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Numerische Simulation auf massiv parallelen Rechnern

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Convergence of Asynchronous Jacobi-Newton-Iterations

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Abstract

Asynchronous iterations often converge under different conditions than their synchronous counterparts. In this paper we will study the global convergence of Jacobi-Newton-like methods for nonlinear equations $Fx = 0$. It is a known fact, that the synchronous algorithm converges monotonically, if F is a convex M-function and the starting values x^0 and y^0 meet the condition $Fx^0 \leq 0 \leq Fy^0$. In the paper it will be shown, which modifications are necessary to guarantee a similar convergence behavior for an asynchronous computation.

1 Introduction

Throughout this paper the natural partial ordering in $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ is used. For any $x, y \in \mathbb{R}^n$ with $x \leq y$ the set $\langle x, y\rangle = \{z \in \mathbb{R}^n : x \leq z \leq y\}$ is called order interval.
A mapping $G : Q \subseteq \mathbb{R}^n \to \mathbb{R}^n$ (or $\mathbb{R}^{n\times n}$) is called isotone on $Q_0 \subseteq Q$ if $G(x) \leq G(y)$ holds for all $x, y \in Q$ with $x \leq y$.
The notation $x^k \uparrow x^*$ means that the sequence $\{x^k\}$ is monotonically increasing and $\lim_{k\to\infty} x^k = x^*$; analogously $x^k \downarrow x^*$ is defined.

The following theorem on the convergence of sequential Jacobi-Newton-iterations is known from [5]:

Theorem 1 *Let $F : Q \subseteq \mathbb{R}^n \to \mathbb{R}^n$. Assume that there exist $x^0, y^0 \in Q$ so that $x^0 \leq y^0$, $\langle x^0, y^0\rangle \subseteq Q$ and $F(x^0) \leq 0 \leq F(y^0)$, and that F is Frechét-differentiable on $\langle x^0, y^0\rangle$. Moreover, suppose that $F'(x)$ is a nonsingular M-matrix for each $x \in \langle x^0, y^0\rangle$, and that $F'(x) : \langle x^0, y^0\rangle \to \mathbb{R}^{n\times n}$ is an isotone mapping in x. Then the sequences defined by*

$$\begin{array}{rcll} y_i^{k+1} &=& y_i^k - (\partial_i F_i(y^k))^{-1} F_i(y^k), & i = 1, \ldots, n \\ x_i^{k+1} &=& x_i^k - (\partial_i F_i(y^k))^{-1} F_i(x^k), & i = 1, \ldots, n \end{array}, \qquad k = 0, 1, \ldots, \tag{1}$$

satisfy $y^k \downarrow x^*$ and $x^k \uparrow x^*$, where x^* is the unique solution of $F(x) = 0$ in $\langle x^0, y^0\rangle$.

Note that the monotone behavior of the iterates is crucial for proving convergence.

This iteration can be parallelized by assigning each processor P_j with updating a subset J_j of components. The parallel iteration may be done synchronously or asynchronously. Asynchronous iterations recently attracted much attention because they may have significantly lower computing times (see e.g. [4], also for more references). Asynchronous implementations on parallel computers usually always fit into the following definition, [1]:

Definition 1 *Let $Q \subseteq \mathbb{R}^n$, $Q = Q_1 \times \ldots \times Q_n$, and let $H : Q \to Q$. For $k = 0, 1, \ldots$ consider non-empty sets $I^k \subseteq \{1, \ldots, n\}$ and n-tuples $(s_1(k), \ldots, s_n(k))$ of nonnegative integers. Suppose that the following three conditions hold:*

$$s_i(k) \leq k \quad \textit{for each } i \in \{1, \ldots, n\},\ k = 0, 1, \ldots,$$

$$lim_{k \to \infty} s_i(k) = \infty \quad \textit{for each } i \in \{1, \ldots, n\},$$

$$|\{k \in \mathbb{N} : i \in I^k\}| = \infty \quad \textit{for each } i \in \{1, \ldots, n\}.$$

Then the iterative method which, starting with an initial guess $x^0 \in Q$, calculates the iterates x^k according to

$$x_i^{k+1} = \begin{cases} H_i(x_1^{s_1(k)}, \ldots, x_n^{s_n(k)}) =: H_i(x^{s(k)}) & \textit{if } i \in I^k \\ x_i^k & \textit{if } i \notin I^k \end{cases}, \qquad k = 0, 1, \ldots,$$

is termed asynchronous iteration for H.

Furthermore it will be assumed that, while updating a variable x_i, its last iterate is known:

$$x_i^{k+1} = \begin{cases} H_i(x_1^{s_1(k)}, \ldots, x_{i-1}^{s_{i-1}(k)}, x_i^k, x_{i+1}^{s_{i+1}(k)}, \ldots, x_n^{s_n(k)}) & \text{if } i \in I^k \\ x_i^k & \text{if } i \notin I^k \end{cases}, \qquad k = 0, 1, \ldots.$$

This assumption is fulfilled if the subsets J_j are pairwise disjoint.

