A duality approach to gap functions for variational inequalities and equilibrium problems

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M.Sc. Lkhamsuren Altangerel
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Gutachter: Prof. Dr. Gert Wanka
Prof. Dr. Petra Weidner
Prof. Dr. Rentsen Enkhbat

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To my parents

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Lkhamsuren Altangerel

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Abstract

This work aims to investigate some applications of the conjugate duality for scalar and vector optimization problems to the construction of gap functions for variational inequalities and equilibrium problems. The basic idea of the approach is to reformulate variational inequalities and equilibrium problems into optimization problems depending on a fixed variable, which allows us to apply duality results from optimization problems.

Based on some perturbations, first we consider the conjugate duality for scalar optimization. As applications, duality investigations for the convex partially separable optimization problem are discussed.

Afterwards, we concentrate our attention on some applications of conjugate duality for convex optimization problems in finite and infinite-dimensional spaces to the construction of a gap function for variational inequalities and equilibrium problems. To verify the properties in the definition of a gap function weak and strong duality are used.

The remainder of this thesis deals with the extension of this approach to vector variational inequalities and vector equilibrium problems. By using the perturbation functions in analogy to the scalar case, different dual problems for vector optimization and duality assertions for these problems are derived. This study allows us to propose some set-valued gap functions for the vector variational inequality. Finally, by applying the Fenchel duality on the basis of weak orderings, some variational principles for vector equilibrium problems are investigated.

Keywords

conjugate duality, perturbation function, convex partially separable optimization problems, variational inequalities, equilibrium problems, gap functions, dual gap functions, vector optimization, conjugate map, vector variational inequality, vector equilibrium problem, dual vector equilibrium problem, variational principle, weak vector variational inequality, Minty weak vector variational inequality

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Introduction

In connection with studying free boundary value problems, variational inequalities were first investigated by Stampacchia [80]. There is a huge literature on the subjects of variational inequalities and related problems. Specially, the books by Kinderlehrer and Stampacchia [51] and by Baiocchi and Capelo [13] provide a thorough introduction to the applications of variational inequalities in infinite-dimensional spaces. Moreover, for an overview of theory, algorithms and applications of finite-dimensional variational inequalities we refer to the survey paper by Harker and Pang [47] and the comprehensive books by Facchinei and Pang [30], [31].

From theoretical and practical point of view, the reformulation of variational inequalities into equivalent optimization problems is one of the interesting subjects in nonlinear analysis. This approach is based on the so-called gap or merit functions. Some related well known results are due to Auchmuty [7], Auslender [8], Fukushima [33], Peng [70], Yamashita, Taji and Fukushima [91], Zhu and Marcotte [99] for the variational inequality; Chen, Yang and Goh [21] for the extended variational inequality; Giannessi [38], [39] for the quasivariational inequality and Yang [93] for the prevariational inequality problems, respectively. We refer also to the survey papers by Fukushima [34] and by Larsson and Patriksson [57]. Depending on the used approaches, different classes of gap functions for variational inequalities are known as Auslender's [8]; dual [61]; regularized [7], [33], [99]; "D" or "difference" [70], [91] and Giannessi's [38], respectively.

Among the mentioned approaches, the gap function due to Giannessi [38] has been associated to the Lagrange duality for optimization problems. In order to obtain variational principles for the variational inequality problem, Auchmuty [7] used the saddle point characterization of the solution and later some properties of such gap functions were discussed by Larsson and Patriksson [57]. Duality aspects for variational inequalities (such problems are called inverse variational inequalities) were investigated by Mosco [66] and later by Chen, Yang and Goh [21]. By applying the approach due to Zhu and Marcotte [99], some relations between gap functions for the extended variational inequality and the Fenchel duality for optimization problems were studied by Chen, Yang and Goh [21]. In the context of convex optimization and variational inequalities the connections between properties of gap functions and duality have been interpreted (see [21], [48]).

According to Blum and Oettli [15], equilibrium problems provide an unified framework to the study of different problems in optimization, saddle and fixed point theory, variational inequalities etc. Some results from these fields have been extended to equilibrium problems. By using the approach of Auchmuty [7], variational principles for equilibrium problems were investigated by Blum and Oettli [14].

On the other hand, various duality schemes for equilibrium problems were developed by Konnov and Schaible [52]. Here the authors deal with the relations between solution sets of the primal and "dual problems" under generalized convexity and generalized monotonicity of the functions. One can notice that the so-called Minty variational inequality follows from the classical dual scheme for the equilibrium problem.

The vector variational inequality in a finite-dimensional space was introduced first by Giannessi [37] and some gap functions for variational inequalities have been extended to the vector case. By defining some set-valued mappings, gap functions in the sense of Auslender were extended from the scalar case to vector variational inequalities by Chen, Yang and Goh [23]. The authors introduced also a generalization of Giannessi's gap function if the ground set of vector variational inequalities is given by linear inequality constraints.

In analogy to the scalar case, vector equilibrium problems can be considered as a generalization of vector variational inequalities, vector optimization and equilibrium problems (see [4], [45] and [69]). Therefore some results established for these special cases have been extended to vector equilibrium problems. By using set-valued mappings as a generalization of the scalar case (cf. [7] and [14]) and by extending the gap functions for vector variational inequalities, variational principles for vector equilibrium problems were investigated by Ansari, Konnov and Yao [5], [6] (see also [53]).

The aim of this work is to investigate a new approach on gap functions for variational inequalities and equilibrium problems on the basis of the conjugate duality for scalar and vector optimization problems. The proposed approach is considered first for variational inequalities in finite-dimensional Euclidean spaces, afterwards this is extended to the equilibrium problems in topological vector spaces.

Before discussing the construction of some new gap functions for variational inequalities, we consider the conjugate duality for scalar optimization in connection with some additional perturbations. Dual problems related to such perturbations are so-called Fenchel-type and Fenchel-Lagrange type dual problems, respectively. Closely related to this study, we reformulate the strong duality theorem in [16] and as applications we discuss duality investigations for the convex partially separable optimization problem.

Dual problems arising from the different perturbations (see [16] and [90]) in convex optimization allow us to apply them to the construction of gap functions. The construction of a new gap function for variational inequalities is based on the following basic ideas:

- ⋄ to reduce variational inequalities into optimization problems depending on a fixed variable in the sense that both problems have the same solutions;
- ⋄ to state the corresponding dual problems;
- ⋄ to introduce a function as being the negative optimal value of the stated dual problem for any fixed variable;
- ⋄ to prove that the introduced function is a gap function for variational inequalities.

To verify the properties of a gap function for variational inequalities, weak and strong duality results are used. Under certain conditions as well as continuity and monotonicity, by using the relations between gap functions and Minty variational inequality problem, the so-called dual gap functions for the variational inequality problem are investigated.

We mention that the construction based on the Fenchel duality does not depend on the ground set of the problem. Based on this remark, our approach is extended to more general cases including variational inequalities, namely to equilibrium problems. Duality results we are going to use are recent developments on the conjugate duality theory in the settings of locally convex spaces due to Boţ and Wanka [18].

The introduced gap functions for equilibrium problems provide a convenient way of explaining as special cases the conjugate duality results for convex optimization problems and some gap functions for variational inequalities. Involving conjugate functions in the formulation of gap functions for variational inequalities and equilibrium problems, the techniques of convex analysis can be used to compute them.

By introducing some set-valued mappings in connection with the conjugate duality for vector optimization, we show that a similar approach like in the scalar case can be applied to the vector variational inequality problem. For this reason, we mention the conjugate duality theory developed by Tanino and Sawaragi [75], [82] based on the Pareto efficiency. We remark that although the objective function of the primal problem is vector-valued, by using this theory, the objective functions of the dual problems turn out to be set-valued. By applying different perturbations as in the scalar case (see [90]), we obtain some dual problems of this kind. In analogy to the scalar case, we call them the Lagrange, the Fenchel and the Fenchel-Lagrange dual problem for vector optimization, respectively.

As done in the scalar case, we show that the approach used by the construction of a gap function for the vector variational inequality can be applied to the study of variational principles for vector equilibrium problems. In order to investigate these variational principles, we use the conjugate duality theory for vector optimization problems on the basis of weak orderings developed by Tanino [84].

The thesis is organized as follows. The first chapter deals with the study of the conjugate duality for scalar optimization via perturbations. Moreover, as applications we extend duality investigations for the convex partially separable optimization problem.

In the second chapter, we apply the conjugate duality discussed in Chapter 1 to the construction of gap functions for variational inequalities and equilibrium problems. In order to introduce new gap functions, for any fixed variable we use some equivalent reformulations of variational inequalities and equilibrium problems as optimization problems which allow us to apply duality results for convex optimization.

The third chapter is devoted to the conjugate duality for vector optimization and its applications to the vector variational inequality. First, we investigate dual vector optimization problems arising from the different perturbations like in the scalar case (see [90]). Afterwards, we concentrate on special cases of the stated dual vector optimization problems which have some advantages for applications. In conclusion, we define some new gap functions for the vector variational inequality on the basis of the duality results discussed in this chapter.

In the last chapter we focus our attention on the investigation of variational principles for vector equilibrium problems related to the conjugate duality. In order to describe new variational principles, by using the Fenchel duality for vector optimization based on weak orderings, we introduce set-valued mappings depending on the data, not on the solution set of the vector equilibrium problems. A similar way is applied to variational principles for the dual vector equilibrium problem. As special cases we discuss gap functions for weak vector variational inequalities.

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Chapter 1

Conjugate duality for scalar optimization

1.1 An analysis of the conjugate duality

The concept of perturbed problems and conjugate functions provides an unified framework to duality in optimization. The related original works are due to Fenchel [32] and Moreau [65]. Later this theory was developed by Rockafellar [71] and by Ekeland and Temam [27] in finite- and infinite-dimensional spaces, respectively. For a comprehensive introduction to the conjugate duality we refer also to the book [73] by Rockafellar.

In [90], Wanka and Boţ considered three different dual problems for scalar optimization problems with inequality constraints: the well known Lagrange and Fenchel dual problems and a "combination" of both, which is the so-called Fenchel-Lagrange dual problem. More details about this approach and its applications can be found in [16], [17], [85], [89] and [90].

By construction, "weak duality" always holds, i.e. the optimal objective values of the mentioned dual problems are less than or equal to the optimal objective value of the primal problem. Under convexity assumptions, a constraint qualification guarantees the so-called "strong duality", in fact that the optimal objective values of the primal and the dual problems coincide and that the dual problems have optimal solutions.

By using the indicator function of a set, we investigate some additional dual problems. These dual problems will be so-called Fenchel-type and Fenchel-Lagrange-type dual problems, respectively. Under convexity and regularity assumptions, we study the relations between their optimal objective values.

1.1.1 Fenchel-type dual problems

Let $X \subseteq \mathbb{R}^n$ be a nonempty set and $f : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, $g = (g_1, ..., g_m)^T : \mathbb{R}^n \to \mathbb{R}^m$ be given functions. We consider the optimization problem

$$(P) \qquad \inf_{x \in X \cap G} f(x), \quad G = \{x \in \mathbb{R}^n |\ g(x) \underset{\mathbb{R}^m_+}{\leq} 0\}.$$

Further we assume that $\mathrm{dom}\ f \cap X \cap G \neq \varnothing$, where $\mathrm{dom}\ h = \{x \in \mathbb{R}^n |\ h(x) < +\infty\}$ is the effective domain of the function $h : \mathbb{R}^n \to \overline{\mathbb{R}}$. For $x, y \in \mathbb{R}^m$, $x \underset{\mathbb{R}^m_+}{\leq} y$ means

$$y - x \in \mathbb{R}^m_+ = \{z = (z_1, ..., z_m)^T \in \mathbb{R}^m |\ z_i \geq 0,\ i = \overline{1, m}\}.$$

Let us remark that throughout this work the elements in the finite-dimensional Euclidean spaces are supposed to be column vectors and $x^T y = \sum\limits_{i=1}^{n} x_i y_i$ denotes as usual the inner product of the vectors $x, y \in \mathbb{R}^n$.
By using a general perturbation approach and the theory of conjugate functions different dual problems to (P) have been derived (see [16], [90])

$$(D_L) \qquad \sup_{\substack{q \underset{\mathbb{R}^m_+}{\geqq} 0}} \inf_{x \in X} [f(x) + q^T g(x)],$$

$$(D_F) \qquad \sup_{p \in \mathbb{R}^n} \Big\{ -f^*(p) + \inf_{x \in X \cap G} p^T x \Big\}$$

and

$$(D_{FL}) \qquad \sup_{\substack{p \in \mathbb{R}^n \\ q \underset{\mathbb{R}^m_+}{\geqq} 0}} \Big\{ -f^*(p) + \inf_{x \in X} [p^T x + q^T g(x)] \Big\}.$$

By $h^*_C : \mathbb{R}^n \to \overline{\mathbb{R}}$ we denote the conjugate function of the function h relative to the set C defined by $h^*_C(\xi) = \sup\limits_{x \in C} [\xi^T x - h(x)]$. If $C = \mathbb{R}^n$, then h^*_C becomes the classical (Fenchel-Moreau) conjugate, which will be denoted by h^*. The problems (D_L) and (D_F) are the classical Lagrange and Fenchel dual problems, respectively. The dual problem (D_{FL}) is called the Fenchel-Lagrange dual and it is a "combination" of the Fenchel and Lagrange dual problems.

In this subsection we aim to discuss some dual problems to (P), which have a similar form as (D_F). By using the indicator function we can reduce the problem (P) to the equivalent form

$$(P^\delta) \qquad \inf_{x \in \mathbb{R}^n} (f + \delta_X + \delta_G)(x),$$

where $\delta_C(x)$ is the indicator function of a given set C defined by

$$\delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{if } x \notin C. \end{cases}$$

Obviously, the optimal objective values of (P) and (P^δ) coincide. Let us notice that to (D_F) associates the perturbation function $\Phi_F : \mathbb{R}^n \times \mathbb{R}^n \to \overline{\mathbb{R}}$ given by (see [16])

$$\Phi_F(x, y) = \begin{cases} f(x + y), & \text{if } x \in X \cap G, \\ +\infty, & \text{otherwise.} \end{cases}$$

We assume that each term of $f^\delta := f + \delta_X + \delta_G$ takes the same perturbation variable and let us consider all possible perturbations that we can do in the objective function f^δ. Introducing the corresponding perturbation functions, we can state different dual problems. But some dual problems related to these perturbation functions coincide with each other or they lead to (P) or (D_F). In other words, based on the following perturbation functions, we formulate some different dual problems.

(i) $\Phi_{F_1} : \mathbb{R}^n \times \mathbb{R}^n \to \overline{\mathbb{R}}$,
$\Phi_{F_1}(x, y) = f(x + y) + \delta_X(x + y) + \delta_X(x) + \delta_G(x)$,

(ii) $\Phi_{F_2} : \mathbb{R}^n \times \mathbb{R}^n \to \overline{\mathbb{R}}$,
$\Phi_{F_2}(x, y) = f(x + y) + \delta_X(x) + \delta_G(x + y) + \delta_G(x)$,

(iii) $\Phi_{F_3} : \mathbb{R}^n \times \mathbb{R}^n \to \overline{\mathbb{R}}$,
$\Phi_{F_3}(x, y) = f(x + y) + \delta_X(x + y) + \delta_G(x)$,

(iv) $\Phi_{F_4} : \mathbb{R}^n \times \mathbb{R}^n \to \overline{\mathbb{R}}$,
$\Phi_{F_4}(x, y) = f(x + y) + \delta_X(x) + \delta_G(x + y)$,

where $y \in \mathbb{R}^n$ is the perturbation variable.

The dual problems to (P^δ) can be defined by

$$(D_{F_i}) \qquad \sup_{p \in \mathbb{R}^n} \Big\{ -\Phi^*_{F_i}(0, p) \Big\},$$

where

$$\Phi^*_{F_i}(0, p) = \sup_{x, y \in \mathbb{R}^n} \Big\{ p^T y - \Phi_{F_i}(x, y) \Big\},\ i = \overline{1, 4}.$$

Calculating the conjugate functions $\Phi^*_{F_i},\ i = \overline{1, 4}$, we obtain the following four dual problems

$$(D_{F_1}) \qquad \sup_{p \in \mathbb{R}^n} \Big\{ -f^*_X(p) + \inf_{x \in X \cap G} p^T x \Big\},$$

$$(D_{F_2}) \qquad \sup_{p \in \mathbb{R}^n} \Big\{ -f^*_G(p) + \inf_{x \in X \cap G} p^T x \Big\},$$

$$(D_{F_3}) \qquad \sup_{p \in \mathbb{R}^n} \Big\{ -f^*_X(p) + \inf_{x \in G} p^T x \Big\},$$

$$(D_{F_4}) \qquad \sup_{p \in \mathbb{R}^n} \Big\{ -f^*_G(p) + \inf_{x \in X} p^T x \Big\}$$

or, equivalently,

$$(D_{F_1}) \qquad \sup_{p \in \mathbb{R}^n} \Big\{ \inf_{x \in X} [f(x) - p^T x] + \inf_{x \in X \cap G} p^T x \Big\},$$

$$(D_{F_2}) \qquad \sup_{p \in \mathbb{R}^n} \Big\{ \inf_{x \in G} [f(x) - p^T x] + \inf_{x \in X \cap G} p^T x \Big\},$$

$$(D_{F_3}) \qquad \sup_{p \in \mathbb{R}^n} \Big\{ \inf_{x \in X} [f(x) - p^T x] + \inf_{x \in G} p^T x \Big\},$$

$$(D_{F_4}) \qquad \sup_{p \in \mathbb{R}^n} \Big\{ \inf_{x \in G} [f(x) - p^T x] + \inf_{x \in X} p^T x \Big\}.$$

One can notice that the problems $(D_{F_i}),\ i = \overline{1, 4}$, are so-called Fenchel-type dual problems. As said before, the weak duality is always fulfilled and even more we have the following relation

$$\begin{array}{ccccc} v(D_{F_3}) & & & & \\ & \leq v(D_{F_1}) & & & \\ v(D_F) & & \leq v(P^\delta) = v(P), & & \\ & \leq v(D_{F_2}) & & & \\ v(D_{F_4}) & & & & \end{array} \tag{1.1}$$

where we denote by $v(P),\ v(D_F)$ and $v(D_{F_i})$ the optimal objective value of $(P),\ (D_F)$ and $(D_{F_i}),\ i = \overline{1, 4}$, respectively.

All inequalities in (1.1) can be strict. The next example shows this for some of them.

Example 1.1 Let $X = [0, +\infty)$ and the functions $f, g : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x, & x \geq 0, \\ 0, & \text{otherwise}, \end{cases} \qquad g(x) = 1 - x^2,$$

respectively. In order to find optimal objective values of $(D_{F_i}), i = \overline{1,4}$, one has to calculate the following terms:

(i) $\inf\limits_{x\in X}[f(x)-px] = \inf\limits_{x\geq 0} x(1-p) = \begin{cases} 0, & p \leq 1, \\ -\infty, & \text{otherwise}; \end{cases}$

(ii) $\inf\limits_{x\in G}[f(x)-px] = \min\left\{\inf\limits_{x\geq 1} x(1-p), \inf\limits_{x\leq -1}(-px)\right\} = \begin{cases} p, & 0 \leq p \leq 0.5, \\ 1-p, & 0.5 \leq p \leq 1, \\ -\infty, & \text{otherwise}; \end{cases}$

(iii) $\inf\limits_{x\in X\cap G} px = \inf\limits_{\substack{x\geq 0 \\ 1-x^2\leq 0}} px = \inf\limits_{x\geq 1} px = \begin{cases} p, & p \geq 0, \\ -\infty, & \text{otherwise}; \end{cases}$

(iv) $\inf\limits_{x\in X} px = \inf\limits_{x\geq 0} px = \begin{cases} 0, & p \geq 0, \\ -\infty, & \text{otherwise}; \end{cases}$

(v) $\inf\limits_{x\in G} px = \min\left\{\inf\limits_{x\geq 1} px, \inf\limits_{x\leq -1} px\right\} = \begin{cases} 0, & p = 0, \\ -\infty, & \text{otherwise}. \end{cases}$

By using the above calculations, the optimal objective values of $(D_{F_i}),\ i = \overline{1,4}$, can be obtained as follows.

$$\begin{aligned}
v(D_{F_1}) &= \sup_{p\in\mathbb{R}}\left\{\inf_{x\in X}[f(x)-px] + \inf_{x\in X\cap G} px\right\} = \sup_{0\leq p\leq 1} p = 1; \\
v(D_{F_2}) &= \sup_{p\in\mathbb{R}}\left\{\inf_{x\in G}[f(x)-px] + \inf_{x\in X\cap G} px\right\} \\
&= \max\left\{\sup_{0\leq p\leq 0.5} 2p, \sup_{0.5\leq p\leq 1} 1\right\} = 1; \\
v(D_{F_3}) &= \sup_{p\in\mathbb{R}}\left\{\inf_{x\in X}[f(x)-px] + \inf_{x\in G} px\right\} = 0; \\
v(D_{F_4}) &= \sup_{p\in\mathbb{R}}\left\{\inf_{x\in G}[f(x)-px] + \inf_{x\in X} px\right\} \\
&= \max\left\{\sup_{0\leq p\leq 0.5} p, \sup_{0.5\leq p\leq 1} (1-p)\right\} = 0.5.
\end{aligned}$$

As

$$\begin{aligned}
\inf_{x\in\mathbb{R}^n}[f(x)-px] &= \min\left\{\inf_{x\geq 0} x(1-p), \inf_{x\leq 0}(-px)\right\} \\
&= \begin{cases} 0, & 0 \leq p \leq 1, \\ -\infty, & \text{otherwise}, \end{cases}
\end{aligned}$$

the optimal objective value of the Fenchel dual problem is

$$v(D_F) = \sup_{p\in\mathbb{R}^n}\left\{-f^*(p) + \inf_{x\in X\cap G} px\right\} = \sup_{0\leq p\leq 1} p = 1.$$

Moreover, $v(P) = \inf\limits_{x\in X\cap G} f(x) = 1$. Hence, it holds

$$v(D_{F_3}) < v(D_{F_4}) < v(D_{F_1}) = v(D_{F_2}) = v(D_F) = v(P).$$

The following assertion deals with the equality relations between optimal objective values of the primal and above dual problems.

Theorem 1.1 *(see [71]) Assume that X and G are convex sets and that f is a convex function.*

(i) *If* $ri(X \cap dom\ f) \cap ri(G) \neq \varnothing$, *then* $v(D_{F_3}) = v(P)$,

(ii) *If* $ri(G \cap dom\ f) \cap ri(X) \neq \varnothing$, *then* $v(D_{F_4}) = v(P)$,

(iii) *If* $ri(X \cap G) \cap ri(dom\ f) \neq \varnothing$, *then* $v(D_F) = v(P)$,

(iv) *If* $ri(X) \cap ri(G) \cap ri(dom\ f) \neq \varnothing$, *then*

$$v(D_{F_1}) = v(D_{F_2}) = v(D_{F_3}) = v(D_{F_4}) = v(D_F) = v(P).$$

1.1.2 Fenchel-Lagrange-type dual problems

Before considering further dual problems to (P), we remark that to (D_{FL}) (see [16]) corresponds the perturbation function $\Phi_{FL} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ defined by

$$\Phi_{FL}(x, y, z) = \begin{cases} f(x + y), & \text{if } x \in X \text{ and } g(x) \underset{\mathbb{R}^m_+}{\leq} z, \\ +\infty, & \text{otherwise.} \end{cases}$$

In order to apply the same approach to (D_{FL}), we transform the problem (P) into the following equivalent form

$$(P^{\delta}_X) \qquad \inf_{x \in G} (f + \delta_X)(x).$$

Each term of the objective function for (P^{δ}_X) is supposed to take the same perturbation variable. According to all possible perturbation functions, we can obtain some dual problems having similar form as (D_{FL}). Since some of them are reduced to (D_{FL}) or (D_L), we introduce only three additional perturbation functions.

(i) $\Phi_{FL_1} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$,

$$\Phi_{FL_1}(x, y, z) = \begin{cases} f(x + y) + \delta_X(x + y) + \delta_X(x), & \text{if } g(x) \underset{\mathbb{R}^m_+}{\leq} z, \\ +\infty, & \text{otherwise.} \end{cases}$$

(ii) $\Phi_{FL_2} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$,

$$\Phi_{FL_2}(x, y, z) = \begin{cases} f(x) + \delta_X(x + y), & \text{if } g(x) \underset{\mathbb{R}^m_+}{\leq} z, \\ +\infty, & \text{otherwise.} \end{cases}$$

(iii) $\Phi_{FL_3} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$,

$$\Phi_{FL_3}(x, y, z) = \begin{cases} f(x + y) + \delta_X(x + y), & \text{if } g(x) \underset{\mathbb{R}^m_+}{\leq} z, \\ +\infty, & \text{otherwise.} \end{cases}$$

Consequently, we define the following three dual problems

$$(D_{FL_1}) \qquad \sup_{\substack{p \in \mathbb{R}^n \\ q \underset{\mathbb{R}^m_+}{\geq} 0}} \Big\{ \inf_{x \in X} [f(x) - p^T x] + \inf_{x \in X} [p^T x + q^T g(x)] \Big\},$$

$$(D_{FL_2}) \qquad \sup_{\substack{p \in \mathbb{R}^n \\ q \underset{\mathbb{R}^m_+}{\geq} 0}} \Big\{ \inf_{x \in X} [-p^T x] + \inf_{x \in \mathbb{R}^n} [p^T x + f(x) + q^T g(x)] \Big\},$$

$$(D_{FL_3}) \qquad \sup_{\substack{p \in \mathbb{R}^n \\ q \underset{\mathbb{R}^m_+}{\geq} 0}} \Big\{ \inf_{x \in X} [f(x) - p^T x] + \inf_{x \in \mathbb{R}^n} [p^T x + q^T g(x)] \Big\}.$$

As in the previous subsection, we call the problems (D_{FL_i}), $i = \overline{1, 3}$, Fenchel-Lagrange-type dual problems. If we denote by $v(D_L), v(D_{FL})$ and $v(D_{FL_i})$ the optimal objective value of (D_L), (D_{FL}) and (D_{FL_i}), $i = \overline{1,3}$, respectively, then it holds

$$\begin{array}{ccccc} v(D_{FL_2}) & & & & \\ & & & & \\ v(D_{FL}) & & & \leq v(D_L). & \\ & & \leq v(D_{FL_1}) & & \\ v(D_{FL_3}) & & & & \end{array} \tag{1.2}$$

Example 1.2 (see [16]) Let $X = [0, \infty)$ and functions $f : \mathbb{R} \to \overline{\mathbb{R}}$, $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -x^2, & x \geq 0, \\ +\infty, & \text{otherwise}, \end{cases} \qquad g(x) = x^2 - 1,$$

respectively. It is easy to verify that

$$v(D_{FL_1}) = (D_{FL_3}) = -\infty.$$

Moreover, in [16] it was shown that

$$v(D_L) = -1 \text{ and } v(D_{FL}) = -\infty.$$

It remains to compute $v(D_{FL_2})$. As

$$\begin{aligned} \inf_{x\in\mathbb{R}}[px + f(x) + qg(x)] &= \inf_{x\geq 0}[px - x^2 + q(x^2-1)] \\ &= \begin{cases} -q, & q \geq 1,\ p \geq 0, \\ -\infty, & 0 \leq q < 1 \text{ or } q = 1,\ p < 0, \\ -\frac{p^2}{4(q-1)} - q, & q > 1,\ p \leq 0 \end{cases} \end{aligned}$$

and in view of

$$\inf_{x\geq 0}(-px) = \begin{cases} 0, & p \leq 0, \\ -\infty, & \text{otherwise}, \end{cases}$$

one has

$$\begin{aligned} v(D_{FL_2}) &= \sup_{\substack{p\in\mathbb{R} \\ q\geq 0}} \left\{ \inf_{x\in X}(-px) + \inf_{x\in\mathbb{R}^n}[px + f(x) + qg(x)] \right\} \\ &= \max\left\{ \sup_{\substack{p=0 \\ q\geq 1}}(-q), \sup_{\substack{p\leq 0 \\ q>1}}\left(-\frac{p^2}{4(q-1)} - q \right) \right\} = -1. \end{aligned}$$

In conclusion, we have

$$v(D_{FL_1}) = v(D_{FL_3}) = v(D_{FL}) < v(D_{FL_2}) = v(D_L).$$

In order to state the equality relations between optimal objective values of duals introduced in this subsection, let us mention two auxiliary assertions (see Fenchel's duality theorem in [71]).

Lemma 1.1 *Assume that X is a convex set and that f, g_i, $i = \overline{1,m}$, are convex functions. Let $ri(X) \cap ri(dom\ f) \neq \varnothing$. Then for each $q \in \mathbb{R}^m$, $q \underset{\mathbb{R}^m_+}{\geq} 0$, it holds*

$$\begin{aligned} \inf_{x\in X}[f(x) + q^T g(x)] &= \sup_{p\in\mathbb{R}^n} \left\{ \inf_{x\in X}[f(x) - p^T x] + \inf_{x\in\mathbb{R}^n}[p^T x + q^T g(x)] \right\} \\ &= \sup_{p\in\mathbb{R}^n} \left\{ \inf_{x\in\mathbb{R}^n}[f(x) - p^T x] + \inf_{x\in X}[p^T x + q^T g(x)] \right\}. \end{aligned} \tag{1.3}$$

Lemma 1.2 *Assume that X is a convex set and that $f,\ g_i,\ i=\overline{1,m}$, are convex functions. Let $ri(X)\cap ri(dom\ f)\neq\varnothing$. Then for each $q\in\mathbb{R}^m,\ q\underset{\mathbb{R}^m_+}{\geq}0$, it holds*

$$\inf_{x\in X}[f(x)+q^Tg(x)]=\sup_{p\in\mathbb{R}^n}\left\{\inf_{x\in X}[-p^Tx]+\inf_{x\in\mathbb{R}^n}[f(x)+p^Tx+q^Tg(x)]\right\}. \tag{1.4}$$

Theorem 1.2 *Assume that X is a convex set and that $f,\ g_i,\ i=\overline{1,m}$, are convex functions. Let $ri(X)\cap ri(dom\ f)\neq\varnothing$. Then, it holds*

$$v(D_{FL_1})=v(D_{FL_2})=v(D_{FL_3})=v(D_{FL})=v(D_L).$$

Proof: As (1.3), (1.4) hold for each $q\in\mathbb{R}^m,\ q\underset{\mathbb{R}^m_+}{\geq}0$, and by (1.2), one has

$$v(D_{FL_1})=v(D_{FL_2})=v(D_{FL_3})=v(D_{FL})=v(D_L).$$

□

Taking into account a constraint qualification in Theorem 1.1 and Theorem 1.2, the optimal objective values of all dual problems investigated in this section turn out to be equal to each other. Specially, we can reformulate Theorem 2.8 in [16] given in the case dom $f=X$. Recall that the constraint qualification in [16] can be rewritten as

$$(CQ)\qquad \exists x'\in\mathrm{ri}(X)\cap\mathrm{ri}(\mathrm{dom}\ f):\ \left[\begin{array}{l} g_i(x')\leq 0,\ i\in L,\\ g_i(x')<0,\ i\in N.\end{array}\right.$$

Here

$$L=\{i\in\{1,...,m\}\mid g_i \text{ is an affine function}\}$$

and

$$N=\{i\in\{1,...,m\}\mid g_i \text{ is not an affine function}\}.$$

Theorem 1.3 *(see Theorem 2.8 in [16])*
Let X be a convex set and $f,\ g_i,\ i=\overline{1,m}$, be convex functions. Assume that the constraint qualification (CQ) is fulfilled. If $v(P)$ is finite then $(D_L),\ (D_F),\ (D_{FL})$ have optimal solutions and it holds

$$v(P)=v(D_L)=v(D_F)=v(D_{FL}).$$

1.2 Convex partially separable optimization problems

The study of convex partially separable optimization problems first might be appeared in [43] and [44]. According to [19], [25] and [76] by Schmidt et al., the convexity and some other conditions arising in spline approximation problems usually lead to such type problems.

In [76] and references therein, Lagrange dual problems for convex partially separable optimization problem and its particular cases were established and strong duality assertions were derived. In most of these cases, Lagrange dual problems are unconstrained and if solutions of them are known, then the solutions of the primal problems can be explicitly computed by the so-called return-formula. This is the idea which has been applied by solving tridiagonally separable optimization problems and then by different convex and monotone spline approximations problems. For details, we refer to [76] and [77].

This section aims to extend duality investigations for convex partially separable optimization problems. By using the duality results discussed in Section 1.1, we obtain different dual problems for the convex partially separable optimization problem. Moreover, we derive optimality conditions for the mentioned problem and its particular cases.

1.2.1 Problem formulation and preliminaries

Assume that $F_i : \mathbb{R}^{l_i} \to \mathbb{R}$ and $G_i : \mathbb{R}^{l_i} \to \mathbb{R}^m,\ i = \overline{1,n}$, are convex functions and $W_i \subseteq \mathbb{R}^{l_i},\ \ i = \overline{1,n}$, are convex sets. Let $A_i \in \mathbb{R}^{l_i \times (n+1)},\ l_i \in \mathbb{N}$ be given matrices. Let us introduce the following optimization problem

$$(P^{cps}) \quad \inf_{u \in W} \sum_{i=1}^{n} F_i(A_i u),$$

where

$$W = \left\{ u = (u_0, ..., u_n)^T \in \mathbb{R}^{n+1} \middle| \ \sum_{i=1}^{n} G_i(A_i u) \underset{\mathbb{R}^m_+}{\leq} 0,\ A_i u \in W_i,\ i = \overline{1,n} \right\}.$$

(P^{cps}) is called the convex partially separable optimization problem.

Introducing the auxiliary variables $v_i = A_i u \in \mathbb{R}^{l_i},\ i = \overline{1,n}$, (P^{cps}) can be rewritten as

$$(P^{cps}) \quad \inf_{v \in V} \sum_{i=1}^{n} F_i(v_i),$$

where

$$V = \left\{ v \in \mathbb{R}^k \middle| \ \sum_{i=1}^{n} G_i(v_i) \underset{\mathbb{R}^m_+}{\leq} 0,\ v_i - A_i u = 0,\ v_i \in W_i,\ i = \overline{1,n} \right\},$$

with $v = (u, v_1, ..., v_n) \in \mathbb{R}^{n+1} \times \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n}$ and $k = n + 1 + l_1 + ... + l_n$.

In order to obtain different dual problems to (P^{cps}), let us consider the convex optimization problem

$$(\widetilde{P}) \quad \inf_{x \in \widetilde{G}} \widetilde{f}(x), \quad \widetilde{G} = \{x \in \widetilde{X} |\ \widetilde{g}(x) \underset{\mathbb{R}^t_+}{\leq} 0,\ \widetilde{h}(x) = 0\},$$

where $\widetilde{X} \subseteq \mathbb{R}^l$ is a convex set, $\widetilde{f} : \mathbb{R}^l \to \overline{\mathbb{R}},\ \widetilde{g} = (\widetilde{g}_1, ..., \widetilde{g}_t)^T : \mathbb{R}^l \to \mathbb{R}^t,\ \widetilde{h} = (\widetilde{h}_1, ..., \widetilde{h}_w)^T : \mathbb{R}^l \to \mathbb{R}^w$ are given such that $\widetilde{f},\ \widetilde{g}_i,\ i = \overline{1,t}$, are convex functions and $\widetilde{h}_j,\ j = \overline{1,w}$, are affine functions.

Based on the previous section, it is clear that the corresponding dual problems to $(\widetilde{P})$ become

$$(\widetilde{D}_L) \quad \sup_{\substack{q_1 \underset{\mathbb{R}^t_+}{\geq} 0 \\ q_2 \in \mathbb{R}^w}} \inf_{x \in \widetilde{X}} \{\widetilde{f}(x) + q_1^T \widetilde{g}(x) + q_2^T \widetilde{h}(x)\},$$

$$(\widetilde{D}_F) \quad \sup_{p \in \mathbb{R}^l} \left\{ -\widetilde{f}^*(p) + \inf_{x \in \widetilde{G}} p^T x \right\}$$

and

$$(\widetilde{D}_{FL}) \quad \sup_{\substack{p \in \mathbb{R}^l, q_1 \underset{\mathbb{R}^t_+}{\geq} 0 \\ q_2 \in \mathbb{R}^w}} \left\{ -\widetilde{f}^*(p) + \inf_{x \in \widetilde{X}} [p^T x + q_1^T \widetilde{g}(x) + q_2^T \widetilde{h}(x)] \right\},$$

respectively. For $(\widetilde{P})$ the constraint qualification (CQ) looks like

$$(\widetilde{CQ}) \qquad \exists x' \in \mathrm{ri}(\widetilde{X}) \cap \mathrm{ri}(\mathrm{dom}\ \widetilde{f}) : \left[\begin{array}{l} \widetilde{g}_i(x') \leq 0,\ i \in \widetilde{L}, \\ \widetilde{g}_i(x') < 0,\ i \in \widetilde{N}, \\ \widetilde{h}_j(x') = 0,\ j = \overline{1,w}. \end{array} \right.$$

Here

$$\widetilde{L} = \{i \in \{1, ..., t\} \mid \widetilde{g}_i \text{ is an affine function}\}$$

and

$$\widetilde{N} = \{i \in \{1, ..., t\} \mid \widetilde{g}_i \text{ is not an affine function}\}.$$

Denoting by $v(\widetilde{P})$ the optimal objective value of $(\widetilde{P})$ and by $v(\widetilde{D}_L),\ v(\widetilde{D}_F),\ v(\widetilde{D}_{FL})$ the optimal objective values of $(\widetilde{D}_L),\ (\widetilde{D}_F)$ and $(\widetilde{D}_{FL})$, respectively, we have the following assertion.

