

# The extended Hertzian Approach for lateral loading

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## Abstract

Motivated by the structure of the normal surface stress of the extended Hertzian approach [1] given due to terms of the form  $r^{2n}(a^2-r^2)^{1/2}$  ( $n=0, 2, 4, 6, \dots$ ) it seems attractive to evaluate the complete elastic field also for shear loadings of this form. The reason for this lays in the demand for analytical tools for the description of mixed loading conditions as they appear for example in scratch experiments.

- [1] N. Schwarzer, "Elastic Surface Deformation due to Indenters with Arbitrary symmetry of revolution", J. Phys. D: Appl. Phys., 37 (2004) 2761-2772

## Aim of the paper

Thus, the aim of this paper is to derive the complete potential functions for surface shearing loads of the type:

$$\begin{aligned}\tau_{xz0}(r, \varphi) &= \sum_{n=0}^N c_{\tau xn} r^n \sqrt{a^2 - r^2} \\ \tau_{yz0}(r, \varphi) &= \sum_{n=0}^N c_{\tau yn} r^n \sqrt{a^2 - r^2}\end{aligned}\tag{1}$$

with arbitrary constants  $c$ . The evaluation for  $n=0,2,4,6$  has been explicitly performed by the author and is given in this paper. However, it should be noted here, that by following the instructions of the mathematical procedures given also complete potential functions for even higher  $n$  can be derived rather easily.

## Stress and Displacement in a Transversal Isotropic Half-Space

In this chapter we will represent the principle equations for the potential function formulation for transverse isotropy. We use the notation of Fabrikant [2] and take the  $z$  axis as the axis of material symmetry. The following stress strain relations may be given than

$$\sigma_{xx} = A_{11} \frac{\partial u}{\partial x} + (A_{11} - 2A_{66}) \frac{\partial v}{\partial y} + A_{13} \frac{\partial w}{\partial z},\tag{2}$$

$$\sigma_{yy} = (A_{11} - 2A_{66}) \frac{\partial u}{\partial x} + A_{11} \frac{\partial v}{\partial y} + A_{13} \frac{\partial w}{\partial z},\tag{3}$$

$$\sigma_{zz} = A_{13} \frac{\partial u}{\partial x} + A_{13} \frac{\partial v}{\partial y} + A_{33} \frac{\partial w}{\partial z}, \quad (4)$$

$$\tau_{xy} = A_{66} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \quad (5)$$

$$\tau_{xz} = A_{44} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right), \quad (6)$$

$$\tau_{yz} = A_{44} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right). \quad (7)$$

Here  $u$ ,  $v$  and  $w$  are the displacements in  $x$ -,  $y$ - and  $z$ -direction and  $A_{11}$ ,  $A_{13}$ ,  $A_{33}$ ,  $A_{44}$ ,  $A_{66}$  are the elastic constants of the transversal isotropic medium. Further the symbols  $\sigma_{xx}$ ,  $\sigma_{yy}$ ,  $\sigma_{zz}$  denote the normal and  $\tau_{xy}$ ,  $\tau_{xz}$ ,  $\tau_{yz}$  the shearing stress components.

We define the following operators

$$\Lambda = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \quad ; \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad (8)$$

with

$$i = \sqrt{-1}.$$

For practical purpose it is useful to choose cylindrical co-ordinates, so we may rewrite these operators in cylinder co-ordinates using

$$x = r \cos \varphi; \quad y = r \sin \varphi; \quad z = z$$

and

$$r = \sqrt{x^2 + y^2}; \quad \varphi = \arctan \left[ \frac{y}{x} \right]$$

and obtain with

$$\frac{\partial r}{\partial x} = \frac{x}{r}; \quad \frac{\partial r}{\partial y} = \frac{y}{r}; \quad \frac{\partial \varphi}{\partial x} = -\frac{y}{r^2}; \quad \frac{\partial \varphi}{\partial y} = \frac{x}{r^2}$$

the following results:

$$\begin{aligned} \Lambda &= \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} = \left( \frac{\partial r}{\partial x} + i \frac{\partial r}{\partial y} \right) \frac{\partial}{\partial r} + \left( \frac{\partial \varphi}{\partial x} + i \frac{\partial \varphi}{\partial y} \right) \frac{\partial}{\partial \varphi} \\ &= \left( \frac{x}{r} + i \frac{y}{r} \right) \frac{\partial}{\partial r} + \left( \frac{-y}{r^2} + i \frac{x}{r^2} \right) \frac{\partial}{\partial \varphi} = e^{i\varphi} \frac{\partial}{\partial r} + \frac{e^{i\left(\varphi + \frac{\pi}{2}\right)}}{r} \frac{\partial}{\partial \varphi} \\ &= e^{i\varphi} \left( \frac{\partial}{\partial r} + \frac{i}{r} \frac{\partial}{\partial \varphi} \right), \end{aligned}$$

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2},$$

as well as

$$\Lambda^2 = e^{2i\varphi} \left( \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + 2i \left( \frac{1}{r} \frac{\partial^2}{\partial r \partial \varphi} - \frac{1}{r^2} \frac{\partial}{\partial \varphi} \right) - \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right).$$