2 A modified Jacobi-Newton-operator and some convergence theorems

The basic Jacobi-Newton-operator for the upper bound is $H_i(y) = y_i - (\partial_i F_i(y))^{-1} F_i(y)$, $i = 1, \ldots, n$. We propose the following modification:

$$H_i(y) = \begin{cases} y_i - (\partial_i F_i(y))^{-1} F_i(y) & \text{if } F_i(y) \geq 0 \\ y_i & \text{otherwise} \end{cases}, \quad i = 1, \ldots, n. \tag{2}$$

The analogously modified operator for the lower bound is:

$$H_i(x) = \begin{cases} x_i - (\partial_i F_i(y))^{-1} F_i(x) & \text{if } F_i(x) \leq 0 \\ x_i & \text{otherwise} \end{cases}, \quad i = 1, \ldots, n. \tag{3}$$

The modification is necessary because an asynchronous iteration for the basic operator may not only cause non-monotone sequences of iterates, but also that an iterate out of the domain of definition Q of the function F is computed.

The following two theorems for the modified asynchronous iteration are the main results of this paper.

Theorem 2 *Assume that F, x^0 and y^0 fulfil the conditions of Theorem 1. Then the iterates $\{x^k\}$ and $\{y^k\}$ of an asynchronous iteration for the operators (2) and (3) satisfy $y^k \downarrow x^*$ and $x^k \uparrow x^*$, respectively.*

Proof: For $k = 0, 1, \ldots$ let the sets L^k and M^k be defined by $L^k = \{i \in I^k : F_i(y^{s(k)}) \geq 0\}$ and $M^k = \{i \in I^k : F_i(x^{s(k)}) \leq 0\}$, respectively. Then the iterates x^k and y^k, $k = 0, 1, \ldots$, are calculated according to

$$y_i^{k+1} = \begin{cases} y_i^k - (\partial_i F_i(y^{s(k)}))^{-1} F_i(y^{s(k)}) & \text{if } i \in L^k \\ y_i^k & \text{otherwise} \end{cases} \tag{4}$$

and

$$x_i^{k+1} = \begin{cases} x_i^k - (\partial_i F_i(y^{s(k)}))^{-1} F_i(x^{s(k)}) & \text{if } i \in M^k \\ x_i^k & \text{otherwise} \end{cases} . \tag{5}$$

By assumption, F is a M-function on $\langle x^0, y^0 \rangle$. Hence, since $F(x^0) \leq 0 \leq F(y^0)$, there exists a unique solution $x^* \in \langle x^0, y^0 \rangle$ of $F(x) = 0$. On the other hand, since F' is an isotone mapping, Theorem 13.3.2 of [5] ensures that F is order-convex. Consequently,

$$F(y) - F(x) \geq F'(x)(y - x), \tag{6}$$

whenever x and y are comparable, that means, $x \leq y$ or $y \leq x$.

The proof consists of 4 parts. We have to show that
(a) $x^k \leq x^{k+1}$, $y^{k+1} \leq y^k$, $k = 0, 1, \ldots$,
(b) $x^k \leq x^* \leq y^k$, $k = 0, 1, \ldots$,
(c) $|\{k \in \mathbb{N} : i \in L^k\}| = \infty$, $|\{k \in \mathbb{N} : i \in M^k\}| = \infty$, $i \in \{1, \ldots, n\}$, and
(d) $\lim_{k\to\infty} y^k = x^*$, $\lim_{k\to\infty} x^k = x^*$.

(a) Since $F'(y)$ is a nonsingular M-matrix, $\partial_i F_i(y) > 0$ for each $y \in \langle x^0, y^0 \rangle$. Using this relation, the monotone behavior of the iterates immediately follows from the definition of the operators (2) and (3).

(b) We show by induction that

$$F(x^j) \leq 0 \leq F(y^j), \quad j = 0, 1, \ldots . \tag{7}$$

Then the assertion follows from the inverse isotonicity of F. By assumption, (7) holds for $j = 0$. Suppose that (7) is true for $j = k$.
First consider the upper iterate y^{k+1}. By (6) we have

$$F_i(y^{k+1}) \geq F_i(y^k) + \sum_{j \in L^k} \partial_j F_i(y^k)(y_j^{k+1} - y_j^k), \quad i = 1, \ldots, n,$$

and hence, because $\partial_j F_i(y^k) \leq 0$ if $i \neq j$,

$$\begin{aligned} F_i(y^{k+1}) &\geq F_i(y^k) \geq 0, & i \notin L^k, \\ F_i(y^{k+1}) &\geq F_i(y^k) + \partial_i F_i(y^k)(y_i^{k+1} - y_i^k), & i \in L^k. \end{aligned}$$

Thus, in the case $i \in L^k$, from (4) we obtain

$$F_i(y^{k+1}) \geq F_i(y^k) - \partial_i F_i(y^k)(\partial_i F_i(y^{s(k)}))^{-1} F_i(y^{s(k)}).$$

Using $y^k \leq y^{s(k)}$, it follows from the isotonicity of F' that

$$\partial_i F_i(y^k) \leq \partial_i F_i(y^{s(k)}), \tag{8}$$

and, since F is off-diagonally antitone, that $F_i(y^{s(k)}) \leq F_i(y^k)$. Hence, $F_i(y^{k+1}) \geq 0$ holds for $i \in L^k$, too.