Proposition 1.1 *(see Theorem 1.3)*
Assume that the constraint qualification $(\widetilde{CQ})$ is fulfilled. If $v(\widetilde{P})$ is finite then $(\widetilde{D}_L),\ (\widetilde{D}_F),\ (\widetilde{D}_{FL})$ have optimal solutions and it holds

$$v(\widetilde{P}) = v(\widetilde{D}_L) = v(\widetilde{D}_F) = v(\widetilde{D}_{FL}).$$

1.2.2 Duality for convex partially separable optimization problems

For the convex partially separable optimization problem (P^{cps}) we obtain the following dual problems, which follows from $(\widetilde{D}_L),\ (\widetilde{D}_F)$ and $(\widetilde{D}_{FL})$, respectively:

$$(D_L^{cps}) \qquad \sup_{\substack{q_i \in \mathbb{R}^{l_i}, i=\overline{1,n} \\ \sum_{i=1}^n A_i^T q_i = 0 \\ q_{n+1} \underset{\mathbb{R}^m_+}{\geq} 0}} \left\{ \sum_{i=1}^n \inf_{v_i \in W_i} [F_i(v_i) + q_{n+1}^T G_i(v_i) + q_i^T v_i] \right\},$$

$$(D_F^{cps}) \qquad \sup_{p_i \in \mathbb{R}^{l_i}, i=\overline{1,n}} \left\{ -\sum_{i=1}^n F_i^*(p_i) + \inf_{v \in V} \sum_{i=1}^n p_i^T v_i \right\}$$

and

$$(D_{FL}^{cps}) \sup_{\substack{q_i, p_i \in \mathbb{R}^{l_i}, i=\overline{1,n} \\ \sum_{i=1}^n A_i^T q_i = 0 \\ q_{n+1} \underset{\mathbb{R}^m_+}{\geq} 0}} \left\{ -\sum_{i=1}^n F_i^*(p_i) + \sum_{i=1}^n \inf_{v_i \in W_i} [(p_i + q_i)^T v_i + q_{n+1}^T G_i(v_i)] \right\}.$$

The functions F_i^* are the conjugates of $F_i,\ i = \overline{1,n}$.
Indeed, let us observe that the convex partially separable optimization problem (P^{cps}) is a particular case of $(\widetilde{P})$, namely taking

$$\left\{ \begin{array}{l} \widetilde{X} = \mathbb{R}^{n+1} \times W_1 \times \cdots \times W_n \quad \widetilde{G} = V, \\ \widetilde{f} : \mathbb{R}^k \to \mathbb{R}, \quad \widetilde{f}(v) = \sum_{i=1}^n F_i(v_i), \\ \widetilde{g} : \mathbb{R}^k \to \mathbb{R}^m, \quad \widetilde{g}(v) = \sum_{i=1}^n G_i(v_i), \\ \widetilde{h} : \mathbb{R}^k \to \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n}, \\ \widetilde{h}(v) = (v_1 - A_1 u, \cdots, v_n - A_n u)^T, \\ v = (u, v_1, ..., v_n) \in \mathbb{R}^{n+1} \times \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n}. \end{array} \right. \tag{1.5}$$

Lagrange duality. Substituting (1.5) in $(\widetilde{D}_L)$, we have

$$\begin{aligned}
&\sup_{\substack{q_1 \underset{\mathbb{R}^m_+}{\geqq} 0 \\ q_2 \in \mathbb{R}^{l_1}\times\cdots\times\mathbb{R}^{l_n} \\ q_2=(q_{21},\ldots,q_{2n})}} \inf_{v\in\widetilde{X}} \left\{ \sum_{i=1}^n F_i(v_i) + \sum_{i=1}^n q_1^T G_i(v_i) + \sum_{i=1}^n q_{2i}^T(v_i - A_i u) \right\} \\
= &\sup_{\substack{q_1 \underset{\mathbb{R}^m_+}{\geqq} 0 \\ q_2 \in \mathbb{R}^{l_1}\times\cdots\times\mathbb{R}^{l_n}}} \inf_{\substack{u\in\mathbb{R}^{n+1} \\ v_i\in W_i, i=\overline{1,n}}} \Big\{ \sum_{i=1}^n F_i(v_i) + \sum_{i=1}^n q_1^T G_i(v_i) \\
&\qquad + \sum_{i=1}^n q_{2i}^T v_i - \sum_{i=1}^n q_{2i}^T(A_i u) \Big\} \\
= &\sup_{\substack{q_1 \underset{\mathbb{R}^m_+}{\geqq} 0 \\ q_2 \in \mathbb{R}^{l_1}\times\cdots\times\mathbb{R}^{l_n}}} \Big\{ \inf_{u\in\mathbb{R}^{n+1}} u^T\Big(-\sum_{i=1}^n A_i^T q_{2i}\Big) + \sum_{i=1}^n \inf_{v_i\in W_i} [F_i(v_i) \\
&\qquad + q_1^T G_i(v_i) + q_{2i}^T v_i] \Big\}.
\end{aligned}$$

Because of $$\inf_{u\in\mathbb{R}^{n+1}} u^T\Big(-\sum_{i=1}^n A_i^T q_{2i}\Big) = \begin{cases} 0, & \text{if } \sum\limits_{i=1}^n A_i^T q_{2i} = 0, \\ -\infty, & \text{otherwise,} \end{cases} \tag{1.6}$$

we get (D_L^{cps}), by taking the dual variables $q_i := q_{2i},\ i = \overline{1,n}$, and $q_{n+1} := q_1$.

Fenchel duality. For $p = (p_u, p_{v_1}, ..., p_{v_n})$, we calculate $\widetilde{f}^*(p)$ that appears in the formulation of $(\widetilde{D}_F)$. By definition, it holds

$$\begin{aligned}
\widetilde{f}^*(p) &= \sup_{v\in\mathbb{R}^k} \{p^T v - \widetilde{f}(v)\} = \sup_{v\in\mathbb{R}^k} \left\{ p^T v - \sum_{i=1}^n F_i(v_i) \right\} \\
&= \sup_{\substack{u\in\mathbb{R}^{n+1} \\ v_i\in\mathbb{R}^{l_i}, i=\overline{1,n}}} \left\{ p_u^T u + \sum_{i=1}^n p_{v_i}^T v_i - \sum_{i=1}^n F_i(v_i) \right\} \\
&= \sup_{u\in\mathbb{R}^{n+1}} p_u^T u + \sum_{i=1}^n \sup_{v_i\in\mathbb{R}^{l_i}} \{p_{v_i}^T v_i - F_i(v_i)\}.
\end{aligned}$$

Thus, in view of

$$\sup_{u\in\mathbb{R}^{n+1}} p_u^T u = \begin{cases} 0, & \text{if } p_u = 0, \\ +\infty, & \text{otherwise,} \end{cases} \tag{1.7}$$

and taking into account that $\inf\limits_{v\in V} p^T v = \inf\limits_{v\in V} \sum\limits_{i=1}^n p_{v_i}^T v_i$, (D_F^{cps}) is immediately obtained, where $p_i := p_{v_i},\ i = \overline{1,n}$, are the corresponding dual variables.

Fenchel-Lagrange duality. As we have seen before

$$\widetilde{f}^*(p) = \sup_{u\in\mathbb{R}^{n+1}} p_u^T u + \sum_{i=1}^n F_i^*(p_{v_i}).$$

By (1.7), we can omit p_u in the second term of $(\widetilde{D}_{FL})$. Thus, this looks like

$$\begin{aligned}
&\inf_{v\in\widetilde{X}}\left\{\sum_{i=1}^{n}p_{v_i}^T v_i+\sum_{i=1}^{n}q_1^T G_i(v_i)+\sum_{i=1}^{n}q_{2i}^T(v_i-A_i u)\right\}\\
=\ &\inf_{\substack{u\in\mathbb{R}^{n+1}\\ v_i\in W_i, i=\overline{1,n}}}\left\{\sum_{i=1}^{n}p_{v_i}^T v_i+\sum_{i=1}^{n}q_1^T G_i(v_i)+\sum_{i=1}^{n}q_{2i}^T v_i-\sum_{i=1}^{n}u^T(A_i^T q_{2i})\right\}\\
=\ &\inf_{u\in\mathbb{R}^{n+1}}u^T\Big(-\sum_{i=1}^{n}A_i^T q_{2i}\Big)+\sum_{i=1}^{n}\inf_{v_i\in W_i}[(p_{v_i}+q_{2i})^T v_i+q_1^T G_i(v_i)].
\end{aligned}$$

In view of (1.6) and replacing $p_{v_i},\ q_{2i},\ i=\overline{1,n}$, and q_1 by $p_i,\ q_i,\ i=\overline{1,n}$, and q_{n+1}, respectively, we get (D_{FL}^{cps}). By using Proposition 1.1, one can derive for (P^{cps}) and its duals the following necessary and sufficient optimality conditions.

Theorem 1.4 *(**Optimality conditions for** (P^{cps}) **and** (D_L^{cps}))*
(***a***) *Assume that the constraint qualification $(\widetilde{CQ})$ is fulfilled (with the denotations given in (1.5)). Let $\bar{u}\in\mathbb{R}^{n+1}$ be an optimal solution to (P^{cps}). Then there exists an element $\bar{q}=(\bar{q}_1,...,\bar{q}_n,\bar{q}_{n+1})\in\mathbb{R}^{l_1}\times\cdots\times\mathbb{R}^{l_n}\times\mathbb{R}^m,\ \bar{q}_{n+1}\underset{\mathbb{R}^m_+}{\geq}0,\ \sum\limits_{i=1}^{n}A_i^T\bar{q}_i=0$ such that the following optimality conditions are satisfied:*

(i) $F_i(\bar{v}_i)+\bar{q}_{n+1}^T G_i(\bar{v}_i)+\bar{q}_i^T\bar{v}_i=\inf\limits_{v_i\in W_i}\{F_i(v_i)+\bar{q}_{n+1}^T G_i(v_i)+\bar{q}_i^T v_i\},\ i=\overline{1,n},$

(ii) $\bar{q}_{n+1}^T\Big(\sum\limits_{i=1}^{n}G_i(\bar{v}_i)\Big)=0,$

(iii) $\bar{v}_i=A_i\bar{u},\ i=\overline{1,n}.$

(***b***) *Let $\bar{u}\in W$ and $\bar{q}=(\bar{q}_1,...,\bar{q}_n,\bar{q}_{n+1})\in\mathbb{R}^{l_1}\times\cdots\times\mathbb{R}^{l_n}\times\mathbb{R}^m$ be feasible to (D_L^{cps}), satisfying $(i)-(iii)$. Then $\bar{u}$ and $\bar{q}$ are optimal solutions to (P^{cps}) and (D_L^{cps}), respectively, and strong duality holds.*

Proof: Let $\bar{u}$ be an optimal solution to (P^{cps}). Then $v(P^{cps})=\sum\limits_{i=1}^{n}F_i(\bar{v}_i)\in\mathbb{R}$, where $\bar{v}_i=A_i\bar{u},\ i=\overline{1,n}$. Therefore, by Proposition 1.1, there exists $\bar{q}=(\bar{q}_1,...,\bar{q}_n,\bar{q}_{n+1})\in\mathbb{R}^{l_1}\times\cdots\times\mathbb{R}^{l_n}\times\mathbb{R}^m$, an optimal solution to (D_L^{cps}) such that $\bar{q}_{n+1}\underset{\mathbb{R}^m_+}{\geq}0,\ \sum\limits_{i=1}^{n}A_i^T\bar{q}_i=0$, and strong duality holds

$$\sum_{i=1}^{n}F_i(\bar{v}_i)=\sum_{i=1}^{n}\inf_{v_i\in W_i}\{F_i(v_i)+\bar{q}_{n+1}^T G_i(v_i)+\bar{q}_i^T v_i\}.$$

After some transformations we get

$$\begin{aligned}
0\ =\ &\sum_{i=1}^{n}\{F_i(\bar{v}_i)+\bar{q}_{n+1}^T G_i(\bar{v}_i)+\bar{q}_i^T\bar{v}_i\\
-\ &\inf_{v_i\in W_i}[F_i(v_i)+\bar{q}_{n+1}^T G_i(v_i)+\bar{q}_i^T v_i]\}\\
+\ &\bar{q}_{n+1}^T\Big(-\sum_{i=1}^{n}G_i(\bar{v}_i)\Big)+\bar{u}^T\Big(-\sum_{i=1}^{n}A_i^T\bar{q}_i\Big).
\end{aligned}$$

Taking into account that $\bar{u},\bar{q}$ are feasible to (P^{cps}) and (D_L^{cps}), respectively, and since the inequality

$$F_i(\bar{v}_i)+\bar{q}_{n+1}^T G_i(\bar{v}_i)+\bar{q}_i^T\bar{v}_i\geq\inf_{v_i\in W_i}[F_i(v_i)+\bar{q}_{n+1}^T G_i(v_i)+\bar{q}_i^T v_i],i=\overline{1,n},$$

is true, $(i) - (iii)$ follows.
In order to prove the statement (**b**) the same calculations can be done in the opposite direction. $\square$

Theorem 1.5 ***(Optimality conditions for*** (P^{cps}) ***and*** (D_F^{cps})***)***
(***a***) *Assume that the constraint qualification* $(\widetilde{CQ})$ *is fulfilled. Let* $\bar{u} \in \mathbb{R}^{n+1}$ *be an optimal solution to* (P^{cps})*. Then there exists an element* $\bar{p} = (\bar{p}_1, ..., \bar{p}_n) \in \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n}$ *such that the following optimality conditions are satisfied:*

(i) $F_i(\bar{v}_i) + F_i^*(\bar{p}_i) = \bar{p}_i^T \bar{v}_i,\ i = \overline{1, n},$

(ii) $\sum\limits_{i=1}^n \bar{p}_i^T \bar{v}_i = \inf\limits_{v \in V} \sum\limits_{i=1}^n \bar{p}_i^T v_i,$

(iii) $\bar{v}_i = A_i \bar{u},\ i = \overline{1, n}.$

(***b***) *Let* $\bar{u} \in W$ *and* $\bar{p} \in \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n}$ *be such that* $(i) - (iii)$ *are satisfied. Then* $\bar{u}$ *and* $\bar{p}$ *are optimal solutions to* (P^{cps}) *and* (D_F^{cps})*, respectively, and strong duality holds.*

Proof: Let $\bar{u}$ be an optimal solution to (P^{cps}). Then $v(P^{cps}) = \sum\limits_{i=1}^n F_i(\bar{v}_i) \in \mathbb{R}$, where $\bar{v}_i = A_i \bar{u},\ i = \overline{1, n}$. Therefore, by Proposition 1.1, there exists $\bar{p} = (\bar{p}_1, ..., \bar{p}_n) \in \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n}$, an optimal solution to (D_F^{cps}), and it holds

$$\sum_{i=1}^n F_i(\bar{v}_i) = -\sum_{i=1}^n F_i^*(\bar{p}_i) + \inf_{v \in V} \sum_{i=1}^n \bar{p}_i^T v_i.$$

The last relation can be rewritten as

$$0 = \sum_{i=1}^n \{F_i(\bar{v}_i) + F_i^*(\bar{p}_i) - \bar{p}_i^T \bar{v}_i\} + \sum_{i=1}^n \bar{p}_i^T \bar{v}_i - \inf_{v \in V} \sum_{i=1}^n \bar{p}_i^T v_i. \tag{1.8}$$

Since the inequalities

$$\begin{aligned} F_i(\bar{v}_i) + F_i^*(\bar{p}_i) &\geq \bar{p}_i^T \bar{v}_i,\ i = \overline{1, n} \text{ (Young inequality)}, \\ \sum_{i=1}^n \bar{p}_i^T \bar{v}_i &- \inf_{v \in V} \sum_{i=1}^n \bar{p}_i^T v_i \geq 0 \end{aligned}$$

are always true, all terms in (1.8) must be equal to zero. Therefore $(i)-(iii)$ follows. In order to get the second part of the theorem one has to make the same calculations, but in the opposite direction. $\square$

Theorem 1.6 ***(Optimality conditions for*** (P^{cps}) ***and*** (D_{FL}^{cps})***)***
(***a***) *Assume that the constraint qualification* $(\widetilde{CQ})$ *is fulfilled. Let* $\bar{u} \in \mathbb{R}^{n+1}$ *be an optimal solution to* (P^{cps})*. Then there exists an element* $(\bar{p}, \bar{q}),\ \bar{p} = (\bar{p}_1, ..., \bar{p}_n) \in \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n},\ \bar{q} = (\bar{q}_1, ..., \bar{q}_n, \bar{q}_{n+1}) \in \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n} \times \mathbb{R}^m,\ \bar{q}_{n+1} \underset{\mathbb{R}^m_+}{\geq} 0,\ \sum\limits_{i=1}^n A_i^T \bar{q}_i = 0$ *such that the following optimality conditions are satisfied:*

(i) $F_i(\bar{v}_i) + F_i^*(\bar{p}_i) = \bar{p}_i^T \bar{v}_i,\ i = \overline{1, n},$

(ii) $(\bar{p}_i + \bar{q}_i)^T \bar{v}_i + \bar{q}_{n+1}^T G_i(\bar{v}_i) = \inf\limits_{v_i \in W_i} \{(\bar{p}_i + \bar{q}_i)^T v_i + \bar{q}_{n+1}^T G_i(v_i)\},\ i = \overline{1, n},$

(iii) $\bar{q}_{n+1}^T \Big(\sum\limits_{i=1}^n G_i(\bar{v}_i) \Big) = 0,$

(iv) $\bar{v}_i = A_i\bar{u},\ i = \overline{1,n}$.

(***b***) *Let* $\bar{u} \in W$ *and* $(\bar{p},\bar{q}),\ \bar{p} = (\bar{p}_1,...,\bar{p}_n) \in \mathbb{R}^{l_1} \times ... \times \mathbb{R}^{l_n},\ \bar{q} = (\bar{q}_1,...,\bar{q}_n,\bar{q}_{n+1}) \in \mathbb{R}^{l_1} \times ... \times \mathbb{R}^{l_n} \times \mathbb{R}^m$ *be feasible to* (D^{cps}_{FL})*, satisfying* $(i)-(iv)$*. Then* $\bar{u}$ *and* $(\bar{p},\bar{q})$ *are optimal solutions to* (P^{cps}) *and* (D^{cps}_{FL})*, respectively, and strong duality holds.*

Proof: Let $\bar{u}$ be an optimal solution to (P^{cps}). Then $v(P^{cps}) = \sum\limits_{i=1}^{n} F_i(\bar{v}_i) \in \mathbb{R}$, where $\bar{v}_i = A_i\bar{u},\ i = \overline{1,n}$. Therefore by Proposition 1.1, there exists $(\bar{p},\bar{q}),\ \bar{p} = (\bar{p}_1,...,\bar{p}_n) \in \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n},\ \bar{q} = (\bar{q}_1,...,\bar{q}_n,\bar{q}_{n+1}) \in \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n} \times \mathbb{R}^m$, an optimal solution to (P^{cps}_{FL}) such that $\bar{q}_{n+1} \underset{\mathbb{R}^m_+}{\geq} 0,\ \sum\limits_{i=1}^{n} A_i^T\bar{q}_i = 0$, and it holds

$$\sum_{i=1}^{n} F_i(\bar{v}_i) = -\sum_{i=1}^{n} F_i^*(\bar{p}_i) + \sum_{i=1}^{n} \inf_{v_i \in W_i}\{(\bar{p}_i + \bar{q}_i)^T v_i + \bar{q}_{n+1}^T G(v_i)\rangle\}.$$

The last equality is rewritable as

$$\begin{aligned} 0 \ &= \ \sum_{i=1}^{n}\{F_i(\bar{v}_i) + F_i^*(\bar{p}_i) - \bar{p}_i^T\bar{v}_i\} + \sum_{i=1}^{n}\{(\bar{p}_i + \bar{q}_i)^T\bar{v}_i + \bar{q}_{n+1}^T G_i(\bar{v}_i) \\ &- \inf_{v_i \in W_i}[(\bar{p}_i + \bar{q}_i)^T v_i + \bar{q}_{n+1}^T G_i(v_i)]\} + \bar{q}_{n+1}^T\Big(-\sum_{i=1}^{n} G_i(\bar{v}_i)\Big) + \bar{u}^T\Big(-\sum_{i=1}^{n} A_i^T\bar{q}_i\Big). \end{aligned}$$

Because $\bar{u}$ and $(\bar{p},\bar{q})$ are feasible to (P^{cps}) and (D^{cps}_{FL}), respectively, and since the inequalities

$$\begin{aligned} F_i(\bar{v}_i) + F_i^*(\bar{p}_i) \ &\geq \ \bar{p}_i^T\bar{v}_i,\ i = \overline{1,n} \text{ (Young inequality)}, \\ (\bar{p}_i + \bar{q}_i)^T\bar{v}_i + \bar{q}_{n+1}^T G_i(\bar{v}_i) \ &\geq \ \inf_{v_i \in W_i}[(\bar{p}_i + \bar{q}_i)^T v_i + \bar{q}_{n+1}^T G_i(v_i)],\ i = \overline{1,n}, \end{aligned}$$

are true, we obtain $(i)-(iv)$.
The second part of the theorem follows by making the same calculations, but in the opposite direction. □

1.2.3 Special cases

The convex partially separable optimization problem with affine constraints. Consider the problem

$$(P^{lps})\quad \inf_{u \in W} \sum_{i=1}^{n} F_i(A_i u),$$

where

$$W = \left\{ u \in \mathbb{R}^{n+1} \middle| \ \sum_{i=1}^{n} D_i A_i u = b,\ A_i u \in W_i,\ i = \overline{1,n} \right\}$$

and $D_i \in \mathbb{R}^{m \times l_i},\ i = \overline{1,n},\ b \in \mathbb{R}^m$ are given.

It is obvious that (P^{lps}) is a special case of $(\widetilde{P})$, whose feasible set containing only affine constraints. The dual problems to (P^{lps}) look like

$$(D^{lps}_L)\qquad \sup_{\substack{q_i \in \mathbb{R}^{l_i},\ i=\overline{1,n} \\ \sum\limits_{i=1}^{n} A_i^T q_i = 0 \\ q_{n+1} \in \mathbb{R}^m}} \left\{ q_{n+1}^T b - \sum_{i=1}^{n} (F_i)^*_{W_i}(D_i^T q_{n+1} + q_i) \right\},$$

$$(D^{lps}_F)\qquad \sup_{p_i \in \mathbb{R}^{l_i},\ i=\overline{1,n}} \left\{ -\sum_{i=1}^{n} F_i^*(p_i) + \inf_{u \in W} \sum_{i=1}^{n} p_i^T(A_i u) \right\}$$

and

$$(D_{FL}^{lps}) \qquad \sup_{\substack{q_i, p_i \in \mathbb{R}^{l_i},\ i=\overline{1,n} \\ \sum\limits_{i=1}^{n} A_i^T q_i = 0 \\ q_{n+1} \in \mathbb{R}^m}} \Bigg\{ q_{n+1}^T b - \sum_{i=1}^{n} F_i^*(p_i) + \sum_{i=1}^{n} \inf_{v_i \in W_i} (p_i + q_i + D_i^T q_{n+1})^T v_i \Bigg\}.$$

As we have seen in Subsection 1.2.2, optimality conditions for all these three dual problems can be derived. Let us give the case concerning the Fenchel-Lagrange dual problem.

Proposition 1.2 ***(Optimality conditions for*** (P^{lps}) ***and*** (D_{FL}^{lps})***)***
(***a***) *Assume that the constraint qualification* $(\widetilde{CQ})$ *is fulfilled. Let* $\bar{u} \in \mathbb{R}^{n+1}$ *be an optimal solution to* (P^{lps})*. Then there exists an element* $(\bar{p}, \bar{q}),\ \bar{p} = (\bar{p}_1, ..., \bar{p}_n) \in \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n},\ \bar{q} = (\bar{q}_1, ..., \bar{q}_n, \bar{q}_{n+1}) \in \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n} \times \mathbb{R}^m,\ \sum\limits_{i=1}^{n} A_i^T \bar{q}_i = 0$ *such that the following optimality conditions are satisfied:*

(i) $F_i(\bar{v}_i) + F_i^*(\bar{p}_i) = \bar{p}_i^T \bar{v}_i,\ i = \overline{1,n},$

(ii) $(\bar{p}_i + \bar{q}_i + D_i^T \bar{q}_{n+1})^T \bar{v}_i = \inf\limits_{v_i \in W_i} (\bar{p}_i + \bar{q}_i + D_i^T \bar{q}_{n+1})^T v_i,\ i = \overline{1,n},$

(iii) $\bar{v}_i = A_i \bar{u},\ i = \overline{1,n}.$

(***b***) *Let* $\bar{u} \in W$ *and* $(\bar{p}, \bar{q}),\ \bar{p} = (\bar{p}_1, ..., \bar{p}_n) \in \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n},\ \bar{q} = (\bar{q}_1, ..., \bar{q}_n, \bar{q}_{n+1}) \in \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n} \times \mathbb{R}^m$ *be feasible to* (D_{FL}^{lps})*, satisfying* $(i) - (iii)$*. Then* $\bar{u}$ *and* $(\bar{p}, \bar{q})$ *are optimal solutions to* (P^{lps}) *and* (D_{FL}^{lps})*, respectively, and strong duality holds.*

The tridiagonally separable optimization problem. Let us now treat the problem

$$(P^{ts}) \quad \inf_{u \in W} \sum_{i=1}^{n} F_i(u_{i-1}, u_i),$$

where

$$W = \Bigg\{ u = (u_0, ..., u_n) \in \underbrace{\mathbb{R}^s \times \cdots \times \mathbb{R}^s}_{n+1} \Bigg| \sum_{i=1}^{n} (B_i u_{i-1} + C_i u_i) = b,\ (u_{i-1}, u_i) \in W_i \subseteq \mathbb{R}^{2s},\ i = \overline{1,n} \Bigg\}$$

and $B_i,\ C_i \in \mathbb{R}^{m \times s},\ i = \overline{1,n},\ b \in \mathbb{R}^m$ are given.
For (P^{ts}) we can use the dual schemes for (P^{lps}). Indeed, one can notice that $A_i = \begin{pmatrix} E_i^T \\ E_{i+1}^T \end{pmatrix}$ and $A_i u = \begin{pmatrix} u_{i-1} \\ u_i \end{pmatrix}$. Here $E_i^T = (\emptyset, ..., I, ..., \emptyset) \in \mathbb{R}^{s \times s(n+1)}$ is a matrix, where $\emptyset$ and I denote the quadratic matrixes with $a_{ij} = 0,\ i,j = \overline{1,s}$, and $a_{ii} = 1,\ a_{ij} = 0$ for $i \neq j,\ i,j = \overline{1,s}$, respectively. If we take D_i as (B_i, C_i), then one has

$$D_i v_i = (B_i, C_i) \begin{pmatrix} u_{i-1} \\ u_i \end{pmatrix} = B_i u_{i-1} + C_i u_i \quad \text{and} \quad D_i^T q_{n+1} = \begin{pmatrix} B_i^T q_{n+1} \\ C_i^T q_{n+1} \end{pmatrix}.$$

On the other hand, as $\sum_{i=1}^{n} A_i^T q_i = 0$, we obtain that

$$\begin{aligned} q_{i1} &= 0, \\ q_{i2} &+ q_{i+1,i} = 0,\ i = \overline{1,..,n-1}, \quad \text{where} \quad q_i = \begin{pmatrix} q_{i1} \\ q_{i2} \end{pmatrix} \in \mathbb{R}^{2s}. \\ q_{n2} &= 0, \end{aligned}$$

Replacing $\bar{q}_i := q_{i+1,1},\ i = 1, ..., n-1$, it follows that

$$D_i q_{n+1} + q_i = \begin{pmatrix} B_i^T q_{n+1} + \bar{q}_{i-1} \\ C_i^T q_{n+1} - \bar{q}_i \end{pmatrix},\ i = \overline{1,n}, \quad \text{and} \quad \bar{q}_0 = \bar{q}_n = 0.$$

Consequently, the duals to (P^{lps}) become in this situation

$$(D_L^{ts}) \qquad \sup_{\substack{q_i \in \mathbb{R}^s, i=\overline{0,n} \\ q_0 = q_n = 0 \\ q_{n+1} \in \mathbb{R}^m}} \left\{ q_{n+1}^T b - \sum_{i=1}^{n} (F_i)_{W_i}^* (q_{i-1} + B_i^T q_{n+1}, -q_i + C_i^T q_{n+1}) \right\},$$

$$(D_F^{ts}) \qquad \sup_{\substack{(p_{i1}, p_{i2}) \in \mathbb{R}^{2s} \\ i=\overline{1,n}}} \left\{ -\sum_{i=1}^{n} F_i^*(p_{i1}, p_{i2}) + \inf_{u \in W} \sum_{i=1}^{n} [p_{i1}^T u_{i-1} + p_{i2}^T u_i] \right\},$$

and

$$(D_{FL}^{ts}) \qquad \sup_{\substack{(p_{i1}, p_{i2}) \in \mathbb{R}^{2s} \\ q_i \in \mathbb{R}^s, i=\overline{0,n} \\ q_0 = q_n = 0 \\ q_{n+1} \in \mathbb{R}^m}} \Bigg\{ q_{n+1}^T b - \sum_{i=1}^{n} F_i^*(p_{i1}, p_{i2}) + \sum_{i=1}^{n} \inf_{(u_{i-1}, u_i) \in W_i}$$
$$[(p_{i1} - q_{i-1} - B_i^T q_{n+1})^T u_{i-1} + (p_{i2} + q_i - C_i^T q_{n+1})^T u_i] \Bigg\},$$

respectively. The next proposition provides optimality conditions for (P^{ts}) and (D_{FL}^{ts}).

Proposition 1.3 *(**Optimality conditions for** (P^{ts}) **and** (D_{FL}^{ts}))*
($\boldsymbol{a}$) *Assume that the constraint qualification $(\widetilde{CQ})$ is fulfilled. Let $\bar{u} \in \mathbb{R}^{n+1}$ be an optimal solution to (P^{ts}). Then there exists an element $(\bar{p}, \bar{q})$, $\bar{p} = (\bar{p}_1, ..., \bar{p}_n) \in \underbrace{\mathbb{R}^{2s} \times \cdots \times \mathbb{R}^{2s}}_{n}$, $\bar{q} = (\bar{q}_0, \bar{q}_1, ..., \bar{q}_n, \bar{q}_{n+1}) \in \underbrace{\mathbb{R}^s \times \cdots \times \mathbb{R}^s}_{n+1} \times \mathbb{R}^m$, $\bar{q}_0 = \bar{q}_n = 0$ such that the following optimality conditions are satisfied:*

(i) $F_i(\bar{u}_{i-1}, \bar{u}_i) + F_i^*(\bar{p}_{i1}, \bar{p}_{i2}) = \bar{p}_{i1}^T \bar{u}_{i-1} + \bar{p}_{i2}^T \bar{u}_i,\ i = \overline{1,n},$

(ii) $(\bar{p}_{i1} - \bar{q}_{i-1} - {B_i}^T \bar{q}_{n+1})^T \bar{u}_{i-1} + (\bar{p}_{i2} + \bar{q}_i - {C_i}^T \bar{q}_{n+1})^T \bar{u}_i$
$= \inf_{(u_{i-1}, u_i) \in W_i} [(\bar{p}_{i1} - \bar{q}_{i-1} - {B_i}^T \bar{q}_{n+1})^T u_{i-1} + (\bar{p}_{i2} + \bar{q}_i - {C_i}^T \bar{q}_{n+1})^T u_i],$
$i = \overline{1,n}.$

($\boldsymbol{b}$) *Let $\bar{u} \in W$ and $(\bar{p}, \bar{q})$, $\bar{p} = (\bar{p}_1, ..., \bar{p}_n) \in \underbrace{\mathbb{R}^{2s} \times \cdots \times \mathbb{R}^{2s}}_{n}$, $\bar{q} = (\bar{q}_0, \bar{q}_1, ..., \bar{q}_n, \bar{q}_{n+1}) \in \underbrace{\mathbb{R}^s \times \cdots \times \mathbb{R}^s}_{n+1} \times \mathbb{R}^m$ be feasible to (D_{FL}^{ts}), satisfying (i) − (ii). Then $\bar{u}$ and $(\bar{p}, \bar{q})$ are optimal solutions to (P^{ts}) and (D_{FL}^{ts}), respectively, and strong duality holds.*

Convex interpolation with cubic C^1 splines. At the end of this section we show how it is possible to reformulate the convex interpolation problem with C^1 splines as a tridiagonally separable optimization problem. For more details about other examples associated to the spline approximation including the mentioned one, we refer to [19], [25], [76] and [77]. The role of the duality by solving this problem will also be discussed.
Let $(x_i, y_i)^T \in \mathbb{R}^2,\ i = \overline{0,n}$, be given data points defined on the grid

$$\Delta_n :\ x_0 < x_1 < ... < x_n.$$

A cubic spline S on Δ_n can be given for $[x_{i-1}, x_i]$ by the formula

$$\begin{aligned} S(x) &= y_{i-1} + m_{i-1}(x - x_{i-1}) \\ &+ (3\tau_i - 2m_{i-1} - m_i)\frac{(x - x_{i-1})^2}{h_i} + (m_{i-1} + m_i - 2\tau_i)\frac{(x - x_{i-1})^3}{h_i^2} \end{aligned}$$

with $h_i = x_i - x_{i-1},\ \tau_i = \frac{y_i - y_{i-1}}{h_i},\ i = \overline{1,n}$. It holds $S \in C^1[x_0, x_n]$ and $S(x_i) = y_i,\ S'(x_i) = m_i,\ i = \overline{0,n}$.
The points $(x_0, y_0), ..., (x_n, y_n)$ associated to Δ_n are said to be in convex position if

$$\tau_1 \le \tau_2 \le \cdots \le \tau_n. \tag{1.9}$$

By (1.9) the necessary and sufficient condition which guarantees the convexity of S on $[0,1]$ leads to the following problem

$$(m_{i-1}, m_i)^T \in W_i \tag{1.10}$$

where

$$W_i = \{(m_{i-1}, m_i)^T \in \mathbb{R}^2 |\ 2m_{i-1} + m_i \le 3\tau_i \le m_{i-1} + 2m_i\},\ i = \overline{1,n}. \tag{1.11}$$

If the inequality $a_i \le b_i,\ i = \overline{1,n}$ where $a_0 = -\infty,\ b_0 = +\infty$ and $a_i = \max\{\tau_i, \frac{1}{2}(3\tau_i - b_{i-1})\},\ b_i = 3\tau_i - 2a_{i-1},\ i = \overline{1,n}$, is fulfilled, then the problem (1.10) is solvable (see [76]), but not uniquely in general. In order to select an unique convex interpolant one has to minimize the mean curvature of S. It is easy to verify that (see [76])

$$\begin{aligned} \int_{x_0}^{x_n} S''(x)^2 dx &= \sum_{i=1}^{n} \frac{4}{h_i^2}\{m_i^2 + m_i m_{i-1} + m_{i-1}^2 - 3\tau_i(m_i + m_{i-1}) + 3\tau_i^2\} \\ &= \sum_{i=1}^{n} F_i(m_{i-1}, m_i), \end{aligned}$$

and therefore we get the following optimization problem

$$(P^{sca}) \qquad \min_{\substack{(m_{i-1}, m_i)^T \in W_i \\ i = \overline{1,n}}} \sum_{i=1}^{n} F_i(m_{i-1}, m_i),$$

where $W_i,\ i = \overline{1,n}$, is given by (1.11). Obviously, (P^{csa}) is a particular case of (P^{ts}). As we have seen, the Lagrange dual problem to (P^{csa}) is

$$(D_L^{csa}) \qquad \sup_{\substack{q \in \mathbb{R}^{n+1} \\ q = (q_0, q_1, ..., q_n)^T \\ q_0 = q_n = 0}} -\left\{\sum_{i=1}^{n} (F_i)^*_{W_i}(q_{i-1}, -q_i)\right\},$$

where (see [25], [76]),

$$(F_i)^*_{W_i}(\xi,\eta) = \begin{cases} \tau_i(\xi+\eta) + \frac{h_i}{12}(\xi^2 - \xi\eta + \eta^2), & \text{if } \ \xi \le 0, \eta \ge 0, \\ \tau_i(\xi+\eta) + \frac{h_i}{12}(\frac{\xi}{2} - \eta)^2 & \text{if } \ 0 \le \xi \le 2\eta, \\ \tau_i(\xi+\eta) + \frac{h_i}{12}(\xi - \frac{\eta}{2})^2, & \text{if } \ 2\xi \le \eta \le 0, \\ \tau_i(\xi+\eta), & \text{if } \ \xi \ge 2\eta, 2\xi \ge \eta. \end{cases}$$

The problem (P^{csa}) was solved in the literature by using of the so-called return-formula (see [76])

$$(u_{i-1}, u_i)^T = \mathrm{grad}[(F_i)^*_{W_i}(\bar{q}_{i-1}, -\bar{q}_i)],$$

where $(\bar{q}_0, \bar{q}_1, ..., \bar{q}_n)^T \in \mathbb{R}^{n+1}$ is an optimal solution to (D_L^{csa}).

Later in Subsection 2.1.4 we discuss for a more general problem than (P^{sca}), the relations between the optimality conditions related to the Fenchel-Lagrange duality (cf. Theorem 1.6) and so-called generalized variational inequalities. The generalized variational inequality is closely related to the inclusion problem of finding a zero of set-valued mappings. Whence, well-known proximal point and splitting algorithms for solving the inclusion problems can be used to compute the solutions of the problems arising from such optimality conditions.

Chapter 2

Variational inequalities and equilibrium problems

This chapter deals with some applications of conjugate duality for convex optimization problems in finite and infinite-dimensional spaces to the construction of gap functions for variational inequalities and equilibrium problems. The basic idea of the approach is to reformulate variational inequalities and equilibrium problems into optimization problems depending on a fixed variable, which allows us to apply duality results from optimization problems.

2.1 Variational inequalities

In this section we consider new gap functions for variational inequalities based on conjugate duality for convex optimization problems. By using dual problems investigated in [90] (see Chapter 1), we propose some new gap functions for variational inequalities. Under certain assumptions, we discuss a further class of gap functions for the variational inequality problem, the so-called dual gap functions.

2.1.1 Problem formulation and some remarks on gap functions

Let $K \subseteq \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ be a vector-valued function. The variational inequality problem consists in finding a point $x \in K$ such that

$$(VI) \qquad F(x)^T(y - x) \geq 0, \ \forall y \in K.$$

Although it is supposed mostly in the literature that K is a closed, convex set and F is a continuous vector-valued function, we will make such assumptions only if they are required. As mentioned before, one of the approaches for solving the problem (VI) is to reformulate it into an equivalent optimization problem.

By using the conjugate duality theory presented in the previous chapter, we discuss the construction of gap functions for variational inequalities. Before doing this, we recall the definition of a gap function and some well-known gap functions for the problem (VI).