The solution for the equilibrium condition of the theory of elasticity in the transversal isotropic case ([2], p. 72)

$$\frac{1}{2}(A_{11} + A_{66})\Delta u^c + A_{44} \frac{\partial^2 u}{\partial z^2} + \frac{1}{2}(A_{11} - A_{66})\Lambda^2 \bar{u}^c + (A_{13} + A_{44})\Lambda \frac{\partial w}{\partial z} = 0,$$

$$A_{44}\Delta w + A_{33} \frac{\partial^2 w}{\partial z^2} + \frac{1}{2}(A_{13} + A_{44}) \frac{\partial}{\partial z} (\bar{\Lambda} u^c + \Lambda \bar{u}^c) = 0,$$

with

$$u + iv \equiv u^c$$

may be given now as follows

$$u + iv \equiv u^c = \Lambda(F_1 + F_2 + iF_3); \quad w = m_1 \frac{\partial F_1}{\partial z} + m_2 \frac{\partial F_2}{\partial z}. \quad (9)$$

The functions  $F_1, F_2, F_3$  satisfy the Laplacian-like relation

$$\Delta F_k + \gamma_k^2 \frac{\partial^2 F_k}{\partial z^2} = 0 \quad ; \quad k = 1, 2, 3. \quad (10)$$

While  $\gamma_3$  is given through  $\gamma_3^2 = A_{44}/A_{66}$  the  $\gamma_k$  ( $k=1, 2$ ) have to be obtained from  $\gamma_k^2 = n_k$ , whereas  $n_k$  denote the two (real or conjugate complex) roots of the equation

$$A_{11}A_{44}n^2 + [A_{13}(A_{13} + 2A_{44}) - A_{11}A_{33}]n + A_{33}A_{44} = 0. \quad (11)$$

The constants  $m_k$  ( $k=1, 2$ ) are related to the  $\gamma_k$  as

$$\frac{A_{11}\gamma_k^2 - A_{44}}{A_{13} + A_{44}} = \frac{(A_{13} + A_{44})\gamma_k^2}{A_{33} - \gamma_k^2 A_{44}} = m_k \quad (12)$$

In addition we define the constant  $H$  as

$$H = \frac{(\gamma_1 + \gamma_2)A_{11}}{2\pi(A_{11}A_{33} - A_{13}^2)} \quad (13)$$

In order to simplify the stress field Fabrikant [2] used the following combinations

$$\sigma_1 = \sigma_{xx} + \sigma_{yy} = \sigma_{rr} + \sigma_{\varphi\varphi}, \quad \sigma_2 = \sigma_{xx} - \sigma_{yy} + 2i\tau_{xy} = e^{2i\varphi} (\sigma_{rr} - \sigma_{\varphi\varphi} + 2i\tau_{r\varphi}), \quad \tau_z = \tau_{xz} + i\tau_{yz} = e^{i\varphi} (\tau_{rz} + i\tau_{\varphi z}).$$

Thus, one is able to rewrite the stress strain relations as

$$\sigma_1 = (A_{11} - A_{66})(\bar{\Lambda}u + \Lambda\bar{u}) + 2A_{13} \frac{\partial w}{\partial z}, \quad (14)$$

$$\sigma_2 = 2A_{66}\Lambda u, \quad (15)$$

$$\sigma_{zz} = \frac{1}{2}A_{13}(\bar{\Lambda}u + \Lambda\bar{u}) + A_{33}\frac{\partial w}{\partial z}, \quad (16)$$

$$\tau_z = A_{44}\left(\frac{\partial u}{\partial z} + \Lambda w\right) \quad (17)$$

and, by using the identity (10), obtains finally

$$\sigma_1 = 2A_{66}\frac{\partial^2}{\partial z^2}\left(\left[\gamma_1^2 - (1+m_1)\gamma_3^2\right]F_1 + \left[\gamma_2^2 - (1+m_2)\gamma_3^2\right]F_2\right), \quad (18)$$

$$\sigma_2 = 2A_{66}\Lambda^2(F_1 + F_2 + iF_3), \quad (19)$$

$$\sigma_{zz} = A_{44}\frac{\partial^2}{\partial z^2}\left(\gamma_1^2(1+m_1)F_1 + \gamma_2^2(1+m_2)F_2\right), \quad (20)$$

$$\tau_z = A_{44}\Lambda\frac{\partial}{\partial z}\left((1+m_1)F_1 + (1+m_2)F_2 + iF_3\right). \quad (21)$$

A very good discussion about the formulae presented as well as the properties of the constants they obtain was given by Hanson and Wang [3].

### The Point Force Solution For the Transverse Isotropic Half-Space $z \geq 0$

Again we follow here the results of Fabrikant [2], pp. 77 – 79. A concentrated force with the components T ( $T=t_x+it_y$ ) and P shall be applied at the point  $N_0(x_0, y_0, z=0)=N_0(r_0, \varphi_0, z=0)$ .