Next consider x^{k+1}. We show first that $x_i^{k+1} \leq y_i^k$, $i = 1, \ldots, n$. Obviously this holds for $i \notin M^k$, because $x_i^{k+1} = x_i^k$. In the case $i \in M^k$ we start from the relation $y_i^k \geq y_i^k - (\partial_i F_i(y^{s(k)}))^{-1} F_i(y^k)$, which holds by inductive assumption. Adding (5) to this relation, we find

$$y_i^k \geq x_i^{k+1} + (y_i^k - x_i^k) + (\partial_i F_i(y^{s(k)}))^{-1}(F_i(x^{s(k)}) - F_i(y^k)).$$

Since F is off-diagonally antitone, $x^{s(k)} \leq x^k$ ensures

$$F_i(x^{s(k)}) \geq F_i(x^k). \tag{9}$$

Hence

$$y_i^k \geq x_i^{k+1} + (y_i^k - x_i^k) + (\partial_i F_i(y^{s(k)}))^{-1}(F_i(x^k) - F_i(y^k)). \tag{10}$$

Due to (6), the inductive assumption and again off-diagonal isotonicity, it follows that

$$F_i(x^k) - F_i(y^k) \geq \sum_{j=1}^{n} \partial_j F_i(y^k)(x_j^k - y_j^k) \geq \partial_i F_i(y^k)(x_i^k - y_i^k).$$

Together with (10) we get

$$y_i^k \geq x_i^{k+1} + [1 - (\partial_i F_i(y^{s(k)}))^{-1} \partial_i F_i(y^k)](y_i^k - x_i^k),$$

and, because of (8), $y_i^k \geq x_i^{k+1}$. Now, an analogous argument as for y^{k+1} leads to $F(x^{k+1}) \leq 0$: By (6) we have

$$F_i(x^{k+1}) \leq F_i(x^k) + \sum_{j \in M^k} \partial_j F_i(x^{k+1})(x_j^{k+1} - x_j^k), \quad i = 1, \ldots, n,$$

and hence, because $\partial_j F_i(x^{k+1}) \leq 0$ if $i \neq j$,

$$\begin{array}{ll} F_i(x^{k+1}) \leq F_i(x^k) \leq 0, & i \notin M^k, \\ F_i(x^{k+1}) \leq F_i(x^k) + \partial_i F_i(x^{k+1})(x_i^{k+1} - x_i^k), & i \in M^k. \end{array}$$

Thus, in the case $i \in M^k$, from (5) we obtain

$$F_i(x^{k+1}) \leq F_i(x^k) - \partial_i F_i(x^{k+1})(\partial_i F_i(y^{s(k)}))^{-1} F_i(x^{s(k)}).$$

Using $x^{k+1} \le y^k \le y^{s(k)}$, it follows from the isotonicity of F' that

$$\partial_i F_i(x^{k+1}) \le \partial_i F_i(y^{s(k)}).$$

Hence, together with (9), $F_i(x^{k+1}) \le 0$ holds for $i \in M^k$, too.

(c) Let i be an arbitrary, but fixed element of $\{1, \dots, n\}$. We number through the elements of the infinite set $\{k \in \mathbb{N} : i \in I^k\}$, so that $\{k \in \mathbb{N} : i \in I^k\} = \{k^i_j, j = 1, 2, \dots\}$, where the sequence $\{k^i_j\}$ is strictly monotonically increasing. Analogously we represent the sets $\{k \in \mathbb{N} : i \in L^k\}$ and $\{k \in \mathbb{N} : i \in M^k\}$ by the sequences $\{l^i_j\}$ and $\{m^i_j\}$, respectively. We show by induction that for any element l^i_j there exists a successor l^i_{j+1}. At the beginning $l^i_1 = k^i_1$ holds, because of

$$F_i(y^{s(k^i_1)}) = F_i(y_1^{s_1(k^i_1)}, \dots, y_{i-1}^{s_{i-1}(k^i_1)}, y_i^0, y_{i+1}^{s_{i+1}(k^i_1)}, \dots, y_n^{s_n(k^i_1)}) \ge F_i(y^0) \ge 0. \tag{11}$$