Definition 2.1 *A function $\gamma : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be a gap function for the problem (VI) if it satisfies the following properties*

(i) $\gamma(y) \geq 0, \ \forall y \in K$;

(ii) $\gamma(x) = 0$ *if and only if x solves the problem (VI).*

Definition 2.2 *(Auslender's gap function, [8])*

$$\gamma_A^{VI}(x) := \max_{y\in K} F(x)^T(x-y).$$

Let us now assume that the ground set K is defined by

$$K = \{x \in \mathbb{R}^n |\ g_i(x) \leq 0,\ i = 1, 2, .., m\}, \tag{2.1}$$

where $g_i : \mathbb{R}^n \to \mathbb{R}$ are convex functions and $g(x) = (g_1(x), ..., g_m(x))^T$. Giannessi proposed the following gap function which explicitly incorporates the constraints that define the ground set K.

Definition 2.3 *(Giannessi's gap function, [39])*

$$\gamma_G^{VI}(x) := \inf_{\lambda \underset{\mathbb{R}^m_+}{\geq} 0} \sup_{y\in\mathbb{R}^n} \Big\{F(x)^T(x-y) - \lambda^T g(y)\Big\}.$$

Notice that the formulation of Giannessi's gap function is inspired by the Lagrange duality for the optimization problem

$$(P^{VI}; x) \qquad \inf_{y\in K} F(x)^T(y-x),$$

where K is given by (2.1) and $x \in \mathbb{R}^n$ is fixed. It is easy to see that

$$\gamma_G^{VI}(x) \equiv \gamma_L^{VI}(x) := -v(D_L^{VI}; x),$$

where $v(D_L^{VI}; x)$ denotes the optimal objective value of the Lagrange dual problem to $(P^{VI}; x)$. Now let us state the Fenchel dual problem to $(P^{VI}; x)$ and define a function in the similar way, i.e.

$$\gamma_F^{VI}(x) := -v(D_F^{VI}; x).$$

Since the conjugate of the objective function for $(P^{VI}; x)$ is

$$\sup_{y\in\mathbb{R}^n} [p^T y - F(x)^T(y-x)] = \begin{cases} F(x)^T x, & \text{if } p = F(x), \\ +\infty, & \text{otherwise}, \end{cases} \tag{2.2}$$

the Fenchel dual problem to $(P^{VI}; x)$ turns out to be (cf. (D_F) in Subsection 1.1.1)

$$(D_F^{VI}; x) \qquad \sup_{p=F(x)} \Big\{ -F(x)^T x + \inf_{y\in K} p^T y\Big\} = \inf_{y\in K} F(x)^T(y-x).$$

Whence we define

$$\gamma_F^{VI}(x) := -v(D_F^{VI}; x) = -\inf_{y\in K} F(x)^T(y-x) = \sup_{y\in K} F(x)^T(x-y).$$

γ_F^{VI} is nothing else than Auslender's gap function. Let us notice that, by using the Fenchel duality, we can define a gap function for an arbitrary ground set K. Assuming again that the ground set K is given by (2.1), in view of (2.2), the Fenchel-Lagrange dual problem to $(P^{VI}; x)$ becomes

$$\begin{aligned}(D_{FL}^{VI}; x) \qquad & \sup_{\substack{p=F(x)\\ q \underset{\mathbb{R}^m_+}{\geq} 0}} \Big\{ -F(x)^T x + \inf_{y\in\mathbb{R}^n} [p^T y + q^T g(y)]\Big\} \\ = \qquad & \sup_{q \underset{\mathbb{R}^m_+}{\geq} 0} \inf_{y\in\mathbb{R}^n} [F(x)^T(y-x) + q^T g(y)]\Big\}.\end{aligned}$$

Therefore, the function $\gamma_{FL}^{VI}(x) := -v(D_{FL}^{VI}; x)$ also reduces to Giannessi's gap function. The result can be summarized as follows.

Proposition 2.1

(i) *For the problem* (VI), *it holds* $\gamma_F^{VI}(y) = \gamma_A^{VI}(y),\ \forall y \in \mathbb{R}^n$.

(ii) *If the ground set is given by (2.1), then it holds*

$$\gamma_{FL}^{VI}(y) = \gamma_G^{VI}(y),\ \forall y \in \mathbb{R}^n.$$

2.1.2 Gap functions for the mixed variational inequality

The problem (VI) can be generalized to the following mixed variational inequality problem which consists in finding a point $x \in K$ such that

$$(MVI) \qquad F(x)^T(y-x) + f(y) - f(x) \geq 0,\ \forall y \in K,$$

where $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a proper, convex function. Some results related to (MVI) can be found in [27] and [55]. As said before, to the problem (MVI) one can associate the following primal problem

$$(P^{MVI}; x) \qquad \inf_{y \in K} \varphi(y),$$

where the function $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ is defined by

$$\varphi(y) := F(x)^T(y-x) + f(y) - f(x) \tag{2.3}$$

and $x \in \mathbb{R}^n$ is fixed. One can derive the conjugate of φ by

$$\begin{aligned} \varphi^*(p) &= \sup_{y \in \mathbb{R}^n} [p^T y - \varphi(y)] = \sup_{y \in \mathbb{R}^n} [p^T y - F(x)^T(y-x) - f(y) + f(x))] \\ &= f^*(p - F(x)) + F(x)^T x + f(x). \end{aligned} \tag{2.4}$$

Therefore the Fenchel dual problem to $(P^{MVI}; x)$ is

$$(D_F^{MVI}; x) \qquad \sup_{p \in \mathbb{R}^n} \Big\{ - f^*(p - F(x)) - F(x)^T x - f(x) + \inf_{y \in K} p^T y \Big\}.$$

In analogy to the problem (VI), we can introduce the following function

$$\gamma_F^{MVI}(x) := -v(D_F^{MVI}; x) = \inf_{p \in \mathbb{R}^n} \Big\{ f^*(p - F(x)) + F(x)^T x + f(x) + \delta_K^*(-p) \Big\}.$$

Theorem 2.1 *Let* $ri(K) \cap ri(dom\ f) \neq \varnothing$ *and* K *be a convex set. Then* γ_F^{MVI} *is a gap function for the problem* (MVI).

Proof:

(i) Let $x \in K$ be fixed. By weak duality it holds

$$v(D_F^{MVI}; x) \leq v(P^{MVI}; x) \leq 0.$$

Whence $\gamma_F^{MVI}(x) = -v(D_F^{MVI}; x) \geq 0$.

(ii) If $\gamma_F^{MVI}(x) = 0$, then $0 = v(D_F^{MVI}; x) \leq v(P^{MVI}; x) \leq 0$ and so $v(P^{MVI}; x) = 0$. This means that x solves the problem (MVI). On the other hand, if $x \in K$ is a solution to the problem (MVI), then $v(P^{MVI}; x) = 0$. Taking into account Theorem 1.1(iii), we conclude that

$$\gamma_F^{MVI}(x) = -v(D_F^{MVI}; x) = -v(P^{MVI}; x) = 0.$$

□

Remark 2.1 If one takes $K = \mathbb{R}^n$ in the formulation of the problem (MVI), then this reduces to the extended variational inequality problem. By using γ_F^{MVI} we obtain the same gap function for the extended variational inequality as in [21]. Indeed, because of

$$\delta^*_{\mathbb{R}^n}(-p) = \sup_{x\in\mathbb{R}^n}[-p^T x] = \begin{cases} 0, & \text{if } p = 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned} \gamma_F^{EVI}(x) &= \inf_{p\in\mathbb{R}^n}\Big\{f^*(p - F(x)) + F(x)^T x + f(x) + \delta^*_{\mathbb{R}^n}(-p)\Big\} \\ &= f^*(-F(x)) + F(x)^T x + f(x). \end{aligned}$$

Example 2.1 Let $K \subseteq \mathbb{R}^n$ be a convex set and $a \in \mathbb{R}^n$ be a given point. Consider the following so-called best approximation problem of finding $x \in K$ such that

$$(P^{app}) \qquad \|y - a\| \geq \|x - a\|, \ \forall y \in K,$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$ defined by $\|x\| = \sqrt{x^T x}$.
The point x is nothing else than the *projection of* a *onto* K. It is easy to verify that the problem (P^{app}) is equivalent to the problem of finding $x \in K$ such that

$$-2a^T(y - x) + \|y\|^2 - \|x\|^2 \geq 0, \ \forall y \in K.$$

This problem is a particular case of (MVI), taking $F : \mathbb{R}^n \to \mathbb{R}^n$, $F(z) = -2a$ and $f : \mathbb{R}^n \to \mathbb{R}$, $f(z) = \|z\|^2$. Let $x \in \mathbb{R}^n$ be fixed. Since

$$\begin{aligned} f^*(p - F(x)) &= f^*(p + 2a) = \sup_{y\in\mathbb{R}^n}[(p + 2a)^T y - f(y)] \\ &= \sup_{y\in\mathbb{R}^n}[(p + 2a)^T y - y^T y] = \frac{1}{4}(p + 2a)^T(p + 2a), \forall p \in \mathbb{R}^n, \end{aligned}$$

the gap function for this mixed variational inequality problem for any $x \in \mathbb{R}^n$ turns out to be

$$\gamma_F^{P^{app}}(x) = \|x\|^2 - 2a^T x + \inf_{p\in\mathbb{R}^n}\Big\{\frac{1}{4}(p + 2a)^T(p + 2a) + \delta^*_K(-p)\Big\}.$$

Let us notice that in order to calculate the gap function $\gamma_F^{P^{app}}$, one must first solve the optimization problem with a linear objective function $\delta^*_K(-p) = \sup\limits_{y\in K}(-p)^T y$ and afterwards minimize the sum of a quadratic function and δ^*_K over the whole space $\mathbb{R}^n$. This can be a much easier task than minimizing the norm function over the convex set K.

Let the ground set K be given by (2.1). By using the formulations of the duals (D_L) and (D_{FL}), we can introduce for $x \in \mathbb{R}^n$ the following functions

$$\begin{aligned} \gamma_L^{MVI}(x) : &= -v(D_L^{MVI}; x) \\ &= -\sup_{q \underset{\mathbb{R}^m_+}{\geq} 0}\inf_{y\in\mathbb{R}^n}\Big\{F(x)^T(y - x) + f(y) - f(x) + q^T g(y)\Big\} \\ &= \inf_{q \underset{\mathbb{R}^m_+}{\geq} 0}\sup_{y\in\mathbb{R}^n}\Big\{F(x)^T(x - y) - f(y) + f(x) - q^T g(y)\Big\} \\ &= \inf_{q \underset{\mathbb{R}^m_+}{\geq} 0}\Big\{F(x)^T x + f(x) + \sup_{y\in\mathbb{R}^n}[-F(x)^T y - f(y) - q^T g(y)]\Big\} \\ &= \inf_{q \underset{\mathbb{R}^m_+}{\geq} 0}\Big\{F(x)^T x + f(x) + (f + q^T g)^*(-F(x))\Big\} \end{aligned}$$

and, in view of (2.4)

$$\begin{aligned}\gamma_{FL}^{MVI}(x) : &= -v(D_{FL}^{MVI};x)\\ &= -\sup_{\substack{p\in\mathbb{R}^n\\ q \underset{\mathbb{R}^m_+}{\geq} 0}}\Big\{-f^*(p-F(x))-F(x)^Tx-f(x)-(q^Tg)^*(-p)\Big\}\\ &= \inf_{\substack{p\in\mathbb{R}^n\\ q \underset{\mathbb{R}^m_+}{\geq} 0}}\Big\{f^*(p-F(x))+F(x)^Tx+f(x)+(q^Tg)^*(-p)\Big\},\end{aligned}$$

respectively.

Theorem 2.2 *Assume that the constraint qualification (CQ) (cf. Subsection 1.1.2) is fulfilled. Then γ_L^{MVI} and γ_{FL}^{MVI} are gap functions for the problem (MVI).*

Proof:

(i) It is easily verified by weak duality (see the proof of Theorem 2.1(i)).

(ii) As $\gamma_L^{MVI}(x) = \gamma_{FL}^{MVI}(x) = 0$, x is a solution to (MVI). Conversely, let the problem (MVI) be solved by x and the constraint qualification (CQ) be fulfilled. Then by Theorem 1.3, it holds strong duality. This implies that

$$\gamma_L^{MVI}(x) = \gamma_{FL}^{MVI}(x) = -v(D_L^{MVI};x) = -v(D_{FL}^{MVI};x) = -v(P^{MVI};x) = 0.$$

□

Remark 2.2 Because of $v(P^{MVI};x) \leq 0$, where x is fixed, by the strong duality results in Chapter 1, the dual problems to $(P^{MVI};x)$ have optimal solutions. Consequently, under the assumptions of Theorem 2.1 and Theorem 2.2, one can use $''\min''$ instead of $''\inf''$ for the proposed gap functions.

Example 2.2 Let $F \equiv 0$. Assume that the ground set is given by (2.1) and the assumptions of Theorem 2.1 and Theorem 2.2 are fulfilled. The problem (MVI) reduces to finding an optimal solution $x \in K$ to the convex optimization problem

$$(P^c) \qquad \inf_{y\in K} f(y).$$

The gap functions γ_F^{MVI}, γ_L^{MVI} and γ_{FL}^{MVI} for $x \in \mathbb{R}^n$ become

$$\begin{aligned}\gamma_F^{P^c}(x) &= f(x) + \inf_{p\in\mathbb{R}^n}\Big\{f^*(p)+\delta_K^*(-p)\Big\},\\ \gamma_L^{P^c}(x) &= f(x) + \inf_{q \underset{\mathbb{R}^m_+}{\geq} 0}(f+q^Tg)^*(0)\end{aligned}$$

and

$$\gamma_{FL}^{P^c}(x) = f(x) + \inf_{\substack{p\in\mathbb{R}^n\\ q \underset{\mathbb{R}^m_+}{\geq} 0}}\Big\{f^*(p)+(q^Tg)^*(-p)\Big\},$$

respectively. Let us remark that

$$\begin{aligned}&(D_F^c) \qquad \sup_{p\in\mathbb{R}^n}\Big\{-f^*(p)-\delta_K^*(-p)\Big\},\\ &(D_L^c) \qquad \sup_{q \underset{\mathbb{R}^m_+}{\geq} 0}\Big\{-(f+q^Tg)^*(0)\Big\}\end{aligned}$$

and

$$(D^c_{FL}) \qquad \sup_{\substack{p\in\mathbb{R}^n \\ q \underset{\mathbb{R}^m_+}{\geq} 0}} \Big\{ - f^*(p) - (q^T g)^*(-p) \Big\}$$

are the Fenchel, the Lagrange and the Fenchel-Lagrange dual problem to (P^c) (cf. Subsection 1.1.1), respectively. From whence the property (i) in the definition of a gap function is nothing else than the weak duality between the primal and each of the dual problems. The second requirement claims that $x \in K$ is an optimal solution to (P^c) if and only if $\gamma_F^{P^c}(x) = \gamma_L^{P^c}(x) = \gamma_{FL}^{P^c}(x) = 0$, which is nothing else than

$$\begin{aligned} f(x) &= \sup_{p\in\mathbb{R}^n} \Big\{ - f^*(p) - \delta_K^*(-p) \Big\} = \sup_{q \underset{\mathbb{R}^m_+}{\geq} 0} \inf_{x\in\mathbb{R}^n} \Big\{ f(x) + q^T g(x) \Big\} \\ &= \sup_{\substack{p\in\mathbb{R}^n \\ q \underset{\mathbb{R}^m_+}{\geq} 0}} \Big\{ - f^*(p) - (q^T g)^*(-p) \Big\}. \end{aligned}$$

In other words, it expresses the strong duality assertions between the primal problem (P^c) and the dual problems (D^c_F), (D^c_L) and (D^c_{FL}), respectively.

Next let us study the relations between the gap functions for (MVI) introduced above.

Proposition 2.2 *Let the ground set K be given by (2.1). Then it holds*

$$\begin{matrix} \gamma_L^{MVI}(x) \\ \\ \gamma_F^{MVI}(x) \end{matrix} \leq \gamma_{FL}^{MVI}(x), \ \forall x \in \mathbb{R}^n.$$

Proof: Let $x \in \mathbb{R}^n$ be fixed. According to Propositions 2.1 and 2.2 in [16] (see also [90]), one has

$$v(D_{FL}^{MVI}; x) \leq \begin{matrix} v(D_L^{MVI}; x) \\ \\ v(D_F^{MVI}; x) \end{matrix},$$

or, equivalently,

$$\begin{matrix} -v(D_L^{MVI}; x) \\ \\ -v(D_F^{MVI}; x) \end{matrix} \leq -v(D_{FL}^{MVI}; x).$$

which leads to the desired conclusion. □

One of the desirable properties of gap functions is the convexity. Under certain assumptions this property is fulfilled. First we have to introduce the following definition.

Definition 2.4 *A vector-valued function $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be*

(i) monotone if for any points $x, y \in \mathbb{R}^n$, we have $[F(x) - F(y)]^T(x-y) \geq 0$;

(ii) pseudo-monotone if for any points $x, y \in \mathbb{R}^n$, we have $F(y)^T(x-y) \geq 0$ implies

$$F(x)^T(x-y) \geq 0.$$

Proposition 2.3 *(Convexity of γ_F^{MVI})*
Assume that K is a convex set and $F : \mathbb{R}^n \to \mathbb{R}^n$ is an affine and monotone vector-valued function. Then γ_F^{MVI} is convex.

Proof: Let us verify first that the function

$$(x, p) \mapsto f^*(p - F(x)) + F(x)^T x + f(x) + \delta_K^*(-p) \tag{2.5}$$

is convex with respect to (x, p). As F is affine and monotone and f is convex, the function $(x, p) \mapsto F(x)^T x + f(x)$ is convex. On the other hand, the conjugate functions f^* and δ_K^* are also convex. F being affine, then $(x, p) \mapsto p - F(x)$ is affine. So $(x, p) \mapsto f^*(p - F(x))$ is convex as it is the composition of a convex function with an affine one. In conclusion, the function given by (2.5) is convex. Therefore, by Theorem 1 in [73], γ_F^{MVI} is convex. □

Proposition 2.4 *(Convexity of γ_L^{MVI} and γ_{FL}^{MVI})*
Assume that $F : \mathbb{R}^n \to \mathbb{R}^n$ is an affine and monotone vector-valued function. Then γ_L^{MVI} and γ_{FL}^{MVI} are convex.

Proof: Because of the functions

$$(f + q^T g)^*(-F(x)) = \sup_{y \in \mathbb{R}^n} [-F(x)^T y - f(y) - q^T g(y)]$$

and $(q^T g)_X^*(-p) = \sup_{y \in X} [-p^T y - q^T g(y)]$ are convex as the pointwise supremum of affine functions with respect to (x, q) and (p, q), respectively, the convexity of γ_L^{MVI} and γ_{FL}^{MVI} follows from Theorem 1 in [73]. □

2.1.3 Dual gap functions for the problem (VI)

In this subsection we introduce another class of gap functions for the problem (VI), the so-called dual gap functions. Before doing this, let us mention the following lemma which was proved first by Minty for a monotone vector-valued function.

Lemma 2.1 *(see [47] and [51]) Let K be a nonempty, closed and convex subset of $\mathbb{R}^n$. Assume that $F : \mathbb{R}^n \to \mathbb{R}^n$ is a pseudo-monotone and continuous vector-valued function on K. Then $x \in K$ solves the problem (VI) if and only if $x \in K$ and*

$$(VI') \qquad F(y)^T (y - x) \geq 0, \ \forall y \in K.$$

Whence, under the assumptions of Lemma 2.1, the function $\gamma_A^{VI'} : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by

$$\gamma_A^{VI'}(x) := \sup_{y \in K} F(y)^T (x - y)$$

is a gap function for the problem (VI) and it is called the *dual gap function for* (VI). Remark that $\gamma_A^{VI'}$ is the gap function for the problem (VI') in the sense of Auslender and has been studied, for instance, in [61] and [97]. Using its duals $(D_F^{VI'}; x)$, $(D_L^{VI'}; x)$ and $(D_{FL}^{VI'}; x)$ we can formulate for the optimization problem

$$(P^{VI'}; x) \qquad \inf_{y \in K} F(y)^T (y - x),$$

where $x \in \mathbb{R}^n$ is fixed, the corresponding functions as follows

$$\begin{aligned}
\gamma_F^{VI'}(x) : &= -v(D_F^{VI'}; x) = \inf_{p \in \mathbb{R}^n} \Big\{ \sup_{y \in \mathbb{R}^n} [p^T y + F(y)^T (x - y)] + \delta_K^*(-p) \Big\}, \\
\gamma_L^{VI'}(x) : &= -v(D_L^{VI'}; x) = \inf_{q \underset{\mathbb{R}^m_+}{\geq} 0} \sup_{y \in \mathbb{R}^n} \Big\{ F(y)^T (x - y) - q^T g(y) \Big\}. \\
\gamma_{FL}^{VI'}(x) : &= -v(D_{FL}^{VI'}; x) = \inf_{\substack{p \in \mathbb{R}^n \\ q \underset{\mathbb{R}^m_+}{\geq} 0}} \Big\{ \sup_{y \in \mathbb{R}^n} [p^T y + F(y)^T (x - y)] + (q^T g)^*(-p) \Big\}.
\end{aligned}$$

In case of the functions $\gamma_L^{VI'}$ and $\gamma_{FL}^{VI'}$, K is given by (2.1). Before we show that the proposed functions are gap functions for the problem (VI), let us prove some relations between them.

Proposition 2.5 *It holds*

$$\gamma_A^{VI'}(x) \leq \gamma_F^{VI'}(x),\ \forall x \in \mathbb{R}^n.$$

Proof: Let $x \in \mathbb{R}^n$ be fixed. For any $p \in \mathbb{R}^n$ it holds

$$\sup_{z \in \mathbb{R}^n} [p^T z - F(z)^T(z-x)] \geq p^T y - F(y)^T(y-x),\ \forall y \in \mathbb{R}^n,$$

or, equivalently,

$$\sup_{z \in \mathbb{R}^n} [p^T z - F(z)^T(z-x)] - p^T y \geq F(y)^T(x-y),\ \forall y \in \mathbb{R}^n.$$

Taking the supremum in both sides over all $y \in K$ one gets

$$\sup_{z \in \mathbb{R}^n} [p^T z - F(z)^T(z-x)] + \delta_K^*(-p) \geq \sup_{y \in K} F(y)^T(x-y).$$

After taking the infimum in the left hand side over all $p \in \mathbb{R}^n$ we conclude that $\gamma_F^{VI'}(x) \geq \gamma_A^{VI'}(x),\ \forall x \in \mathbb{R}^n.$ □

Proposition 2.6 *Let the ground set be given by* (2.1)*. Then it holds*

$$\gamma_A^{VI'}(x) \leq \begin{array}{c} \gamma_L^{VI'}(x) \\ \\ \gamma_F^{VI'}(x) \end{array} \leq \gamma_{FL}^{VI'}(x),\ \forall x \in \mathbb{R}^n.$$

Proof: Like in Proposition 2.2, by Propositions 2.1 and 2.2 in [16] (see also [90]), one can conclude that

$$\begin{array}{c} \gamma_L^{VI'}(x) \\ \\ \gamma_F^{VI'}(x) \end{array} \leq \gamma_{FL}^{VI'}(x),\ \forall x \in \mathbb{R}^n.$$

On the other hand by Proposition 2.5, one has $\gamma_A^{VI'}(x) \leq \gamma_F^{VI'}(x),\ \forall x \in \mathbb{R}^n$. Let $x \in \mathbb{R}^n$ and $q \underset{\mathbb{R}^m_+}{\geq} 0$ be fixed. Because of $-q^T g(y) \geq 0, \forall y \in K$, adding $F(y)^T(x-y)$ in both sides, we have

$$F(y)^T(x-y) - q^T g(y) \geq F(y)^T(x-y).$$

Taking the supremum over all $y \in K$, we obtain that

$$\sup_{y \in \mathbb{R}^n} \{F(y)^T(x-y) - q^T g(y)\} \geq \sup_{y \in K} \{F(y)^T(x-y) - q^T g(y)\} \geq \sup_{y \in K} F(y)^T(x-y).$$

After taking the infimum in the left side over all $q \underset{\mathbb{R}^m_+}{\geq} 0$, it follows that $\gamma_L^{VI'}(x) \geq \gamma_A^{VI'}(x),\ \forall x \in \mathbb{R}^n$. Thus the proof is completed. □

At next we show that under monotonicity assumptions the functions introduced above can be related also to Auslender's and Giannessi's gap functions.

Proposition 2.7 *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a monotone vector-valued function. Then it holds*

$$\gamma_A^{VI'}(x) \leq \gamma_F^{VI'}(x) \leq \gamma_A^{VI}(x),\ \forall x \in \mathbb{R}^n.$$

Proof: By Proposition 2.5 there is $\gamma_A^{VI'}(x) \leq \gamma_F^{VI'}(x),\ \forall x \in \mathbb{R}^n$. Taking into account the monotonicity of F, it holds

$$[F(y) - F(x)]^T(y - x) \geq 0,\ \forall x, y \in \mathbb{R}^n,$$

or

$$F(y)^T(y - x) \geq F(x)^T(y - x),\ \forall x, y \in \mathbb{R}^n.$$

Let $x \in \mathbb{R}^n$ and $p \in \mathbb{R}^n$ be fixed. Adding $-p^T y$ and taking the infimum in both sides over all $y \in \mathbb{R}^n$, we get

$$\inf_{y \in \mathbb{R}^n}[-p^T y + F(y)^T(y - x)] \geq \inf_{y \in \mathbb{R}^n}[-p^T y + F(x)^T(y - x)],$$

or, equivalently,

$$\sup_{y \in \mathbb{R}^n}[p^T y - F(y)^T(y - x)] \leq \sup_{y \in \mathbb{R}^n}[p^T y - F(x)^T(y - x)]. \tag{2.6}$$

Then, after adding $\delta_K^*(-p)$ and taking the infimum in both sides over all $p \in \mathbb{R}^n$, we get

$$\begin{aligned} & \inf_{p \in \mathbb{R}^n}\left\{\sup_{y \in \mathbb{R}^n}[p^T y - F(y)^T(y - x)] + \delta_K^*(-p)\right\} = \gamma_F^{VI'}(x) \\ \leq\ & \inf_{p \in \mathbb{R}^n}\left\{\sup_{y \in \mathbb{R}^n}[p^T y - F(x)^T(y - x)] + \delta_K^*(-p)\right\} = \gamma_F^{VI}(x). \end{aligned}$$

In view of Proposition 2.1(i), one has $\gamma_F^{VI'}(x) \leq \gamma_A^{VI}(x),\ \forall x \in \mathbb{R}^n$. □

Example 2.3 Consider the optimization problem

$$(P_1) \qquad \min_{(x_1,x_2)^T \in \mathbb{B}}(x_1 + x_2),$$

where $\mathbb{B} = \{(x_1, x_2)^T \in \mathbb{R}^2 |\ x_1^2 + x_2^2 \leq 1\}$. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(y_1, y_2) = (1, 1)^T$. Then $x = (x_1, x_2)^T \in \mathbb{B}$ is an optimal solution to the optimization problem (P_1) if and only if $(x_1, x_2)^T \in \mathbb{B}$ is a solution to the variational inequality

$$F(x)^T(y - x) \geq 0, \forall y = (y_1, y_2)^T \in \mathbb{B}.$$

Since F is a constant function, one can easy see that Auslender's gap functions for (VI) and for (VI') are equal having the following formulation for $(x_1, x_2)^T \in \mathbb{R}^2$

$$\begin{aligned} \gamma_A^{VI}(x_1, x_2) &= \gamma_A^{VI'}(x_1, x_2) = \sup_{(y_1,y_2)^T \in \mathbb{B}}(1, 1)\begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \end{pmatrix} \\ &= x_1 + x_2 + \sup_{(y_1,y_2)^T \in \mathbb{B}}(-y_1 - y_2) = x_1 + x_2 + \sqrt{2}. \end{aligned}$$

According to Proposition 2.7, $\gamma_F^{VI'}$ turns out to be

$$\gamma_F^{VI'}(x_1, x_2) = x_1 + x_2 + \sqrt{2}, (x_1, x_2)^T \in \mathbb{R}^2.$$

We also show by direct computation that this is true. Indeed, for $(p_1, p_2)^T \in \mathbb{R}^2$, we have

$$\begin{aligned} & \sup_{(y_1,y_2)^T \in \mathbb{R}^2}\left[(p_1, p_2)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + F(y_1, y_2)^T\begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \end{pmatrix}\right] \\ =\ & \sup_{(y_1,y_2)^T \in \mathbb{R}^2}(p_1 y_1 + p_2 y_2 + x_1 - y_1 + x_2 - y_2) \\ =\ & x_1 + x_2 + \sup_{(y_1,y_2)^T \in \mathbb{R}^2}[(p_1 - 1)y_1 + (p_2 - 1)y_2] \\ =\ & \begin{cases} x_1 + x_2, & p_1 = p_2 = 1, \\ +\infty, & \text{otherwise}, \end{cases} \end{aligned}$$

from whence

$$\begin{aligned}\gamma_F^{VI'}(x_1,x_2) &= \inf_{\substack{p_1=1\\ p_2=1}} \left\{ x_1+x_2+\sup_{(y_1,y_2)^T\in\mathbb{B}} (-p_1y_1-p_2y_2)\right\}\\ &= x_1+x_2+\sup_{(y_1,y_2)^T\in\mathbb{B}}(-y_1-y_2) = x_1+x_2+\sqrt{2}.\end{aligned}$$

Let us prove now that $\gamma_F^{VI'}$ fulfills the properties in the definition of a gap function. As $x_1+x_2\geq -\sqrt{2}$, $\forall (x_1,x_2)^T\in\mathbb{B}$, relation (i) follows. Assume now that $\gamma_F^{VI'}(x_1,x_2)=0$, for $(x_1,x_2)^T\in\mathbb{B}$. As $x_1=-x_2-\sqrt{2}$, we have from $x_1^2+x_2^2\leq 1$ that $2x_2^2+2\sqrt{2}x_2+1\leq 0$. This is equivalent to $(\sqrt{2}x_2+1)^2\leq 0$ and therefore $x_1=x_2=-\frac{\sqrt{2}}{2}$. Since $\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)^T$ is an optimal solution to (P_1), it solves the variational inequality associated to (P_1).

Proposition 2.8 *Let $F:\mathbb{R}^n\to\mathbb{R}^n$ be a monotone vector-valued function and the ground set K be given by (2.1). Then it holds*

$$\gamma_L^{VI'}(x)\leq\gamma_{FL}^{VI'}(x)\leq\gamma_G^{VI}(x),\ \forall x\in\mathbb{R}^n.$$

Proof: By Proposition 2.6 one has

$$\gamma_L^{VI'}(x)\leq\gamma_{FL}^{VI'}(x),\ \forall x\in\mathbb{R}^n.$$

Let $x,p\in\mathbb{R}^n$ and $q\underset{\mathbb{R}^m_+}{\geq}0$ be fixed. Since F is monotone, in the same way we can obtain the relation (2.6). Hence, adding $(q^Tg)^*(-p)$ and taking the infimum in both sides over all $p\in\mathbb{R}^n$ and $q\underset{\mathbb{R}^m_+}{\geq}0$, it follows that

$$\begin{aligned}&\inf_{\substack{p\in\mathbb{R}^n\\ q\underset{\mathbb{R}^m_+}{\geq}0}}\left\{\sup_{y\in\mathbb{R}^n}[p^Ty-F(y)^T(y-x)]+(q^Tg)^*(-p)\right\}=\gamma_{FL}^{VI'}(x)\\ \leq\ &\inf_{\substack{p\in\mathbb{R}^n\\ q\underset{\mathbb{R}^m_+}{\geq}0}}\left\{\sup_{y\in\mathbb{R}^n}[p^Ty-F(x)^T(y-x)]+(q^Tg)^*(-p)\right\}=\gamma_{FL}^{VI}(x).\end{aligned}$$

Taking into account Proposition 2.1(ii) we conclude that

$$\gamma_{FL}^{VI'}(x)\leq\gamma_G^{VI}(x),\ \forall x\in\mathbb{R}^n.$$

□

Theorem 2.3 *Let K be a nonempty, closed, convex set and $F:\mathbb{R}^n\to\mathbb{R}^n$ be a monotone and continuous vector-valued function. Then $\gamma_F^{VI'}$ is a gap function for the problem (VI).*

Proof:

(i) Let $x\in K$ be fixed. By weak duality it holds

$$\gamma_F^{VI'}(x)=-v(D_F^{VI'};x)\geq -v(P^{VI'};x)\geq 0.$$

(ii) If $\gamma_F^{VI'}(x)=0$, then

$$0=v(D_F^{VI'};x)\leq v(P^{VI'};x)\leq 0.$$

Thus $v(P^{VI'};x)=0$ and so x is a solution to (VI'). By Lemma 2.1, x is also a solution to (VI). Conversely, let $x\in K$ be a solution to the problem (VI). Then it holds $\gamma_A^{VI}(x)=0$. By Proposition 2.7 and according to (i) we conclude that $\gamma_F^{VI'}(x)=0$. □

Theorem 2.4 *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a monotone and continuous vector-valued function. Assume that for the problem (VI) the constraint qualification (CQ) is fulfilled. Then $\gamma_L^{VI'}$ and $\gamma_{FL}^{VI'}$ are gap functions for the problem (VI).*

Proof:

(i) Let $x \in K$ be fixed. By weak duality and in view of Proposition 2.6 it holds

$$\gamma_{FL}^{VI'}(x) \geq \gamma_L^{VI'}(x) = -v(D_L^{VI'}; x) \geq -v(P^{VI'}; x) \geq 0.$$

(ii) To show the second property in the definition of a gap function, we apply the same way for $\gamma_L^{VI'}$ and $\gamma_{FL}^{VI'}$. Therefore verify this only for $\gamma_L^{VI'}$. As $\gamma_L^{VI'}(x) = 0$, we get that $0 = v(D_L^{VI'}; x) \leq v(P^{VI'}; x) \leq 0$. Consequently $v(P^{VI'}; x) = 0$. In other words, x solves (VI'). By Lemma 2.1, x is a solution to (VI). Let $x \in K$ be a solution to the problem (VI) and the constraint qualification (CQ) be fulfilled. Then it holds $\gamma_G^{VI}(x) = 0$. By Proposition 2.8 and in view of (i), it follows that $\gamma_L^{VI'}(x) = 0$. □

Remark 2.3 Since the functions

$$\sup_{y \in \mathbb{R}^n} \left\{ p^T y + F(y)^T (x - y) \right\} \quad \text{and} \quad \sup_{y \in K} \left\{ F(y)^T (x - y) - q^T g(y) \right\}$$

are convex as the pointwise supremum of affine functions with respect to (p, x) and (q, x), respectively, by Theorem 1 in [73] one can easily verify the convexity of the functions $\gamma_F^{VI'}$, $\gamma_L^{VI'}$ and $\gamma_{FL}^{VI'}$.

Example 2.4 Consider the optimization problem

$$(P_2) \qquad \min_{(x_1, x_2)^T \in \mathbb{B}} (x_1^2 - x_2),$$

where $\mathbb{B} = \{(x_1, x_2)^T \in \mathbb{R}^2 | \; x_1^2 + x_2^2 \leq 1\}$. One can verify that $(x_1, x_2)^T \in \mathbb{B}$ fulfills

$$y_1^2 - y_2 \geq x_1^2 - x_2, \forall (y_1, y_2)^T \in \mathbb{B} \tag{2.7}$$

if and only if it holds

$$2y_1^2 - 2x_1 y_1 \geq y_2 - x_2, \forall (y_1, y_2)^T \in \mathbb{B}. \tag{2.8}$$

By some widely-used inequalities it follows that (2.7) implies (2.8). For the reverse implication, let $(x_1, x_2)^T \in \mathbb{B}$ be satisfied (2.8) and consider an arbitrary pair $(y_1, y_2)^T \in \mathbb{B}$. Applying (2.8) for $\left(\frac{x_1+y_1}{2}, \frac{x_2+y_2}{2}\right)^T \in \mathbb{B}$, one gets immediately (2.7). Let us notice that (2.8) can be equivalently written as

$$2y_1(y_1 - x_1) - (y_2 - x_2) \geq 0, \forall (y_1, y_2)^T \in \mathbb{B}.$$

Considering $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(y_1, y_2) = (2y_1, -1)^T$, it follows that $x = (x_1, x_2)^T \in \mathbb{B}$ is an optimal solution to (P_2) if and only if $x = (x_1, x_2)^T \in \mathbb{B}$ solves the following variational inequality

$$F(y)^T (y - x) \geq 0, \forall y = (y_1, y_2)^T \in \mathbb{B}.$$

As $\mathbb{B}$ is a convex and closed set and F is a monotone and continuous mapping, by Lemma 2.1 this is equivalent to the problem of finding $x = (x_1, x_2)^T \in \mathbb{B}$ such that

$$F(x)^T (y - x) \geq 0, \forall y = (y_1, y_2)^T \in \mathbb{B}.$$

For $(x_1, x_2)^T \in \mathbb{R}^2$, Auslender's gap function turns out to be

$$\begin{aligned}\gamma_A^{VI}(x_1, x_2) &= \sup_{(y_1,y_2)^T \in \mathbb{B}} (2x_1, -1) \begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \end{pmatrix} \\ &= 2x_1^2 - x_2 + \sup_{(y_1,y_2)^T \in \mathbb{B}} (-2x_1 y_1 + y_2) = 2x_1^2 - x_2 + \sqrt{4x_1^2 + 1}.\end{aligned}$$

By Theorem 1.3, all gap functions introduced in Subsection 2.1.1 are equal with γ_A^{VI}. For $(x_1, x_2)^T \in \mathbb{B}$, one has $\gamma_A^{VI}(x_1, x_2) \geq 1 - x_2 \geq 0$. On the other hand, for an $(x_1, x_2)^T \in \mathbb{B}$ with $\gamma_A^{VI}(x_1, x_2) = 0$, x_2 must be equal to 1 and x_1 must be equal to 0. As $(0, 1)^T$ is the optimal solution of the problem (P_2), we succeeded to prove that γ_A^{VI} is really a gap function.