The field of stresses and displacements in the elastic half-space may be evaluated from the following potential functions:

$$\begin{aligned} F_1 &= \frac{H\gamma_1}{(m_1-1)}\left[P\ln[R_1+z_1] + \frac{\gamma_2}{2}(\bar{T}\Lambda + T\bar{\Lambda})\chi_1\right], \\ F_2 &= \frac{H\gamma_2}{(m_2-1)}\left[P\ln[R_2+z_2] + \frac{\gamma_1}{2}(\bar{T}\Lambda + T\bar{\Lambda})\chi_2\right], \\ F_3 &= i\frac{\gamma_3}{4\pi A_{44}}(T\bar{\Lambda} - \bar{T}\Lambda)\chi_3, \end{aligned} \quad (22)$$

with

$$\begin{aligned} R_k &= R(z_k), \quad z_k = \frac{z}{\gamma_k}, \\ \chi_k &= z_k \ln[R_k + z_k] - R_k, \quad k=1,2,3 \end{aligned}$$

where R is given as

$$R(x, x_0, y, y_0, z) = \sqrt{(x-x_0)^2 + (y-y_0)^2 + z^2}, \quad (23)$$

respectively in cylinder co-ordinates

$$R(r, r_0, \varphi, \varphi_0, z) = \sqrt{r^2 + r_0^2 - 2rr_0 \cos[\varphi - \varphi_0] + z^2}. \quad (24)$$

## Calculation of the Potential Function

In order to obtain a variety of potential functions for different indenter shapes also in the case of tangential loading, we start with the following “extended Hertzian” approach for a singularity free shear load distribution below the indenter:

$$\tau_{z0}(r, \varphi) = c_n r^n \sqrt{a^2 - r^2}. \quad (25)$$

where the following normalisation condition needs to be satisfied:

$$t_x + it_y = \int_0^{2\pi} \int_0^a \tau_{z0} r dr d\varphi. \quad (26)$$

The resulting potentials then can be calculated from equation (22) with P=0 and

$$\begin{aligned} \chi(r, \varphi, z) &= \int_0^{2\pi} \int_0^a \tau_{z0}(r_0, \varphi_0) \left[ z \ln(R(r, r_0, \varphi, \varphi_0, z) + z) - R(r, r_0, \varphi, \varphi_0, z) \right] r_0 dr_0 d\varphi_0 \\ &\equiv z * \Psi - \Phi \end{aligned} \quad (27)$$

To solve the double integral we make use of Fabrikant’s method [2] and replace the terms containing R after differentiation with respect to z and obtaining 1/R by an integral representation of the form:

$$\frac{1}{R} = \frac{2}{\pi} \int_0^{l_1} \frac{dx}{\{(r^2 - x^2)(\rho^2 - g^2(x))\}^{1/2}} L\left(\frac{x^2}{r\rho}, \varphi - \varphi_0\right), \quad (28)$$

where

$$\begin{aligned} L\left(\frac{x^2}{r\rho}, \varphi - \varphi_0\right) &= \frac{1 - \left(\frac{x^2}{r\rho}\right)^2}{1 + \left(\frac{x^2}{r\rho}\right)^2 - 2\frac{x^2}{r\rho} \cos(\varphi - \varphi_0)} \\ g^2(x) &= x^2 \left[ 1 + \frac{z^2}{r^2 - x^2} \right] \\ l_1 &= \frac{1}{2} \left( \sqrt{(r+a)^2 + z^2} - \sqrt{(r-a)^2 + z^2} \right). \end{aligned} \quad (29)$$

Integration with respect to  $\varphi_0$  yields:

$$\frac{\partial}{\partial z} \Psi(r, \varphi, z) = 4 \int_0^{l_1} \frac{dx}{(r^2 - x^2)^{1/2}} \int_{g(x)}^a \left[ \frac{\rho^n (a^2 - \rho^2)^{1/2}}{(\rho^2 - g^2(x))^{1/2}} \right] \rho d\rho. \quad (30)$$

Because the process of further calculation would become relatively cumbersome for odd numbers of n we here consider only even n. The second integration results in:

$$\frac{a \sqrt{\pi} \Gamma\left(-\frac{n}{2} - \frac{1}{2}\right) {}_2F_1\left(-\frac{1}{2}, \frac{n+2}{2}; \frac{n+3}{2}; \frac{g^2}{a^2}\right) g^{n+1}}{2\Gamma\left(-\frac{n}{2}\right)} +$$

$$\frac{a^{n+2} \sqrt{\pi} \Gamma\left(\frac{n+1}{2}\right) {}_2F_1\left(\frac{1}{2}, -\frac{n}{2} - 1; \frac{1}{2} - \frac{n}{2}; \frac{g^2}{a^2}\right)}{4\Gamma\left(\frac{n}{2} + 2\right)} \quad (31)$$

where we have set  $g=g(x)$  and  ${}_2F_1$  stands for the hypergeometric function defined as:

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}$$

In order to avoid expressions of hypergeometric and gamma functions, the further evaluation will be performed for concrete even  $n$  up to  $n=6$ . This would give us the opportunity to use approaches as:

$$\tau_{rz0}(r) = (c_{\tau 0} + c_{\tau 2} r^2 + c_{\tau 4} r^4 + c_{\tau 6} r^6) * \sqrt{a^2 - r^2}$$

for the shearing traction below the indenter. The results are given below.

If  $T=t_x+it_y$  gives the total force the necessary normalisation for the tangential stress distribution on the surface is determined due to equation (26) with the result:

$$\tau_{rz0}(r, a) = \frac{T 315 \sqrt{a^2 - r^2} [1 + c_{\tau 2} r^2 + c_{\tau 4} r^4 + c_{\tau 6} r^6]}{2\pi a^3 (105 + 42c_{\tau 2} a^2 + 24c_{\tau 4} a^4 + 16c_{\tau 6} a^6)} \quad (32)$$

where we have set all  $c_{\text{odd}}=c_{n>6}=0$  and  $c_{\tau 0}=1$ .