Assume that $l^i_j = k^i_p$.
By Definition 1, for all $h \in \{1, \dots, n\}$ there exists a number $q_h \in \mathbb{N}$, so that $s_h(k) \ge l^i_j + 1$ for all $k \ge q_h$. Set $q = \max_h q_h$ and $r = \min\{t \in \mathbb{N} : k^i_t \ge q\}$. The number r exists because the k^i_t form an infinite sequence.
Then $l^i_{j+1} \in \{k^i_{p+1}, \dots, k^i_r\}$; since, if $l^i_{j+1} > k^i_{r-1}$, then

$$F_i(y^{s(k^i_r)}) = F_i(y_1^{s_1(k^i_r)}, \dots, y_{i-1}^{s_{i-1}(k^i_r)}, y_i^{l^i_j+1}, y_{i+1}^{s_{i+1}(k^i_r)}, \dots, y_n^{s_n(k^i_r)}) \ge F_i(y^{l^i_j+1}) \ge 0.$$

In the same manner it can be shown that the sequence $\{m^i_j\}$ is infinite.

(d) We consider the sequence $\{y^k\}$. Up to now it is shown that there exists $\lim_{k\to\infty} y^k_i = \lim_{j\to\infty} y_i^{l^i_j} = z_i \ge x^*_i$, $i = 1, \dots, n$. Since $F'(x)$ is isotone in x, we find that

$$y_i^{l^i_j} - y_i^{l^i_{j+1}} = (\partial_i F_i(y^{s(l^i_j)}))^{-1} F_i(y^{s(l^i_j)}) \ge (\partial_i F_i(y^0))^{-1} F_i(y^{s(l^i_j)}) \ge 0, \quad i = 1, \dots, n.$$

Therefore $\lim_{j\to\infty}[y_i^{l^i_j} - y_i^{l^i_{j+1}}] = 0$ and $(\partial_i F_i(y^0))^{-1} > 0$ imply $\lim_{j\to\infty} F_i(y^{s(l^i_j)}) = 0$, $i = 1, \dots, n$. Consequently, using the continuity of F together with Definition 1, we get

$$F_i(\lim_{j\to\infty} y^{s(l^i_j)}) = F_i(\lim_{k\to\infty} y^k) = F_i(z) = 0, \quad i = 1, \dots, n,$$

which shows that $z = x^*$. The proof for $\{x^k\}$ can be done analogously. $\square$

The conditions on the initial guesses x^0, y^0 can be weakend. Theorem 2 also holds if instead of $F(x^0) \le 0 \le F(y^0)$ only $x^0 \le x^* \le y^0$ is demanded. We formulate this as a theorem:

Theorem 3 *Let $F : Q \subseteq \mathbb{R}^n \to \mathbb{R}^n$. Assume that there exist $x^*, x^0, y^0 \in Q$ so that*

$$F(x^*) = 0, \quad x^0 \le x^* \le y^0, \quad \langle x^0, y^0 \rangle \subseteq Q.$$

Suppose that F is Frechét-differentiable on $\langle x^0, y^0 \rangle$, that $F'(x)$ is a nonsingular M-matrix for each $x \in \langle x^0, y^0 \rangle$, and that $F'(x) : \langle x^0, y^0 \rangle \to \mathbb{R}^{n\times n}$ is an isotone mapping in x. Then the iterates $\{x^k\}$ and $\{y^k\}$ of an asynchronous iteration for the operators (2) and (3) satisfy $y^k \downarrow x^*$ and $x^k \uparrow x^*$, respectively.

Proof: We proceed as in the proof of Theorem 2. Clearly, part (a) holds. For the sake of brevity we denote by $(y^{s(k)}; z_i)$ the vector $y^{s(k)}$, for which the i-th component was replaced by z_i:

$$(y^{s(k)}; z_i) = (y_1^{s_1(k)}, \ldots, y_{i-1}^{s_{i-1}(k)}, z_i, y_{i+1}^{s_{i+1}(k)}, \ldots, y_n^{s_n(k)}).$$

To show assertion (b), we proceed by induction. By assumption, we have that $x^0 \le x^* \le y^0$. Assume that $x^j \le x^* \le y^j$, $j = 0, 1, \ldots, k$.
First consider y^{k+1}. Obviously, if $i \notin L^k$, then $y_i^{k+1} = y_i^k \ge x_i^*$. If $i \in L^k$, using (6) and (4), we get

$$F_i((y^{s(k)}; y_i^{k+1})) \ge F_i((y^{s(k)}; y_i^k)) + \partial_i F_i((y^{s(k)}; y_i^k))(y_i^{k+1} - y_i^k) = 0. \tag{12}$$

Suppose that $y_i^{k+1} < x_i^*$. Then $F_i((y^{s(k)}; y_i^{k+1})) < F_i(x^*) = 0$, because F is a M-function. This is a contradiction, hence $y_i^{k+1} \ge x_i^*$ is shown.
Next consider x^{k+1}. Again it is clear that $x_i^{k+1} = x_i^k \le x_i^*$, if $i \notin M^k$. If $i \in M^k$, using (5) and the isotonicity of F', we get

$$x_i^{k+1} = x_i^k - (\partial_i F_i(y^{s(k)}))^{-1} F_i(x^{s(k)}) \le x_i^k - (\partial_i F_i(x^*))^{-1} F_i(x^{s(k)}). \tag{13}$$