Let us try to find out the dual gap functions for the variational inequality problem associated to (P_2). As mentioned before, by Theorem 1.3 for all $(x_1, x_2)^T \in \mathbb{R}^2$ one has

$$\gamma_A^{VI'}(x_1, x_2) = \gamma_F^{VI'}(x_1, x_2) = \gamma_L^{VI'}(x_1, x_2) = \gamma_{FL}^{VI'}(x_1, x_2).$$

Let us calculate $\gamma_F^{VI'}$. By definition, for $(x_1, x_2)^T \in \mathbb{R}^2$, one has

$$\gamma_F^{VI'}(x_1, x_2) =$$

$$\inf_{(p_1,p_2)^T \in \mathbb{R}^2} \left[\sup_{(y_1,y_2)^T \in \mathbb{R}^2} (p_1 y_1 + p_2 y_2 - 2y_1^2 + 2x_1 y_1 + y_2 - x_2) + \delta_{\mathbb{B}}^*(-p_1, -p_2) \right].$$

As

$$\sup_{(y_1,y_2)^T \in \mathbb{R}^2} (p_1 y_1 + p_2 y_2 - 2y_1^2 + 2x_1 y_1 + y_2) = \begin{cases} \frac{(p_1 + 2x_1)^2}{8}, & \text{if } p_2 = -1, \\ +\infty, & \text{otherwise}, \end{cases}$$

and

$$\delta_{\mathbb{B}}^*(-p_1, -p_2) = \sqrt{p_1^2 + p_2^2},$$

we have

$$\gamma_F^{VI'}(x_1, x_2) = -x_2 + \inf_{p \in \mathbb{R}} \left\{ \frac{(p + 2x_1)^2}{8} + \sqrt{p^2 + 1} \right\}.$$

Since for $(x_1, x_2)^T \in \mathbb{B}$, one has $\gamma_F^{VI'}(x_1, x_2) \geq 1 - x_2 \geq 0$, property (i) in the definition of a gap function is fulfilled. On the other hand, if for $(x_1, x_2)^T \in \mathbb{B}$, $\gamma_F^{VI'}(x_1, x_2) = 0$, then x_2 must be equal to 1 and

$$\inf_{p \in \mathbb{R}} \left\{ \frac{(p + 2x_1)^2}{8} + \sqrt{p^2 + 1} \right\} = 1.$$

This can be true if $x_1 = 0$ and then the infimum is attained for $p = 0$. As $(0, 1)^T$ is the optimal solution to the problem (P_2), this proves that $\gamma_F^{VI'}$ is a gap function.

2.1.4 Optimality conditions and generalized variational inequalities

This subsection aims to consider some reformulations of the optimality conditions arising from the conjugate duality, the so-called generalized variational inequalities. The generalized variational inequality problem is closely related to the inclusion problem of finding a zero of set-valued mappings. Therefore, various methods as well as proximal point and splitting algorithms for solving the inclusion problems can be applied to problems concerning the optimality conditions.

Let $K \subseteq \mathbb{R}^n$ be a nonempty, closed and convex set, $F : K \rightrightarrows \mathbb{R}^n$ be a set-valued mapping. The generalized variational inequality is the problem of finding a point $x \in K$ such that

$$(GVI) \qquad \exists p \in F(x),\ p^T(y-x) \geq 0,\ \forall y \in K.$$

It is well known that (GVI) is closely related to the inclusion problem of finding a zero of a set-valued mapping $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$

$$(IP) \qquad 0 \in T(z),\ z \in \mathbb{R}^n.$$

In the case $K = \mathbb{R}^n$ and $F \equiv T$, (GVI) reduces to (IP). On the other hand, $x \in \mathbb{R}^n$ solves (GVI) if and only if

$$(IP_{gvi}) \qquad 0 \in F(x) + N_K(x),$$

where N_K is the normal cone operator given by

$$N_K(x) = \begin{cases} \{z \in \mathbb{R}^n |\ z^T(y-x) \leq 0,\ \forall y \in K\}, & \text{if } x \in K; \\ \varnothing, & \text{otherwise.} \end{cases}$$

The case concerning the Fenchel dual problem. Let $G \subseteq \mathbb{R}^n$ be a nonempty set and $u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a given function. We consider the optimization problem

$$(P_u) \qquad \inf_{x \in G} u(x).$$

One of the dual problems mentioned in Chapter 1 is the Fenchel dual problem as being

$$\sup_{p \in \mathbb{R}^n} \Big\{ -u^*(p) + \inf_{x \in G} p^T x \Big\}.$$

Proposition 2.9 *Let G be a convex set and $u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function. Assume that $ri(G) \cap ri(dom\ u) \neq \varnothing$. Then $\bar{x} \in G$ is an optimal solution to (P_u) if and only if it is a solution to the generalized variational inequality problem, i.e. there exists $\bar{p} \in \partial u(\bar{x})$ such that*

$$(GVI_u) \qquad \bar{p}^T(x - \bar{x}) \geq 0,\ \forall x \in G,$$

where $\partial u(x)$ is the subdifferential of the function u at $x \in \mathbb{R}^n$ defined by

$$\partial u(x) = \{z \in \mathbb{R}^n |\ u(y) - u(x) \geq z^T(y-x),\ \forall y \in \mathbb{R}^n\}.$$

Proof: Let $\bar{x} \in G$ be an optimal solution to (P_u). By the assumptions and in view of Theorem 2.10 (a) in [16], there exists $\bar{p} \in \mathbb{R}^n$ such that

$$u(\bar{x}) + u^*(\bar{p}) = \bar{p}^T \bar{x} \qquad \text{and} \qquad \bar{p}^T \bar{x} = \inf_{x \in G} \bar{p}^T x,$$

or, equivalently, $\bar{p} \in \partial u(\bar{x})$ such that

$$\bar{p}^T(x - \bar{x}) \geq 0,\ \forall x \in G.$$

This means that $\bar{x}$ is a solution to (GVI_u). The converse conclusion can be easily verified by using Theorem 2.10(b) in [16]. □

Consequently, (GVI_u) reduces to the following inclusion problem of finding $\bar{x} \in \mathbb{R}^n$ such that

$$(IP_u) \qquad 0 \in \partial u(\bar{x}) + N_G(\bar{x}).$$

It is well known that under maximal monotonicity assumptions of both set-valued mappings, the so-called splitting algorithm can be applied to (IP_u). Such algorithms can be found as survey in [28, Chapter 3] and in the related papers [35], [36] and [72]. Let us now recall the definition of maximal monotonicity of a set-valued mapping and some related results.

Definition 2.5

(i) A set-valued mapping $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be monotone if

$$(z - z')^T (x - x') \geq 0, \ \forall x, x' \in \mathbb{R}^n, \ z \in T(x), \ z' \in T(x');$$

(ii) A set-valued mapping $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be maximal monotone if it is monotone and its graph

$$G(T) = \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^n | \ u \in T(x)\}$$

is not strictly contained in the graph of any other monotone operator.

Proposition 2.10 *(see [71]) Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, lower semicontinuous function and $G \subseteq \mathbb{R}^n$ be a nonempty closed, convex set. Then set-valued maps $\partial f : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $N_G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ are maximal monotone.*

Let us now consider a further example in connection with variational inequalities. According to the mixed variational inequality (see Subsection 2.1.2), we can state the following assertion.

Proposition 2.11 *Let $ri(K) \cap ri(dom\ f) \neq \varnothing$ and K be a convex set. Then $\bar{x} \in K$ is a solution to (MVI) if and only if there exists $\bar{p} \in F(\bar{x}) + \partial f(\bar{x})$ such that*

$$(GVI_{mvi}) \qquad \bar{p}^T (y - \bar{x}) \geq 0, \ \forall y \in K.$$

Proof: Let $\bar{x} \in K$ be a solution to (GVI). Since $\inf\limits_{y \in K} \varphi(y) = 0 < +\infty$ (see (2.3)), by Theorem 2.10 (a) in [16], $\exists \bar{p} \in \mathbb{R}^n$ such that

$$f^*(\bar{p} - F(\bar{x})) + f(\bar{x}) = \bar{p}^T \bar{x} - F(\bar{x})^T \bar{x} \text{ and } \bar{p}^T \bar{x} = \inf_{y \in K} \bar{p}^T y.$$

In other words, $\bar{x}$ is a solution to (GVI_{mvi}). The converse direction follows from Theorem 2.10 (b). □

Corollary 2.1 *(cf. Theorem 2.1) Let $ri(K) \cap ri(dom\ f) \neq \varnothing$ and K be a convex set. Then $\bar{x} \in K$ is a solution to (GVI_{mvi}) if and only if $\gamma_F^{MVI}(\bar{x}) = 0$.*

(GVI_{mvi}) is equivalent to the inclusion problem of finding $\bar{x} \in \mathbb{R}^n$ such that

$$0 \in \partial f(\bar{x}) + F(\bar{x}) + N_K(\bar{x}).$$

Proposition 2.12 *(see [72]) Let $K \subseteq \mathbb{R}^n$ be a nonempty closed convex set and $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous, monotone vector-valued function. Then $F + N_K$ is a maximal monotone operator.*

The case concerning the Fenchel-Lagrange dual problem. Let in Subsection 1.2.1, the feasible set of the problem (P^{cps}) be given by

$$W = \left\{ u = (u_0, ..., u_n)^T \in \mathbb{R}^{n+1} \middle| \ A_i u \in W_i, \ i = \overline{1,n} \right\}.$$

In this case, the Fenchel-Lagrange dual problem to (P^{cps}) becomes

$$(D_{FL}^{cps}) \qquad \sup_{\substack{q_i, p_i \in \mathbb{R}^{l_i}, i = \overline{1,n} \\ \sum\limits_{i=1}^{n} A_i^T q_i = 0}} \left\{ - \sum_{i=1}^{n} F_i^*(p_i) + \sum_{i=1}^{n} \inf_{v_i \in W_i} (p_i + q_i)^T v_i \right\}.$$

Proposition 2.13 *Assume that* $\exists u' \in \mathbb{R}^{n+1}$ *such that* $A_i u' \in ri(W_i)$, $i = \overline{1,n}$ *(cf.* $(\widetilde{CQ})$*). Then a vector* $\bar{u} \in W$ *is an optimal solution to* (P^{cps}) *if and only if* $\forall i \in \{1,...,n\}$, $\bar{v}_i = A_i\bar{u} \in \mathbb{R}^{l_i}$ *is a solution to the following generalized variational inequality problem:* $\exists \bar{p}_i \in \partial F_i(\bar{v}_i)$ *such that*

$$(GVI^i_{cps}) \qquad (\bar{p}_i + \bar{q}_i)^T(v_i - \bar{v}_i) \geq 0,\ \forall v_i \in W_i,$$

where $\sum\limits_{i=1}^{n} A_i^T \bar{q}_i = 0$.

Proof: Let $\bar{u} \in W$ be an optimal solution to (P^{cps}). Then, by Theorem 1.6(a), there exists $\bar{p} = (\bar{p}_1, ..., \bar{p}_n) \in \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n}$ and
$\bar{q} = (\bar{q}_1, ..., \bar{q}_n) \in \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n}$, $\sum\limits_{i=1}^{n} A_i^T \bar{q}_i = 0$ such that

$$F_i(\bar{v}_i) + F_i^*(\bar{p}_i) = \bar{p}_i^T \bar{v}_i, \quad \text{and} \quad (\bar{p}_i + \bar{q}_i)^T \bar{v}_i = \inf_{v_i \in W_i} (\bar{p}_i + \bar{q}_i)^T v_i, \ \text{for } \bar{v}_i = A_i\bar{u},\ i = \overline{1,n}.$$

In other words, $\bar{v}_i$ is solution to (GVI^i_{cps}). In order to show the opposite direction, we apply Theorem 1.6(b). □

$\forall i \in \{1, ..., n\}$, (GVI^i_{cps}) leads to the inclusion problem of finding $\bar{v}_i \in \mathbb{R}^{l_i}$ such that

$$(IP^i_{cps}) \qquad 0 \in \bar{q}_i + N_{W_i}(\bar{v}_i) + \partial F_i(\bar{v}_i),$$

where $\bar{q}_i \in \mathbb{R}^{l_i}$ fulfills $\sum\limits_{i=1}^{n} A_i^T \bar{q}_i = 0$.

2.2 Gap functions for equilibrium problems

As discussed in Section 2.1, by using the Fenchel duality we can introduce a gap function for variational inequalities with an arbitrary ground set K. Following this idea, the approach from the previous section can be applied to more general cases including variational inequalities, namely equilibrium problems. Dealing with weaker sufficient conditions for Fenchel duality regarding convex optimization problems in the settings of locally convex spaces in [18], we extend the construction of a gap function from finite-dimensional variational inequalities to equilibrium problems in topological vector spaces.

2.2.1 Problem formulation and preliminaries

Let X be a real topological vector space and $K \subseteq X$ be a nonempty closed and convex set. Assume that $f : X \times X \to \mathbb{R} \cup \{+\infty\}$ is a bifunction satisfying $f(x, x) = 0,\ \forall x \in K$. The equilibrium problem consists in finding $x \in K$ such that

$$(EP) \qquad f(x, y) \geq 0,\ \forall y \in K.$$

A function $\gamma : X \to \overline{\mathbb{R}}$ is said to be a gap function for (EP) [63, Definition 2.1] if it satisfies the properties

(i) $\gamma(y) \geq 0,\ \forall y \in K$;

(ii) $\gamma(x) = 0$ and $x \in K$ if and only if x is a solution to (EP).

Some gap functions have been extended from variational inequalities to (EP). For instance, so-called regularized gap functions were investigated by Blum and Oettli

[15]. Such gap functions are summarized in Subsection 2.2.3. Moreover, the natural extension of the gap function in the sense of Auslender can be written as follows

$$\gamma_A^{EP}(x) := \sup_{y \in K} [-f(x,y)].$$

In this section we aim to apply the Fenchel duality in the settings of locally convex spaces to the construction of gap functions for equilibrium problems. Before doing this, let us recall some related definitions and results.

Let X be a real locally convex space and X^* be its topological dual, the set of all continuous linear functionals over X endowed with the weak* topology $w(X^*, X)$. By $\langle x^*, x\rangle$ we denote the value of $x^* \in X^*$ at $x \in X$. For the nonempty set $C \subseteq X$, the indicator function $\delta_C : X \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise,} \end{cases}$$

while the support function is $\sigma_C(x^*) = \sup\limits_{x \in C} \langle x^*, x\rangle$. Considering now a function $h : X \to \mathbb{R} \cup \{+\infty\}$, we denote by dom $h = \Big\{x \in X |\ h(x) < +\infty\Big\}$ its effective domain and by

$$\text{epi } h = \Big\{(x,r) \in X \times \mathbb{R} |\ h(x) \leq r\Big\}$$

its epigraph. A function $h : X \to \mathbb{R} \cup \{+\infty\}$ is called proper if dom $h \neq \varnothing$. The (Fenchel-Moreau) conjugate function of h is $h^* : X^* \to \mathbb{R} \cup \{+\infty\}$ defined by

$$h^*(p) = \sup_{x \in X} [\langle p, x\rangle - h(x)].$$

Definition 2.6 *Let the functions $h_i : X \to \mathbb{R} \cup \{+\infty\},\ i = 1, ..., m,$ be given.*

(i) The function $h_1 \Box \cdots \Box h_m : X \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$h_1 \Box \cdots \Box h_m(x) = \inf \Big\{ \sum_{i=1}^m h_i(x_i) |\ \sum_{i=1}^m x_i = x \Big\}$$

is called the infimal convolution function of $h_1, ..., h_m$.

(ii) The infimal convolution $h_1 \Box \cdots \Box h_m$ is called to be exact at $x \in X$ if there exist some $x_i \in X,\ i = 1, ..., m,$ such that $\sum\limits_{i=1}^m x_i = x$ and

$$h_1 \Box \cdots \Box h_m(x) = h_1(x_1) + ... + h_m(x_m).$$

Furthermore, we say that $h_1 \Box \cdots \Box h_m$ is exact if it is exact at every $x \in X$.

Let $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ and $\psi : X \to \mathbb{R} \cup \{+\infty\}$ be proper, convex and lower semicontinuous functions such that dom $\varphi \cap$ dom $\psi \neq \varnothing$. We consider the following optimization problem

$$(P) \qquad \inf_{x \in X} \Big\{\varphi(x) + \psi(x)\Big\}.$$

The Fenchel dual problem to (P) is

$$(D) \qquad \sup_{p \in X^*} \Big\{ - \varphi^*(-p) - \psi^*(p)\Big\}.$$

In [18] a new weaker regularity condition has been introduced in a more general case in order to guarantee the existence of strong duality between a convex optimization problem and its Fenchel dual, namely that the optimal objective values of the primal and the dual are equal and the dual has an optimal solution. This regularity condition for (P) can be written as

(FRC) $\quad \varphi^* \square \psi^*$ is lower semicontinuous and

$$\text{epi } (\varphi^* \square \psi^*) \cap \Big(\{0\} \times \mathbb{R}\Big) = \Big(\text{epi}(\varphi^*) + \text{epi}(\psi^*)\Big) \cap \Big(\{0\} \times \mathbb{R}\Big),$$

or, equivalently,

(FRC) $\quad \varphi^* \square \psi^*$ is a lower semicontinuous function and exact at 0.

Let us denote by $v(P)$ the optimal objective value of the optimization problem (P). The following theorem states the existence of strong duality between (P) and (D).

Proposition 2.14 *(see[18]) Let (FRC) be fulfilled. Then $v(P) = v(D)$ and (D) has an optimal solution.*

Remark that considering the perturbation function $\Phi : X \times X \to \mathbb{R} \cup \{+\infty\}$ defined by $\Phi(x, z) = \varphi(x) + \psi(x + z)$, one can obtain the Fenchel dual (D). Indeed, the function Φ fulfills $\Phi(x, 0) = \varphi(x) + \psi(x)$, $\forall x \in X$ and choosing (D) as being (cf. [27])

$$(D) \qquad \sup_{p \in X^*} \Big\{ - \Phi^*(0, p) \Big\},$$

this problem becomes actually the well-known Fenchel dual problem.

2.2.2 Gap functions based on Fenchel duality

In this subsection we construct gap functions for equilibrium problems by using a similar approach like the one considered for finite-dimensional variational inequalities in Section 2.1. Here, the Fenchel duality will play an important role. We assume that X is a real locally convex space and $K \subseteq X$ is a nonempty closed and convex set. Further, let $f : X \times X \to \mathbb{R} \cup \{+\infty\}$ be a given function such that $K \times K \subseteq \text{dom } f$ and $f(x, x) = 0$, $\forall x \in K$. Let $x \in X$ be given. Then (EP) can be reduced to the optimization problem

$$(P^{EP}; x) \qquad \inf_{y \in K} f(x, y).$$

We mention that $x^* \in K$ is a solution to (EP) if and only if it is an optimal solution to $(P^{EP}; x^*)$. Now let us reformulate $(P^{EP}; x)$ using the indicator function $\delta_K(y)$ as

$$(P^{EP}; x) \qquad \inf_{y \in X} \Big\{ f(x, y) + \delta_K(y) \Big\}.$$

Then we can write the Fenchel dual to $(P^{EP}; x)$ as being

$$\begin{aligned} (D^{EP}; x) \qquad & \sup_{p \in X^*} \Big\{ - \sup_{y \in X} [\langle p, y \rangle - f(x, y)] - \delta_K^*(-p) \Big\} \\ = \; & \sup_{p \in X^*} \Big\{ - f_y^*(x, p) - \delta_K^*(-p) \Big\}, \end{aligned}$$

where $f_y^*(x,p) := \sup\limits_{y\in X}[\langle p,y\rangle - f(x,y)]$ is the conjugate of $y \mapsto f(x,y)$ for a given $x \in X$. Let us introduce for any $x \in X$ the following function

$$\begin{aligned}\gamma_F^{EP}(x) := -v(D^{EP};x) &= -\sup_{p\in X^*}\Big\{-f_y^*(x,p) - \delta_K^*(-p)\Big\}\\ &= \inf_{p\in X^*}\Big\{f_y^*(x,p) + \sigma_K(-p)\Big\}.\end{aligned}$$

For $(P^{EP};x)$, the regularity condition (FRC) can be written as follows

$(FRC^{EP};x)$ $\quad f_y^*(x,\cdot)\Box\sigma_K$ is a lower semicontinuous function and exact at 0.

Theorem 2.5 *Assume that $\forall x \in K$ the regularity condition $(FRC^{EP};x)$ is fulfilled. Let for each $x \in K$, $y \mapsto f(x,y)$ be convex and lower semicontinuous. Then γ_F^{EP} is a gap function for (EP).*

Proof:

(i) By weak duality it holds

$$v(D^{EP};x) \le v(P^{EP};x) \le 0,\ \forall x \in K.$$

Therefore one has $\gamma_F^{EP}(x) = -v(D^{EP};x) \ge 0,\ \forall x \in K$.

(ii) If $\bar{x} \in K$ is a solution to (EP), then $v(P^{EP};\bar{x}) = 0$. On the other hand, by Proposition 2.14 the strong duality between $(P^{EP};\bar{x})$ and $(D^{EP};\bar{x})$ holds. In other words

$$v(D^{EP};\bar{x}) = v(P^{EP};\bar{x}) = 0.$$

This means that $\gamma_F^{EP}(\bar{x}) = 0$. Conversely, let $\gamma_F^{EP}(\bar{x}) = 0$ for $\bar{x} \in K$. Then

$$0 = v(D^{EP};\bar{x}) \le v(P^{EP};\bar{x}) \le 0.$$

Therefore $\bar{x}$ is a solution to (EP). □

Remark 2.4 According to Theorem 2.5, under the assumption $(FRC^{EP};x)$, $\forall x \in K$ the gap function introduced above coincides with γ_A^{EP}. The advantage of considering γ_F^{EP} may come when computing it. In order to do this one has to minimize the sum of the conjugate of a given function, for whose calculation the well-developed apparatus existent in the field of convex analysis can be helpful, with the support function of a nonempty closed convex set. On the other hand, in γ_A^{EP} for fixed $x \in K$ one has to compute the maximization problem over the set K which can be a harder work. This aspect is underlined in Example 2.5.

Even if the assumption that $(FRC^{EP};x)$ must be fulfilled for all $x \in K$ seems complicate let us notice that it is valid under the natural assumption $\operatorname{int} K \ne \emptyset$. For a comprehensive study on regularity conditions for Fenchel duality we refer to [18].

Example 2.5 Let $X = \mathbb{R}^2$, $K = \{(x_1,x_2)^T \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \le 1\}$ and $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x_1,x_2,y_1,y_2) = y_1^2 - x_1^2 - y_2 + x_2$. Consider the equilibrium problem of finding $(x_1,x_2)^T \in K$ such that

$$y_1^2 - y_2 \ge x_1^2 - x_2, \forall (y_1,y_2)^T \in K.$$

Instead of using γ_A^{EP} we determine γ_F^{EP}, as the calculations are easier. By definition, for $(x_1,x_2)^T \in \mathbb{R}^2$, one has

$$\gamma_F^{EP}(x_1,x_2) =$$

$$\inf_{(p_1,p_2)^T\in\mathbb{R}^2}\left[\sup_{(y_1,y_2)^T\in\mathbb{R}^2}(p_1y_1+p_2y_2-y_1^2+x_1^2+y_2-x_2)+\delta_K^*(-p_1,-p_2)\right].$$

As

$$\sup_{(y_1,y_2)^T\in\mathbb{R}^2}(p_1y_1+p_2y_2-y_1^2+y_2)=\begin{cases}\frac{p_1^2}{4}, & \text{if } p_2=-1,\\ +\infty, & \text{otherwise,}\end{cases}$$

and

$$\delta_K^*(-p_1,-p_2)=\sqrt{p_1^2+p_2^2},$$

we have

$$\gamma_F^{EP}(x_1,x_2)=x_1^2-x_2+\inf_{p\in\mathbb{R}}\left\{\frac{p^2}{4}+\sqrt{p^2+1}\right\}=x_1^2-x_2+1.$$

Since for $(x_1,x_2)^T\in K$, one has $\gamma_F^{EP}(x_1,x_2)\geq 1-x_2\geq 0$, property (i) in the definition of a gap function is fulfilled. On the other hand, if for an $(x_1,x_2)^T\in K$, $\gamma_F^{EP}(x_1,x_2)=0$, then x_2 must be equal to 1 and x_1 must be equal to 0. As $(0,1)^T$ is the only solution to the equilibrium problem considered within this example, γ_F^{EP} is a gap function.

An alternative proof of the fact that γ_F^{EP} is a gap function comes from verifying the fulfillment of the hypotheses of Theorem 2.5, which are surely fulfilled. As int $K\neq\varnothing$, the regularity condition $(FRC^{EP};x)$ is obviously valid for all $x\in K$.

Example 2.6 Let $X=\mathbb{R}^2$, $K=\{0\}\times\mathbb{R}_+$ and $f:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}\cup\{+\infty\}$, $f=\delta_{\mathbb{R}^2_+\times\mathbb{R}^2_+}$. One can see that $K\times K\subseteq \operatorname{dom} f$, $f(x,x)=0,\forall x\in K$ and that for all $x\in K$ the mapping $y\mapsto f(x,y)$ is convex and lower semicontinuous. We show that although int $K\neq\varnothing$ fails, the regularity condition $(FRC^{EP};x)$ is fulfilled for all $x\in K$.

Let $x\in K$ be fixed. For all $p\in\mathbb{R}^2$ we have

$$f_y^*(x,p)=\sup_{y\in\mathbb{R}^2_+}p^Ty=\delta_{-\mathbb{R}^2_+}(p)$$

and

$$\sigma_K(p)=\sup_{y\in\{0\}\times\mathbb{R}_+}p^Ty=\delta_{\mathbb{R}\times(-\mathbb{R}_+)}(p).$$

As $f_y^*(x,\cdot)\Box\sigma_K=\delta_{\mathbb{R}\times(-\mathbb{R}_+)}$, it is obvious that this function is lower semicontinuous and exact at 0. The regularity condition $(FRC^{EP};x)$ is fulfilled for all $x\in K$ and one can apply Theorem 2.5.

Remark 2.5 In the following we stress the connections between the gap function we have just introduced and convex optimization. Therefore let $K\subseteq X$ be a convex and closed set and $u:X\to\mathbb{R}\cup\{+\infty\}$ be a convex and lower semicontinuous function with $K\subseteq \operatorname{dom} u$. We consider the following optimization problem with geometrical constraints

$$(P_u)\qquad\qquad \inf_{x\in K}u(x).$$

Take $f:X\times X\to\mathbb{R}\cup\{+\infty\}$, $f(x,y)=u(y)-u(x)$ and assume, by convention, that $(+\infty)-(+\infty)=+\infty$. For all $x\in X$ the gap function γ_F^{EP} becomes $\gamma_F^{EP}(x)=\inf_{p\in X^*}\left\{u^*(p)+\sigma_K(-p)\right\}+u(x)$. Assuming that $u^*\Box\sigma_K$ is lower semicontinuous and exact at 0, the hypotheses of Theorem 2.5 are fulfilled and, so, γ_F^{EP} turns out to be a gap function for the equilibrium problem which consists in finding $x\in K$ such that

$$f(x,y)=u(y)-u(x)\geq 0,\forall y\in K\Leftrightarrow u(y)\geq u(x),\forall y\in K.$$

Since

$$(D_u) \qquad \sup_{p\in X^*}\{-u^*(p)-\sigma_K(-p)\}$$

is the Fenchel dual problem to (P_u), we observe that the property (i) in the definition of a gap function is nothing else than weak duality between these problems. The second requirement asks $x \in K$ to be a solution to (P_u) if and only if $\gamma_F^{EP}(x)=0$, which is nothing else than $u(x)=\sup\limits_{p\in X^*}\{-u^*(p)-\sigma_K(-p)\}$.

In the second part of the subsection we assume that dom $f = X\times X$ and under this assumption we deal with the so-called dual equilibrium problem (cf. [52]) which is closely related to (EP) and consists in finding $x\in K$ such that

$$(DEP) \qquad f(y,x)\leq 0,\ \forall y\in K,$$

or, equivalently,

$$(DEP) \qquad -f(y,x)\geq 0,\ \forall y\in K.$$

By K^{EP} and K^{DEP} we denote the solution sets of the problems (EP) and (DEP), respectively. In order to suggest another gap function for (EP) we need some definitions and results.

Definition 2.7 *The bifunction $f : X\times X\to\mathbb{R}$ is said to be*

(i) monotone if, for each pair of points $x,y\in X$, we have

$$f(x,y)+f(y,x)\leq 0;$$

(ii) pseudomonotone if, for each pair of points $x,y\in X$, we have

$$f(x,y)\geq 0 \textit{ implies } f(y,x)\leq 0.$$

Definition 2.8 *Let $K\subseteq X$ and $\varphi : X\to\mathbb{R}$. The function φ is said to be*

(i) quasiconvex on K if, for each pair of points $x,y\in K$ and for all $\alpha\in[0,1]$, we have

$$\varphi(\alpha x+(1-\alpha)y)\leq \max\Big\{\varphi(x),\varphi(y)\Big\};$$

(ii) explicitly quasiconvex on K if it is quasiconvex on K and for each pair of points $x,y\in K$ such that $\varphi(x)\neq\varphi(y)$ and for all $\alpha\in(0,1)$, we have

$$\varphi(\alpha x+(1-\alpha)y)< \max\Big\{\varphi(x),\varphi(y)\Big\}.$$

(iii) (explicitly) quasiconcave on K if $-\varphi$ is (explicitly) quasiconvex on K.

Definition 2.9 *Let $K\subseteq X$ and $\varphi : X\to\mathbb{R}$. The function φ is said to be u-hemicontinuous on K if, for all $x,y\in K$ and $\alpha\in[0,1]$, the function $\tau(\alpha)=\varphi(\alpha x+(1-\alpha)y)$ is upper semicontinuous at 0.*

Proposition 2.15 *(cf. [52, Proposition 2.1])*

(i) If f is pseudomonotone, then $K^{EP}\subseteq K^{DEP}$.

(ii) If $f(\cdot,y)$ is u-hemicontinuous on K for all $y\in K$ and $f(x,\cdot)$ is explicitly quasiconvex on K for all $x\in K$ then $K^{DEP}\subseteq K^{EP}$.

By using (DEP), in the same way as before, we introduce a new gap function for (EP). Let $x \in K$ be a solution to (DEP). This is equivalent to that x is an optimal solution to the optimization problem

$$(P^{DEP}; x) \qquad \inf_{y \in K} [-f(y, x)].$$

Now we consider $(P^{DEP}; x)$ for all $x \in X$. The corresponding Fenchel dual problem to $(P^{DEP}; x)$ is

$$(D^{DEP}; x) \qquad \sup_{p \in X^*} \Big\{ -\sup_{y \in X} [\langle p, y \rangle + f(y, x)] - \delta_K^*(-p) \Big\},$$

if we rewrite $(P^{DEP}; x)$ again using δ_K similarly as done for $(P^{EP}; x)$. Let us define the function

$$\begin{aligned} \gamma_F^{DEP}(x) : &= -v(D^{DEP}; x) \\ &= -\sup_{p \in X^*} \Big\{ -\sup_{y \in X} [\langle p, y \rangle + f(y, x)] - \delta_K^*(-p) \Big\} \\ &= \inf_{p \in X^*} \Big\{ \sup_{y \in X} [\langle p, y \rangle + f(y, x)] + \sigma_K(-p) \Big\}. \end{aligned}$$

Assuming that for all $x \in K$ the function $y \mapsto -f(y, x)$ is convex and lower-semicontinuous one can give, in analogy to Theorem 2.5, some weak regularity conditions such that γ_F^{DEP} becomes a gap function for (DEP). Next result shows under which conditions γ_F^{DEP} becomes a gap function for the equilibrium problem (EP).

Proposition 2.16 *Assume that f is a monotone bifunction. Then it holds*

$$\gamma_F^{DEP}(x) \leq \gamma_F^{EP}(x), \ \forall x \in X.$$

Proof: By the monotonicity of f, we have

$$f(x, y) + f(y, x) \leq 0, \ \forall x, y \in X,$$

or, equivalently, $f(y, x) \leq -f(x, y), \ \forall x, y \in X$. Let $p \in X^*$ be fixed. Adding $\langle p, y \rangle$ and taking the supremum in both sides over all $y \in X$ yields

$$\sup_{y \in X} [\langle p, y \rangle + f(y, x)] \leq \sup_{y \in X} [\langle p, y \rangle - f(x, y)].$$

After adding $\sigma_K(-p)$ and taking the infimum in both sides over $p \in X^*$, we conclude that $\gamma_F^{DEP}(x) \leq \gamma_F^{EP}(x), \ \forall x \in X.$ □

Theorem 2.6 *Let the assumptions of Theorem 2.5, Proposition 2.15(ii) and Proposition 2.16 be fulfilled. Then γ_F^{DEP} is a gap function for (EP).*

Proof:

(i) By weak duality it holds

$$\gamma_F^{DEP}(x) = -v(D^{DEP}; x) \geq -v(P^{DEP}; x) \geq 0, \ \forall x \in K.$$

(ii) Let $\bar{x}$ be a solution to $(EP.)$ By Theorem 2.5, $\bar{x}$ is solution to (EP) if and only if $\gamma_F^{EP}(\bar{x}) = 0$. In view of (i) and Proposition 2.16, we get

$$0 \leq \gamma_F^{DEP}(\bar{x}) \leq \gamma_F^{EP}(\bar{x}) = 0.$$

Whence $\gamma_F^{DEP}(\bar{x}) = 0$. Let now $\gamma_F^{DEP}(\bar{x}) = 0$. By weak duality it holds

$$0 = v(D^{DEP}; \bar{x}) \leq v(P^{DEP}; \bar{x}) \leq 0.$$

Consequently $v(P^{DEP}; \bar{x}) = 0$. That means $\bar{x} \in K^{DEP}$. Hence, according to Proposition 2.15(ii), $\bar{x}$ is a solution to (EP). □

2.2.3 Regularized gap functions

The current subsection purposes to summarize some gap functions for (EP) (see [14] and [63]) in the same way as in Subsection 2.2.2. Throughout this subsection we assume that X is a real reflexive Banach space and f is defined on $K \times K$ and takes real values, fulfilling $f(x,x) = 0,\ \forall x \in K$. This can be seen as special case of our general framework, i.e. the situation when $\text{dom} f = K \times K$. Further, let $h : K \times K \to \mathbb{R}$ be a bifunction such that for each $x \in K,\ y \mapsto h(x,y)$ is convex, differentiable and

(a) $h(x,y) \geq 0,\ \forall x, y \in K$;

(b) $h(x,x) = 0,\ \forall x \in K$;

(c) $h'_y(x,x) = 0,\ \forall x \in K$, where h'_y means the derivative of h in the sense of Gâteaux (cf. Definition 2.10) with respect to the second variable.

The gap functions we consider in this section will be defined on the set K. In order to remain consistent with the definition of the gap function we gave in the introduction, one may consider the gap function as taking the value $+\infty$ outside K.

Definition 2.10 *A functional $g : X \to \mathbb{R}$ is said to be differentiable (in the sense of Gâteaux) at the point $x \in X$ if there exists $g'(x) \in X^*$ such that*

$$\lim_{t \to 0} \frac{g(x+th) - g(x)}{t} = \langle g'(x), h \rangle$$

is finite.

Proposition 2.17 *[62, cf. Proposition 2.1]*
Let $f(x,y)$ be a convex, differentiable bifunction with respect to y and $h(x,y)$ be a function fulfilling the conditions $(a)-(c)$. Then $\bar{x}$ is a solution to (EP) if and only if it is a solution to the auxiliary equilibrium problem of finding $\bar{x} \in K$ such that

$$(EP_h) \qquad f(\bar{x}, y) + h(\bar{x}, y) \geq 0,\ \forall y \in K.$$

Proof: Since in [62] has been used the alternative formulation, namely the variables were exchanged in (EP), let us show how the proof looks at our case. Indeed, it is clear that if $\bar{x}$ is a solution to (EP), then it is also a solution to (EP_h). Let $\bar{x}$ be a solution to (EP_h). Then $\bar{x}$ is an optimal solution to the optimization problem

$$\inf_{y \in K} [f(\bar{x}, y) + h(\bar{x}, y)]. \tag{2.9}$$

Since K is convex, $\bar{x}$ is an optimal solution to (2.9) if and only if

$$\langle f'_y(\bar{x}, \bar{x}) + h'_y(\bar{x}, \bar{x}), y - \bar{x} \rangle \geq 0,\ \forall y \in K,$$

or, equivalently,

$$\langle f'_y(\bar{x}, \bar{x}), y - \bar{x} \rangle \geq 0,\ \forall y \in K.$$

In view of the convexity of $f(\bar{x}, \cdot)$ we obtain that

$$f(\bar{x}, y) - f(\bar{x}, \bar{x}) \geq \langle f'_y(\bar{x}, \bar{x}), y - \bar{x} \rangle \geq 0,\ \forall y \in K.$$

This means that $f(\bar{x}, y) \geq 0,\ \forall y \in K$. □

Corollary 2.2 *Let $f(x,y)$ be a concave, differentiable bifunction with respect to x. Then $\bar{x}$ is a solution to (DEP) if and only if it is a solution to the dual auxiliary equilibrium problem of finding $\bar{x} \in K$ such that*

$$(DEP_h) \qquad -f(y,\bar{x}) + h(\bar{x},y) \geq 0, \ \forall y \in K.$$

Proof: Since $-f(x,y)$ is convex and differentiable with respect to x, choosing $-f(y,x)$ instead of $f(x,y)$, we can apply Proposition 2.17. □

In [14], Blum and Oettli proposed the following gap function for (EP)

$$\gamma_h^{EP}(x) := \sup_{y \in K}[-f(x,y) - h(x,y)],$$

while instead of (c) was taken the condition

($\bar{c}$) $h(x,(1-\lambda)x + \lambda y) = o(\lambda), \ \lambda \in [0,1].$

Gap functions of such type have been investigated also for finite-dimensional variational inequalities, see for instance in [7], [21] and [99], whose important property under certain assumptions is the differentiability. Recently, in a finite-dimensional space, the differentiability of such type of a gap function for (EP) has been considered in [63].