Substitution of (31) into (30) leaves us with one remaining integration to obtain the first derivative of  $\Psi$  with respect to  $z$ . For  $n=0$  the calculation has been first performed by Fabrikant [4], Appendix A (normal load) and Hanson [5] (tangential load). For other  $n$  we give the results of the integration below. Here the function  $\Phi$  can be obtained due to

$\Phi = \int \frac{\partial \Psi}{\partial z} z dz$ . Further integrating of  $\frac{\partial \Psi}{\partial z}$  with respect to  $z$  provides us with the complete

function  $\Psi$  already given in [1]. The latter potential is also necessary for the extended Hertzian approach with pure normal loading.

**n=2:**

$$\frac{\partial \Psi}{\partial z} = \frac{\pi}{32} \left[ (8a^4 + 8(x^2 - 2z^2)a^2 - 9x^4 - 24z^4 + 72x^2z^2) \tan^{-1}\left(\frac{l_1}{\sqrt{x^2 - l_1^2}}\right) + \right.$$

$$\frac{1}{(x^2 - l_1^2)^{3/2}} (6l_1^7 - (8a^2 + 3(x^2 + 8z^2))l_1^5 +$$

$$4(-3x^4 + 24z^2x^2 - 8z^4 + 4a^2(x-z)(x+z))l_1^3 +$$

$$\left. x^2(9x^4 - 72z^2x^2 + 24z^4 - 8a^2(x^2 - 2z^2))l_1) \right],$$

$$\Phi = \frac{1}{384} \pi \left( 3 (128 a^4 + 420 r^2 a^2 + 105 r^4) \tan^{-1} \left( \frac{l_1}{\sqrt{r^2 - l_1^2}} \right) r^2 - \right. \\ \left. 6 \tan^{-1} \left( \frac{l_1}{\sqrt{r^2 - l_1^2}} \right) (8 l_1^6 - 4 (4 a^2 + 15 r^2 - 30 l_2^2) l_1^4 + 15 (7 r^4 + 8 l_2^4) l_1^2 + 8 l_2^6 - 4 (4 a^2 + 15 r^2) l_2^4 + 105 r^4 l_2^2) + \right. \\ \left. \frac{1}{l_1} (\sqrt{r^2 - l_1^2} (64 a^6 - 64 l_2^2 a^4 + 48 l_2^4 a^2 - 108 l_1^6 + \right. \\ \left. 12 l_1^4 (18 a^2 + 35 r^2 - 70 l_2^2) + l_1^2 (-96 a^4 - 315 r^4 + 30 l_2^2 (16 a^2 + 21 r^2 - 12 l_2^2)))) \right),$$

**n=4:**

$$\frac{\partial \Phi}{\partial z} = \frac{\pi}{384} \\ \left( 3 (16 a^6 + 8 (x^2 - 2 z^2) a^4 + 6 (3 x^4 - 24 z^2 x^2 + 8 z^4) a^2 - 25 x^6 + 80 z^6 - \right. \\ \left. 600 x^2 z^4 + 450 x^4 z^2) \tan^{-1} \left( \frac{l_1}{\sqrt{x^2 - l_1^2}} \right) + \right. \\ \left. \frac{1}{(x^2 - l_1^2)^{5/2}} (-40 l_1^{11} + 2 (18 a^2 + 35 x^2 + 90 z^2) l_1^9 + \right. \\ 3 (8 a^4 - 6 (3 x^2 + 8 z^2) a^2 - 15 (x^4 - 2 z^2 x^2 + 8 z^4)) l_1^7 + \\ ((48 z^2 - 72 x^2) a^4 - 6 (9 x^4 - 120 z^2 x^2 + 32 z^4) a^2 + \\ 23 (5 x^6 - 90 z^2 x^4 + 120 z^4 x^2 - 16 z^6)) l_1^5 + \\ x^2 (24 (3 x^2 - 4 z^2) a^4 + 42 (3 x^4 - 24 z^2 x^2 + 8 z^4) a^2 - \\ 35 (5 x^6 - 90 z^2 x^4 + 120 z^4 x^2 - 16 z^6)) l_1^3 + \\ 3 x^4 (-8 (x^2 - 2 z^2) a^4 - 6 (3 x^4 - 24 z^2 x^2 + 8 z^4) a^2 + \\ \left. 5 (5 x^6 - 90 z^2 x^4 + 120 z^4 x^2 - 16 z^6)) l_1 \right),$$