From (6) we obtain

$$F_i(x^{s(k)}) \ge F_i(x^*) + \sum_{j=1}^{n} \partial_j F_i(x^*)(x_j^{s_j(k)} - x_j^*) \ge \partial_i F_i(x^*)(x_i^k - x_i^*).$$

Hence, together with (13), $x_i^{k+1} \le x_i^*$ is shown.
In the remainder of the proof we show parts (c) and (d) together. Consider first the sets L^k and the sequence $\{y^k\}$. We split the set of indices into three parts, $\{1, \ldots, n\} = A + B + C^0$, where

$$\begin{aligned} A &= \{i \in \{1, \ldots, n\} : y_i^0 = x_i^*\}, \\ B &= \{i \in \{1, \ldots, n\} : y_i^0 > x_i^* \text{ and } F_i(y_0) \ge 0\}, \\ C^0 &= \{i \in \{1, \ldots, n\} : y_i^0 > x_i^* \text{ and } F_i(y_0) < 0\}. \end{aligned}$$

If $i \in A$, then $y_i^k = x_i^*$ for all $k = 0, 1, \ldots$, because of $F_i((y^{s(k)}; x_i^*)) \le F_i(x^*) = 0$.
Without loss of generality we assume $B \ne \emptyset$. Otherwise it would follow from $F(y^0) \le 0$ that $y^0 \le x^*$, hence $A = \{1, \ldots, n\}$, which would finish the proof.
If $i \in B$, then, because of (11), $l_1^i = k_1^i$. Assume that $l_j^i = k_p^i$. Then, by (12), we have $F_i((y^{s(l_j^i)}; y_i^{l_j^i+1})) \ge 0$. By Definition 1, for all $h \in \{1, \ldots, n\}$ there exists a number q_h, so that $s_h(k) \ge s_h(l_j^i)$ for all $k \ge q_h$. Set $q = \max_h q_h$ and $r = \min\{t \in \mathbb{N} : k_t^i > q\}$. Then $l_{j+1}^i \in \{k_{p+1}^i, \ldots, k_r^i\}$; since, if $l_{j+1}^i > k_{r-1}^i$, then

$$F_i(y^{s(k_r^i)}) = F_i(y_1^{s_1(k_r^i)}, \ldots, y_{i-1}^{s_{i-1}(k_r^i)}, y_i^{l_j^i+1}, y_{i+1}^{s_{i+1}(k_r^i)}, \ldots, y_n^{s_n(k_r^i)}) \ge F_i((y^{s(l_j^i)}; y_i^{l_j^i+1}) \ge 0.$$

Thereby we have shown that $|\{k \in \mathbb{N} : i \in L^k\}| = \infty$ for all indices i, which are contained in B.
If $C^0 = \emptyset$, then part (d) of the proof of Theorem 2 can be used to complete the proof.

Assume now that $C^0 \neq \emptyset$. We will show that there exist an index $j^0 \in C^0$ and a number $c(j^0)$, so that $F_{j^0}(y^{c(j^0)}) \geq 0$.
Suppose that for all $i \in C^0$ we have that $F_i(y^k) < 0$, $k = 0, 1, \ldots$. Then $\lim_{k\to\infty} y^k = \tilde{y}$ exists. If $i \in A$, then $\tilde{y}_i = x_i^*$; if $i \in C^0$, then $\tilde{y}_i = y_i^0$. For $i \in B$ the same argument as in part (d) of proof of Theorem 2 leads to $F_i(\tilde{y}) = 0$. Since $F_i(\tilde{y}) \leq 0$, $i \in C^0$, it follows that $F(\tilde{y}) \leq 0$. This results in $\tilde{y} \leq x^*$, which contradicts $\tilde{y}_i = y_i^0 > x^*$ for $i \in C^0$.
By Definition 1, for all $h \in \{1, \ldots, n\}$ there exists a number q_h, so that $s_h(k) \geq c(j^0)$ for all $k \geq q_h$. As usual, $q = \max_h q_h$ and $r = \min\{t \in \mathbb{N} : k_t^i \geq q\}$ are defined. Then $l_1^{j^0} \leq k_r^{j^0}$; since, if $l_1^{j^0} > k_{r-1}^{j^0}$, then