Theorem 2.7 *Let the assumptions of Proposition 2.17 be fulfilled. Then γ_h^{EP} is a gap function for (EP).*

Proof:

(i) $\gamma_h^{EP}(x) = \sup_{y \in K}[-f(x,y) - h(x,y)] \geq -f(x,x) - h(x,x) = 0, \ \forall x \in K.$

(ii) If $\bar{x}$ is a solution to (EP), then by (a) we have

$$f(\bar{x},y) + h(\bar{x},y) \geq 0, \ \forall y \in K.$$

Whence $\gamma_h^{EP}(\bar{x}) = \sup_{y \in K}[-f(\bar{x},y) - h(\bar{x},y)] \leq 0$. Therefore, by (i), we obtain $\gamma_h^{EP}(\bar{x}) = 0$. Let now $\gamma_h^{EP}(\bar{x}) = 0$. Consequently

$$f(\bar{x},y) + h(\bar{x},y) \geq 0, \ \forall y \in K.$$

By Proposition 2.17, this is true if and only if $f(\bar{x},y) \geq 0, \ \forall y \in K$. □

On the other hand, γ_h^{EP} is closely related to the function $\gamma_h^{DEP} : K \to \mathbb{R} \cup \{+\infty\}$ defined by (see [14])

$$\gamma_h^{DEP}(x) := \sup_{y \in K}[f(y,x) - h(x,y)].$$

Proposition 2.18 *Let $f : K \times K \to \mathbb{R}$ be a monotone bifunction. Then it holds*

$$\gamma_h^{DEP}(x) \leq \gamma_h^{EP}(x), \ \forall x \in K. \tag{2.10}$$

Proof: By the monotonicity of f, we have

$$f(x,y) + f(y,x) \leq 0, \ \forall x,y \in K,$$

or, equivalently,

$$f(y,x) \leq -f(x,y), \ \forall x,y \in K.$$

After adding $-h(x,y)$ and taking the infimum in both sides over $y \in K$, we conclude that $\gamma_h^{DEP}(x) \leq \gamma_h^{EP}(x), \ \forall x \in K.$ □

Theorem 2.8 *Let $f : K \times K \to \mathbb{R},\ (x, y) \mapsto f(x, y)$ be concave with respect to x and convex with respect to y. Assume that f is a monotone differentiable bifunction and the assumptions of Proposition 2.15(ii) are fulfilled. Then γ_h^{DEP} is a gap function for (EP).*

Proof:

(i) $\gamma_h^{DEP}(x) = \sup\limits_{y \in K}[f(y, x) - h(x, y)] \geq f(x, x) - h(x, x) = 0,\ \forall x \in K.$

(ii) By Theorem 2.7, $\bar{x}$ is a solution to (EP) if and only if $\gamma_h^{EP}(\bar{x}) = 0$. According to (2.10) it holds

$$0 \leq \gamma_h^{DEP}(\bar{x}) \leq \gamma_h^{EP}(\bar{x}) = 0.$$

In other words $\gamma_h^{DEP}(\bar{x}) = 0$. Let now $\gamma_h^{DEP}(\bar{x}) = 0$. Then

$$-f(y, \bar{x}) + h(\bar{x}, y) \geq 0,\ \forall y \in K.$$

Taking into account Corollary 2.2 and Proposition 2.15(ii) we conclude that $f(\bar{x}, y) \geq 0,\ \forall y \in K.$ □

2.2.4 Applications to variational inequalities

In this subsection we apply the approach proposed in Subsection 2.2.2 to variational inequalities in a real Banach space. We assume that X is a real Banach space. The variational inequality problem consists in finding $x \in K$ such that

$$(VI) \qquad \langle F(x), y - x\rangle \geq 0,\ \forall y \in K,$$

where $F : K \to X^*$ is a given mapping and $K \subseteq X$ is a closed and convex set. Considering $f : X \times X \to \mathbb{R} \cup \{+\infty\}$,

$$f(x, y) = \begin{cases} \langle F(x), y - x\rangle, & \text{if } (x, y) \in K \times X, \\ +\infty, & \text{otherwise,} \end{cases}$$

the problem (VI) can be seen as a particular case of the equilibrium problem (EP).

For $x \in K$, (VI) can be rewritten as the optimization problem

$$(P^{VI}; x) \qquad \inf_{y \in X} \Big\{\langle F(x), y - x\rangle + \delta_K(y)\Big\},$$

in the sense that $\bar{x}$ is a solution to (VI) if and only if it is an optimal solution to $(P^{VI}; \bar{x})$. In view of γ_F^{EP}, we introduce the function based on Fenchel duality for (VI) by

$$\begin{aligned} \gamma_F^{VI}(x) &= \inf_{p \in X^*} \Big\{ \sup_{y \in X}[\langle p, y\rangle - \langle F(x), y - x\rangle] + \sigma_K(-p)\Big\} \\ &= \inf_{p \in X^*} \Big\{ \sup_{y \in X}\langle p - F(x), y\rangle + \sigma_K(-p)\Big\} + \langle F(x), x\rangle, \forall x \in K. \end{aligned}$$

From

$$\sup_{y \in X}\langle p - F(x), y\rangle = \begin{cases} 0, & \text{if } p = F(x), \\ +\infty, & \text{otherwise,} \end{cases}$$

follows that

$$\gamma_F^{VI}(x) = \inf_{p = F(x)} \sup_{y \in K}\langle -p, y\rangle + \langle F(x), x\rangle = \sup_{y \in K}\langle F(x), x - y\rangle, \forall x \in K.$$

In accordance to the definition of γ_F^{EP} in Subsection 2.2.2, we have that for $x \notin K$, $\gamma_F^{VI}(x) = -\infty$.

Let us notice that for all $x \in K$, $y \mapsto f(x, y)$ is an affine function, thus continuous. On the other hand, the set $\mathrm{epi}(f_y^*(x, \cdot)) + \mathrm{epi}(\sigma_K) = \{F(x)\} \times [\langle F(x), x\rangle, +\infty) + \mathrm{epi}(\sigma_K)$ is closed for all $x \in K$. This means that for all $x \in K$, $f_y^*(x, \cdot) \square \sigma_K$ is lower semicontinuous and exact everywhere in X^* (cf. [18]). Thus the hypotheses of Theorem 2.5 are verified and γ_F^{VI} turns out to be a gap function for the problem (VI). γ_F^{VI} is actually so-called Auslender's gap function (see [2] and [8]).

The problem (VI) can be associated to the following variational inequality introduced by Minty which consists in finding $x \in K$ such that

$$(VI') \qquad \langle F(y), y - x\rangle \geq 0, \ \forall y \in K.$$

As in Subsection 2.2.2, before we introduce another gap function for (VI), let us consider some definitions and assertions.

Definition 2.11 *A mapping $F : K \to X^*$ is said to be*

(i) monotone if, for each pair of points $x, y \in K$, we have

$$\langle F(y) - F(x), y - x\rangle \geq 0;$$

(ii) pseudo-monotone if, for each pair of points $x, y \in K$, we have

$$\langle F(x), y - x\rangle \geq 0 \textit{ implies } \langle F(y), y - x\rangle \geq 0;$$

(iii) continuous on finite-dimensional subspaces if for any finite-dimensional subspace M of X with $K \cap M \neq \varnothing$ the restricted mapping $F : K \cap M \to X^*$ is continuous from the norm topology of $K \cap M$ to the weak topology of X^*.*

Proposition 2.19 *(see [95, Lemma 3.1]) Let $F : K \to X^*$ be a pseudo-monotone mapping which is continuous on finite-dimensional subspaces. Then $x \in K$ is a solution to (VI) if and only if it is a solution to (VI').*

Minty's variational inequality (VI') is equivalent to the equilibrium problem which consists in finding $x \in K$ such that

$$-f(y, x) \geq 0, \forall y \in K.$$

As

$$-f(y, x) = \begin{cases} \langle F(y), y - x\rangle, & \text{if } (x, y) \in X \times K, \\ -\infty, & \text{otherwise,} \end{cases}$$

using the formula of γ_F^{DEP}, we get

$$\gamma_F^{VI'}(x) := \inf_{p \in X^*} \Big\{ \sup_{y \in K} [\langle p, y\rangle - \langle F(y), y - x\rangle] + \sigma_K(-p) \Big\}.$$

We can see that $\gamma_F^{VI'}$ is nothing else than the so-called dual gap function for (VI) which considered in Subsection 2.1.3 in finite-dimensional spaces. In fact, let $x \in X^*$ be fixed. According to the inequality

$$\sup_{y \in K} [\langle p, y\rangle - \langle F(y), y - x\rangle] \geq \langle p, z\rangle - \langle F(z), z - x\rangle, \ \forall z \in K,$$

it is easy to obtain that

$$\sup_{y \in K} [\langle p, y\rangle - \langle F(y), y - x\rangle] + \sup_{y \in K} [-\langle p, y\rangle] \geq \sup_{y \in K} \langle F(y), x - y\rangle.$$

After taking the infimum in the left hand side over all $p \in X^*$ and since the infimum is attained at $0 \in X^*$, one has (cf. $\gamma_A^{VI'}$ in Subsection 2.1.3)

$$\gamma_F^{VI'}(x) = \sup_{y \in K} \langle F(y), x - y \rangle.$$

Proposition 2.20 *Let $F : K \to X^*$ be a monotone mapping. Then it holds*

$$\gamma_F^{VI'}(x) \leq \gamma_F^{VI}(x), \ \forall x \in K.$$

Theorem 2.9 *Let $F : K \to X^*$ be a monotone mapping which is continuous on finite-dimensional subspaces. Then $\gamma_F^{VI'}$ is a gap function for (VI).*

Proof:

(i) $\gamma_F^{DEP}(x) \geq 0$ implies that $\gamma_F^{VI'}(x) \geq 0, \ \forall x \in K$, as this is a special case.

(ii) By the definition of a gap function, $\bar{x} \in K$ is a solution to (VI) if and only if $\gamma_F^{VI}(\bar{x}) = 0$. Taking into account (i) and Proposition 2.20, one has

$$0 \leq \gamma_F^{VI'}(\bar{x}) \leq \gamma_F^{VI}(\bar{x}) = 0.$$

In other words, $\gamma_F^{VI'}(\bar{x}) = 0$. Let now $\gamma_F^{VI'}(\bar{x}) = 0$. We can easily see that $\bar{x} \in K$ is a solution to (VI'). This follows using an analogous argumentation as in the proof of Theorem 2.6. Whence, according to Proposition 2.19, $\bar{x}$ solves (VI). □

Chapter 3

Conjugate duality for vector optimization with applications

3.1 Conjugate duality for vector optimization

Among the references dealing with conjugate duality for vector optimization problems, we mention the papers [81], [82] by Tanino and Sawaragi and the books [42], [75] as playing an important role in this chapter. Tanino and Sawaragi developed the conjugate duality for vector optimization by introducing new concepts of conjugate maps and set-valued subgradients based on Pareto efficiency. Furthermore, by using the concept of supremum of a set (cf. [83]) on the basis of weak orderings, the conjugate duality theory was extended to a partially ordered topological vector space by Tanino [84] and to set-valued vector optimization problems by Song [78], respectively.

This chapter begins with recalling the concepts of conjugate maps, set-valued subgradients and duality results for vector optimization given in [75]. For convenience, we use some notations and definitions from [42]. Afterwards, we propose dual vector optimization problems having set-valued objective maps, which arise from different perturbations in analogy to the scalar case in [90]. In addition, different dual problems stated by using conjugate maps with vector variables are also discussed.

3.1.1 Preliminaries

Let C be a pointed closed and convex cone in $\mathbb{R}^n$. For any $\xi, \mu \in \mathbb{R}^n$, we use the following ordering relations:

$$\begin{aligned} \xi \underset{C}{\leq} \mu \quad &\Leftrightarrow \quad \mu - \xi \in C; \\ \xi \underset{C\setminus\{0\}}{\leq} \mu \quad &\Leftrightarrow \quad \mu - \xi \in C\setminus\{0\}; \\ \xi \underset{C\setminus\{0\}}{\not\leq} \mu \quad &\Leftrightarrow \quad \mu - \xi \notin C\setminus\{0\}. \end{aligned}$$

The notions $\underset{C}{\geq}$, $\underset{C\setminus\{0\}}{\geq}$ and $\underset{C\setminus\{0\}}{\not\geq}$ are used in an alternative way.

Definition 3.1 *A point $y \in \mathbb{R}^n$ is said to be a maximal point of a set $Y \subseteq \mathbb{R}^n$ if $y \in Y$ and there is no $y' \in Y$ such that $y \underset{C\setminus\{0\}}{\leq} y'$.*

The set of all maximal points of Y is called the maximum of Y and is denoted by $\max\limits_{C\setminus\{0\}} Y$. The minimum of Y is defined analogously. Further we take the cone C being the nonnegative orthant

$$\mathbb{R}^n_+ = \Big\{x = (x_1, ..., x_n)^T \in \mathbb{R}^n |\ x_i \geq 0,\ i = \overline{1,n}\Big\}.$$

Lemma 3.1 *[75, cf. Proposition 3.1.3] Let $Y_1, Y_2 \subseteq \mathbb{R}^n$. Then*

(i) $\max\limits_{\mathbb{R}^n_+\setminus\{0\}} (Y_1 + Y_2) \subseteq \max\limits_{\mathbb{R}^n_+\setminus\{0\}} Y_1 + \max\limits_{\mathbb{R}^n_+\setminus\{0\}} Y_2$;

(ii) $\min\limits_{\mathbb{R}^n_+\setminus\{0\}} (Y_1 + Y_2) \subseteq \min\limits_{\mathbb{R}^n_+\setminus\{0\}} Y_1 + \min\limits_{\mathbb{R}^n_+\setminus\{0\}} Y_2$.

Definition 3.2 *[42, cf. Definition 8.2.2]*

(i) Let $Y \subseteq \mathbb{R}^n$ be a given set. The set $\min\limits_{\mathbb{R}^n_+\setminus\{0\}} Y$ is said to be externally stable if

$$Y \subseteq \min_{\mathbb{R}^n_+\setminus\{0\}} Y + \mathbb{R}^n_+.$$

(ii) Similarly, the set $\max\limits_{\mathbb{R}^n_+\setminus\{0\}} Y$ is said to be externally stable if

$$Y \subseteq \max_{\mathbb{R}^n_+\setminus\{0\}} Y - \mathbb{R}^n_+.$$

Lemma 3.2 *[75, Lemma 6.1.1] Let $F_1 : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ and $F_2 : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be set-valued maps and $X \subseteq \mathbb{R}^n$. Then*

$$\max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \Big[F_1(x) + F_2(x)\Big] \subseteq \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \Big[F_1(x) + \max_{\mathbb{R}^p_+\setminus\{0\}} F_2(x)\Big].$$

If $\max\limits_{\mathbb{R}^p_+\setminus\{0\}} F_2(x)$ is externally stable for every $x \in X$, then the converse inclusion also holds.

Corollary 3.1 *[75, Corollary 6.1.3] Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a set-valued map and $X \subseteq \mathbb{R}^n$. If $\max\limits_{\mathbb{R}^p_+\setminus\{0\}} F(x)$ is externally stable for every $x \in X$, then*

$$\max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} F(x) = \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \max_{\mathbb{R}^p_+\setminus\{0\}} F(x).$$

Before describing the conjugate duality for vector optimization let us recall the concepts of conjugate maps and the set-valued subgradient.

Definition 3.3 *[42, Definition 8.2.1]*
Let $h : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a set-valued map.

(i) The set-valued map $h^* : \mathbb{R}^{p\times n} \rightrightarrows \mathbb{R}^p$ defined by

$$h^*(U) = \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in\mathbb{R}^n} \Big[Ux - h(x)\Big],\ U \in \mathbb{R}^{p\times n}$$

is called the conjugate map of h.

(ii) The conjugate map of h^*, h^{**} is called the biconjugate map of h, i.e.

$$h^{**}(x) = \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{U\in\mathbb{R}^{p\times n}} \Big[Ux - h^*(U)\Big],\ x \in \mathbb{R}^n.$$

(iii) U is said to be a subgradient of the set-valued map h at $(\bar{x};\bar{y})$ if $\bar{y} \in h(\bar{x})$ and

$$\bar{y} - U\bar{x} \in \min_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in\mathbb{R}^n} [h(x) - Ux].$$

The set of all subgradients of h at $(x;y)$ is denoted by $\partial h(x;y)$ and is called *the subdifferential of h at $(x;y)$*. If $\partial h(x;y) \neq \varnothing,\ \forall y \in h(x)$, then h is said to be *subdifferentiable at x*.

When $\varphi : \mathbb{R}^n \to \mathbb{R}^p$ is a vector-valued function, then the conjugate map φ^* of φ is defined by

$$\varphi^*(T) = \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{Tx - \varphi(x)|\ x \in \mathbb{R}^n\Big\},\ T \in \mathbb{R}^{p\times n}.$$

Let $f : \mathbb{R}^n \to \mathbb{R}^p \cup \{+\infty\}$ be an extended vector-valued function. Here $+\infty$ is the imaginary point whose every component is $+\infty$. We consider the following unconstrained vector optimization problem

$$(P^{VO}) \qquad \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{f(x)|\ x \in \mathbb{R}^n\Big\}.$$

In other words, (P^{VO}) is the problem of finding $\bar{x} \in \mathbb{R}^n$ such that

$$f(x) \underset{\mathbb{R}^p_+\setminus\{0\}}{\not\leq} f(\bar{x}),\ \forall x \in \mathbb{R}^n.$$

Let $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p \cup \{+\infty\}$ be another vector-valued function such that

$$\Phi(x, 0) = f(x),\ \forall x \in \mathbb{R}^n,$$

which is the so-called perturbation function. *The value function* is a set-valued map $\Psi : \mathbb{R}^m \rightrightarrows \mathbb{R}^p \cup \{+\infty\}$ defined by

$$\Psi(y) = \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\Phi(x, y)|\ x \in \mathbb{R}^n\Big\}.$$

Clearly $\Psi(0) = \min\limits_{\mathbb{R}^p_+\setminus\{0\}} f(\mathbb{R}^n)$ is the minimal frontier of the problem (P^{VO}). The problem (P^{VO}) can be stated as the primal optimization problem

$$(P^{VO}) \qquad \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\Phi(x, 0)|\ x \in \mathbb{R}^n\Big\}.$$

The conjugate map of Φ, denoted by $\Phi^* : \mathbb{R}^{p\times n}\times\mathbb{R}^{p\times m} \rightrightarrows \mathbb{R}^p\cup\{+\infty\}$, is a set-valued map defined in the usual manner:

$$\Phi^*(U, V) = \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{Ux + Vy - \Phi(x, y)|\ x \in \mathbb{R}^n,\ y \in \mathbb{R}^m\Big\}.$$

Then the conjugate dual optimization problem can be defined as being

$$(D^{VO}) \qquad \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{V\in\mathbb{R}^{p\times m}} \Big[-\Phi^*(0, V)\Big].$$

Since $-\Phi^*$ is a set-valued map, the problem (D^{VO}) is not an ordinary vector optimization problem. In other words, it can be reformulated as follows.

Find $V^* \in \mathbb{R}^{p\times m}$ such that

$$-\Phi^*(0, V^*) \cap \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{V\in\mathbb{R}^{p\times m}} \Big[-\Phi^*(0, V)\Big] \neq \varnothing.$$

Theorem 3.1 *[75, Proposition 6.1.12] (Weak duality)*

$$\Phi(x,0) \notin -\Phi^*(0,V) - \mathbb{R}^p_+ \backslash \{0\},\ \forall x \in \mathbb{R}^n,\ \forall\ V \in \mathbb{R}^{p\times m}.$$

Definition 3.4 *The primal problem (P^{VO}) is said to be stable with respect to the perturbation function Φ if the value function Ψ is subdifferentiable at $y=0$.*

Theorem 3.2 *[75, Theorem 6.1.1] (Strong duality)*

(i) The primal problem (P^{VO}) is stable with respect to Φ if and only if for each solution x^* to the primal problem (P^{VO}) there exists a solution V^* to the dual problem (D^{VO}) such that

$$\Phi(x^*,0) \in -\Phi^*(0,V^*). \tag{3.1}$$

(ii) Conversely, if $x^* \in \mathbb{R}^n$ and $V^* \in \mathbb{R}^{p\times m}$ satisfy (3.1), then x^* is a solution to (P^{VO}) and V^* is a solution to (D^{VO}).

3.1.2 Perturbation functions and stability

Let $f:\mathbb{R}^n \to \mathbb{R}^p,\ g:\mathbb{R}^n \to \mathbb{R}^m$ be vector-valued functions and $X \subseteq \mathbb{R}^n$. Consider the vector optimization problem

$$(VO) \qquad \min_{\mathbb{R}^p_+\backslash\{0\}} \Big\{ f(x)|\ x \in G \Big\},$$

where

$$G = \Big\{ x \in X |\ g(x) \leqq_{\mathbb{R}^m_+} 0 \Big\}.$$

In analogy to the scalar case, let us introduce now the following perturbation functions (cf. [16], [90])

$$\begin{aligned}
\Phi_1 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p \cup \{+\infty\},\quad \Phi_1(x,u) &= \begin{cases} f(x), & x \in X,\ g(x) \leqq_{\mathbb{R}^m_+} u, \\ +\infty, & \text{otherwise}; \end{cases} \\
\Phi_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p \cup \{+\infty\},\quad \Phi_2(x,v) &= \begin{cases} f(x+v), & x \in G, \\ +\infty, & \text{otherwise}; \end{cases} \\
\Phi_3 : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p \cup \{+\infty\},& \\
\Phi_3(x,v,u) &= \begin{cases} f(x+v), & x \in X,\ g(x) \leqq_{\mathbb{R}^m_+} u, \\ +\infty, & \text{otherwise}. \end{cases}
\end{aligned}$$

The corresponding value functions are defined by

$$\begin{aligned}
\Psi_1 : \mathbb{R}^m \rightrightarrows \mathbb{R}^p,\ \Psi_1(u) &= \min_{\mathbb{R}^p_+\backslash\{0\}} \Big\{ \Phi_1(x,u)|\ x \in \mathbb{R}^n \Big\} \\
&= \min_{\mathbb{R}^p_+\backslash\{0\}} \Big\{ f(x)|\ x \in X,\ g(x) \leqq_{\mathbb{R}^m_+} u \Big\}; \\
\Psi_2 : \mathbb{R}^n \rightrightarrows \mathbb{R}^p,\ \Psi_2(v) &= \min_{\mathbb{R}^p_+\backslash\{0\}} \Big\{ \Phi_2(x,v)|\ x \in \mathbb{R}^n \Big\} \\
&= \min_{\mathbb{R}^p_+\backslash\{0\}} \Big\{ f(x+v)|\ x \in G \Big\};
\end{aligned}$$

and

$$\begin{aligned}\Psi_3 : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^p,\ \Psi_3(v,u) &= \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\Phi_3(x,v,u)|\ x \in \mathbb{R}^n\Big\} \\ &= \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{f(x+v)|\ x \in X,\ g(x) \underset{\mathbb{R}^m_+}{\leq} u\Big\},\end{aligned}$$

respectively. In view of Definition 3.4, the problem (VO) is said to be stable with respect to the perturbation function $\Phi_i,\ i=1,2,3$, if the value function $\Psi_i,\ i=1,2,3$, is subdifferentiable at 0.

Definition 3.5 *Let $Z \subseteq \mathbb{R}^n$ be a convex set.*

(i) The set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is said to be convex, if for any $x_1, x_2 \in Z,\ x_1 \neq x_2$ and $\xi \in [0,1]$, we have

$$\xi G(x_1) + (1-\xi)G(x_2) \subseteq G(\xi x_1 + (1-\xi)x_2) + \mathbb{R}^p_+.$$

(ii) The set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is said to be strictly convex, if for any $x_1, x_2 \in Z,\ x_1 \neq x_2$ and $\xi \in (0,1)$, we have

$$\xi G(x_1) + (1-\xi)G(x_2) \subseteq G(\xi x_1 + (1-\xi)x_2) + int\ \mathbb{R}^p_+.$$

Lemma 3.3 *Let $X \subseteq \mathbb{R}^n$ be a convex set and $f_i,\ i=\overline{1,p},\ g_j,\ j=\overline{1,m}$, be convex functions. If $\forall u \in \mathbb{R}^m$ (resp., $\forall v \in \mathbb{R}^n$ and $\forall (v,u) \in \mathbb{R}^n \times \mathbb{R}^m$) the set $\Psi_1(u)$ (resp., $\Psi_2(v)$ and $\Psi_3(v,u)$) is externally stable, then the value function Ψ_1 (resp., Ψ_2 and Ψ_3) is convex.*

Proof: Let us verify it only for Ψ_1. In the same way one can prove the result for Ψ_2 and Ψ_3. Let $u_1, u_2 \in \mathbb{R}^m$ and $\lambda \in [0,1]$. Then

$$\lambda\Psi_1(u_1) + (1-\lambda)\Psi_1(u_2) \subseteq \lambda H_1(u_1) + (1-\lambda)H_1(u_2),$$

where $H_1(u)$ is defined by $H_1(u) = \Big\{f(x)|\ x \in X,\ g(x) \underset{\mathbb{R}^m_+}{\leq} u\Big\}$. By using the convexity and the external stability we have

$$\begin{aligned}\lambda H_1(u_1) + (1-\lambda)H_1(u_2) &\subseteq \Big\{f(\lambda x + (1-\lambda)z)|\ \lambda x + (1-\lambda)z \in X, \\ &\qquad g(\lambda x + (1-\lambda)z) \underset{\mathbb{R}^m_+}{\leq} \lambda u_1 + (1-\lambda)u_2\Big\} + \mathbb{R}^p_+ \\ &= H_1(\lambda u_1 + (1-\lambda)u_2) + \mathbb{R}^p_+ \\ &\subseteq \Psi_1(\lambda u_1 + (1-\lambda)u_2) + \mathbb{R}^p_+.\end{aligned}$$

Consequently, one has

$$\lambda\Psi_1(u_1) + (1-\lambda)\Psi_1(u_2) \subseteq \Psi_1(\lambda u_1 + (1-\lambda)u_2) + \mathbb{R}^p_+.$$

□

Let us show some stability criteria with respect to the above perturbation functions. Similar results can be found in [82].

Proposition 3.1 *(cf. [82]) Let $X \subseteq \mathbb{R}^n$ be a convex set and $g_j,\ j=\overline{1,m}$, be convex functions. Assume that the functions $f_i,\ i=\overline{1,p}$, are strictly convex and $\forall u \in \mathbb{R}^m$ the set $\Psi_1(u)$ is externally stable. If there exists $x_0 \in X$ such that $g(x_0) \underset{int\ \mathbb{R}^m_+}{\leq} 0$, then the problem (VO) is stable with respect to Φ_1.*

Proof: We define the set

$$\text{epi } \Psi_1 = \{(u,z) \in \mathbb{R}^m \times \mathbb{R}^p |\ z \in \Psi_1(u) + \mathbb{R}^p_+\}.$$

In analogy to Lemma 3.3 one can show that Ψ_1 is a strictly convex set-valued map. Whence epi Ψ_1 is a convex set. Let the set-valued map $H_1 : \mathbb{R}^m \rightrightarrows \mathbb{R}^p$ be defined by

$$H_1(u) = \{f(x)|\ x \in X,\ g(x) \underset{\mathbb{R}^m_+}{\leq} u\}.$$

Then it holds

$$\Psi_1(u) + \mathbb{R}^p_+ = H_1(u) + \mathbb{R}^p_+,\ u \in \mathbb{R}^m.$$

The inclusion $\Psi_1(u) + \mathbb{R}^p_+ \subseteq H_1(u) + \mathbb{R}^p_+$ is clear. The converse one follows from the external stability of the set $\Psi_1(u),\ u \in \mathbb{R}^m$. Consequently, we have

$$\text{epi } \Psi_1 = \{(u,z) \in \mathbb{R}^m \times \mathbb{R}^p |\ z \in H_1(u) + \mathbb{R}^p_+\}.$$

Hence $(u,z) \in$ epi Ψ_1 if and only if

$$\exists\ x \in X \text{ such that } f(x) \underset{\mathbb{R}^p_+}{\leq} z \text{ and } g(x) \underset{\mathbb{R}^m_+}{\leq} u. \tag{3.2}$$

By assumption, if $\exists\ x_0 \in X$ such that $g(x_0) \underset{\text{int } \mathbb{R}^m_+}{\leq} 0$, then $\exists \varepsilon \in \text{int } \mathbb{R}^m_+$ such that $g(x_0) \underset{\text{int } \mathbb{R}^m_+}{\leq} -\varepsilon$. By using the notation

$$]a,b[_k := \prod_{i=1}^{k} \{x_i \in \mathbb{R}|\ a_i < x_i < b_i,\ a_i,\ b_i \in \mathbb{R}\},\ \text{for } a,b \in \mathbb{R}^k,$$

which is the extension of an open interval in $\mathbb{R}^k$, we define the set

$$M :=]-\varepsilon, \varepsilon[_m \times]f(x_0), f(x_0) + \delta[_p \subseteq \mathbb{R}^m \times \mathbb{R}^p,$$

where $\varepsilon \in \text{int } \mathbb{R}^m_+$ and $\delta \in \text{int } \mathbb{R}^p_+$ are given.
Let $(u,z) \in M$. This means that $g(x_0) \underset{\text{int } \mathbb{R}^m_+}{\leq} -\varepsilon \underset{\text{int } \mathbb{R}^m_+}{\leq} u$ and $f(x_0) \underset{\text{int } \mathbb{R}^p_+}{\leq} z$. According to (3.2), $(u,z) \in$ epi Ψ_1. Therefore int(epi $\Psi_1) \neq \varnothing$.

Let $\widehat{z} \in \Psi_1(0)$ be fixed. Then $\exists \widehat{x} \in G$ such that $\widehat{z} = f(\widehat{x}) \in \Psi_1(0) = \min\limits_{\mathbb{R}^p_+ \setminus \{0\}} \{f(x)|\ x \in G\}$. Let us show that $(0,\widehat{z})$ is a boundary point of epi Ψ_1. Indeed, it is clear that $(0,\widehat{z}) \in$ epi Ψ_1. Assume that $(0,\widehat{z}) \in$ int(epi $\Psi_1)$. Then there exists a neighborhood $U \times V$ of $(0,\widehat{z})$ such that $U \times V \subseteq$ epi Ψ_1. In other words, $\exists \bar{z} = \widehat{z} - k,\ k \in \text{int } \mathbb{R}^p_+$ such that $\bar{z} \in V$ and it holds $(0,\bar{z}) \in$ epi Ψ_1. This means that $\bar{z} \in H_1(0) + \mathbb{R}^p_+$. Therefore $\exists \bar{x} \in G$ such that $f(\bar{x}) \underset{\mathbb{R}^p_+}{\leq} \bar{z}$. On the other hand $\bar{z} \underset{\text{int } \mathbb{R}^p_+}{\leq} \widehat{z}$ which leads to a contradiction. Whence $(0,\widehat{z}) \in$ epi $\Psi_1 \setminus$ int(epi Ψ_1).

By a well-known separation theorem, there exists $(\xi,\mu) \in \mathbb{R}^m \times \mathbb{R}^p,\ (\xi,\mu) \neq (0,0)$ such that

$$\xi^T u + \mu^T z \geq \mu^T \widehat{z},\ \forall (u,z) \in \text{epi } \Psi_1.$$

If $z_1 \in \Psi_1(u_1) + \mathbb{R}^p_+$, then for any $z \in z_1 + \text{int } \mathbb{R}^p_+$ it holds $z \in \Psi_1(u_1) + \text{int } \mathbb{R}^p_+$. Whence $\mu \in \mathbb{R}^p_+$. Let $\mu = 0$. Then $\xi^T u \geq 0,\ \forall (u,z) \in$ epi Ψ_1. Clearly, $(m, f(x)) \in$ epi $\Psi_1,\ \forall x \in G,\ \forall m \in \mathbb{R}^m_+$. As $\xi^T m \geq 0$, we have $\xi \in \mathbb{R}^m_+$. If $\xi \neq 0$, then $\xi^T g(x) \geq 0,\ \forall x \in X$. But by assumption, for $x_0 \in X$ it holds $\xi^T g(x_0) < 0$. In other words $\xi = 0$, which is a contradiction. This means that $0 \underset{\mathbb{R}^p_+\setminus\{0\}}{\leq} \mu$. Moreover, one can assume that $\mu \in S^p := \Big\{\mu = (\mu_1, ..., \mu_p)^T \in \mathbb{R}^p |\ \mu_i \geq 0,\ i = \overline{1,p},\ \sum\limits_{i=1}^{p} \mu_i = 1\Big\}$.

Since Ψ_1 is strictly convex, we can show that

$$\xi^T u + \mu^T z > \mu^T \widehat{z},\ \forall (u,z) \in \text{epi}\ \Psi_1,\ u \neq 0. \tag{3.3}$$

In fact, if this is not fulfilled, $\exists (u', z') \in \text{epi}\ \Psi_1,\ u' \neq 0$, such that $\xi^T u' + \mu^T z' = \mu^T \widehat{z}$. As

$$\begin{aligned}\frac{1}{2}z' + \frac{1}{2}\widehat{z} \in \frac{1}{2}\Psi_1(u') + \frac{1}{2}\Psi_1(0) + \mathbb{R}^p_+ \quad &\subseteq \quad \Psi_1\Big(\frac{1}{2}u'\Big) + \text{int}\ \mathbb{R}^p_+ + \mathbb{R}^p_+ \\ &\subseteq \quad \Psi_1\Big(\frac{1}{2}u'\Big) + \text{int}\ \mathbb{R}^p_+,\end{aligned}$$

$\exists k' \in \text{int}\ \mathbb{R}^p_+$ such that

$$\frac{1}{2}z' + \frac{1}{2}\widehat{z} - k' \in \Psi_1\Big(\frac{1}{2}u'\Big),$$

and this implies that $\Big(\frac{1}{2}u', \frac{1}{2}z' + \frac{1}{2}\widehat{z} - k'\Big) \in \text{epi}\ \Psi_1$ and, so,

$$\xi^T\Big(\frac{1}{2}u'\Big) + \mu^T\Big(\frac{1}{2}z' + \frac{1}{2}\widehat{z}\Big) - \mu^T k' \geq \mu^T \widehat{z},$$

or, equivalently,

$$\frac{1}{2}\mu^T\widehat{z} + \frac{1}{2}\mu^T\widehat{z} - \mu^T k \geq \mu^T\widehat{z}\ \Leftrightarrow\ \mu^T k' \leq 0.$$

But, as $\mu^T k' > 0$, we get a contradiction. Therefore (3.3) holds.

Let us notice that $\partial\Psi_1(0; z) \neq \varnothing$ means $\forall z \in \Psi_1(0),\ \exists \Lambda \in \mathbb{R}^{p\times m}$ such that

$$z \in \min_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{u\in\mathbb{R}^m} [\Psi_1(u) - \Lambda u].$$

By assumption $\widehat{z} \in \Psi_1(0)$ and for $\widehat{\Lambda} = [-\xi, ..., -\xi]^T \in \mathbb{R}^{p\times m}$ we verify that this relation holds. Let us assume that

$$\widehat{z} \notin \min_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{u\in\mathbb{R}^m} [\Psi_1(u) + (\xi^T u)_p],$$

where $(\xi^T u)_p = (\xi^T u, ..., \xi^T u)^T \in \mathbb{R}^p$. Then $\exists \bar{u} \in \mathbb{R}^m,\ \bar{u} \neq 0$ and $\exists \bar{z} \in \Psi_1(\bar{u})$ such that

$$\bar{z} + (\xi^T \bar{u})_p \underset{\mathbb{R}^p_+\setminus\{0\}}{\leq} \widehat{z}.$$

Since $\mu \in S^p$, it holds

$$\mu^T\bar{z} + \xi^T\bar{u} \leq \mu^T\widehat{z}.$$

On the other hand, in view of (3.3) we see that $\mu^T\bar{z} + \xi^T\bar{u} > \mu^T\widehat{z}$. This leads to a contradiction. In other words $\partial\Psi_1(0; \widehat{z}) \neq \varnothing$. $\square$

Proposition 3.2 *Let $X \subseteq \mathbb{R}^n$ be a convex set and $g_j,\ j = \overline{1,m}$, be convex functions. Assume that the functions $f_i,\ i = \overline{1,p}$, are strictly convex. If $\forall v \in \mathbb{R}^n$ the set $\Psi_2(v)$ is externally stable, then the problem (VO) is stable with respect to Φ_2.*

Proof: Let $H_2 : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a set-valued map defined by $H_2(v) = \{f(x+v)|\ x \in G\}$. Introducing the set

$$\text{epi }\ \Psi_2 = \{(v,z) \in \mathbb{R}^n \times \mathbb{R}^p|\ z \in \Psi_2(v) + \mathbb{R}^p_+\},$$

and according to the external stability of $\Psi_2(v),\ v \in \mathbb{R}^n$ it holds

$$\Psi_2(v) + \mathbb{R}^p_+ = H_2(v) + \mathbb{R}^p_+,\ v \in \mathbb{R}^n.$$

Whence

$$\text{epi }\ \Psi_2 = \{(v,z) \in \mathbb{R}^n \times \mathbb{R}^p|\ z \in H_2(v) + \mathbb{R}^p_+\}.$$

Moreover, epi Ψ_2 is a convex set. Let us notice that $(v,z) \in$ epi Ψ_2 means that

$$\exists x \in G \ \text{ such that } \ f(x+v) \underset{\mathbb{R}^p_+}{\leq} z. \tag{3.4}$$

As f is strictly convex, it is also continuous. Let $x_0 \in G$ be fixed. Since f is continuous at $x_0 \in G$, for any $\varepsilon \underset{\text{int }\mathbb{R}^p_+}{\geq} 0$, there exists $\delta > 0$ such that

$$f(x) \underset{\text{int }\mathbb{R}^p_+}{\leq} f(x_0) + \varepsilon,\ \text{for any } x \in U_\delta(x_0) \subseteq G,$$

where $U_\delta(x_0) := \{x \in \mathbb{R}^n|\ \|x - x_0\| < \delta\}$ denotes an open neighborhood of x_0. We define the set

$$N := \underbrace{(0,\delta) \times \cdots \times (0,\delta)}_{n} \times]f(x_0) + \varepsilon, f(x_0) + 2\varepsilon[_p \subseteq \mathbb{R}^n \times \mathbb{R}^p.$$

Let $(v,z) \in N$. Then it holds

$$f(x_0 + v) \underset{\text{int }\mathbb{R}^p_+}{\leq} f(x_0) + \varepsilon \underset{\text{int }\mathbb{R}^p_+}{\leq} z.$$

In view of (3.4), we have $(v,z) \in$ epi Ψ_2. This means that int(epi $\Psi_2) \neq \varnothing$.