$$\Phi = \frac{1}{24576\sqrt{r^2 - l_1^2}} \left( \pi \left( l_1 (-2048 a^8 - 45045 r^8 - 1920 l_2^2 + \right. \right. \\ 128 (53 a^2 + 210 r^2 - 670 l_1^2) l_2^6 - 32 (248 a^4 + 2835 r^4 + 8982 l_1^4 - (2992 a^2 + 8505 r^2) l_1^2) l_2^4 + \\ 32 (-4008 l_1^6 + 6 (632 a^2 + 1673 r^2) l_1^4 - 5 (256 a^4 + 1995 r^4) l_1^2 + 5 (32 a^6 + 693 r^6)) l_2^2 + \\ \left. l_1^2 (4096 a^6 + 125895 r^6 + 6 l_1^2 (-2560 a^4 - 20279 r^4 - 824 l_1^4 + 4 (640 a^2 + 1907 r^2) l_1^2)) \right) + \\ 3 \tan^{-1} \left( \frac{l_1}{\sqrt{r^2 - l_1^2}} \right) \sqrt{r^2 - l_1^2} (15015 r^8 + 32 (20 l_1^8 - 8 (8 a^2 + 35 r^2 - 100 l_2^2) l_1^6 + (64 a^4 + 945 r^4 + 24 l_2^2 \\ (-32 a^2 - 105 r^2 + 95 l_2^2)) l_1^4 + (4 l_2^2 (64 a^4 + 735 r^4 + 200 l_2^4 - 6 (32 a^2 + 105 r^2) l_2^2) - 1155 r^6) \\ l_1^2 + 20 l_2^2 - 8 (8 a^2 + 35 r^2) l_2^6 + (64 a^4 + 945 r^4) l_2^4 - 1155 r^6 l_2^2)) \right) \Bigg),$$

**n=6:**

$$\frac{\partial \Phi}{\partial z} = \frac{\pi}{24576} \left( 3 (640 a^8 + 256 (x^2 - 2 z^2) a^6 + 96 (3 x^4 - 24 z^2 x^2 + 8 z^4) a^4 + 160 \right. \\ (5 x^6 - 90 z^2 x^4 + 120 z^4 x^2 - 16 z^6) a^2 - \\ \left. 35 (35 x^8 - 1120 z^2 x^6 + 3360 z^4 x^4 - 1792 z^6 x^2 + 128 z^8) \right) \\ \tan^{-1} \left( \frac{l_1}{\sqrt{x^2 - l_1^2}} \right) + \frac{1}{(x^2 - l_1^2)^{7/2}} \\ (1680 l_1^{15} - 40 (32 a^2 + 119 r^2 + 224 z^2) l_1^{13} + \\ (-576 a^4 + 320 (11 x^2 + 18 z^2) a^2 + 70 (67 x^4 + 160 z^2 x^2 + 288 z^4)) l_1^{11} - \\ (768 a^6 - 288 (5 x^2 + 8 z^2) a^4 + 160 (23 x^4 + 18 z^2 x^2 + 72 z^4) a^2 + \\ 35 (31 x^6 + 544 z^2 x^4 - 864 z^4 x^2 + 768 z^6)) l_1^9 + \\ 16 (-385 x^8 + 320 a^2 x^6 + 192 a^6 x^2 - 1408 z^8 - \\ 32 (23 a^2 - 616 x^2) z^6 + 96 (2 a^4 + 65 x^2 a^2 - 385 x^4) z^4 - \\ 32 (3 a^6 + 27 x^2 a^4 + 135 x^4 a^2 - 385 x^6) z^2) l_1^7 + \\ 2 x^2 (-2304 (x - z) (x + z) a^6 - 96 (15 x^4 - 144 z^2 x^2 + 44 z^4) a^4 - \\ 928 (5 x^6 - 90 z^2 x^4 + 120 z^4 x^2 - 16 z^6) a^2 + \\ 203 (35 x^8 - 1120 z^2 x^6 + 3360 z^4 x^4 - 1792 z^6 x^2 + 128 z^8)) l_1^5 + \\ 2 x^4 (768 (2 x^2 - 3 z^2) a^6 + 480 (3 x^4 - 24 z^2 x^2 + 8 z^4) a^4 + \\ 800 (5 x^6 - 90 z^2 x^4 + 120 z^4 x^2 - 16 z^6) a^2 - \\ 175 (35 x^8 - 1120 z^2 x^6 + 3360 z^4 x^4 - 1792 z^6 x^2 + 128 z^8)) l_1^3 + \\ 3 x^6 (-256 (x^2 - 2 z^2) a^6 - 96 (3 x^4 - 24 z^2 x^2 + 8 z^4) a^4 - \\ 160 (5 x^6 - 90 z^2 x^4 + 120 z^4 x^2 - 16 z^6) a^2 + \\ \left. 35 (35 x^8 - 1120 z^2 x^6 + 3360 z^4 x^4 - 1792 z^6 x^2 + 128 z^8)) l_1 \right),$$