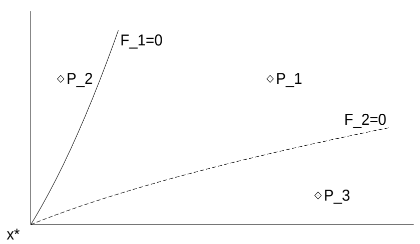
$$F_{j^0}(y^{s(k_r^{j^0})}) = F_{j^0}(y_1^{s_1(k_r^{j^0})}, \ldots, y_{j^0-1}^{s_{j^0-1}(k_r^{j^0})}, y_{j^0}^0, y_{j^0+1}^{s_{j^0+1}(k_r^{j^0})}, \ldots, y_n^{s_n(k_r^{j^0})}) \geq F_{j^0}(y^{c(j^0)}) \geq 0.$$

Analogously to the case $i \in B$ we can prove now that $|\{k \in \mathbb{N} : j^0 \in L^k\}| = \infty$.
Now we set $C^1 = C^0 \backslash \{j^0\}$, and proceed for C^1 as for C^0. If necessary, we set $C^2 = C^1 \backslash \{j^1\}$, and so on. This process is finite, since C^0 does not contain more then $n-1$ elements. This concludes the proof for L^k and $\{y^k\}$.
To prove parts (c) and (d) for the sets M^k and the sequence $\{x^k\}$ we proceed analogously. To do this, we only need the following counterpart of (12): If $i \in M^k$, then

$$\begin{aligned} F_i((x^{s(k)}; x_i^{k+1})) &\leq F_i((x^{s(k)}; x_i^k)) + \partial_i F_i((x^{s(k)}; x_i^{k+1}))(x_i^{k+1} - x_i^k) \\ &= F_i((x^{s(k)}; x_i^k)) - \partial_i F_i((x^{s(k)}; x_i^{k+1}))(\partial_i F_i((y^{s(k)}; y_i^k)))^{-1} F_i((x^{s(k)}; x_i^k)) \\ &\leq 0, \end{aligned} \tag{14}$$

which can be obtained using (6), (5), part (b) and the isotonicity of F'. □

Remark: Under the conditions of Theorem 2 the function F and the initial guesses x^0 and y^0 fulfil the assumptions of Theorem 3, too. So we could have omitted the proof of Theorem 2. Nethertheless it was additionally given because in part (b) supplementary to the assertion relation (7) was shown. This enabled us to prove part (c) much easier than in the proof of Theorem 3.



Remark: The assumption $x^0 \leq x^* \leq y^0$ is really weaker than $F(x^0) \leq 0 \leq F(y^0)$. For the case $F : Q \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ this is illustrated by the figure. If y^0 lies between the curves $F_1 = 0$ and $F_2 = 0$ (like P_1), then it fulfils both conditions, but if it is located above $F_1 = 0$ (like P_2) or below $F_2 = 0$ (like P_3), it only satisfies the first condition.

If $F'(x)$ is not isotone on $\langle x^0, y^0\rangle$, but upper bounds M_{ii} for the partial derivatives $\partial_i F_i$ are available, e.g., via interval arithmetic, then Theorems 2 and 3 still hold for the operators which arise, if in (2) and (3) the derivatives $\partial_i F_i(y)$ are replaced by the bounds $M_{ii}(x, y)$. This is given in the following theorem:

Theorem 4 *Let $F : Q \subseteq \mathbb{R}^n \to \mathbb{R}^n$ and $x^0, y^0 \in Q$ with $x^0 \leq y^0$, $\langle x^0, y^0\rangle \subseteq Q$. Suppose that F is Frechét-differentiable on $\langle x^0, y^0\rangle$ and that $F'(x)$ is a nonsingular M-matrix for each $x \in \langle x^0, y^0\rangle$. Moreover, assume that for $i = 1, \ldots, n$ there exist $M_{ii}(x^0, y^0) \in \mathbb{R}$ so that $\partial_i F_i(z) \leq M_{ii}(x^0, y^0)$ for each $z \in \langle x^0, y^0\rangle$. In addition assume either $F(x^0) \leq 0 \leq F(y^0)$ (as in Theorem 2) or that there exists $x^* \in \langle x^0, y^0\rangle$ with $F(x^*) = 0$ (as in Theorem 3). Then the sequences defined by*

$$y_i^{k+1} = \begin{cases} y_i^k - (M_{ii}(x^{s(k)}, y^{s(k)}))^{-1} F_i(y^{s(k)}) & \text{if } i \in L^k \\ y_i^k & \text{otherwise} \end{cases}, \quad k = 0, 1, \ldots, \tag{15}$$

and

$$x_i^{k+1} = \begin{cases} x_i^k - (M_{ii}(x^{s(k)}, y^{s(k)}))^{-1} F_i(x^{s(k)}) & \text{if } i \in M^k \\ x_i^k & \text{otherwise} \end{cases}, \quad k = 0, 1, \ldots, \tag{16}$$

satisfy $y^k \downarrow x^*$ and $x^k \uparrow x^*$, respectively.

