$$

Let us assume that we can define the extension operators $t_i$

$$t_i \ : \ V_{i,J} \longrightarrow W_{i,J}$$

such that

$$\begin{aligned} t_i\varphi^h &= u^h, \\ u^h(x) &= \varphi^h(x), \ x \in \Gamma_i, \\ \|t_i\varphi^h\|_{H^1(\Omega_i)} &\le c_5\|\varphi^h\|_{H^{1/2}(\Gamma_i)} \quad \forall\varphi^h \in V_{i,J}, \end{aligned} \tag{2.2}$$

with $c_5$ independent of $h$ and $H$. Here $\|\varphi^h\|_{H^{1/2}(\Gamma_i)}$ is the norm [10] in the Sobolev space $H^{1/2}(\Gamma_i)$

$$\|\varphi^h\|^2_{H^{1/2}(\Gamma_i)} = H\int\limits_{\Gamma_i}(\varphi^h(x))^2dx + \int\limits_{\Gamma_i}\int\limits_{\Gamma_i}\frac{(\varphi^h(x)-\varphi^h(y))^2}{|x-y|^2}dxdy.$$

Then, we can define the extension operator $t$

$$t \ : \ H_h(S) \to H_h(\Omega),$$

where $H_h(S)$ is the space of traces of functions from $H_h(\Omega)$ at $S$

$$S = \bigcup_{i=1}^{n}\Gamma_i$$

and for any $\varphi^h \in H_h(S)$

$$\begin{aligned} t\varphi^h &= u^h, \\ u^h(x) &= \varphi^h(x), \quad x \in S, \\ \|t\varphi^h\|_{H^1(\Omega)} &\le c_5\|\varphi^h\|_{H^{1/2}(S)}. \end{aligned}$$

Here

$$\|\varphi^h\|^2_{H^{1/2}(S)} = \sum_{i=1}^{n}\|\varphi^h\|^2_{H^{1/2}(\Gamma_i)}.$$

The operator $t_i$ from (2.2) is constructed in the Section 3.

Let $\Sigma$ satisfies to the following inequalities

$$c_6\|\varphi^h\|^2_{H^{1/2}(S)} \le (\Sigma\varphi,\varphi) \le c_7\|\varphi^h\|^2_{H^{1/2}(S)} \quad \forall\varphi^h \in H_h(S), \tag{2.3}$$

where $c_6, c_7$ independent of $h$ and $H$. Then, according to [11], we can define the preconditioning operator $B$ as follows

$$B^{-1} = \begin{bmatrix} 0 & & & \\ & B_1^{-1} & & \\ & & \ddots & \\ & & & B_n^{-1} \end{bmatrix} + t\Sigma^{-1}t^*. \tag{2.4}$$

Here 0 is the null-matrix which corresponds to nodes of the triangulation $\Omega^h$ at $S$ and $B_i$ is from (2.1).

The following theorem is valid

**Theorem 2.1** *If the operator $B$ is from (2.4), then the constants $c_1, c_2$ from (1.6) are independent of $h$ and $H$.*

## 3 Multilevel explicit extension operators

The main goal of this section is to construct the robust operator $t_i$ from (2.2). During this section, we omit the subscript $i$.

To design the extension operator

$$t \ : \ V_J \to W_J,$$

we follow to [5, 14]. Denote by $\varphi_i^{(k)}, i = 1, 2, \ldots, N_k$, the nodal basis of $V_k$ and denote by $\Phi_i^{(k)}$ the one-dimensional subspace spanned by this function $\varphi_i^{(k)}$. Define

$$Q_i^{(k)} \ : \ L_2(\Gamma) \to \Phi_i^{(k)}$$

the $L_2$-orthoprojection from $L_2(\Gamma)$ on to $\Phi_i^{(k)}$ and denote

$$\tilde{Q}_k = \sum_{i=1}^{N_k} Q_i^{(k)}, \quad k = 0, 1, \ldots, J-1.$$

For $k = J$ we define $\tilde{Q}_J$ as the $L_2$-orthoprojection from $L_2(\Gamma)$ on to $V_j$.

The following lemmas are valid [14].

**Lemma 3.1** *There exists a positive constant $c_8$, independent of $h$ and $H$, such that for any $\varphi^h \in V_J$*

$$\|\varphi_0^h\|^2_{H^{1/2}(\Gamma)} + \frac{1}{H}\|\varphi_1^h\|^2_{L_2(\Gamma)} + |\varphi_1^h|^2_{H^{1/2}(\Gamma)} \le c_8 \|\varphi^h\|^2_{H^{1/2}(\Gamma)},$$

*where*

$$\varphi_0^h = \tilde{Q}_0 \varphi^h, \quad \varphi_1^h = \varphi^h - \varphi_0^h. \tag{3.1}$$

*Here*

$$|\varphi^h|^2_{H^{1/2}(\Gamma)} = \int\limits_\Gamma \int\limits_\Gamma \frac{(\varphi^h(x) - \varphi^h(y))^2}{|x-y|^2} dxdy.$$