$$\begin{aligned}
\Phi = & \frac{1}{491520} \left( \pi \left( 15 (32768 a^8 + 568320 r^2 a^6 + 1854720 r^4 a^4 + 1381380 r^6 a^2 + 153153 r^8) \tan^{-1} \left( \frac{l_1}{\sqrt{r^2 - l_1^2}} \right) r^2 + \right. \right. \\
& \frac{1}{l_1} \left( \sqrt{r^2 - l_1^2} (-75744 l_1^{10} + 192 (1645 a^2 + 5599 r^2 - 20470 l_2^2) l_1^8 - \right. \\
& 12 (40960 a^4 + 327327 r^4 + 1345120 l_2^4 - 10 (44672 a^2 + 108955 r^2) l_2^2) l_1^6 + \\
& 20 (16384 a^6 + 261261 r^6 - 463680 l_2^6 + 672 (662 a^2 + 1375 r^2) l_2^4 - 10 (16384 a^4 + 81081 r^4) l_2^2) l_1^4 - \\
& 5 (12288 a^8 + 459459 r^8 + 120960 l_2^8 - 6720 (56 a^2 + 99 r^2) l_2^6 + 224 (1808 a^4 + 6435 r^4) l_2^4 - 10 \\
& (16384 a^6 + 135135 r^6) l_2^2) l_1^2 + 256 a^2 (96 a^8 - 192 l_2^2 a^6 + 476 l_2^4 a^4 - 380 l_2^6 a^2 + 105 l_2^8)) - \\
& 30 \tan^{-1} \left( \frac{l_1}{\sqrt{r^2 - l_1^2}} \right) l_1 (896 l_1^{10} - 960 (4 a^2 + 21 r^2) l_1^8 + 96 (64 a^4 + 1155 r^4) l_1^6 - \\
& 16 (256 a^6 + 15015 r^6) l_1^4 + 225225 r^8 l_1^2 + 896 l_2^{10} - 320 (12 a^2 + 63 r^2 - 238 l_2^2) l_2^8 + \\
& 32 (192 a^4 + 3465 r^4 + 70 l_1^2 (-48 a^2 - 171 r^2 + 220 l_1^2)) l_2^6 + \\
& 16 (-256 a^6 - 15015 r^6 + 30800 l_1^6 + 66 (64 a^4 + 735 r^4) l_1^2) l_2^4 + \\
& \left. \left. \left. \left. \left. (225225 r^8 + 32 l_1^4 (70 (-48 a^2 - 171 r^2 + 34 l_1^2) l_2^2 + 33 (64 a^4 + 735 r^4) l_1^2)) l_2^2 \right) \right) \right) \right) \right).
\end{aligned}$$

Other derivatives of the complete function  $\Psi$  necessary to calculate the elastic field of stress, strain and displacement yield rather long and cumbersome formulae. It can be obtained and simplified from our formulae by using the results of Fabrikant [2], p. 324. In addition, the equations are also woven into software packages ([6] and [7]), which can be tested for free and allow the examination and application of the new potential functions.

## Application

As a demonstrating example we consider a hypothetical load-depth-curve of a so called ACCU-Tip nanoindentation [8] into a very thin film of only 50nm (Young's modulus 450GPa, Poisson's ratio 0.2) on silicon (Young's modulus 165GPa, Poisson's ratio 0.223). The ACCU-tip shall be of a rather non-spherical shape (fig. 1).

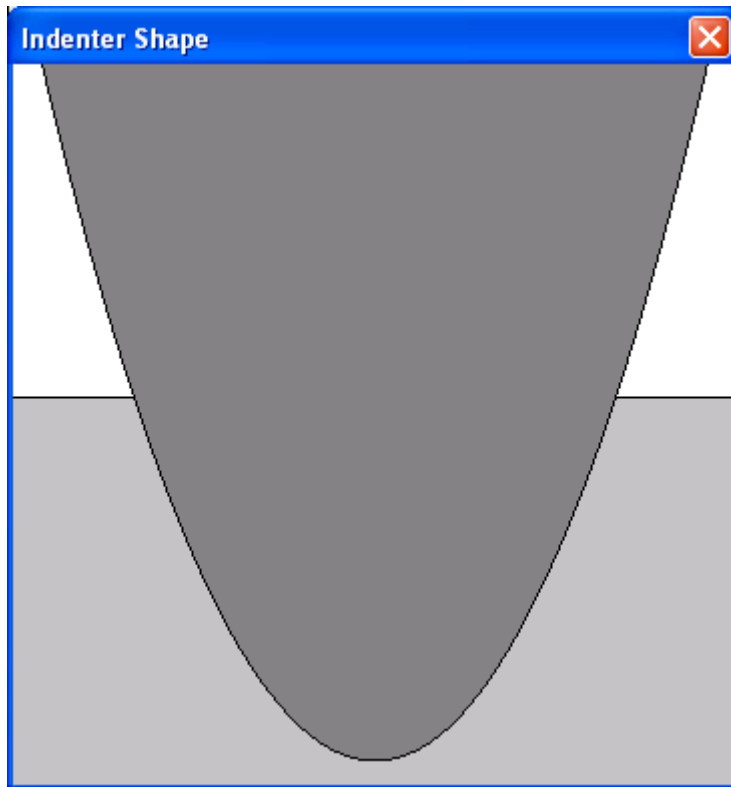


Fig. 1: Example for Indenter (ACCU-Tip) with rather non-spherical shape function within the contact zone. The shape function can be given due to  $z(r)=r^2/0.05\mu\text{m}+10^{12}\cdot r^4/(\mu\text{m})^3$ .

The normal load applied shall be  $20\mu\text{N}$  resulting in a contact radius of  $a=10,24\text{nm}$  and figure 2 show the resulting load depth curve assuming that the penetration is completely elastic. Now an additional lateral load shall be applied, thus scratching the indenter over the surface in order to induce plastic surface damage. Assuming that plastic flow is responsible for the damage the analysis of the critical lateral and normal load requires the evaluation of the von Mises stress. Figure 3 shows the von Mises stress on the surface around the contact area. In addition it might be interesting also to investigate the normal stress in moving direction of the indenter because here high tensile stresses behind the indenter could lead to mode I fracture (figure 4). Also interesting could be the distribution of work due shape-change within the stressed surface (figure 5).



Fig. 2: Load-depth-curve for the ACCU-tip of fig. 1 pressed into a coating-substrate-compound with a maximumload of 20μN.