Proof: Relation (6) does not hold under the assumptions of this theorem, but it can be replaced by

$$F_i(y) \leq F_i(x) + M_{ii}(x, y)(y_i - x_i), \quad i = 1, \ldots, n, \tag{17}$$

which holds for $x, y \in \langle x^0, y^0\rangle$ whenever $x \leq y$. Using (15), (16) and (17) instead of (4), (5) and (6), the assertion is proved by the same arguments as in the proofs of Theorems 2 and 3. □

It is possible to view Theorems 2 and 3 as special cases of the following general theorem on asynchronous iterations for enclosing fixed points of isotone operators, stated in [2]:

Theorem 5 *Let $Q \subseteq \mathbb{R}^n$, $Q = Q_1 \times \ldots \times Q_n$. Suppose that $H : Q \to Q$ is continuous, isotone and has an unique fixed point $x^* \in Q$. Assume that there exist $x^0, y^0 \in Q$ so that*

$$x^0 \leq y^0, \quad x^0 \leq H(x^0), \quad H(y^0) \leq y^0.$$

Then the sequences of the iterates $\{x^k\}$ and $\{y^k\}$ of an asynchronous iteration for H satisfy

$$x^k \leq x^* \leq y^k, \quad k = 0, 1, \ldots, \quad \text{and} \quad \lim_{k\to\infty} x^k = x^*, \quad \lim_{k\to\infty} y^k = x^*.$$

Note that here no monotone behavior of the iterates is stated. In general this would require the additional assumption that the sequences $\{s_i(k)\}$, $i = 1, \ldots, n$, are increasing, but in the case of Theorems 2–4 it is a trivial consequence of the special form of the operators (2), (3), (15) and (16).
To show that Theorems 2 and 3 are immediate corollaries of Theorem 5, it now suffices to prove the following lemma:

Lemma 6 *Under assumptions of Theorems 2 and 3 the operators (2) and (3) are isotone on $\langle x^*, y^0\rangle$ and $\langle x^0, x^*\rangle$, respectively.*

Proof: First consider operator (2). Let $x^* \leq y^2 \leq y^1 \leq y^0$. We have to show that $H(y^2) \leq H(y^1)$.
In case that $F_i(y^1) < 0$ it is clear that $H_i(y^2) \leq y_i^2 \leq y_i^1 = H_i(y^1)$.
On the other hand, if $F_i(y^1) \geq 0$, then we distinguish between the following two cases. If $F_i(y^2) \geq 0$, then the isotonicity of F' and (6) imply

$$\begin{aligned} H_i(y^1) - H_i(y^2) &= y_i^1 - y_i^2 + (\partial_i F_i(y^2))^{-1} F_i(y^2) - (\partial_i F_i(y^1))^{-1} F_i(y^1) \\ &\geq y_i^1 - y_i^2 + (\partial_i F_i(y^1))^{-1}(F_i(y^2) - F_i(y^1)) \\ &\geq 0. \end{aligned}$$

If $F_i(y^2) < 0$, we proceed as follows. From (12) we know $F_i((y^1; H_i(y^1))) \geq 0$. Suppose that $y_i^2 = H_i(y^2) > H_i(y^1)$, then we get $F_i(y^2) \geq F_i((y^1; y_i^2)) > F_i((y^1; H_i(y^1))) \geq 0$, because F is a M-function. This is a contradiction.
Next consider operator (3). Let $x^0 \leq x^1 \leq x^2 \leq x^*$. The operator depends not only on the lower iterate x, but on the upper iterate y, too. Because during the asynchronous iteration the sequence $\{y^k\}$ is monotonically decreasing, while the sequence $\{x^k\}$ is monotonically increasing, $x^* \leq y^2 \leq y^1 \leq y^0$ is fulfilled for the corresponding values of y^1 and y^2. The aim now is to show that $H(x^1, y^1) \leq H(x^2, y^2)$.
If $F_i(x^1) > 0$, then, obviously, $H_i(x^1, y^1) = x_i^1 \leq x_i^2 \leq H_i(x^2, y^2)$.
If, on the contrary, $F_i(x^1) \leq 0$, then we again distinguish between two cases. If $F_i(x^2) \leq 0$, then the isotonicity of F' and (6) imply

$$\begin{aligned} H_i(x^1, y^1) - H_i(x^2, y^2) &= x_i^1 - x_i^2 + (\partial_i F_i(y^2))^{-1} F_i(x^2) - (\partial_i F_i(y^1))^{-1} F_i(x^1) \\ &\leq x_i^1 - x_i^2 + (\partial_i F_i(y^2))^{-1}(F_i(x^2) - F_i(x^1)) \\ &\leq x_i^1 - x_i^2 + (\partial_i F_i(y^2))^{-1} \partial_i F_i(x^2)(x_i^2 - x_i^1) \\ &\leq 0. \end{aligned}$$

If $F_i(x^2) > 0$, then the assumption $H_i(x^2, y^2) = x^2 < H_i(x^1, y^1)$, using (14), leads to the contradiction $F_i(x^2) \leq F_i((x^1; x_i^2)) < F_i((x^1; H_i(x^1, y^1))) \leq 0$. □

Remark: Note that the unmodified Jacobi-Newton-operators, defined by (1), are isotone only on the sets $\{z \in \mathbb{R}^n : F(z) \geq 0\}$ and $\{z \in \mathbb{R}^n : F(z) \leq 0\}$, respectively, so that the assumptions of the mentioned general theorem are not fulfilled. Also from this point of view the proposed modification seems to be useful.