Let $\widehat{z} \in \Psi_2(0) = \min\limits_{\mathbb{R}^p_+\setminus\{0\}} \{f(x)|\ x \in G\}$ be fixed. Then $(0,\widehat{z})$ is a boundary point of epi Ψ_2, i.e. $(0,\widehat{z}) \in$ epi $\Psi_2 \setminus$ int(epi Ψ_2).

By a well-known separation theorem, there exists $(\xi,\mu) \in \mathbb{R}^n \times \mathbb{R}^p,\ (\xi,\mu) \neq (0,0)$ such that

$$\xi^T v + \mu^T z \geq \mu^T \widehat{z},\ \forall (v,z) \in \text{epi }\ \Psi_2.$$

If $z_1 \in \Psi_2(v_1) + \mathbb{R}^p_+$, then for any $z \in z_1 + \text{int }\mathbb{R}^p_+$, it holds $z \in \Psi_2(v_1) + \text{int }\mathbb{R}^p_+$. Whence $\mu \in \mathbb{R}^p_+$. Let $\mu = 0$. Then $\xi^T v \geq 0,\ \forall (v,z) \in$ epi Ψ_2. Since f is strictly convex, one has

$$f\Big(x_0 + \frac{1}{2}(x - x_0)\Big) = f\Big(\frac{1}{2}x + \frac{1}{2}x_0\Big) \underset{\text{int }\mathbb{R}^p_+}{\leq} \frac{1}{2}f(x) + \frac{1}{2}f(x_0),\ \forall x \in G,\ x \neq x_0.$$

In other words $\Big(\frac{1}{2}(x - x_0), \frac{1}{2}(f(x) + f(x_0))\Big) \in$ epi $\Psi_2,\ \forall x \in G,\ x \neq x_0$. Whence $\xi^T(\frac{1}{2}(x - x_0)) \geq 0$, or, equivalently, $\xi^T(x - x_0) \geq 0,\ \forall x \in G,\ x \neq x_0$.

Choosing $\widetilde{x}_i \in U_\delta(x_0),\ i = \overline{1,n}$, such that $\widetilde{x}_i^j = \begin{cases} x_0^j, & j \neq i, \\ x_0^i + \frac{\delta}{2}, & j = i \end{cases}$ and $\bar{x}_i \in U_\delta(x_0),\ i = \overline{1,n}$, such that $\bar{x}_i^j = \begin{cases} x_0^j, & j \neq i, \\ x_0^i - \frac{\delta}{2}, & j = i \end{cases}$, it follows that $\xi = 0$, which is a contradiction. Therefore $0 \underset{\mathbb{R}^p_+ \setminus \{0\}}{\leq} \mu$ and it can be assumed that $\mu \in S^p$.

Since Ψ_2 is strictly convex, one can show that

$$\xi^T v + \mu^T z > \mu^T \widehat{z}, \ \forall (v,z) \in \text{epi } \Psi_2, \ v \neq 0. \tag{3.5}$$

By assumption $\widehat{z} \in \Psi_2(0)$ and for $\widehat{T} = [-\xi, ..., -\xi]^T \in \mathbb{R}^{p \times n}$ let us now show that $\partial \Psi_2(0; \widehat{z}) \neq \varnothing$ holds. Assume that

$$\widehat{z} \notin \min_{\mathbb{R}^p_+ \setminus \{0\}} \bigcup_{v \in \mathbb{R}^n} [\Psi_2(v) + (\xi^T v)_p],$$

where $(\xi^T v)_p = (\xi^T v, ..., \xi^T v)^T \in \mathbb{R}^p$. Then $\exists \bar{v} \in \mathbb{R}^n, \ \bar{v} \neq 0$ and $\exists \bar{z} \in \Psi_2(\bar{v})$ such that

$$\bar{z} + (\xi^T \bar{v})_p \underset{\mathbb{R}^p_+ \setminus \{0\}}{\leq} \widehat{z}.$$

As $\mu \in S^p$, it holds $\mu^T \bar{z} + \xi^T \bar{v} \leq \mu^T \widehat{z}$. Taking into account (3.5), one has $\mu^T \bar{z} + \xi^T \bar{v} > \mu^T \widehat{z}$, which leads to a contradiction. □

Proposition 3.3 *Let $X \subseteq \mathbb{R}^n$ be a convex set and $g_j, \ j = \overline{1,m}$, be convex functions. Assume that the functions $f_i, \ i = \overline{1,p}$, are strictly convex and $\forall (v,u) \in \mathbb{R}^n \times \mathbb{R}^m$ the set $\Psi_3(v,u)$ is externally stable. If there exists $x_0 \in X$ such that $g(x_0) \underset{int \ \mathbb{R}^m_+}{\leq} 0$, then the problem (VO) is stable with respect to Φ_3.*

Proof: Let $H_3 : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^p$ be a set-valued map defined by $H_3(v,u) = \{f(x+v) | \ x \in X, \ g(x) \underset{\mathbb{R}^m_+}{\leq} u\}$. As usual, the set epi Ψ_3 defined by

$$\text{epi } \Psi_3 = \{(v,u,z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p | \ z \in \Psi_3(v,u) + \mathbb{R}^p_+\}$$

is convex. By the external stability of Ψ_3, it holds

$$\Psi_3(u,v) + \mathbb{R}^p_+ = H_3(u,v) + \mathbb{R}^p_+, \ v \in \mathbb{R}^n, \ u \in \mathbb{R}^m.$$

Therefore $(v,u,z) \in$ epi Ψ_3 means that

$$\exists x \in X \text{ such that } f(x+v) \underset{\mathbb{R}^p_+}{\leq} z \text{ and } g(x) \underset{\mathbb{R}^m_+}{\leq} u.$$

Since f is continuous at $x_0 \in X$, for any $\varepsilon \underset{\text{int } \mathbb{R}^p_+}{\geq} 0$, there exists $\sigma > 0$ such that

$$f(x) \underset{\text{int } \mathbb{R}^p_+}{\leq} f(x_0) + \varepsilon, \text{ for any } x \in U_\sigma(x_0) \subseteq X.$$

On the other hand, as $g(x_0) \underset{\text{int } \mathbb{R}^m_+}{\leq} 0$ there exists $\delta \underset{\text{int } \mathbb{R}^m_+}{\geq} 0$ such that $g(x_0) \underset{\text{int } \mathbb{R}^m_+}{\leq} -\delta$.

Let us define the set

$$S := \underbrace{(0,\sigma) \times \cdots \times (0,\sigma)}_{n} \times]-\delta, \delta[_m \times]f(x_0)+\varepsilon, f(x_0)+2\varepsilon[_p \subseteq \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p.$$

Let $(v,u,z) \in S$. This means that

$$f(x_0+v) \underset{\text{int } \mathbb{R}^p_+}{\leq} f(x_0)+\varepsilon \underset{\text{int } \mathbb{R}^p_+}{\leq} z \text{ and } g(x_0) \underset{\text{int } \mathbb{R}^m_+}{\leq} -\delta \underset{\text{int } \mathbb{R}^m_+}{\leq} u.$$

In other words $(v, u, z) \in \text{epi } H_3$. Consequently, we have $\text{int}(\text{epi } \Psi_3) \neq \varnothing$.

Let $\widehat{z} \in \Psi_3(0,0) = \min\limits_{\mathbb{R}^p_+ \setminus \{0\}} \{f(x)|\ x \in G\}$ be fixed. Since a point $(0, 0, \widehat{z})$ is a boundary point of epi Ψ_3, by a well-known separation theorem, there exists $(\xi_1, \xi_2, \mu) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$, $(\xi_1, \xi_2, \mu) \neq (0, 0, 0)$ such that

$$\xi_1^T v + \xi_2^T u + \mu^T z \geq \mu^T \widehat{z},\ \forall (v, u, z) \in \text{epi } \Psi_3.$$

If $z_1 \in \Psi_3(v_1, u_1) + \mathbb{R}^p_+$, then for any $z \in z_1 + \text{int } \mathbb{R}^p_+$, it holds $z \in \Psi_3(v_1, u_1) + \text{int } \mathbb{R}^p_+$. Whence $\mu \in \mathbb{R}^p_+$. Let $\mu = 0$. Then $\xi_1^T v + \xi_2^T u \geq 0,\ \forall (v, u, z) \in \text{epi } \Psi_3$. Since f is strictly convex, we have $\Big(\frac{1}{2}(x - x_0), m, \frac{1}{2}(f(x) + f(x_0))\Big) \in \text{epi } \Psi_3,\ \forall x \in G,\ x \neq x_0,\ \forall m \in \mathbb{R}^m_+$. Therefore

$$\xi_1^T(\frac{1}{2}(x - x_0)) + \xi_2^T m \geq 0,\ \forall x \in G,\ x \neq x_0,\ \forall m \in \mathbb{R}^m_+.$$

If $m = 0$, then it returns to the case of Ψ_2 (see the proof of Proposition 3.2). Consequently, $\xi_1 = 0$. Whence, we have $\xi_2^T m \geq 0,\ \forall m \in \mathbb{R}^m_+$. It returns to the case of Ψ_1 (see the proof of Proposition 3.1). As a consequence, $\xi_1 = 0$, which leads to a contradiction. Therefore it can be assumed that $\mu \in S^p$.
Moreover, since Ψ_3 is strictly convex, one can show that

$$\xi_1^T v + \xi_2^T u + \mu^T z > \mu^T \widehat{z},\ \forall (v, u, z) \in \text{epi } \Psi_3,\ (u, v) \neq (0, 0). \tag{3.6}$$

By assumption $\widehat{z} \in \Psi_3(0)$ and for $\widehat{T} = [-\xi_1, ..., -\xi_1]^T \in \mathbb{R}^{p \times n}$ and $\widehat{\Lambda} = [-\xi_2, ..., -\xi_2]^T \in \mathbb{R}^{p \times m}$ we verify that $\partial \Psi_3(0, 0; \widehat{z}) \neq \varnothing$ holds. Indeed, let

$$\widehat{z} \notin \min_{\mathbb{R}^p_+ \setminus \{0\}} \bigcup_{(v,u)^T \in \mathbb{R}^n \times \mathbb{R}^m} [\Psi_3(v, u) + (\xi_1^T v)_p + (\xi_2^T u)_p],$$

where $(\xi_1^T v)_p = (\xi_1^T v, ..., \xi_1^T v)^T \in \mathbb{R}^p$ and $(\xi_2^T u)_p = (\xi_2^T u, ..., \xi_2^T u)^T \in \mathbb{R}^p$. Then $\exists (\bar{v}, \bar{u})^T \in \mathbb{R}^n \times \mathbb{R}^m,\ (\bar{u}, \bar{v}) \neq (0, 0)$ and $\exists \bar{z} \in \Psi_3(\bar{v}, \bar{u})$ such that

$$\bar{z} + (\xi_1^T \bar{v})_p + (\xi_2^T \bar{u})_p \underset{\mathbb{R}^p_+ \setminus \{0\}}{\leq} \widehat{z}.$$

Since $\mu \in S^p$, it holds

$$\mu^T \bar{z} + \xi_1^T \bar{v} + \xi_2^T \bar{u} \leq \mu^T \widehat{z}.$$

This contradicts the fact that $\mu^T \bar{z} + \xi_1^T \bar{v} + \xi_2^T \bar{u} > \mu^T \widehat{z}$ (see (3.6)). Consequently, $\partial \Psi_3(0, 0; \widehat{z}) \neq \varnothing$. □

Later for the applications we have to consider the vector optimization problem with linear objective function (cf. Section 3.2). Since the objective function is linear and not strictly convex, we can not apply above stability criteria to this case. But the following result can be given. Let $A \in \mathbb{R}^{p \times n}$. Consider the vector optimization problem

$$(P_A) \qquad \min_{\mathbb{R}^p_+ \setminus \{0\}} \Big\{Ax|\ x \in G\Big\}.$$

Before giving a stability criterion for (P_A) with respect to Φ_2, let us mention the following trivial properties.

Remark 3.1 Let $h : \mathbb{R}^n \to \mathbb{R}^p$ be a vector-valued function and $Y \subseteq \mathbb{R}^n$. The following assertions are true:

(i) $\{h(x)|\ x \in Y\} = \bigcup\limits_{x \in Y} \{h(x)\}$.

(ii) For any $t \in \mathbb{R}^p$ it holds $\{h(x)+t \mid x \in Y\} = \{h(x) \mid x \in Y\} + t$.

(iii) For any set $A \subseteq \mathbb{R}^p$ it holds $\bigcup\limits_{x\in Y} \Big\{A + h(x)\Big\} = A + \bigcup\limits_{x\in Y} \{h(x)\}$.

For the problem (P_A) we can state the following assertion.

Proposition 3.4 *Let the set* $\min\limits_{\mathbb{R}^p_+\setminus\{0\}} \{Ax \mid x \in G\}$ *be externally stable. Then the problem* (P_A) *is stable with respect to* Φ_2.

Proof: Let $f(x) = Ax,\ A \in \mathbb{R}^{p\times n}$. Then, in view of Remark 3.1, we have

$$\begin{aligned}
-\Psi_2^*(T) &= \min_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{v\in\mathbb{R}^n} \Big[\min_{\mathbb{R}^p_+\setminus\{0\}} \{Ax + Av \mid x \in G\} - Tv\Big] \\
&= \min_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{v\in\mathbb{R}^n} \Big[Av - Tv + \min_{\mathbb{R}^p_+\setminus\{0\}} \{Ax \mid x \in G\}\Big] \\
&= \min_{\mathbb{R}^p_+\setminus\{0\}} \Big[\{(A-T)v \mid v \in \mathbb{R}^n\} + \min_{\mathbb{R}^p_+\setminus\{0\}} \{Ax \mid x \in G\}\Big].
\end{aligned}$$

Since the set $\min\limits_{\mathbb{R}^p_+\setminus\{0\}} \{Ax \mid x \in G\}$ is externally stable, one has (cf. Corollary 3.1) $-\Psi_2^*(A) = \min\limits_{\mathbb{R}^p_+\setminus\{0\}} \min\limits_{\mathbb{R}^p_+\setminus\{0\}} \{Ax \mid x \in G\} = \min\limits_{\mathbb{R}^p_+\setminus\{0\}} \{Ax \mid x \in G\}$. In other words, $\forall z \in \min\limits_{\mathbb{R}^p_+\setminus\{0\}} \{Ax \mid x \in G\}$ it holds $z \in -\Psi_2^*(A)$. This means that $\partial\Psi_2(0;z) \neq \varnothing$. $\square$

3.1.3 Dual problems arising from the different perturbations

In this subsection the perturbation functions introduced in Subsection 3.1.2 are used to developing the duality in vector optimization. As a consequence, we obtain different dual problems having set-valued objective maps. Like in the scalar case, let us call them the Lagrange, the Fenchel and the Fenchel-Lagrange dual problem to (VO), respectively.

Lagrange duality. Let us begin with the first perturbation function Φ_1. The following preliminary result deals with the objective map with respect to Φ_1.

Proposition 3.5 *Let* $\Lambda \in \mathbb{R}^{p\times m}$. *Then*

(i) $\Phi_1^*(0,\Lambda) = \max\limits_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\{\Lambda u \mid u \in \mathbb{R}^m_+\} + \{\Lambda g(x) - f(x) \mid x \in X\}\Big\}$.

(ii) If the set $\max\limits_{\mathbb{R}^p_+\setminus\{0\}} \{\Lambda u \mid u \in \mathbb{R}^m_+\}$ *is externally stable, then it holds*

$$\Phi_1^*(0,\Lambda) = \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\max_{\mathbb{R}^p_+\setminus\{0\}} \{\Lambda u \mid u \in \mathbb{R}^m_+\} + \{\Lambda g(x) - f(x) \mid x \in X\}\Big\}.$$

Proof:

(i) Let $\Lambda \in \mathbb{R}^{p\times m}$. Taking into account Remark 3.1

$$\begin{aligned}
\Phi_1^*(0,\Lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\Lambda u - \Phi_1(x,u) \mid x \in \mathbb{R}^n,\ u \in \mathbb{R}^m\Big\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\Lambda u - f(x) \mid x \in X,\ g(x) \leqq_{\mathbb{R}^m_+} u\Big\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \Big\{\Lambda u - f(x) \mid g(x) \leqq_{\mathbb{R}^m_+} u\Big\}.
\end{aligned}$$

Setting $\bar{u} := u - g(x)$, we have

$$\begin{aligned}\Phi_1^*(0,\Lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \Big\{\Lambda g(x) - f(x) + \Lambda\bar{u}|\ \bar{u}\in\mathbb{R}^m_+\Big\}\\ &= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \Big\{\Lambda g(x) - f(x) + \{\Lambda\bar{u}|\ \bar{u}\in\mathbb{R}^m_+\}\Big\}\\ &= \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\{\Lambda u|\ u\in\mathbb{R}^m_+\} + \{\Lambda g(x) - f(x)|\ x\in X\}\Big\}.\end{aligned}$$

(ii) Follows from Lemma 3.2. □

According to Proposition 3.5, we can propose the following dual problem to (VO)

$$\begin{aligned}(D_L^{VO})\qquad & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{\Lambda\in\mathbb{R}^{p\times m}} \Big[-\Phi_1^*(0,\Lambda)\Big]\\ = & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{\Lambda\in\mathbb{R}^{p\times m}} \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\{-\Lambda u|\ u\in\mathbb{R}^m_+\} + \{f(x) - \Lambda g(x)|\ x\in X\}\Big\}.\end{aligned}$$

This dual problem may be considered as a kind of Lagrange-type dual problem. Such interpretation appears evident and natural in the context of the following derivation of the classical Lagrange dual problem to (VO) (cf. [75]).
As applications of Theorem 3.1 and Theorem 3.2 we get weak and strong duality for VO and (D_L^{VO}).

Proposition 3.6 *(weak duality)*

$$f(x) + \xi \underset{\mathbb{R}^p_+\setminus\{0\}}{\not\leq} 0,\ \forall x\in G,\ \forall \xi\in\Phi_1^*(0,\Lambda),$$

where $\Lambda\in\mathbb{R}^{p\times m}$.

Proposition 3.7 *(Strong duality)*

(i) (VO) *is stable with respect to* Φ_1 *if and only if for each solution* x^* *to* (VO) *there exists a solution* Λ^* *to* (D_L^{VO}) *such that*

$$f(x^*) \in -\Phi_1^*(0,\Lambda^*). \tag{3.7}$$

(ii) Conversely, if $x^*\in G$ *and* $\Lambda^*\in\mathbb{R}^{p\times m}$ *satisfy* (3.7)*, then* x^* *is a solution to* (VO) *and* Λ^* *is a solution to* (D_L^{VO}).

Under the external stability criterion of the set $\max\limits_{\mathbb{R}^p_+\setminus\{0\}} \{\Lambda u|\ u\in\mathbb{R}^m_+\}$, for the dual problem with the objective map defined by Proposition 3.5(ii) we can obtain similar results.

Before considering the next perturbation function, let us, as announced, explain how the problem (D_L^{VO}) turns out to be the classical Lagrange dual problem to (VO) (cf. [75]) under a certain restriction on the feasible set of the dual. To do this, we assume that

$$\Lambda\in L := \Big\{\Lambda\in\mathbb{R}^{p\times m}|\ \Lambda u \underset{\mathbb{R}^p_+}{\geqq} 0,\ \forall u\in\mathbb{R}^m_+\Big\} = \Big\{\Lambda\in\mathbb{R}^{p\times m}|\ \Lambda\mathbb{R}^m_+\subseteq\mathbb{R}^p_+\Big\}.$$

Then we conclude immediately that

$$\min_{\mathbb{R}^p_+\setminus\{0\}} \{\Lambda u|\ u \in \mathbb{R}^m_+\} = \{0\},\ \forall \Lambda \in L. \tag{3.8}$$

Because of $\Lambda \in L$, by using (3.8), from Lemma 3.1(i) follows

$$\begin{aligned}
\Phi_1^*(0,-\Lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\{-\Lambda u|\ u \in \mathbb{R}^m_+\} + \{-\Lambda g(x) - f(x)|\ x \in X\}\Big\} \\
&\subseteq \max_{\mathbb{R}^p_+\setminus\{0\}} \{-\Lambda u|\ u \in \mathbb{R}^m_+\} + \max_{\mathbb{R}^p_+\setminus\{0\}} \{-\Lambda g(x) - f(x)|\ x \in X\} \\
&= -\min_{\mathbb{R}^p_+\setminus\{0\}} \{\Lambda u|\ u \in \mathbb{R}^m_+\} + \max_{\mathbb{R}^p_+\setminus\{0\}} \{-\Lambda g(x) - f(x)|\ x \in X\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \{-\Lambda g(x) - f(x)|\ x \in X\}.
\end{aligned}$$

Denoting by $\widetilde{\Phi}(\Lambda) := \max\limits_{\mathbb{R}^p_+\setminus\{0\}} \{-\Lambda g(x) - f(x)|\ x \in X\}$, in this case we get the classical Lagrange dual problem to (VO), as follows

$$\begin{aligned}
(\widetilde{D}^{VO}_L) \qquad & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{\Lambda \in L} \Big[-\widetilde{\Phi}(\Lambda)\Big] \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{\Lambda \in L} \min_{\mathbb{R}^p_+\setminus\{0\}} \{\Lambda g(x) + f(x)|\ x \in X\}.
\end{aligned}$$

Proposition 3.8 *[75, Theorem 5.2.4] (Weak duality)*

$$f(x) + \xi \underset{\mathbb{R}^p_+\setminus\{0\}}{\not\leq} 0,\ \forall x \in G,\ \forall \xi \in \widetilde{\Phi}(\Lambda),$$

where $\Lambda \in L$.

Proposition 3.9 *[42, Theorem 8.3.3] (see also [75, Theorem 5.2.5(i)])*
Let $x^* \in G,\ \Lambda^* \in L$ such that $f(x^*) \in -\widetilde{\Phi}(\Lambda^*)$. Then $f(x^*)$ is simultaneously a minimal point to the primal problem (VO) and a maximal point to the dual problem $(\widetilde{D}^{VO}_L)$.

Fenchel duality. Before stating the next dual problem to (VO), we consider the following assertion relative to its objective map.

Proposition 3.10 *Let $T \in \mathbb{R}^{p\times n}$. Then*

(i) $\Phi_2^*(0,T) = \max\limits_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\{Tv - f(v)|\ v \in \mathbb{R}^n\} + \{-Tx|\ x \in G\}\Big\}.$

(ii) If the set $f^*(T) = \max\limits_{\mathbb{R}^p_+\setminus\{0\}} \{Tv - f(v)|\ v \in \mathbb{R}^n\}$ is externally stable, then it holds

$$\Phi_2^*(0,T) = \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{f^*(T) + \{-Tx|\ x \in G\}\Big\}.$$

Proof:

(i) Let $T \in \mathbb{R}^{p\times n}$. In view of Remark 3.1

$$\begin{aligned}
\Phi_2^*(0,T) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \{Tv - \Phi_2(x,v)|\ x \in \mathbb{R}^n,\ v \in \mathbb{R}^n\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \{Tv - f(x+v)|\ x \in G,\ v \in \mathbb{R}^n\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x \in G} \{Tv - f(x+v)|\ v \in \mathbb{R}^n\}.
\end{aligned}$$

Denoting $\bar{v} := x + v$, one gets

$$\begin{aligned}\Phi_2^*(0,T) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in G} \{T\bar{v} - f(\bar{v}) - Tx \mid \bar{v} \in \mathbb{R}^n\} \\ &= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in G} \Big\{ -Tx + \{T\bar{v} - f(\bar{v}) \mid \bar{v} \in \mathbb{R}^n\}\Big\} \\ &= \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\{Tv - f(v) \mid v \in \mathbb{R}^n\} + \{-Tx \mid x \in G\}\Big\}.\end{aligned}$$

(ii) By using Lemma 3.2, we obtain (ii). □

As a consequence we state the following dual problem to (VO) which will be called the Fenchel dual problem

$$\begin{aligned}(D_F^{VO}) \qquad & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{T\in\mathbb{R}^{p\times n}} \Big[-\Phi_2^*(0,T)\Big] \\ = \; & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{T\in\mathbb{R}^{p\times n}} \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\{f(v) - Tv \mid v \in \mathbb{R}^n\} + \{Tx \mid x \in G\}\Big\}.\end{aligned}$$

Again as consequences of the general theory we have weak and strong duality assertions.

Proposition 3.11 *(weak duality)*

$$f(x) + \xi \underset{\mathbb{R}^p_+\setminus\{0\}}{\not\leq} 0, \; \forall x \in G, \; \forall \xi \in \Phi_2^*(0,T)$$

where $T \in \mathbb{R}^{p\times n}$.

Proposition 3.12 *(Strong duality)*

(i) (VO) *is stable with respect to* Φ_2 *if and only if for each solution* x^* *to* (VO), *there exists a solution* T^* *to* (D_F^{VO}) *such that*

$$f(x^*) \in -\Phi_2^*(0,T^*). \tag{3.9}$$

(ii) Conversely, if $x^* \in G$ *and* $T^* \in \mathbb{R}^{p\times n}$ *satisfy (3.9), then* x^* *is a solution to* (VO) *and* T^* *is a solution to* (D_F^{VO}).

As mentioned before, under the external stability of the set $f^*(T) = \max\limits_{\mathbb{R}^p_+\setminus\{0\}} \{Tv - f(v) \mid v \in \mathbb{R}^n\}$, for the dual problem with the objective map defined by Proposition 3.10(ii) one can also show similar dual assertions.

Fenchel-Lagrange duality. In order to formulate the dual problem to (VO) dealing with the perturbation function Φ_3, one has to find the corresponding dual objective map.

Proposition 3.13 *Let* $\Lambda \in \mathbb{R}^{p\times m}$ *and* $T \in \mathbb{R}^{p\times n}$. *Then*

(i) $\Phi_3^*(0,T,\Lambda) = \max\limits_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ \bigcup\limits_{u\in\mathbb{R}^m_+} \{\Lambda u\} + \bigcup\limits_{v\in\mathbb{R}^n} \{Tv - f(v)\} + \bigcup\limits_{x\in X} \{\Lambda g(x) - Tx\}\Big\}.$

(ii) If the sets $\max\limits_{\mathbb{R}^p_+\setminus\{0\}} \{\Lambda u \mid u \in \mathbb{R}^m_+\}$ *and* $f^*(T)$ *are externally stable, then it holds*

$$\Phi_3^*(0,T,\Lambda) = \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{u\in\mathbb{R}^m_+} \{\Lambda u\} + f^*(T) + \bigcup_{x\in X} \{\Lambda g(x) - Tx\}\Big\}.$$

Proof:

(i) Let $T \in \mathbb{R}^{p\times n}$ and $\Lambda \in \mathbb{R}^{p\times m}$. By applying Remark 3.1

$$\begin{aligned}\Phi_3^*(0,T,\Lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{Tv + \Lambda u - \Phi_3(x,v,u)|\ x\in\mathbb{R}^n,\ v\in\mathbb{R}^n,\ u\in\mathbb{R}^m\Big\}\\ &= \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{Tv + \Lambda u - f(x+v)|\ x\in X,\ v\in\mathbb{R}^n,\ g(x) \underset{\mathbb{R}^m_+}{\leq} u\Big\}\\ &= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X}\bigcup_{v\in\mathbb{R}^n} \Big\{Tv + \Lambda u - f(x+v)|\ g(x) \underset{\mathbb{R}^m_+}{\leq} u\Big\}.\end{aligned}$$

Putting $\bar{u} := u - g(x)$, one has

$$\begin{aligned}\Phi_3^*(0,T,\Lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X}\bigcup_{v\in\mathbb{R}^n} \{Tv + \Lambda g(x) + \Lambda\bar{u} - f(x+v)|\ \bar{u}\in\mathbb{R}^m_+\}\\ &= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X}\bigcup_{v\in\mathbb{R}^n} \Big\{Tv + \Lambda g(x) - f(x+v)\\ &+ \{\Lambda\bar{u}|\ \bar{u}\in\mathbb{R}^m_+\}\Big\}\\ &= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \Big\{\Lambda g(x) + \{\Lambda u|\ u\in\mathbb{R}^m_+\}\\ &+ \{Tv - f(x+v)|\ v\in\mathbb{R}^n\}\Big\}.\end{aligned}$$

Setting $\bar{v} := x + v$, we obtain that

$$\begin{aligned}\Phi_3^*(0,T,\Lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \Big\{\Lambda g(x) + \{\Lambda u|\ u\in\mathbb{R}^m_+\}\\ &+ \{T\bar{v} - Tx - f(\bar{v})|\ \bar{v}\in\mathbb{R}^n\}\Big\}\\ &= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \Big\{\Lambda g(x) - Tx + \{\Lambda u|\ u\in\mathbb{R}^m_+\}\\ &+ \{T\bar{v} - f(\bar{v})|\ \bar{v}\in\mathbb{R}^n\}\Big\}\\ &= \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\{\Lambda u|\ u\in\mathbb{R}^m_+\}\\ &+ \{Tv - f(v)|\ v\in\mathbb{R}^n\} + \{\Lambda g(x) - Tx|\ x\in X\}\Big\}.\end{aligned}$$

(ii) By Lemma 3.2, we can easy verify (ii). □

Consequently, we can obtain the following so-called Fenchel-Lagrange dual problem to (VO)

$$\begin{aligned}(D^{VO}_{FL})\qquad & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{(T,\Lambda)\in\mathbb{R}^{p\times n}\times\mathbb{R}^{p\times m}} \Big[-\Phi_3^*(0,T,\Lambda)\Big]\\ =\ & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{(T,\Lambda)\in\mathbb{R}^{p\times n}\times\mathbb{R}^{p\times m}} \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\{f(v) - Tv|\ v\in\mathbb{R}^n\}\\ +\ & \{-\Lambda u|\ u\in\mathbb{R}^m_+\} + \{Tx - \Lambda g(x)|\ x\in X\}\Big\}.\end{aligned}$$

Proposition 3.14 *(weak duality)*

$$f(x) + \xi \underset{\mathbb{R}^p_+\setminus\{0\}}{\not\leq} 0,\ \forall x\in X,\ \forall \xi\in\Phi_3^*(0,T,\Lambda),$$

where $T \in \mathbb{R}^{p\times n}$ *and* $\Lambda \in \mathbb{R}^{p\times m}$.

Proposition 3.15 *(Strong duality)*

(i) (VO) *is stable with respect to* Φ_3 *if and only if for each solution* x^* *to* (VO) *there exists a solution* (T^*, Λ^*) *to* (D^{VO}_{FL}) *such that*

$$f(x^*) \in -\Phi_3^*(0, T^*, \Lambda^*). \tag{3.10}$$

(ii) Conversely, if $x^* \in X$ *and* $(T^*, \Lambda^*) \in \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m}$ *satisfy (3.10), then* x^* *is a solution to* (VO) *and* (T^*, Λ^*) *is a solution to* (D^{VO}_{FL}).

Similarly as for $(\widetilde{D}^{VO}_L)$, under the same restriction on Λ, we can introduce another dual problem. Indeed, let us suppose that $\Lambda \in L$. Then, according to Lemma 3.1(i) and (3.8), it holds

$$\begin{aligned}
\Phi_3^*(0, T, -\Lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ \bigcup_{u\in\mathbb{R}^m_+} \{-\Lambda u\} + \bigcup_{v\in\mathbb{R}^n} \{Tv - f(v)\} \\
&+ \bigcup_{x\in X} \{-\Lambda g(x) - Tx\} \Big\} \\
&\subseteq \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{u\in\mathbb{R}^m_+} \{-\Lambda u\} \\
&+ \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ \bigcup_{v\in\mathbb{R}^n} \{Tv - f(v)\} + \bigcup_{x\in X} \{-\Lambda g(x) - Tx\} \Big\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ \bigcup_{v\in\mathbb{R}^n} \{Tv - f(v)\} + \bigcup_{x\in X} \{-\Lambda g(x) - Tx\} \Big\}.
\end{aligned}$$

Let us denote by $\widetilde{\Psi}(T, \Lambda) := \max\limits_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ \bigcup\limits_{v\in\mathbb{R}^n} \{Tv - f(v)\} + \bigcup\limits_{x\in X} \{-\Lambda g(x) - Tx\} \Big\}$. If the set $f^*(T)$ is externally stable, then $\widetilde{\Psi}(T, \Lambda)$ can be rewritten as

$$\widetilde{\Psi}(T, \Lambda) = \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ f^*(T) + \bigcup_{x\in X} \{-\Lambda g(x) - Tx\} \Big\}.$$

The proposed map allows us to suggest the dual problem

$$\begin{aligned}
(\widetilde{D}^{VO}_{FL}) \qquad & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{(T,\Lambda)\in\mathbb{R}^{p\times n}\times L} \Big[-\widetilde{\Psi}(T, \Lambda) \Big] \\
= \ & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{(T,\Lambda)\in\mathbb{R}^{p\times n}\times L} \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ \bigcup_{v\in\mathbb{R}^n} \{f(v) - Tv\} + \bigcup_{x\in X} \{Tx + \Lambda g(x)\} \Big\}.
\end{aligned}$$

Proposition 3.16 *(weak duality)*

$$f(x) + \xi \not\leq_{\mathbb{R}^p_+\setminus\{0\}} 0, \ \forall x \in G, \ \forall \xi \in \widetilde{\Psi}(T, \Lambda),$$

where $T \in \mathbb{R}^{p\times n}$ *and* $\Lambda \in L$.

Proof: Let $(T, \Lambda) \in \mathbb{R}^{p\times n} \times L$ be fixed and $\xi \in \widetilde{\Psi}(T, \Lambda)$. In other words

$$\xi \not\leq_{\mathbb{R}^p_+\setminus\{0\}} Tv - f(v) + (-\Lambda g(x) - Tx), \ \forall v \in \mathbb{R}^n, \ \forall x \in X.$$

Choosing $v = x := \bar{x} \in G$, we obtain that

$$f(\bar{x}) + \xi \not\leq_{\mathbb{R}^p_+\setminus\{0\}} -\Lambda g(\bar{x}).$$

On the other hand, since $\Lambda \in L,\ \bar{x} \in G$ it holds $-\Lambda g(\bar{x}) \underset{\mathbb{R}^p_+}{\geq} 0$. Consequently, one has $f(x) + \xi \underset{\mathbb{R}^p_+ \setminus \{0\}}{\nleq} 0$. $\square$

Proposition 3.17 *Let $x^* \in G,\ (T^*, \Lambda^*) \in \mathbb{R}^{p \times n} \times L$ such that $f(x^*) \in -\widetilde{\Psi}(T^*, \Lambda^*)$. Then $f(x^*)$ is simultaneously a minimal point to the primal problem (VO) and a maximal point to the dual problem $(\widetilde{D}^{VO}_{FL})$.*

Proof: Let $x^* \in G,\ (T^*, \Lambda^*) \in \mathbb{R}^{p \times n} \times L$ and $f(x^*) \in -\widetilde{\Psi}(T^*, \Lambda^*)$. The latter means

$$f(x^*) \in \min_{\mathbb{R}^p_+ \setminus \{0\}} \Big\{ \bigcup_{v \in \mathbb{R}^n} \{f(v) - T^* v\} + \bigcup_{x \in X} \{T^* x + \Lambda^* g(x)\} \Big\}. \tag{3.11}$$

If $f(x^*)$ is not a minimal point to the primal problem (VO), then there exists $x \in G$ such that

$$f(x) \underset{\mathbb{R}^p_+ \setminus \{0\}}{\leq} f(x^*).$$

As mentioned before, it holds $\Lambda^* \in L,\ x \in G$ implies $\Lambda^* g(x) \underset{\mathbb{R}^p_+}{\leq} 0$. Consequently, we have $f(x) + \Lambda^* g(x) \underset{\mathbb{R}^p_+ \setminus \{0\}}{\leq} f(x^*)$, or, equivalently,

$$f(x) - T^* x + T^* x + \Lambda^* g(x) \underset{\mathbb{R}^p_+ \setminus \{0\}}{\leq} f(x^*).$$

But

$$f(x) - T^* x + T^* x + \Lambda^* g(x) \in \bigcup_{v \in \mathbb{R}^n} \{f(v) - T^* v\} + \bigcup_{x \in X} \{T^* x + \Lambda^* g(x)\},$$

which is a contradiction to (3.11). Therefore $f(x^*)$ is a minimal point to the problem (VO). Moreover, if $f(x^*)$ is not a solution to $(\widetilde{D}^{VO}_{FL})$, then $\exists \widetilde{y} \in \bigcup\limits_{(T,\Lambda) \in \mathbb{R}^{p \times n} \times L} \Big[-\widetilde{\Psi}(T, \Lambda) \Big]$ such that $f(x^*) \underset{\mathbb{R}^p_+ \setminus \{0\}}{\leq} \widetilde{y}$. Let $(\widetilde{T}, \widetilde{\Lambda}) \in \mathbb{R}^{p \times n} \times L$ such that $\widetilde{y} \in -\widetilde{\Psi}(\widetilde{T}, \widetilde{\Lambda})$. From $\widetilde{\Lambda} g(x^*) \underset{\mathbb{R}^p_+}{\leq} 0$ follows

$$\widetilde{y} \underset{\mathbb{R}^p_+ \setminus \{0\}}{\geq} f(x^*) + \widetilde{\Lambda} g(x^*) = f(x^*) - \widetilde{T} x^* + \widetilde{T} x^* + \widetilde{\Lambda} g(x^*),$$

which contradicts the fact that $\widetilde{y} \in -\widetilde{\Psi}(\widetilde{T}, \widetilde{\Lambda})$ in the same way as before. Accordingly, $f(x^*)$ is a solution to $(\widetilde{D}^{VO}_{FL})$. $\square$

3.1.4 Duality via conjugate maps with vector variables

This subsection aims to investigate some special cases of dual problems based on alternative definitions of the conjugate maps and the subgradient for a set-valued map having vector variables. In Definition 3.3, if we choose $U := [t, ..., t]^T \in \mathbb{R}^{p \times n}$ for $t \in \mathbb{R}^n$, as a variable of the conjugate map, then this reduces to the definition used in this subsection. Remark that duality results for vector optimization developed by Tanino and Sawaragi (see [75] and [82]) are essentially not distinguishable in both cases. Let us recall first definitions of the conjugate maps with vector variables (cf. Definition 3.3).

Definition 3.6 *[42, Definition 7.2.3] (the type II Fenchel transform)*
Let $h : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a set-valued map.