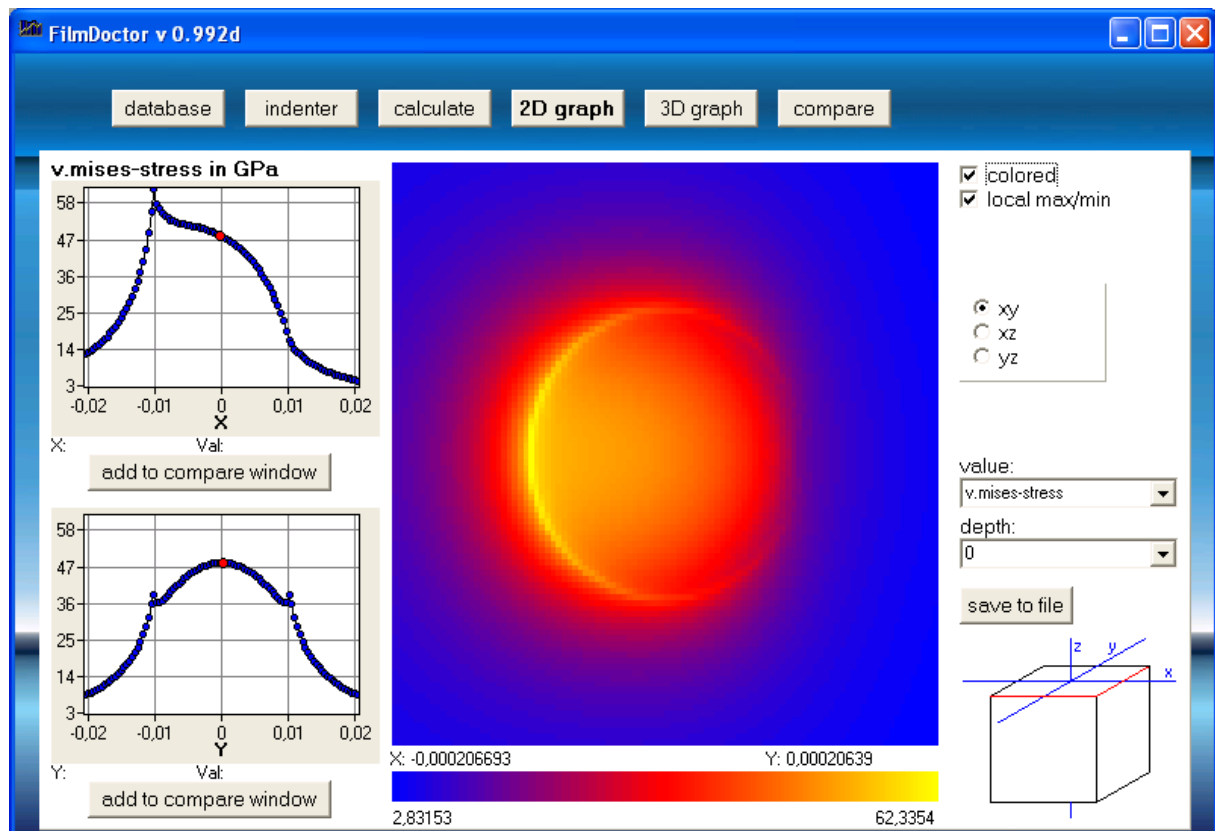


Fig. 3: Resulting von Mises stress for the maximumload shown in fig. 2. The figure shows the stress in the surface, where due to the lateral loading the maximum is to be found.