To conclude this section we remark that in the proof of Lemma 1 we considered asynchronous iterations via the instruction

$$\begin{aligned} y_i^{k+1} &= \begin{cases} H_i(y^{s(k)}) & \text{if } i \in I^k \\ y_i^k & \text{if } i \notin I^k \end{cases} \qquad \text{with } H_i \text{ from (2)}, \\ x_i^{k+1} &= \begin{cases} H_i(x^{s(k)}, y^{s(k)}) & \text{if } i \in I^k \\ x_i^k & \text{if } i \notin I^k \end{cases} \qquad \text{with } H_i \text{ from (3)}, \end{aligned}$$

where the same subsets I^k and n-tuples $s(k)$ are used to compute both, x^{k+1} and y^{k+1}. There are asynchronous implementations which justify the use of the more general scheme

$$\begin{aligned} y_i^{k+1} &= \begin{cases} H_i(y^{s^3(k)}) & \text{if } i \in I_2^k \\ y_i^k & \text{if } i \notin I_2^k \end{cases} && \text{with } H_i \text{ from (2)}, \\ x_i^{k+1} &= \begin{cases} H_i(x^{s^1(k)}, y^{s^2(k)}) & \text{if } i \in I_1^k \\ x_i^k & \text{if } i \notin I_1^k \end{cases} && \text{with } H_i \text{ from (3)}, \end{aligned} \tag{18}$$

see [3]. To discuss this case, we use some ideas of [3], Theorem 3.5. Set $E^1 = \langle x^0, x^* \rangle$, $E^2 = \langle x^*, y^0 \rangle$, $E = E^1 \times E^2 \times E^2$. Denote $x \in E$ by

$$x = (x_1, x_2, x_3) = (x_{1,1}, \dots, x_{1,n}, x_{2,1}, \dots, x_{2,n}, x_{3,1}, \dots, x_{3,n}).$$

In E we define the partial ordering $\leq_E$ as

$$x \leq_E y \iff x_1 \leq y_1,\ x_2 \geq y_2,\ x_3 \geq y_3.$$

Let $G : E \to E$ with

$$G_{1,i}(x) = \begin{cases} x_{1,i} - (\partial_i F_i(x_2))^{-1} F_i(x_1) & \text{if } F_i(x_1) \leq 0 \\ x_{1,i} & \text{otherwise} \end{cases},$$

$$G_{2,i}(x) = G_{3,i}(x) = \begin{cases} x_{3,i} - (\partial_i F_i(x_3))^{-1} F_i(x_3) & \text{if } F_i(x_3) \geq 0 \\ x_{3,i} & \text{otherwise} \end{cases}.$$

Then, under assumptions of Theorems 2 and 3, G is an isotone mapping on E, which is a conclusion of Lemma 1. Additionally we get

$$(x^0, y^0, y^0) \leq_E (x^*, x^*, x^*), \quad (x^0, y^0, y^0) \leq_E G(x^0, y^0, y^0), \quad G(x^*, x^*, x^*) = (x^*, x^*, x^*).$$

Define $I^k = (I_1^k \times \{1\}) \cup (I_2^k \times \{2\}) \cup (I_2^k \times \{3\})$ and

$$s(k) = (s_{1,1}(k), \dots, s_{1,n}(k), s_{2,1}(k), \dots, s_{2,n}(k), s_{3,1}(k), \dots, s_{3,n}(k))$$

with $s_{j,i}(k) = s_i^j(k)$, $j = 1, 2, 3$, $i = 1, \dots, n$. If $\{I_1^k\}$, $\{I_2^k\}$ and $\{s^1(k)\}$, $\{s^2(k)\}$, $\{s^3(k)\}$ meet the conditions of Definition 1, then $\{I^k\}$ and $\{s(k)\}$ satisfy them, too. An asynchronous iteration for G, which uses the sets I^k and the $3n$-tuples $s(k)$, describes iteration (18). Now, using Theorem 3.3 of [3], which is a more general theorem on asynchronous iterations for isotone operators than Theorem 5, we get $(x_1^k, x_2^k, x_3^k) \to (x^*, x^*, x^*)$, $k \to \infty$.

3 Numerical results

For some special functions F, which arise from the discretization of PDE's, this method was implemented on the 32-node distributed memory CM-5 at the Bergische Universität-GH Wuppertal. In comparison with synchronous methods asynchronous iterations were able to save about $10 - 20\%$ of the computing time, especially, when the work load was not balanced.

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