**Lemma 3.2** *There exists a positive constant $c_9$, independent of $h$ and $H$, such that*

$$\|\varphi_0^h\|^2 + \frac{1}{H}\left(\|\tilde{Q}_0\varphi_1^h\|^2_{L_2(\Gamma)} + \sum_{k=1}^{J} 2^k \|(\tilde{Q}_k - \tilde{Q}_{k-1})\varphi_1^h\|^2_{L_2(\Gamma)}\right) \le c_9 \|\varphi^h\|^2_{H^{1/2}(\Gamma)},$$

*where $\varphi_0^h, \varphi_1^h$ from (3.1).*

The construction of the operator $t$ is based on the decomposition from the Lemma 3.2. Denote by $x_i^{(k)}, i = 1, 2, \ldots, L_k$, the nodes of the triangulation $\Omega_k^h$ (we assume that nodes $x_i^{(k)}$ are enumerated first on $\Gamma$ and then inside $\Omega$) and define the extension operator $t$ in the following way. For any $\varphi^h \in V_J$ set

$$\begin{aligned} \psi_0^h &= \tilde{Q}_0 \varphi^h, \\ \psi_k^h &= (\tilde{Q}_k - \tilde{Q}_{k-1})\varphi^h, \quad k = 1, 2, \ldots, J. \end{aligned} \tag{3.2}$$

Then

$$\varphi^h = \psi_0^h + \psi_1^h + \ldots + \psi_J^h.$$

Define the extension $u_k^h \in W_k$ as follows

$$\begin{aligned} u_0^h(x_i^{(0)}) &= \begin{cases} \psi_0^h(x_i^{(0)}), & x_i^{(0)} \in \Gamma, \\ \bar{\psi}, & x_i^{(0)} \notin \Gamma, \end{cases} \\ u_k^h(x_i^{(k)}) &= \begin{cases} \psi_k^h(x_i^{(k)}), & x_i^{(k)} \in \Gamma, \\ 0, & x_i^{(k)} \notin \Gamma, \end{cases} \end{aligned} \tag{3.3}$$

$$k = 1, 2, \ldots, J.$$

Here $\bar{\psi}$ is, for instance, the meanvalue of the function $\psi_0^h$ on $\Gamma$

$$\bar{\psi} = \frac{1}{N_0} \sum_{i=1}^{N_0} \psi_0^h(x_i^{(0)}).$$

Define

$$t\varphi^h = u^h \equiv u_0^h + u_1^h + \ldots + u_J^h \tag{3.4}$$

**Remark 3.1** *We can use the $L_2$-orthoprojections form $L_2(\Gamma)$ on to $V_k$ instead of $\tilde{Q}_k, k = 0, 1, \ldots, J-1$. But in this case the cost of the decomposition (3.2) is expensive (especially for three dimensional problems).*

**Theorem 3.1** *There exists a positive constant $c_{10}$, independent of $h$ and $H$, such that*

$$\|t\varphi^h\|_{H^1(\Omega)} \leq c_{10} \|\varphi^h\|_{H^{1/2}(\Gamma)} \quad \forall \varphi^h \in V_J.$$

*Here the operator $t$ is from (3.2)–(3.4).*

**Remark 3.2** *It is obvious that*

$$Q_i^{(k)} \varphi^h = \frac{(\varphi^h, \varphi_i^{(k)})_{L_2(\Gamma)}}{(\varphi_i^{(k)}, \varphi_i^{(k)})_{L_2(\Gamma)}} \varphi_i^{(k)}$$

*and the cost of the action of $t$ and $t^*$ is proportional to the number of nodes of the grid domain.*

# 4 Interface preconditioning operators

In this section, we construct an optimal interface preconditioner in the space $H_h(S)$ which satisfies (2.3). To do it, we use the idea of Additive Schwarz Method at interface $S$ from [13]. Let $S$ be a union of $K$ nonoverlapping edges $E_i$ of the triangulation $\Omega_0^h$

$$S = \bigcup_{j=1}^{K} \bar{E}_j, \quad E_j \cap E_i = \emptyset, \quad i \neq j.$$

Split $H_h(S)$ into a vector sum of subspaces

$$H_h(S) = U_0 + U_1 + \ldots + U_K, \tag{4.1}$$

where $U_0$ is the coarse space which consists of continuous functions linear on the edges $E_j$, $j = 1, 2, \ldots, K$, and $U_j$, $j = 1, 2, \ldots, K$, correspond to $E_j$ and are defined below.