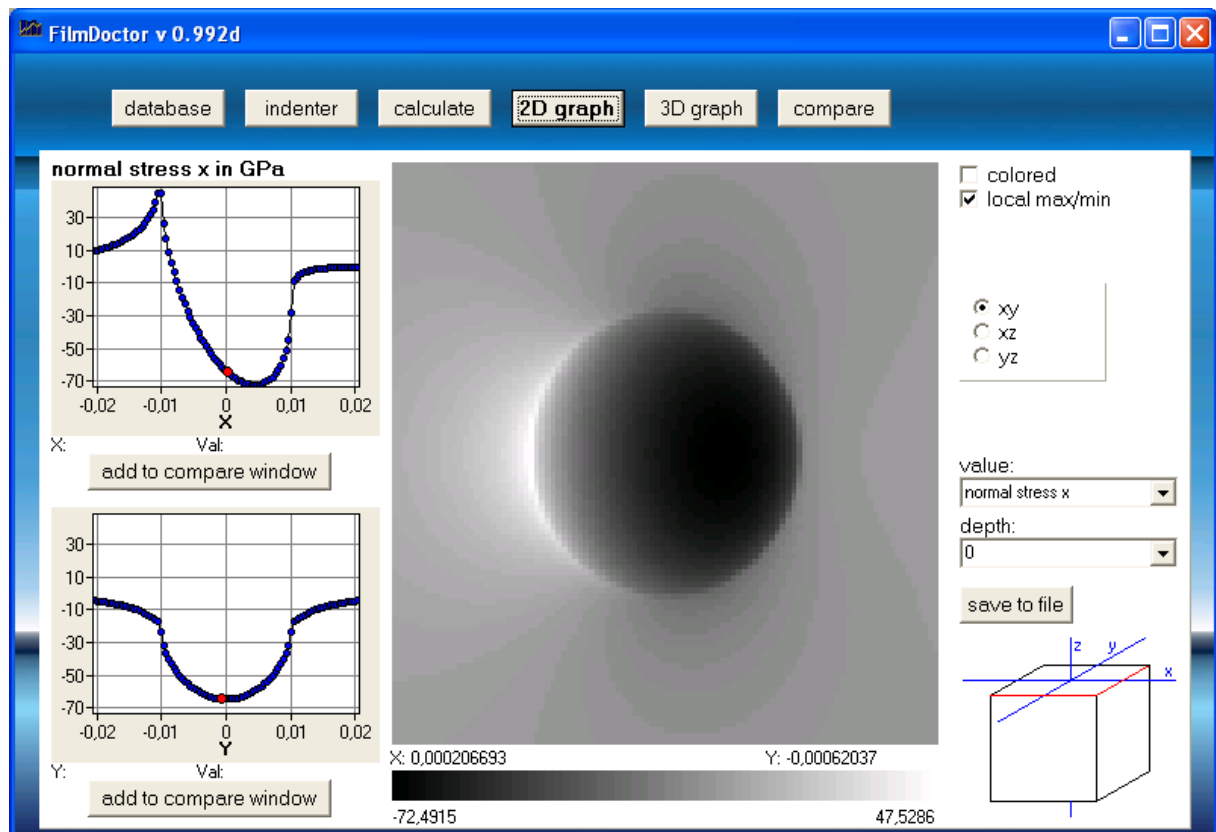


Fig. 4: Resulting normal stress in x-direction for the maximumload shown in fig. 2. The figure shows the stress in the surface, where due to the lateral loading the maximum is to be found.

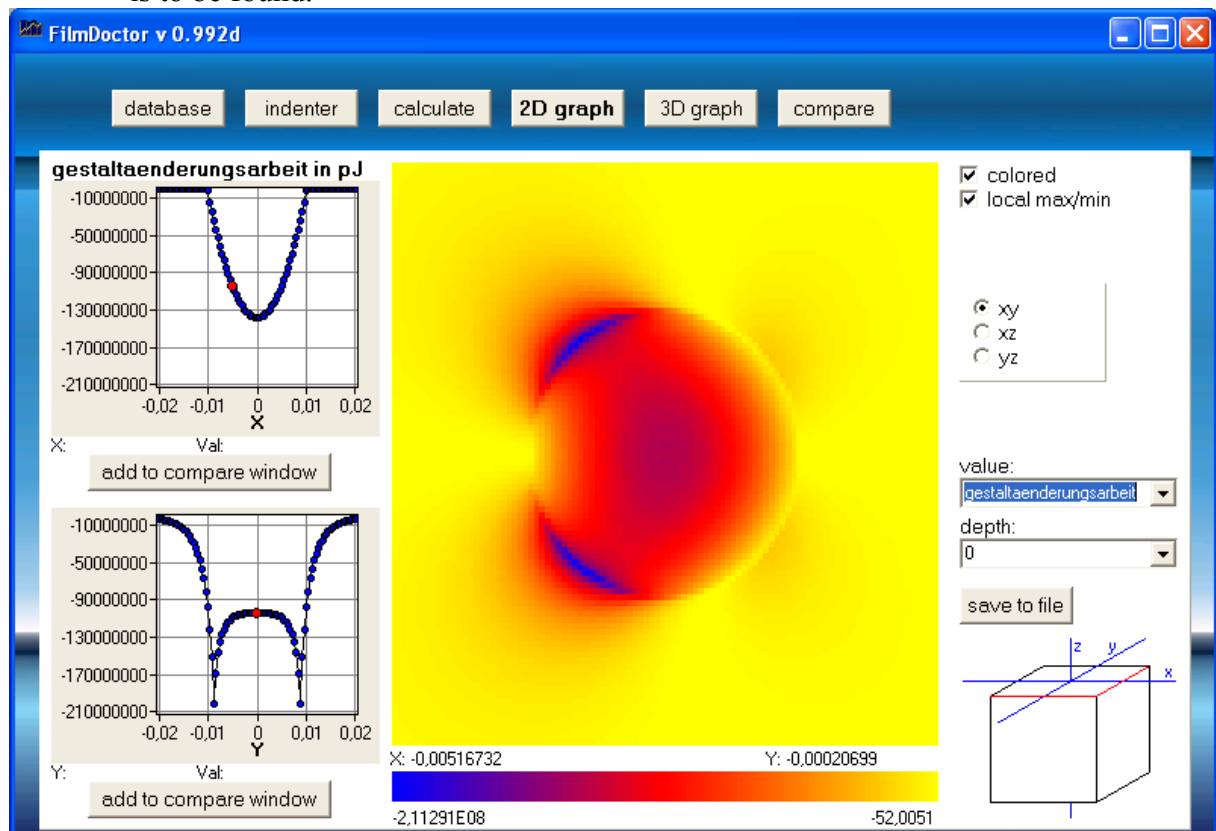


Fig. 5: Resulting work due to shape changing for the maximumload shown in fig. 2. The figure shows the stress in the surface, where due to the lateral loading the maximum is to be found.

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<http://www.esae.de/downloads> (contact: [service@esae.de](mailto:service@esae.de))
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<http://www.esae.de/downloads> (contact [service@esae.de](mailto:service@esae.de))
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