(i) The set-valued map $h^* : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ defined by

$$h_p^*(\lambda) = \max_{\mathbb{R}^p_+ \setminus \{0\}} \bigcup_{x \in \mathbb{R}^n} \Big[(\lambda^T x)_p - h(x)\Big], \; \lambda \in \mathbb{R}^n$$

is called the (type II) conjugate map of h;

(ii) The conjugate map of h_p^*, h_p^{**} is called the biconjugate map of h, i.e.

$$h_p^{**}(x) = \max_{\mathbb{R}^p_+ \setminus \{0\}} \bigcup_{\lambda \in \mathbb{R}^n} \Big[(\lambda^T x)_p - h_p^*(\lambda)\Big], \; x \in \mathbb{R}^n;$$

(iii) $\lambda \in \mathbb{R}^n$ is said to be a subgradient of the set-valued map h at $(\bar{x}; \bar{y})$ if $\bar{y} \in h(\bar{x})$ and

$$\bar{y} - (\lambda^T \bar{x})_p \in \min_{\mathbb{R}^p_+ \setminus \{0\}} \bigcup_{x \in \mathbb{R}^n} [h(x) - (\lambda^T x)_p],$$

where $(\lambda^T x)_p = (\lambda^T x, \ldots, \lambda^T x)^T \in \mathbb{R}^p$.

Like in Subsection 3.1.3, let $f : \mathbb{R}^n \to \mathbb{R}^p$, $g : \mathbb{R}^n \to \mathbb{R}^m$ be vector-valued functions and $X \subseteq \mathbb{R}^n$. Consider the vector optimization problem (VO). Based on the perturbation functions introduced in Subsection 3.1.3, let us suggest some dual problems having vector variables. For convenience, in this subsection we use the following notations.

$$\varphi_1 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p \cup \{+\infty\}, \quad \varphi_1(x, u) = \begin{cases} f(x), & x \in X, \; g(x) \leqq_{\mathbb{R}^m_+} u, \\ +\infty, & \text{otherwise;} \end{cases}$$

$$\varphi_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p \cup \{+\infty\}, \quad \varphi_2(x, v) = \begin{cases} f(x+v), & x \in G, \\ +\infty, & \text{otherwise;} \end{cases}$$

$$\varphi_3 : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p \cup \{+\infty\},$$

$$\varphi_3(x, v, u) = \begin{cases} f(x+v), & x \in X, \; g(x) \leqq_{\mathbb{R}^m_+} u, \\ +\infty, & \text{otherwise.} \end{cases}$$

Let us notice that throughout this subsection instead of φ_{ip}^*, $i = 1, 2, 3$ we write φ_i^*, $i = 1, 2, 3$.

Lagrange duality. By using a dual objective map having vector variable with respect to φ_1, the Lagrange dual problem to (VO) was introduced in [81] (see also [75]). Let us now explain how one can obtain this dual. Such analysis is usable for further calculations.

Lemma 3.4 *Let $\lambda \in \mathbb{R}^m$. Then*

$$\min_{\mathbb{R}^p_+ \setminus \{0\}} \{(\lambda^T x)_p | \; x \in \mathbb{R}^m_+\} = \begin{cases} \{0\}, & \text{if } \lambda \geqq_{\mathbb{R}^m_+} 0; \\ \varnothing, & \text{otherwise,} \end{cases}$$

where $(\lambda^T x)_p = (\lambda^T x, \ldots, \lambda^T x)^T \in \mathbb{R}^p$.

Proof: Let $z \in \min\limits_{\mathbb{R}^p_+ \setminus \{0\}} \{(\lambda^T x)_p | \; x \in \mathbb{R}^m_+\}$. Then $\exists \bar{x} \in \mathbb{R}^m_+$ such that $z = (\lambda^T \bar{x})_p$ and it holds

$$(\lambda^T \bar{x})_p \not\geq_{\mathbb{R}^p_+ \setminus \{0\}} (\lambda^T x)_p, \; \forall x \in \mathbb{R}^m_+,$$

or, equivalently,

$$\lambda^T \bar{x} \leq \lambda^T x, \ \forall x \in \mathbb{R}^m_+.$$

In other words, it holds $\lambda^T \bar{x} = \min\limits_{x \in \mathbb{R}^m_+} \lambda^T x$. Since $\inf\limits_{x \in \mathbb{R}^m_+} \lambda^T x = \begin{cases} 0, & \text{if } \lambda \underset{\mathbb{R}^m_+}{\geq} 0; \\ -\infty, & \text{otherwise,} \end{cases}$

we obtain the conclusion. $\square$

Proposition 3.18 *Let* $\lambda \in \mathbb{R}^m$. *Then*

$$\varphi_1^*(0, \lambda) = \begin{cases} \max\limits_{\mathbb{R}^p_+ \setminus \{0\}} \Big\{ (\lambda^T g(x))_p - f(x) | \ x \in X \Big\}, & \textit{if } \lambda \underset{\mathbb{R}^m_+}{\leq} 0; \\ \varnothing, & \textit{otherwise.} \end{cases}$$

Proof: Let $\lambda \in \mathbb{R}^m$. Then by definition

$$\begin{aligned} \varphi_1^*(0, \lambda) &= \max_{\mathbb{R}^p_+ \setminus \{0\}} \Big\{ (\lambda^T u)_p - \varphi_1(x, u) | \ x \in \mathbb{R}^n, \ u \in \mathbb{R}^m \Big\} \\ &= \max_{\mathbb{R}^p_+ \setminus \{0\}} \Big\{ (\lambda^T u)_p - f(x) | \ x \in X, \ g(x) \underset{\mathbb{R}^m_+}{\leq} u \Big\}. \end{aligned}$$

Setting $\bar{u} := u - g(x)$, we have

$$\begin{aligned} \varphi_1^*(0, \lambda) &= \max_{\mathbb{R}^p_+ \setminus \{0\}} \Big\{ (\lambda^T g(x))_p + (\lambda^T \bar{u})_p - f(x) | \ x \in X, \ \bar{u} \in \mathbb{R}^m_+ \Big\} \\ &= \max_{\mathbb{R}^p_+ \setminus \{0\}} \Big\{ \{ (\lambda^T g(x))_p - f(x) | \ x \in X \} + \{ (\lambda^T \bar{u})_p | \ \bar{u} \in \mathbb{R}^m_+ \} \Big\}. \end{aligned}$$

In view of Lemma 3.1(i) and Lemma 3.4, one has

$$\begin{aligned} \varphi_1^*(0, \lambda) &\subseteq \max_{\mathbb{R}^p_+ \setminus \{0\}} \{ (\lambda^T g(x))_p - f(x) | \ x \in X \} + \max_{\mathbb{R}^p_+ \setminus \{0\}} \{ (\lambda^T u)_p | \ u \in \mathbb{R}^m_+ \} \\ &= \max_{\mathbb{R}^p_+ \setminus \{0\}} \{ (\lambda^T g(x))_p - f(x) | \ x \in X \} - \min_{\mathbb{R}^p_+ \setminus \{0\}} \{ (-\lambda^T u)_p | \ u \in \mathbb{R}^m_+ \} \\ &= \begin{cases} \max\limits_{\mathbb{R}^p_+ \setminus \{0\}} \{ (\lambda^T g(x))_p - f(x) | \ x \in X \}, & \text{if } \lambda \underset{\mathbb{R}^m_+}{\leq} 0; \\ \varnothing, & \text{otherwise.} \end{cases} \end{aligned}$$

It remains to show for $\lambda \underset{\mathbb{R}^m_+}{\leq} 0$ that

$$\max_{\mathbb{R}^p_+ \setminus \{0\}} \{ (\lambda^T g(x))_p - f(x) | \ x \in X \} \subseteq \varphi_1^*(0, \lambda).$$

Let $\bar{y} \in \max\limits_{\mathbb{R}^p_+ \setminus \{0\}} \{ (\lambda^T g(x))_p - f(x) | \ x \in X \}$. This means $\bar{y} \in \{ (\lambda^T g(x))_p - f(x) | \ x \in X \}$ and

$$\bar{y} \underset{\mathbb{R}^p_+ \setminus \{0\}}{\not\leq} (\lambda^T g(x))_p - f(x), \ \forall x \in X. \tag{3.12}$$

Choosing $\bar{u} = 0$, we can verify that

$$\bar{y} = \bar{y} + (\lambda^T \bar{u})_p \in \Big\{ \{ (\lambda^T g(x))_p - f(x) | \ x \in X \} + \{ (\lambda^T u)_p | \ u \in \mathbb{R}^m_+ \} \Big\}.$$

On the other hand, since $(\lambda^T u)_p \underset{\mathbb{R}^p_+}{\leq} 0, \ \forall u \in \mathbb{R}^m_+$, one has $\bar{y} \underset{\mathbb{R}^p_+}{\geq} \bar{y} + (\lambda^T u)_p$ and by (3.12) it holds

$$\bar{y} + (\lambda^T u)_p \underset{\mathbb{R}^p_+ \setminus \{0\}}{\not\leq} (\lambda^T g(x))_p - f(x) + (\lambda^T u)_p, \ \forall x \in X, \ \forall u \in \mathbb{R}^m_+.$$

Consequently, we obtain that

$$\bar{y} \underset{\mathbb{R}^p_+\setminus\{0\}}{\nleq} (\lambda^T g(x))_p - f(x) + (\lambda^T u)_p, \ \forall x \in X, \ \forall u \in \mathbb{R}^m_+.$$

In other words $\bar{y} \in \varphi_1^*(0,\lambda)$. □

In this case the dual problem to (VO) can be written as

$$\begin{aligned}
(\widehat{D}^{VO}_L) \qquad & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{\lambda\in\mathbb{R}^m} \Big[-\varphi_1^*(0,\lambda)\Big] \\
= \ & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{\lambda \underset{\mathbb{R}^m_+}{\leqq} 0} \min_{\mathbb{R}^p_+\setminus\{0\}} \{f(x) - (\lambda^T g(x))_p | \ x \in X\} \\
= \ & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{\lambda \underset{\mathbb{R}^m_+}{\geqq} 0} \min_{\mathbb{R}^p_+\setminus\{0\}} \{f(x) + (\lambda^T g(x))_p | \ x \in X\}.
\end{aligned}$$

Proposition 3.19 *[75, Theorem 6.1.4]*

(i) The problem (VO) is stable with respect to φ_1 if and only if for each solution $\bar{x}$ to (VO), there exists a solution $\bar{\lambda} \in \mathbb{R}^m$ with $\bar{\lambda} \underset{\mathbb{R}^m_+}{\geqq} 0$ to the dual problem $(\widehat{D}^{VO}_L)$ such that

$$f(\bar{x}) \in \min_{\mathbb{R}^p_+\setminus\{0\}} \{f(x) + (\bar{\lambda}^T g(x))_p | \ x \in X\}$$

and $\bar{\lambda}^T g(\bar{x}) = 0$.

(ii) Conversely, if $\bar{x} \in G$ and $\bar{\lambda} \in \mathbb{R}^m$ with $\bar{\lambda} \underset{\mathbb{R}^m_+}{\geqq} 0$ satisfy the above conditions, then $\bar{x}$ and $\bar{\lambda}$ are solutions to (VO) and $(\widehat{D}^{VO}_L)$, respectively.

Remark 3.2 Let $p = 1$ and the assumptions of Theorem 1.3 (see Section 1.1) be fulfilled. Then Proposition 3.19 coincides with the optimality conditions (cf. Theorem 2.9 in [16]) for the Lagrange dual problem in scalar optimization.

Example 3.1 Consider the vector optimization problem

$$(VO_1) \qquad \min_{\mathbb{R}^2_+\setminus\{0\}} \{(x_1,x_2) | \ 0 \leq x_i \leq 1, \ x_i \in \mathbb{R}, \ i = 1,2\}.$$

Let us construct the Lagrange dual problem to (VO_1). Before doing this, in view of $(\widehat{D}^{VO}_L)$, for $\lambda \underset{\mathbb{R}^m_+}{\geqq} 0$, one has to calculate

$$\min_{\mathbb{R}^p_+\setminus\{0\}} \{f(x) + (\lambda^T g(x))_p | \ x \in X\}.$$

Let $\lambda = (\lambda_1,\lambda_2,\lambda_3,\lambda_4)^T \in \mathbb{R}^4$ and the function $g : \mathbb{R}^2 \to \mathbb{R}^4$ be defined by $g(x) = (-x_1, x_1 - 1, -x_2, x_2 - 1)^T$. In other words, we have

$$\min_{\mathbb{R}^2_+\setminus\{0\}} \left\{ \left(\begin{array}{c} x_1 - \lambda_1 x_1 + \lambda_2(x_1 - 1) - \lambda_3 x_2 + \lambda_4(x_2 - 1) \\ x_2 - \lambda_1 x_1 + \lambda_2(x_1 - 1) - \lambda_3 x_2 + \lambda_4(x_2 - 1) \end{array} \right) \Big| \ (x_1,x_2)^T \in \mathbb{R}^2 \right\},$$

or, equivalently,

$$\min_{\mathbb{R}^2_+\setminus\{0\}} \left\{ \left(\begin{array}{c} (\lambda_2 - \lambda_1 + 1)x_1 + (\lambda_4 - \lambda_3)x_2 \\ (\lambda_2 - \lambda_1)x_1 + (\lambda_4 - \lambda_3 + 1)x_2 \end{array} \right) \Big| \ (x_1,x_2)^T \in \mathbb{R}^2 \right\} - \left(\begin{array}{c} \lambda_2 + \lambda_4 \\ \lambda_2 + \lambda_4 \end{array} \right).$$

Let

$$B_1 = \begin{pmatrix} \lambda_2 - \lambda_1 + 1 & \lambda_4 - \lambda_3 \\ \lambda_2 - \lambda_1 & \lambda_4 - \lambda_3 + 1 \end{pmatrix}.$$

Taking into account Theorem 11.20 in [50], if $\exists \mu \in \operatorname{int}\ \mathbb{R}^2_+$ such that (see also Lemma 3.6 in Subsection 3.2.2)

$$\mu^T B_1 = 0^T, \tag{3.13}$$

then $\min\limits_{\mathbb{R}^2_+\setminus\{0\}} \{B_1 x|\ x \in \mathbb{R}^2\} = \{B_1 x|\ x \in \mathbb{R}^2\}$. If (3.13) is not fulfilled, it follows that $\min\limits_{\mathbb{R}^2_+\setminus\{0\}} \{B_1 x|\ x \in \mathbb{R}^2\} = \varnothing$. Moreover, from (3.13) follows

$$\begin{cases} (\lambda_2 - \lambda_1 + 1)\mu_1 + (\lambda_2 - \lambda_1)\mu_2 = 0 \\ (\lambda_4 - \lambda_3)\mu_1 + (\lambda_4 - \lambda_3 + 1)\mu_2 = 0. \end{cases}$$

Consequently, we have

$$\lambda_1 = \lambda_2 + \frac{\mu_1}{\mu_1 + \mu_2}, \quad \lambda_3 = \lambda_4 + \frac{\mu_2}{\mu_1 + \mu_2}.$$

Let us define the set

$$L_1 := \left\{ \lambda \in \mathbb{R}^4 |\ \exists \mu \in \operatorname{int}\ \mathbb{R}^2_+ \text{ such that } \lambda_1 = \lambda_2 + \frac{\mu_1}{\mu_1 + \mu_2}, \quad \lambda_3 = \lambda_4 + \frac{\mu_2}{\mu_1 + \mu_2} \right\}.$$

In conclusion, we obtain the Lagrange dual problem $(\widehat{D}^{VO_1}_L)$ as follows.

$$\max_{\mathbb{R}^2_+\setminus\{0\}} \bigcup_{\substack{\lambda \underset{\mathbb{R}^4_+}{\geqq} 0 \\ \lambda \in L_1}} \left\{ \begin{pmatrix} (\lambda_2 - \lambda_1 + 1)x_1 + (\lambda_4 - \lambda_3)x_2 \\ (\lambda_2 - \lambda_1)x_1 + (\lambda_4 - \lambda_3 + 1)x_2 \end{pmatrix} - \begin{pmatrix} \lambda_2 + \lambda_4 \\ \lambda_2 + \lambda_4 \end{pmatrix} \middle|\ (x_1, x_2)^T \in \mathbb{R}^2 \right\}.$$

Let $\bar{x} = (0,0)^T \in \mathbb{R}^2$ and $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3, \bar{\lambda}_4)^T \in L_1$ be vectors such that $\bar{\lambda} \underset{\mathbb{R}^4_+}{\geqq} 0$ and $\bar{\lambda}^T g(\bar{x}) = 0$. Then, from $\bar{\lambda}^T g(\bar{x}) = 0$ follows $\bar{\lambda}_2 + \bar{\lambda}_4 = 0$. As $\bar{\lambda}_2, \bar{\lambda}_4 \geq 0$, this implies that $\bar{\lambda}_2 = \bar{\lambda}_4 = 0$. Moreover, as $\bar{\lambda} \in L_1$, it holds $\bar{\lambda}_1 = \frac{\mu_1}{\mu_1+\mu_2}$, $\bar{\lambda}_3 = \frac{\mu_2}{\mu_1+\mu_2}$. In other words, $\bar{\lambda}_1 = \alpha := \frac{\mu_1}{\mu_1+\mu_2}$, $\bar{\lambda}_3 = 1 - \alpha,\ 0 < \alpha < 1$. On the other hand, it is clear that

$$\begin{aligned} f(\bar{x}) = (0,0)^T \quad &\in \quad \min_{\mathbb{R}^2_+\setminus\{0\}} \{ f(x) + (\bar{\lambda}\lambda^T g(x))_2 |\ x \in \mathbb{R}^2 \} \\ &= \quad \left\{ \begin{pmatrix} \frac{\mu_2}{\mu_1+\mu_2}(x_1 - x_2) \\ \frac{\mu_1}{\mu_1+\mu_2}(x_2 - x_1) \end{pmatrix} \middle|\ (x_1, x_2)^T \in \mathbb{R}^2 \right\} \\ &= \quad \left\{ \begin{pmatrix} (\alpha - 1)y \\ \alpha y \end{pmatrix} \middle|\ y \in \mathbb{R} \right\},\ 0 < \alpha < 1. \end{aligned}$$

According to Proposition 3.19(ii), $\bar{x} = (0,0)^T$ and $\bar{\lambda} = (\alpha, 0, 1 - \alpha, 0)^T,\ 0 < \alpha < 1$ are solutions to (VO_1) and $(\widehat{D}^{VO_1}_L)$, respectively.

Fenchel duality. Before considering the dual problem, we need the following assertion.

Lemma 3.5 *Let $t \in \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^n$. If the set $\max\limits_{\mathbb{R}^p_+\setminus\{0\}} \{(t^T x)_p |\ x \in Y\} \neq \varnothing$, then*

$$\max_{\mathbb{R}^p_+\setminus\{0\}} \{(t^T x)_p |\ x \in Y\} = \{(\max_{x \in Y} t^T x)_p\}.$$

Proof: Let $t \in \mathbb{R}^n$. By assumption, there exists $\bar{x} \in Y$ such that

$$(t^T\bar{x})_p \underset{\mathbb{R}^p_+\setminus\{0\}}{\not\leq} (t^T x)_p, \ \forall x \in Y,$$

or, equivalently,

$$t^T\bar{x} \geq t^T x, \ \forall x \in Y.$$

Therefore $t^T\bar{x} = \max\limits_{x\in Y} t^T x$. □

Proposition 3.20 *Let $t \in \mathbb{R}^n$. Then*

$$\varphi_2^*(0,t) = \begin{cases} f_p^*(t) - (\min\limits_{x\in G} t^T x)_p, & \textit{if } \max\limits_{\mathbb{R}^p_+\setminus\{0\}} \{(-t^T x)_p |\ x \in G\} \neq \emptyset, \\ \emptyset, & \textit{otherwise.} \end{cases}$$

Proof: Let $t \in \mathbb{R}^n$. By definition

$$\begin{aligned} \varphi_2^*(0,t) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{(t^T v)_p - \varphi_2(x,v)|\ x \in \mathbb{R}^n,\ v \in \mathbb{R}^n\Big\} \\ &= \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{(t^T v)_p - f(x+t)|\ x \in G,\ v \in \mathbb{R}^n\Big\}. \end{aligned}$$

Substituting $\bar{v} := x + v$, we get

$$\begin{aligned} \varphi_2^*(0,t) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{(t^T\bar{v})_p - (t^T x)_p - f(\bar{v})|\ x \in G,\ \bar{v} \in \mathbb{R}^n\Big\} \\ &= \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\{(t^T\bar{v})_p - f(\bar{v})|\ \bar{v} \in \mathbb{R}^n\} + \{(-t^T x)_p|\ x \in G\}\Big\}. \end{aligned}$$

According to Lemma 3.1(i), it follows that

$$\varphi_2^*(0,t) \subseteq \max_{\mathbb{R}^p_+\setminus\{0\}} \{(t^T v)_p - f(v)|\ v \in \mathbb{R}^n\} + \max_{\mathbb{R}^p_+\setminus\{0\}} \{(-t^T x)_p|\ x \in G\}.$$

It is clear that unless $\max\limits_{\mathbb{R}^p_+\setminus\{0\}} \{(-t^T x)_p|\ x \in G\} \neq \emptyset,\ \varphi_2^*(0,t) = \emptyset$.
Since $\max\limits_{\mathbb{R}^p_+\setminus\{0\}} \{(-t^T x)_p|\ x \in G\} \neq \emptyset$, by Lemma 3.5 it holds

$$\max_{\mathbb{R}^p_+\setminus\{0\}} \{(-t^T x)_p|\ x \in G\} = \{(-\min_{x\in G} t^T x)_p\}.$$

In other words

$$\begin{aligned} \varphi_2^*(0,t) &\subseteq \max_{\mathbb{R}^p_+\setminus\{0\}} \{(t^T v)_p - f(v)|\ v \in \mathbb{R}^n\} - (\min_{x\in G} t^T x)_p \\ &= f_p^*(t) - (\min_{x\in G} t^T x)_p. \end{aligned}$$

Let now $\bar{y} \in f_p^*(t) - (\min\limits_{x\in G} t^T x)_p$. Then

$$\bar{y} \in \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\{(t^T v)_p - f(v)|\ v \in \mathbb{R}^n\} - (\min_{x\in G} t^T x)_p\Big\}.$$

This means that

$$\bar{y} \underset{\mathbb{R}^p_+\setminus\{0\}}{\not\leq} (t^T v)_p - f(v) - (\min_{x\in G} t^T x)_p,\ \forall v \in \mathbb{R}^n.$$

Moreover, from

$$(t^T v)_p - f(v) - (\min_{x \in G} t^T x)_p \underset{\mathbb{R}^p_+}{\geq} (t^T v)_p - f(v) - (t^T x)_p, \ \forall x \in G, \ \forall v \in \mathbb{R}^n,$$

follows

$$(t^T v)_p - f(v) - (t^T x)_p \underset{\mathbb{R}^p_+ \setminus \{0\}}{\not\geq} \bar{y}, \ \forall x \in G, \ \forall v \in \mathbb{R}^n.$$

Whence $\bar{y} \in \varphi_2^*(0, t)$. □

The Fenchel dual problem can be stated now as follows.

$$\begin{aligned} (\widehat{D}_F^{VO}) \qquad & \max_{\mathbb{R}^p_+ \setminus \{0\}} \bigcup_{t \in \mathbb{R}^n} \Big[- \varphi_2^*(0, t) \Big] \\ = \ & \max_{\mathbb{R}^p_+ \setminus \{0\}} \bigcup_{t \in \mathbb{R}^n} \Big[- f_p^*(t) + (\min_{x \in G} t^T x)_p \Big] \end{aligned}$$

From Theorem 3.2 and Proposition 3.20 follows the following assertion.

Proposition 3.21

(i) *The problem (VO) is stable with respect to φ_2 if and only if for each solution $\bar{x}$ to (VO), there exists a solution $\bar{t} \in \mathbb{R}^n$ to the dual problem $(\widehat{D}_F^{VO})$ such that*

$$f(\bar{x}) \in -f_p^*(\bar{t}) + (\min_{x \in G} \bar{t}^T x)_p \tag{3.14}$$

and $\bar{t}^T \bar{x} = \min_{x \in G} \bar{t}^T x$.

(ii) *Conversely, if $\bar{x} \in G$ and $\bar{t} \in \mathbb{R}^n$ satisfy the above conditions, then $\bar{x}$ and $\bar{t}$ are solutions to (VO) and $(\widehat{D}_F^{VO})$, respectively.*

Remark 3.3 Let $p = 1$ and the assumptions of Theorem 1.1(iii) be fulfilled. Then Proposition 3.21 is nothing else that the optimality conditions (cf. Theorem 2.10 in [16]) for the Fenchel dual problem in scalar optimization.

Fenchel-Lagrange duality. The last dual problem in this section is constructed by using the perturbation function φ_3.

Proposition 3.22 *Let $t \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^m$. Assume that $\max_{\mathbb{R}^p_+ \setminus \{0\}} \{(\lambda^T g(x) - t^T x)_p | \ x \in X\} \neq \varnothing$, then*

$$\varphi_3^*(0, t, \lambda) = \begin{cases} f_p^*(t) + (\max_{x \in X} [\lambda^T g(x) - t^T x])_p, & \text{if } \lambda \underset{\mathbb{R}^m_+}{\leq} 0; \\ \varnothing, & \text{otherwise.} \end{cases}$$

Proof: Let $t \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^m$. By definition

$$\begin{aligned} \varphi_3^*(0, t, \lambda) \ &= \ \max_{\mathbb{R}^p_+ \setminus \{0\}} \Big\{ (t^T v)_p + (\lambda^T u)_p - \varphi_3(x, v, u) | \ x \in \mathbb{R}^n, v \in \mathbb{R}^n, u \in \mathbb{R}^m \Big\} \\ &= \ \max_{\mathbb{R}^p_+ \setminus \{0\}} \Big\{ (t^T v)_p + (\lambda^T u)_p - f(x + v) | \ x \in X, v \in \mathbb{R}^n, g(x) \underset{\mathbb{R}^m_+}{\leq} u \Big\} \\ &= \ \max_{\mathbb{R}^p_+ \setminus \{0\}} \bigcup_{x \in X} \bigcup_{v \in \mathbb{R}^n} \Big\{ (t^T v)_p + (\lambda^T u)_p - f(x + v) | \ g(x) \underset{\mathbb{R}^m_+}{\leq} u \Big\}. \end{aligned}$$

Taking $\bar{u} := u - g(x)$, one has

$$\begin{aligned}
\varphi_3^*(0,t,\lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X}\bigcup_{v\in\mathbb{R}^n} \Big\{(t^Tv)_p + (\lambda^T g(x))_p + (\lambda^T\bar{u})_p - f(x+v)|\ \bar{u}\in\mathbb{R}^m_+\Big\}\\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X}\bigcup_{v\in\mathbb{R}^n} \Big\{(t^Tv)_p + (\lambda^T g(x))_p - f(x+v) + \{(\lambda^T\bar{u})_p|\ \bar{u}\in\mathbb{R}^m_+\}\Big\}\\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \Big\{(\lambda^T g(x))_p + \{(\lambda^T\bar{u})_p|\ \bar{u}\in\mathbb{R}^m_+\}\\
&+ \ \{(t^Tv)_p - f(x+v)|\ v\in\mathbb{R}^n\}\Big\}.
\end{aligned}$$

Setting now $\bar{v} := x + v$, this implies that

$$\begin{aligned}
\varphi_3^*(0,t,\lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \Big\{(\lambda^T g(x))_p + \{(\lambda^T\bar{u})_p|\ \bar{u}\in\mathbb{R}^m_+\}\\
&+ \ \{(t^T\bar{v})_p - (t^Tx)_p - f(\bar{v})|\ \bar{v}\in\mathbb{R}^n\}\Big\}\\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \Big\{(\lambda^T g(x))_p - (t^Tx)_p + \{(\lambda^T\bar{u})_p|\ \bar{u}\in\mathbb{R}^m_+\}\\
&+ \ \{(t^T\bar{v})_p - f(\bar{v})|\ \bar{v}\in\mathbb{R}^n\}\Big\}\\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\{(\lambda^T u)_p|\ u\in\mathbb{R}^m_+\} + \{(t^Tv)_p - f(v)|\ v\in\mathbb{R}^n\}\\
&+ \ \{(\lambda^T g(x) - t^Tx)_p|\ x\in X\}\Big\}.
\end{aligned}$$

Consequently

$$\begin{aligned}
\varphi_3^*(0,t,\lambda) &\subseteq \max_{\mathbb{R}^p_+\setminus\{0\}} \{(\lambda^T u)_p|\ u\in\mathbb{R}^m_+\}\\
&+ \max_{\mathbb{R}^p_+\setminus\{0\}} \{(t^Tv)_p - f(\bar{v})|\ v\in\mathbb{R}^n\}\\
&+ \max_{\mathbb{R}^p_+\setminus\{0\}} \{(\lambda^T g(x))_p - (t^Tx)_p|\ x\in X\}.
\end{aligned}$$

Moreover, one can easy verify that

$$\max_{\mathbb{R}^p_+\setminus\{0\}} \{(\lambda^T g(x))_p - (t^Tx)_p|\ x\in X\} = \{(\max_{x\in X}[\lambda^T g(x) - t^Tx])_p\}.$$

By Lemma 3.4, we conclude that

$$\varphi_3^*(0,t,\lambda) \subseteq \begin{cases} f_p^*(t) + (\max_{x\in X}[\lambda^T g(x) - t^Tx])_p, & \text{if } \lambda \leqq_{\mathbb{R}^m_+} 0;\\ \emptyset, & \text{otherwise.}\end{cases}$$

Let us now show the converse inclusion. Let $t\in\mathbb{R}^n$, $\lambda \leqq_{\mathbb{R}^m_+} 0$ and $\bar{y} \in f_p^*(t) + (\max_{x\in\mathbb{R}^n}[\lambda^T g(x) - t^Tx])_p$. Then it holds

$$\bar{y} \in \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\{(t^Tv)_p - f(v)|\ v\in\mathbb{R}^n\} + (\max_{x\in X}[\lambda^T g(x) - t^Tx])_p\Big\}.$$

In other words

$$\bar{y} \not\leqq_{\mathbb{R}^p_+\setminus\{0\}} (t^Tv)_p - f(v) + (\max_{x\in X}[\lambda^T g(x) - t^Tx])_p,\ \forall v\in\mathbb{R}^n.$$

Since

$$(t^T v)_p - f(v) + (\lambda^T g(x) - t^T x)_p \underset{\mathbb{R}^p_+}{\leq} (t^T v)_p - f(v) + (\max_{x \in X}[\lambda^T g(x) - t^T x])_p, \ \forall x \in X,$$

we conclude that

$$\bar{y} \underset{\mathbb{R}^p_+ \setminus \{0\}}{\not\leq} (t^T v)_p - f(v) + (\lambda^T g(x) - t^T x)_p, \ \forall x \in X, v \in \mathbb{R}^n,$$

or, equivalently,

$$\bar{y} + (\lambda^T u)_p \underset{\mathbb{R}^p_+ \setminus \{0\}}{\not\leq} (t^T v)_p - f(v) + (\lambda^T g(x) - t^T x)_p + (\lambda^T u)_p, \ \forall x \in X, \ \forall v \in \mathbb{R}^n, \ \forall u \in \mathbb{R}^m_+.$$

On the other hand, because of $(\lambda^T u)_p \underset{\mathbb{R}^p_+}{\leq} 0, \ \forall u \in \mathbb{R}^m_+$, it holds

$$\bar{y} \underset{\mathbb{R}^p_+}{\geq} \bar{y} + (\lambda^T u)_p, \ \forall u \in \mathbb{R}^m_+.$$

Hence we obtain that

$$\bar{y} \underset{\mathbb{R}^p_+ \setminus \{0\}}{\not\leq} (t^T v)_p - f(v) + (\lambda^T g(x) - t^T x)_p + (\lambda^T u)_p, \ \forall x \in X, \ \forall v \in \mathbb{R}^n, \ \forall u \in \mathbb{R}^m_+.$$

Therefore $\bar{y} \in \varphi_3^*(0, t, \lambda)$. □

As a consequence, we can suggest the following dual problem to (VO)

$$\begin{aligned}
(\widehat{D}^{VO}_{FL}) \qquad & \max_{\mathbb{R}^p_+ \setminus \{0\}} \bigcup_{(t,\lambda) \in \mathbb{R}^n \times \mathbb{R}^m} \Big[-\varphi_3^*(0, t, \lambda) \Big] \\
= \ & \max_{\mathbb{R}^p_+ \setminus \{0\}} \bigcup_{\substack{t \in \mathbb{R}^n \\ \lambda \underset{\mathbb{R}^m_+}{\leq} 0}} \Big[-f_p^*(t) + (\min_{x \in X}[t^T x - \lambda^T g(x)])_p \Big] \\
= \ & \max_{\mathbb{R}^p_+ \setminus \{0\}} \bigcup_{\substack{t \in \mathbb{R}^n \\ \lambda \underset{\mathbb{R}^m_+}{\geq} 0}} \Big[-f_p^*(t) + (\min_{x \in X}[t^T x + \lambda^T g(x)])_p \Big].
\end{aligned}$$

According to Theorem 3.2 and Proposition 3.22 one can state the following result.

Proposition 3.23

(i) The problem (VO) is stable with respect to φ_3 if and only if for each solution $\bar{x}$ to (VO), there exists a solution $\bar{t} \in \mathbb{R}^n, \ \bar{\lambda} \in \mathbb{R}^m$ with $\bar{\lambda} \underset{\mathbb{R}^m_+}{\geq} 0$ to the dual problem $(\widehat{D}^{VO}_{FL})$ such that

$$f(\bar{x}) \in -f_p^*(\bar{t}) + (\min_{x \in X}[\bar{t}^T x + \bar{\lambda}^T g(x)])_p. \tag{3.15}$$

Moreover it holds

$$\bar{t}^T \bar{x} + \bar{\lambda}^T g(\bar{x}) = \min_{x \in X}[\bar{t}^T x + \bar{\lambda}^T g(x)] \ \text{ and } \ \bar{\lambda}^T g(\bar{x}) = 0. \tag{3.16}$$

(ii) Conversely, if $\bar{x} \in G$ and $\bar{t} \in \mathbb{R}^n, \ \bar{\lambda} \in \mathbb{R}^m$ with $\bar{\lambda} \underset{\mathbb{R}^m_+}{\geq} 0$ satisfy (3.15)–(3.16), then $\bar{x}$ and $(\bar{t}, \bar{\lambda})$ are solutions to (VO) and $(\widehat{D}^{VO}_{FL})$, respectively.

Remark 3.4 In the scalar case Proposition 3.23 is nothing else than the assertion dealing with the optimality conditions for the Fenchel-Lagrange dual problem (cf. Theorem 2.11 in [16]).

Finally, we show some relations between dual objective maps investigated in this subsection.

Proposition 3.24 *Let $t \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^m$ with $\lambda \underset{\mathbb{R}^m_+}{\leq} 0$. If $\max\limits_{\mathbb{R}^p_+ \setminus \{0\}} \{(-t^T x)_p |\ x \in G\} \neq \varnothing$ and $\max\limits_{\mathbb{R}^p_+ \setminus \{0\}} \{(\lambda^T g(x) - t^T x)_p |\ x \in X\} \neq \varnothing$, then*

$$\varphi_2^*(0,t) \subseteq \varphi_3^*(0,t,\lambda) - \mathbb{R}^p_+.$$

Proof: Let $t \in \mathbb{R}^n$ and $\lambda \underset{\mathbb{R}^m_+}{\leq} 0$. Assume that $z \in \varphi_2^*(0,t) = f_p^*(t) - (\min\limits_{x \in G} t^T x)_p$. Since $g(x) \underset{\mathbb{R}^m_+}{\leq} 0$, for $x \in G$ one has $-\lambda^T g(x) \leq 0,\ \forall x \in G$. After adding $t^T x$ in both sides, we have

$$\min_{x \in X}[t^T x - \lambda^T g(x)] \leq \min_{x \in G}[t^T x - \lambda^T g(x)] \leq \min_{x \in G} t^T x,$$

or, equivalently,

$$-(\min_{x \in G} t^T x)_p \underset{\mathbb{R}^p_+}{\leq} -(\min_{x \in X}[t^T x - \lambda^T g(x)])_p.$$

This means that

$$-(\min_{x \in G} t^T x)_p \in -(\min_{x \in X}[t^T x - \lambda^T g(x)])_p - \mathbb{R}^p_+.$$

Therefore

$$z \in f_p^*(t) - (\min_{x \in X}[t^T x - \lambda^T g(x)])_p - \mathbb{R}^p_+.$$

In other words $z \in \varphi_3^*(0,t,\lambda) - \mathbb{R}^p_+$. □

Proposition 3.25 *Let $t \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^m$ with $\lambda \underset{\mathbb{R}^m_+}{\leq} 0$. If the set $f_p^*(t)$ is externally stable and $\max\limits_{\mathbb{R}^p_+ \setminus \{0\}} \{(\lambda^T g(x) - t^T x)_p |\ x \in X\} \neq \varnothing$, then*

$$\varphi_1^*(0,\lambda) \subseteq \varphi_3^*(0,t,\lambda) - \mathbb{R}^p_+.$$

Proof: Let $t \in \mathbb{R}^n$ and $\lambda \underset{\mathbb{R}^m_+}{\leq} 0$ be fixed. Then it is clear that

$$\begin{aligned}
\varphi_1^*(0,\lambda) &= \max_{\mathbb{R}^p_+ \setminus \{0\}} \{(\lambda^T g(x))_p - f(x) |\ x \in X\} \\
&\subseteq \{(\lambda^T g(x))_p - f(x) |\ x \in X\} \\
&\subseteq \{(t^T x)_p - f(x) |\ x \in \mathbb{R}^n\} + \{-(t^T x - \lambda^T g(x))_p |\ x \in X\}.
\end{aligned}$$

On the other hand, in view of the relation

$$-\{(p^T x - \lambda^T g(x))_p |\ x \in X\} \subseteq -\min_{x \in X}(p^T x - \lambda^T g(x))_p - \mathbb{R}^p_+$$

and by the external stability of $f_p^*(t)$, we have

$$\begin{aligned}
\varphi_1^*(0,\lambda) &\subseteq f_p^*(t) - \mathbb{R}^p_+ - \min_{x \in X}(p^T x - \lambda^T g(x))_p - \mathbb{R}^p_+ \\
&= f_p^*(t) - \min_{x \in X}(p^T x - \lambda^T g(x))_p - \mathbb{R}^p_+ \\
&= \varphi_3^*(0,t,\lambda) - \mathbb{R}^p_+.
\end{aligned}$$

□

3.2 Applications

The dual problems introduced in Section 3.1 allow us to define some new gap functions for the vector variational inequality. In order to prove the properties in the definition of a gap function for this kind of variational inequalities, the duality assertions discussed in Section 3.1 are used.