Denote by

$$\mathring{U}_j = \{\varphi^h \in H_h(S) \mid \varphi^h(x) = 0,\ x \notin E_j\},$$
$$\tilde{U}_j^{(k)} = V_k|_{E_j}, \qquad k = 0, 1, \ldots, J.$$

For any edge $E_j$ we define the explicit extension operator $\tau_j$

$$\tau_j \ : \ \tilde{U}_j^{(J)} \to H_h(S)$$

as follows. Denote by $\varphi_{j,i}^{(k)}$, $i = 1, 2, \ldots, I_j^{(k)}$, the nodal basis of $\tilde{U}_j^{(k)}$ (the functions $\varphi_{j,i}^{(k)}$ are differ from the functions $\varphi_i^{(k)}$ from the Section 3 only at the end points of $E_j$) and denote by $\Phi_{j,i}^{(k)}$ the one-dimensional subspace spanned by this function $\varphi_{j,i}^{(k)}$. Define

$$Q_{j,i}^{(k)} \ : \ L_2(E_j) \to \Phi_{j,i}^{(k)}$$

corresponding $L_2$-orthoprojection. Set

$$\tilde{Q}_j^{(k)} = \sum_{i=1}^{I_j^{(k)}} Q_{j,i}^{(k)}, \quad k = 0, 1, \ldots, J-1\,,$$

and define $\tilde{Q}_J^{(k)}$ as the $L_2$-orthoprojection from $L_2(E_j)$ onto $\tilde{U}_j^{(J)}$. Now we can define the extension operator $\tau_j$ according to (3.2)–(3.4). For any $\varphi^h \in \tilde{U}_j^{(J)}$ set

$$\begin{aligned} \psi_0^h &= \tilde{Q}_j^{(0)}\varphi^h, \\ \psi_k^h &= (\tilde{Q}_j^{(k)} - \tilde{Q}_j^{(k-1)})\varphi^h, \quad k = 1, 2, \ldots, J \end{aligned} \tag{4.2}$$

and

$$u_k^h = \begin{cases} \psi_k^h(x_i^{(k)}), & x_i^{(k)} \in E_j \\ 0, & x_i^{(k)} \notin E_j, \quad k = 0, 1, \ldots, J \end{cases} \tag{4.3}$$

$$\tau_j\varphi^h = u_0^h + u_1^h + \ldots + u_J^h.$$

Define

$$U_j = \mathring{U}_j + \tau_j\tilde{U}_j.$$

Then from the Theorem 3.1 and [13] for the decomposition (4.1) we have the following theorem.

**Theorem 4.1** *There exists a positive constant $c_{11}$, dependent of $h$ and $H$, such that for any function $\varphi^h \in H_h(S)$ there exist $\varphi_j^h \in U_j$, $j = 0, 1, \ldots, K$, such that*

$$\varphi_0^h + \varphi_1^h + \ldots + \varphi_k^h = \varphi^h,$$

$$\|\varphi_0^h\|^2_{H^{1/2}(S)} + \|\varphi_1^h\|^2_{H^{1/2}(S)} + \ldots + \|\varphi_K^h\|^2_{H^{1/2}(S)} \le c_{11}\|\varphi^h\|^2_{H^{1/2}(S)}$$

Let the operator $\Sigma_0$ generates an equivalent norm in $U_0$.

$$c_{12}\|\varphi^h\|^2_{H^{1/2}(S)} \le (\Sigma_0\varphi, \varphi) \le c_{13}\|\varphi^h\|^2_{H^{1/2}(S)} \quad \forall\varphi^h \in U_0, \tag{4.4}$$

where $c_{12}$, $c_{13}$ independent of $h$ and $H$. Define local preconditioners for $U_j$, $j = 1, 2, \ldots, K$. Denote by $\mathring{\Sigma}_j$ and $\widetilde{\Sigma}_j$ the BPX-like preconditioners in the spaces $\mathring{U}_j$ and $\tilde{U}_j$, respectively

$$\mathring{\Sigma}_i^{-1}\varphi^h = \sum_{k=0}^{J} \sum_{\mathrm{supp}\,\varphi_{j,i}^{(k)} \subset E_j} (\varphi^h, \varphi_{j,i}^{(k)})_{L_2(E_j)}\, \varphi_{j,i}^{(k)} \quad \forall \varphi^h \in \mathring{U}_j,$$

$$\Sigma_i^{-1}\varphi^h = \sum_{k=0}^{J} \sum_{\mathrm{supp}\,\varphi_{j,i}^{(k)} \cap E_j \neq \emptyset} (\varphi^h, \varphi_{j,i}^{(k)})_{L_2(E_j)}\, \varphi_{j,i}^{(k)} \quad \forall \varphi^h \in \tilde{U}_j.$$

Then, define the interface preconditioning operator $\Sigma$ in the following way

$$\Sigma^{-1} = \Sigma_0^{+} + \sum_{j=1}^{K} (\mathring{\Sigma}_j^{-1} + \tau_j \widetilde{\Sigma}_j^{-1} \tau_j^{*}). \tag{4.5}$$

Here $\Sigma_0^{+}$ is a pseudo-inverse of $\Sigma_0$ from (4.4), $\tau_j$ is from (4.2), (4.3), and we extend the operator $\mathring{\Sigma}_j^{-1}$ by zero outside $E_j$. The following theorem is valid.

**Theorem 4.2** *If the operator $\Sigma$ is from (4.5) then the constants $c_6, c_7$ form (2.3) are independent of $h$ and $H$.*

**Remark 4.1** *The method suggested in this paper can be generalized evidently for three dimensional problems.*

**Remark 4.2** *Using combination of presented technique and technique from [10], effective preconditioning operators for elliptic problems with jump coefficients can be constructed.*


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