3.2.1 Gap functions for the vector variational inequality

Let $F : \mathbb{R}^n \to \mathbb{R}^{n\times p}$ be a matrix-valued function and $K \subseteq \mathbb{R}^n$. The vector variational inequality problem consists in finding $x \in K$ such that

$$(VVI) \qquad F(x)^T(y-x) \not\leq_{\mathbb{R}^p_+\setminus\{0\}} 0, \ \forall y \in K.$$

Definition 3.7 *(cf. [23] and [42]) A set-valued map $\gamma : K \rightrightarrows \mathbb{R}^p$ is said to be a gap function for (VVI) if it satisfies the following conditions:*

(i) $0 \in \gamma(x)$ *if and only if* $x \in K$ *solves the problem* (VVI);

(ii) $0 \not\geq_{\mathbb{R}^p_+\setminus\{0\}} \gamma(y), \ \forall y \in K.$

For (VVI) the following gap function has been investigated (see [23])

$$\gamma_A^{VVI}(x) = \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ F(x)^T(x-y) |\ y \in K \Big\}.$$

Recall that γ_A^{VVI} is a generalization of Auslender's gap function for the scalar variational inequality problem (cf. [8]).
On the other hand, the dual problems and duality results investigated in Subsection 3.1.3 allow us to introduce some new gap functions for (VVI). We remark that $x \in K$ is a solution to the problem (VVI) if and only if 0 is a minimal point of the set $\Big\{F(x)^T(y-x)|\ y \in K\Big\}$. This means that x is a solution to the following vector optimization problem

$$(P^{VVI}; x) \qquad \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ F(x)^T(y-x) |\ y \in K \Big\}.$$

Let $x \in K$ be fixed. Setting $f_x(y) := F(x)^T(y-x)$ instead of f in (D_F^{VO}), the Fenchel dual problem to $(P^{VVI}; x)$ turns out to be

$$(D_F^{VVI}; x) \qquad \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{T\in\mathbb{R}^{p\times n}} \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ \bigcup_{y\in\mathbb{R}^n} \{(F(x)^T-T)y\} - F(x)^T x + \bigcup_{y\in K}\{Ty\} \Big\}.$$

We define the following map for any $x \in K$

$$\gamma_F^{VVI}(x) := \bigcup_{T\in\mathbb{R}^{p\times n}} \widetilde{\Phi}_2^*(0,T;x),$$

where $\widetilde{\Phi}_2^*(0,T;x)$ is defined by

$$\widetilde{\Phi}_2^*(0,T;x) = \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ \bigcup_{y\in\mathbb{R}^n} \{(T-F(x)^T)y\} + F(x)^T x + \bigcup_{y\in K}\{-Ty\} \Big\}.$$

Theorem 3.3 *Let for any $x \in K$ the problem $(P^{VVI}; x)$ be stable with respect to $\widetilde{\Phi}_2(0,\cdot;x)$. Then γ_F^{VVI} is a gap function for (VVI).*

Proof:

(i) Let $x \in K$ be a solution to the problem (VVI). As the problem $(P^{VVI}; x)$ is stable, by Proposition 3.12(i), there exists a solution $T_x \in \mathbb{R}^{p \times n}$ to $(D_F^{VVI}; x)$ such that
$$f_x(x) = 0 \in -\widetilde{\Phi}_2^*(0, T_x; x). \tag{3.17}$$
In other words, $0 \in \widetilde{\Phi}_2^*(0, T_x; x)$ and this implies that
$$0 \in \bigcup_{T \in \mathbb{R}^{p \times n}} \widetilde{\Phi}_2^*(0, T; x) = \gamma_F^{VVI}(x).$$
Conversely, let $x \in K$ and $0 \in \gamma_F^{VVI}(x)$. Hence, there exists $T_x \in \mathbb{R}^{p \times n}$ such that
$$0 \in \widetilde{\Phi}_2^*(0, T_x; x) \text{ or, equivalently, } 0 = F(x)^T(x - x) \in -\widetilde{\Phi}_2^*(0, T_x; x).$$
According to Proposition 3.12(ii), x is a solution to $(P^{VVI}; x)$ and also to the problem (VVI).

(ii) Let $y \in K$ be fixed. Then in view of Proposition 3.11, for any $T \in \mathbb{R}^{p \times n}$, one has
$$f_y(z) + \xi \underset{\mathbb{R}^p_+ \setminus \{0\}}{\nleq} 0, \ \forall z \in K, \ \forall \xi \in \widetilde{\Phi}_2^*(0, T; y),$$
or equivalently,
$$F(y)^T(z - y) + \xi \underset{\mathbb{R}^p_+ \setminus \{0\}}{\nleq} 0, \ \forall z \in K, \ \forall \xi \in \bigcup_{T \in \mathbb{R}^{p \times n}} \widetilde{\Phi}_2^*(0, T; y) = \gamma_F^{VVI}(y).$$
Setting $z = y$, we get
$$\xi \underset{\mathbb{R}^p_+ \setminus \{0\}}{\nleq} 0, \ \forall \xi \in \gamma_F^{VVI}(y).$$

□

According to Proposition 3.4, we can mention the following result relative to the stability with respect to $\widetilde{\Phi}_2(0, \cdot; x), \ x \in K$.

Proposition 3.26 *Let for any $x \in K$ the set $\min\limits_{\mathbb{R}^p_+ \setminus \{0\}} \{F(x)^T y | \ y \in K\}$ be externally stable. Then the problem $(P^{VVI}; x)$ is stable with respect to $\widetilde{\Phi}_2(0, \cdot; x)$.*

Let us remark that in connection with the Fenchel dual problem we denote γ_F^{VVI} as the Fenchel gap function for the vector variational inequality (VVI). Let now the ground set K be given by
$$K = \Big\{ x \in \mathbb{R}^n | \ g(x) \underset{\mathbb{R}^m_+}{\leq} 0 \Big\},$$
where $g(x) = (g_1(x), ..., g_m(x))^T, \ g_i : \mathbb{R}^n \to \mathbb{R}$. Before introducing two other gap functions, let us state the Lagrange and Fenchel-Lagrange dual problems for $(P^{VVI}; x)$. Taking f_x in $\Phi_1^*(0, \Lambda)$ and $\Phi_3^*(0, T, \Lambda)$, respectively, we have
$$(D_L^{VVI}; x) \qquad \max_{\mathbb{R}^p_+ \setminus \{0\}} \bigcup_{\Lambda \in \mathbb{R}^{p \times m}} \min_{\mathbb{R}^p_+ \setminus \{0\}} \Big\{ \bigcup_{q \in \mathbb{R}^m_+} \{-\Lambda q\} - F(x)^T x + \bigcup_{y \in \mathbb{R}^n} \{F(x)^T y - \Lambda g(y)\} \Big\}$$

and

$$(D_{FL}^{VVI};x) \qquad \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{(T,\Lambda)\in\mathbb{R}^{p\times n}\times\mathbb{R}^{p\times m}} \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ \bigcup_{q\in\mathbb{R}^m_+} \{-\Lambda q\} - F(x)^T x$$
$$+ \bigcup_{y\in\mathbb{R}^n} \{(F(x)^T - T)y\} + \bigcup_{y\in\mathbb{R}^n} \{Ty - \Lambda g(y)\}\Big\}.$$

We introduce the following maps, for any $x \in K$, as follows

$$\gamma_L^{VVI}(x) := \bigcup_{\Lambda\in\mathbb{R}^{p\times m}} \widetilde{\Phi}_1^*(0,\Lambda;x),$$

where we define

$$\widetilde{\Phi}_1^*(0,\Lambda;x) = \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ \bigcup_{q\in\mathbb{R}^m_+} \{\Lambda q\} + F(x)^T x + \bigcup_{y\in\mathbb{R}^n} \{\Lambda g(y) - F(x)^T y\}\Big\}$$

and

$$\gamma_{FL}^{VVI}(x) := \bigcup_{(T,\Lambda)\in\mathbb{R}^{p\times n}\times\mathbb{R}^{p\times m}} \widetilde{\Phi}_3^*(0,T,\Lambda;x),$$

defining

$$\begin{aligned}\widetilde{\Phi}_3^*(0,T,\Lambda;x) \quad &= \quad \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ \bigcup_{q\in\mathbb{R}^m_+} \{\Lambda q\} + F(x)^T x + \bigcup_{y\in\mathbb{R}^n} \{(T - F(x)^T)y\} \\ &\quad + \bigcup_{y\in\mathbb{R}^n} \{\Lambda g(y) - Ty\}\Big\}.\end{aligned}$$

Like in the proof of Theorem 3.3, by applying the duality assertions in Subsection 3.1.3, for (D_L^{VO}) and (D_{FL}^{VO}), respectively, the following theorem can be verified.

Theorem 3.4 *Let for any $x \in K$ the problem $(P^{VVI};x)$ be stable with respect to $\widetilde{\Phi}_1(0,\cdot;x)$ and $\widetilde{\Phi}_3(0,\cdot;x)$, respectively. Then γ_L^{VVI} and γ_{FL}^{VVI} are gap functions for (VVI).*

The origin of these new gap functions for (VVI) justifies to call them as Lagrange gap function γ_L^{VVI} and Fenchel-Lagrange gap function γ_{FL}^{VVI}, respectively.

3.2.2 Gap functions via Fenchel duality

According to the results in Subsection 3.1.4, we can suggest further class of gap functions for (VVI). In this subsection, we restrict the construction of a gap function to the case of Fenchel duality. As mentioned before, for a fixed $x \in K$ we consider the following vector optimization problem relative to (VVI).

$$(P^{VVI};x) \qquad \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{F(x)^T(y-x) |\ y \in K\Big\}.$$

For a fixed $x \in K$, taking $F(x)^T(y-x)$ as the objective function, $(\widehat{D}_F^{VO})$ becomes

$$(\widehat{D}_F^{VVI};x) \qquad \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{t\in\mathbb{R}^n} \Big\{ \min_{\mathbb{R}^p_+\setminus\{0\}} [(F(x)^T(y-x) - (t^T y)_p |\ y \in \mathbb{R}^n] + (\min_{y\in K} t^T y)_p\Big\}.$$

We need the following auxiliary result.

Lemma 3.6 *Let $M \in \mathbb{R}^{p\times n}$. Then*

$$\min_{\mathbb{R}^p_+\setminus\{0\}} \{My|\ y\in\mathbb{R}^n\} = \begin{cases} \{My|\ y \in \mathbb{R}^n\}, & \textit{if } \exists \mu \in \textit{int } \mathbb{R}^p_+ \textit{ such that } \mu^T M = 0^T, \\ \varnothing, & \textit{otherwise.}\end{cases} .$$

Proof: Let $M \in \mathbb{R}^{p \times n}$ be fixed and $\bar{y} \in \mathbb{R}^n$. According to Theorem 11.20 in [50], $M\bar{y} \in \min\limits_{\mathbb{R}^p_+ \setminus \{0\}} \{My |\ y \in \mathbb{R}^n\}$ if and only if $\exists \mu \in \text{int } \mathbb{R}^p_+$ such that

$$\mu^T M\bar{y} \leq \mu^T My, \ \forall y \in \mathbb{R}^n. \tag{3.18}$$

As

$$\inf_{y \in \mathbb{R}^n} \mu^T My = \begin{cases} 0, & \mu^T M = 0^T, \\ -\infty, & \text{otherwise}, \end{cases}$$

$M\bar{y} \in \min\limits_{\mathbb{R}^p_+ \setminus \{0\}} \{My |\ y \in \mathbb{R}^n\}$ if and only if $\exists \mu \in \text{int } \mathbb{R}^p_+$ such that

$$\mu^T M = 0^T. \tag{3.19}$$

This means that under the above assumption each $\bar{y} \in \mathbb{R}^n$ is a solution to (3.18).□

Let $C := [t, ..., t] \in \mathbb{R}^{n \times p}$ and for a fixed $x \in K$ the set $N(x)$ be defined by

$$N(x) := \{t \in \mathbb{R}^n |\ \exists \mu \in \text{int } \mathbb{R}^p_+ \ \text{ such that } \ (F(x) - C)\mu = 0\}.$$

In view of Lemma 3.6, one has

$$(\widehat{D}^{VVI}_F; x) \qquad \max_{\mathbb{R}^p_+ \setminus \{0\}} \bigcup_{t \in N(x)} \Big\{ -F(x)^T x + \{(F(x) - C)^T y |\ y \in \mathbb{R}^n\} + (\min_{y \in K} t^T y)_p \Big\}.$$

Let us introduce for $x \in K$ the following map

$$\widetilde{\gamma}^{VVI}_F(x) := F(x)^T x + \bigcup_{t \in N(x)} \Big[\{(C - F(x))^T y |\ y \in \mathbb{R}^n\} - (\min_{y \in K} t^T y)_p \Big].$$

Theorem 3.5 *Let for any $x \in K$ the set $\min\limits_{\mathbb{R}^p_+ \setminus \{0\}} \{F(x)^T y |\ y \in K\}$ be externally stable. Then $\widetilde{\gamma}^{VVI}_F$ is a gap function for (VVI).*

Proof:

(i) Let $x \in K$ be fixed. As the set $\min\limits_{\mathbb{R}^p_+ \setminus \{0\}} \{F(x)^T y |\ y \in K\}$ is externally stable, by Proposition 3.26 the problem $(P^{VVI}; x)$ is stable. Taking $F(x)^T(y - x)$ instead of $f(y)$ in $f^*_p(t)$, by Lemma 3.6, we have

$$\begin{aligned} f^*_p(t) &= \max_{\mathbb{R}^p_+ \setminus \{0\}} \{(t^T y)_p - F(x)^T (y - x) |\ y \in \mathbb{R}^n\} \\ &= F(x)^T x - \min_{\mathbb{R}^p_+ \setminus \{0\}} \{(F(x) - C)^T y |\ y \in \mathbb{R}^n\} \\ &= F(x)^T x - \{(F(x) - C)^T y |\ y \in \mathbb{R}^n\}, \end{aligned}$$

where $C = [t, ..., t] \in \mathbb{R}^{n \times p}$ and $t \in N(x)$. Then (3.14) is equivalent to

$$0 \in -F(x)^T x + \{(F(x) - C)^T y |\ y \in \mathbb{R}^n\} + (\min_{y \in K} t^T y)_p. \tag{3.20}$$

Let $\bar{x} \in K$ be a solution to (VVI). By Proposition 3.21(i) and (3.20) it follows that $0 \in \widetilde{\gamma}^{VVI}_F(\bar{x})$. Let $\bar{x} \in K$ and $0 \in \widetilde{\gamma}^{VVI}_F(\bar{x})$. Then $\exists \bar{t} \in N(\bar{x})$ such that

$$0 \in F(\bar{x})^T \bar{x} + \{(\bar{C} - F(\bar{x}))^T y |\ y \in \mathbb{R}^n\} - (\min_{y \in K} \bar{t}^T y)_p,$$

where $\bar{C} = [\bar{t}, ..., \bar{t}] \in \mathbb{R}^{n \times p}$. Taking into account Proposition 3.21(ii) and (3.20), $\bar{x}$ is a solution to $(P^{VVI}; \bar{x})$. Consequently, $\bar{x}$ solves the problem (VVI).

(ii) Let $y \in K$. Choosing as $T := [t, ..., t]^T \in \mathbb{R}^{p\times n}$, by Proposition 3.11 and Proposition 3.20, it holds

$$F(y)^T(z-y)+\xi \underset{\mathbb{R}^p_+\setminus\{0\}}{\nleq} 0,\ \forall z \in K,\ \forall \xi \in f_p^*(t)-(\min_{y\in K} t^T y)_p,\ t\in N(y),$$

or, equivalently,

$$F(y)^T(z-y)+\xi \underset{\mathbb{R}^p_+\setminus\{0\}}{\nleq} 0,\ \forall z \in K,\ \forall \xi \in \widetilde{\gamma}_F^{VVI}(y).$$

Setting $z = y$, one has

$$\xi \underset{\mathbb{R}^p_+\setminus\{0\}}{\nleq} 0,\ \forall \xi \in \widetilde{\gamma}_F^{VVI}(y).$$

□

Remark 3.5 In the case $p = 1$, the problem (VVI) reduces to the scalar variational inequality problem of finding $x \in K$ such that

$$(VI) \qquad F(x)^T(x-y) \geq 0,\ y \in K,$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a vector-valued function. Let $x \in K$ be fixed. By the definition of the set $N(x)$, there exists $\mu > 0$ such that $(F(x)-t)\mu = 0$. Therefore it holds $F(x) = t$. Consequently, the gap function for the variational inequality becomes

$$\begin{aligned}\gamma_F^{VI}(x) &= F(x)^T x + \max_{y\in K}(-F(x)^T y)\\ &= \max_{y\in K} F(x)^T(x-y),\end{aligned}$$

which coincides with Auslender's gap function (see [2] and [8]).

Example 3.2 Let $F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ be a constant matrix and

$$K = \{(x_1,x_2)^T \in \mathbb{R}^2 |\ 0 \leq x_i \leq 1,\ x_i \in \mathbb{R},\ i=1,2\}.$$

We consider the vector variational inequality problem of finding $x \in K$ such that

$$(VVI_1) \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}(y-x) \underset{\mathbb{R}^2_+\setminus\{0\}}{\nleq} 0,\ \forall y \in K.$$

Let us describe $\widetilde{\gamma}_F^{VVI}$ for (VVI_1). Let $x = (x_1,x_2)^T \in \mathbb{R}^2$ be fixed. First we consider the set-valued map $W : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ given by (cf. $(\widehat{D}_F^{VVI}; x)$)

$$W(x_1,x_2) = \min_{\mathbb{R}^2_+\setminus\{0\}} \{F(x)^T(y-x)-(t^T y)_2 |\ y \in \mathbb{R}^2\}.$$

Then

$$\begin{aligned}W(x_1,x_2) &= \min_{\mathbb{R}^2_+\setminus\{0\}}\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_1-x_1 \\ y_2-x_2 \end{pmatrix} - \begin{pmatrix} t_1y_1+t_2y_2 \\ t_1y_1+t_2y_2 \end{pmatrix} \middle|\ (y_1,y_2)^T \in \mathbb{R}^2\right\}\\ &= \min_{\mathbb{R}^2_+\setminus\{0\}}\left\{ \begin{pmatrix} y_1-x_1-t_1y_1-t_2y_2 \\ y_2-x_2-t_1y_1-t_2y_2 \end{pmatrix} \middle|\ (y_1,y_2)^T \in \mathbb{R}^2 \right\}\\ &= \min_{\mathbb{R}^2_+\setminus\{0\}}\left\{ \begin{pmatrix} (1-t_1)y_1-t_2y_2 \\ -t_1y_1+(1-t_2)y_2 \end{pmatrix} \middle|\ (y_1,y_2)^T \in \mathbb{R}^2 \right\} - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.\end{aligned}$$

If $\exists \mu = (\mu_1, \mu_2)^T \in \text{int } \mathbb{R}^2_+$ such that $(\mu_1, \mu_2) \begin{pmatrix} 1-t_1 & -t_2 \\ -t_1 & 1-t_2 \end{pmatrix} = 0$, or, equivalently, $\begin{cases} (1-t_1)\mu_1 - t_1\mu_2 & = 0 \\ -t_2\mu_1 + (1-t_2)\mu_2 & = 0, \end{cases}$ then, by Lemma 3.6, it holds

$$W(x_1, x_2) = \left\{ \begin{pmatrix} (1-t_1)y_1 - t_2 y_2 \\ -t_1 y_1 + (1-t_2) y_2 \end{pmatrix} \Big| \, (y_1, y_2)^T \in \mathbb{R}^2 \right\} - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

As $(\mu_1, \mu_2) \in \text{int } \mathbb{R}^2_+$, it must to be $\begin{vmatrix} (1-t_1) & -t_1 \\ -t_2 & (1-t_2) \end{vmatrix} = 0$. As a consequence, one has

$$t_1 + t_2 = 1 \ \text{ and } \ t_2\mu_1 = t_1\mu_2.$$

Whence

$$\begin{aligned} \widetilde{\gamma}_F^{VVI_1}(x) &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \bigcup_{t \in N_1} \left[\left\{ \begin{pmatrix} t(y_2 - y_1) \\ (1-t)(y_1 - y_2) \end{pmatrix} \Big| \, (y_1, y_2)^T \in \mathbb{R}^2 \right\} \right. \\ &\quad - \left. \begin{pmatrix} \min\limits_{0 \le y_1 \le 1} (1-t)y_1 + \min\limits_{0 \le y_2 \le 1} t y_2 \\ \min\limits_{0 \le y_1 \le 1} (1-t)y_1 + \min\limits_{0 \le y_2 \le 1} t y_2 \end{pmatrix} \right], \end{aligned}$$

where the set N_1 is defined by

$$N_1 := \{ t \in \mathbb{R} | \ \exists \mu \in \text{int } \mathbb{R}^2_+ \ \text{ such that } \ (1-t)\mu_1 = t\mu_2 \}.$$

Moreover, as $N_1 = (0, 1)$, we conclude that

$$\widetilde{\gamma}_F^{VVI_1}(x) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \bigcup_{t \in (0,1)} \left\{ \begin{pmatrix} ty \\ (t-1)y \end{pmatrix} \Big| \, y \in \mathbb{R} \right\}.$$

Chapter 4

Variational principles for vector equilibrium problems

In this chapter we focus on the construction of set-valued mappings on the basis of the so-called Fenchel duality, which allow us to propose some new variational principles for vector equilibrium problems. First we present some notions and results regarding conjugate duality in vector optimization based on weak orderings, which are due to Tanino [84] and Song [78]. Under certain assumptions, in order to characterize the solutions for vector equilibrium problems, set-valued mappings on the basis of Fenchel duality depending on the data, but not on the solution sets of vector equilibrium problems, are introduced. In conclusion, by applying these results, we investigate gap functions for the so-called weak vector variational inequalities.

4.1 Preliminaries

Let Y be a real topological vector space partially ordered by a pointed closed convex cone C with a nonempty interior int$C$ in Y. For any $\xi, \mu \in Y$, we use the following ordering relations:

$$\begin{aligned} \xi < \mu \quad &\Leftrightarrow \quad \mu - \xi \in \text{int } C; \\ \xi \not< \mu \quad &\Leftrightarrow \quad \mu - \xi \notin \text{int } C. \end{aligned}$$

The relations $>$ and $\not>$ are defined similarly. Let us now introduce the weak maximum and weak supremum of a set Z in the space $\overline{Y}$ induced by adding to Y two imaginary points $+\infty$ and $-\infty$. We suppose that $-\infty < y < +\infty$ for $y \in Y$. Moreover, we use the following conventions

$$(\pm\infty) + y = y + (\pm\infty) = \pm\infty \text{ for all } y \in Y, \quad (\pm\infty) + (\pm\infty) = \pm\infty,$$

$$\lambda(\pm\infty) = \pm\infty \text{ for } \lambda > 0 \text{ and } \lambda(\pm\infty) = \mp\infty \text{ for } \lambda < 0.$$

The sum $+\infty + (-\infty)$ is not considered, since we can avoid it.

For a given set $Z \subseteq \overline{Y}$, we define *the set* $A(Z)$ *of all points above* Z and *the set* $B(Z)$ *of all points below* Z by

$$A(Z) = \left\{ y \in \overline{Y} \mid y > y' \text{ for some } y' \in Z \right\}$$

and

$$B(Z) = \left\{ y \in \overline{Y} \mid y < y' \text{ for some } y' \in Z \right\},$$

respectively. Clearly $A(Z) \subseteq Y \cup \{+\infty\}$ and $B(Z) \subseteq Y \cup \{-\infty\}$.

Definition 4.1 *A point $\widehat{y} \in \overline{Y}$ is said to be a weak maximal point of $Z \subseteq \overline{Y}$ if $\widehat{y} \in Z$ and $\widehat{y} \notin B(Z)$, that is, if $\widehat{y} \in Z$ and there is no $y \in Z$ such that $\widehat{y} < y$.*

The set of all weak maximal points of Z is called the weak maximum of Z and is denoted by $\operatorname{WMax} Z$.

Definition 4.2 *A point $\widehat{y} \in \overline{Y}$ is said to be a weak supremal point of $Z \subseteq \overline{Y}$ if $\widehat{y} \notin B(Z)$ and $B(\{\widehat{y}\}) \subseteq B(Z)$, that is, if there is no $y \in Z$ such that $\widehat{y} < y$ and if the relation $y' < \widehat{y}$ implies the existence of some $y \in Z$ such that $y' < y$.*

The set of all weak supremal points of Z is called the weak supremum of Z and is denoted by $\operatorname{WSup} Z$. Remark that $\operatorname{WMax} Z = Z \cap \operatorname{WSup} Z$. Moreover it holds $-\operatorname{WMax}(-Z) = \operatorname{WMin} Z$ and $-\operatorname{WSup}(-Z) = \operatorname{WInf} Z$, where a weak minimum and a weak infimum can be defined analogously to the maximum and supremum, respectively. For more properties of these sets we refer to [83] and [84].

Now, we give some definitions of the conjugate mapping and the subgradient of a set-valued mapping based on the weak supremum and the weak maximum of a set. Let X be another real topological vector space and let $\mathcal{L}(\mathrm{X}, \mathrm{Y})$ be the space of all linear continuous operators from X to Y. For $x \in X$ and $l \in \mathcal{L}(\mathrm{X}, \mathrm{Y})$, $\langle l, x \rangle$ denotes the value of l at x.

Definition 4.3 *(see [84]) Let $G : X \rightrightarrows \overline{Y}$ be a set-valued mapping.*

(i) A set-valued mapping $G^* : \mathcal{L}(\mathrm{X}, \mathrm{Y}) \rightrightarrows \overline{Y}$ defined by

$$G^*(T) = WSup \bigcup_{x \in X} \Big[\langle T, x \rangle - G(x)\Big], \ \textit{for } T \in \mathcal{L}(\mathrm{X}, \mathrm{Y})$$

is called the conjugate mapping of G.

(ii) A set-valued mapping $G^{**} : X \rightrightarrows \overline{Y}$ defined by

$$G^{**}(x) = WSup \bigcup_{T \in \mathcal{L}(\mathrm{X}, \mathrm{Y})} \Big[\langle T, x \rangle - G^*(T)\Big], \ \textit{for } x \in X$$

is called the biconjugate mapping of G.

(iii) $T \in \mathcal{L}(\mathrm{X}, \mathrm{Y})$ is said to be a subgradient of the set-valued mapping G at $(x_0; y_0)$ if $y_0 \in G(x_0)$ and

$$\langle T, x_0 \rangle - y_0 \in WMax \bigcup_{x \in X} \Big[\langle T, x \rangle - G(x)\Big].$$

The set of all subgradients of G at $(x_0; y_0)$ is called the subdifferential of G at $(x_0; y_0)$ and is denoted by $\partial G(x_0; y_0)$. If $\partial G(x_0; y_0) \neq \varnothing$ for every $y_0 \in G(x_0)$, then G is said to be subdifferentiable at x_0.

We present now the conjugate duality theory for vector optimization developed by Tanino [84]. Let X and Y be real topological vector spaces. Assume that $\overline{Y}$ is the extended space of Y and h is a function from X to $Y \cup \{+\infty\}$. We consider the vector optimization problem

$$(P) \qquad \operatorname{WInf}\{h(x) |\, x \in X\}.$$

Let U be another real topological vector space, the so-called perturbation space. Let $\Phi : X \times U \to Y \cup \{+\infty\}$ be a perturbation function such that

$$\Phi(x, 0) = h(x), \ \forall x \in X.$$

Then the perturbed problem considered here is

$$(P_u) \qquad \mathrm{WInf}\Big\{\Phi(x,u)|\ x \in X\Big\},$$

where $u \in U$ is a perturbation variable.

Definition 4.4 *The set-valued mapping $W : U \rightrightarrows Y$ defined by*

$$W(u) = WInf(P_u) = WInf\Big\{\Phi(x,u)|\ x \in X\Big\}$$

is called the value mapping of (P).

It is clear that $\mathrm{WInf}(P) = W(0)$. The conjugate mapping of Φ is

$$\Phi^*(T,\Lambda) = \mathrm{WSup}\Big\{\langle T,x\rangle + \langle \Lambda,u\rangle - \Phi(x,u)|\ x \in X,\ u \in U\Big\}$$

for $T \in \mathcal{L}(\mathrm{X},\mathrm{Y})$ and $\Lambda \in \mathcal{L}(\mathrm{U},\mathrm{Y})$. Then

$$\begin{aligned} -\Phi^*(0,\Lambda) &= -\mathrm{WSup}\Big\{\langle \Lambda,u\rangle - \Phi(x,u)|\ x \in X,\ u \in U\Big\} \\ &= \mathrm{WInf}\Big\{\Phi(x,u) - \langle \Lambda,u\rangle|\ x \in X,\ u \in U\Big\}. \end{aligned}$$

A dual problem to (P) can be defined as follows

$$(D) \qquad \mathrm{WSup} \bigcup_{\Lambda \in \mathcal{L}(U,Y)} \Big[-\Phi^*(0,\Lambda)\Big].$$

Since $\Lambda \mapsto -\Phi^*(0,\Lambda)$ is a set-valued mapping, the dual problem is not an usual vector optimization problem.

Proposition 4.1 *[84, Proposition 5.1] (Weak duality)*
For any $x \in X$ and $\Lambda \in \mathcal{L}(U,Y)$ it holds

$$\Phi(x,0) \notin B\Big(-\Phi^*(0,\Lambda)\Big).$$

Definition 4.5 *[84, Definition 5.2]*
The primal problem (P) is said to be stable if the value mapping W is subdifferentiable at 0.

Theorem 4.1 *[84, Theorem 5.1], [78, Theorem 3.1]*
If the problem (P) is stable, then

$$WInf(P) = WSup(D) = WMax(D).$$

Let us notice that the conjugate duality for set-valued vector optimization problems has been investigated by Song [78]. Some stability criteria in connection with this duality can be found in [78], [79] and [84]. As mentioned in Section 3.1, the stability assertion for (P) in [84] deals with the convexity of the perturbation function Φ and a regularity condition.

4.2 Fenchel duality in vector optimization

This section is devoted to the presentation of a special perturbation function which allows us to state the so-called Fenchel duality. Let the spaces X and Y be the same as in Section 4.1. Assume that h is a function from X to $Y \cup \{+\infty\}$ and $G \subseteq X$. We consider the constrained vector optimization problem

$$(P_c) \qquad \text{WInf}\{h(x) |\, x \in G\}.$$

Let us choose the perturbation space $U = X$ and introduce the perturbation function Φ from $X \times X$ to $Y \cup \{+\infty\}$ defined by

$$\Phi(x,u) = \begin{cases} h(x+u), & \text{if } x \in G; \\ +\infty, & \text{otherwise.} \end{cases}$$

Then the perturbed problem turns out to be

$$(P_u) \qquad \text{WInf}\Big\{\Phi(x,u) |\ x \in X\Big\}.$$

To verify the next assertion we use the following trivial properties.

Remark 4.1 Let $g : X \to Y$ be a function and $Z \subseteq X$. The following assertions are true:

(i) For any $y \in Y$ it holds

$$\{g(x) + y |\ x \in Z\} = \{g(x) |\ x \in Z\} + y;$$

(ii) For any set $A \subseteq Y$ it holds

$$\bigcup_{x \in Z} [A + g(x)] = A + \bigcup_{x \in Z} \{g(x)\}.$$

Proposition 4.2 *Let $T \in \mathcal{L}(\mathrm{X},\mathrm{Y})$. Then*

$$\Phi^*(0,T) = \mathit{WSup}\Big\{h^*(T) + \{-\langle T, x\rangle |\ x \in G\}\Big\}.$$

Proof: Let $T \in \mathcal{L}(\mathrm{X},\mathrm{Y})$ be fixed. By definition

$$\begin{aligned} \Phi^*(0,T) &= \text{WSup}\{\langle T,u\rangle - \Phi(x,u) |\ x \in X, u \in X\} \\ &= \text{WSup}\{\langle T,u\rangle - h(x+u) |\ x \in G, u \in X\}. \end{aligned}$$

Setting $\bar{u} := x + u$, by applying Remark 4.1 and Proposition 2.6 in [84], we obtain that

$$\begin{aligned} \Phi^*(0,T) &= \text{WSup}\Big\{\{\langle T,\bar{u}\rangle - h(\bar{u}) |\ \bar{u} \in X\} + \{-\langle T,x\rangle |\ x \in G\}\Big\} \\ &= \text{WSup}\Big\{\ \text{WSup}\{\langle T,\bar{u}\rangle - h(\bar{u}) |\ \bar{u} \in X\} + \{-\langle T,x\rangle |\ x \in G\}\Big\} \\ &= \text{WSup}\Big\{h^*(T) + \{-\langle T,x\rangle |\ x \in G\}\Big\}. \end{aligned}$$

□

Consequently, we can state the dual problem as follows

$$(D_c) \qquad \text{WSup} \bigcup_{T \in \mathcal{L}(\mathrm{X},\mathrm{Y})} \text{WInf}\Big\{ - h^*(T) + \{\langle T,x\rangle |\ x \in G\}\Big\}.$$

Proposition 4.3 *(Weak duality)*
For any $x \in G$ and $T \in \mathcal{L}(\mathrm{X},\mathrm{Y})$ it holds

$$h(x) \notin B\Big(- \Phi^*(0,T)\Big).$$

Proposition 4.4 *If the primal problem is stable, then*

$$\mathit{WInf}(P_c) = \mathit{WSup}(D_c) = \mathit{WMax}(D_c).$$

Remark 4.2 According to Proposition 2.6 in [84], we can use for $\Phi^*(0,T)$ the following equivalent formulations

$$\begin{aligned}
\Phi^*(0,T) &= \mathrm{WSup}\Big\{\{\langle T,u\rangle - h(u)|\ u \in X\} + \{-\langle T,x\rangle|\ x \in G\}\Big\}\\
&= \mathrm{WSup}\Big\{h^*(T) + \{-\langle T,x\rangle|\ x \in G\}\Big\}\\
&= \mathrm{WSup}\Big\{h^*(T) + \mathrm{WSup}\{-\langle T,x\rangle|\ x \in G\}\Big\}.
\end{aligned}$$

The following result deals with the stability of the problem (P_c), if the objective function has the form $h(x) = \langle C,x\rangle,\ C \in \mathcal{L}(\mathrm{X},\mathrm{Y})$.

Proposition 4.5 *Let $C \in \mathcal{L}(\mathrm{X},\mathrm{Y})$ and the objective function $h : X \rightrightarrows Y$ be defined by $h(x) = \langle C,x\rangle$. Then the problem (P_c) is stable.*

Proof: Let $W : X \rightrightarrows Y$ be the value mapping defined by

$$\begin{aligned}
W(y) &= \mathrm{WInf}\{\Phi(x,y)|\ x \in X\}\\
&= \mathrm{WInf}\{\langle C,x+y\rangle|\ x \in G\} = \langle C,y\rangle + \mathrm{WInf}\{\langle C,x\rangle|\ x \in G\}.
\end{aligned}$$

Let $z \in W(0)$ be fixed. Then $\partial W(0;z) \neq 0$ means that $\exists T \in \mathcal{L}(\mathrm{X},\mathrm{Y})$ such that (see Definition 4.3(iii))

$$-z \in \mathrm{WMax} \bigcup_{y\in X} [\langle T,y\rangle - W(y)]. \tag{4.1}$$

One can notice that

$$\mathrm{WMax} \bigcup_{y\in X} [\langle T,y\rangle - W(y)] \subseteq \mathrm{WSup} \bigcup_{y\in X} [\langle T,y\rangle - W(y)] = W^*(T).$$

Let us show that (4.1) holds. By applying Remark 4.1, we have

$$\begin{aligned}
W^*(T) &= \mathrm{WSup} \bigcup_{y\in X} [\langle T,y\rangle - W(y)]\\
&= \mathrm{WSup} \bigcup_{y\in X} [\langle T,y\rangle - \langle C,y\rangle - \mathrm{WInf}\{\langle C,x\rangle|\ x \in G\}]\\
&= \mathrm{WSup}\Big\{ - \mathrm{WInf}\{\langle C,x\rangle|\ x \in G\} + \{\langle T-C,y\rangle|\ y \in X\}\Big\}.
\end{aligned}$$

Taking $T = C$, in view of Corollary 2.3 in [84], one has

$$\begin{aligned}
W^*(C) &= \mathrm{WSup}\,\mathrm{WSup}\{-\langle C,x\rangle|\ x \in G\}\\
&= \mathrm{WSup}\{-\langle C,x\rangle|\ x \in G\} = -\,\mathrm{WInf}\{\langle C,x\rangle|\ x \in G\} = -W(0).
\end{aligned}$$

This means that $\forall z \in W(0)$, it holds $-z \in W^*(C)$. On the other hand, as $\langle C,0\rangle - z \in \bigcup_{y\in X} [\langle C,y\rangle - W(y)]$, it follows that

$$-z \in \mathrm{WMax} \bigcup_{y\in X} [\langle C,y\rangle - W(y)].$$

In other words, W is subdifferentiable at 0. $\square$

4.3 Variational principles for (VEP)

Let X and Y be real topological vector spaces. Assume that K is a nonempty convex set in X and $f : K \times K \to Y$ is a bifunction such that $f(x,x) = 0,\ \forall x \in K$. We consider the vector equilibrium problem which consists in finding $x \in K$ such that

$$(VEP) \qquad f(x,y) \not< 0,\ \forall y \in K.$$

By K^p we denote the solution set of (VEP). We say that a variational principle (see [5]) holds for (VEP) if there exists a set-valued map $G : K \rightrightarrows Y$, depending on the data of (VEP) but not on its solution set such that the solution set of (VEP) coincides with the solution set of the following vector optimization problem

$$(P_G) \qquad \text{WMin} \bigcup_{x \in K} G(x).$$

(P_G) is nothing else than the problem of finding $x_0 \in K$ such that

$$G(x_0) \cap \text{WMin} \bigcup_{x \in K} G(x) \neq \varnothing.$$

Remark that variational principles for (VEP) have been investigated in [5] and [6]. This section aims to show how a similar approach can be extended from the scalar case (see Section 2.2) to vector equilibrium problems. As in the scalar case, we use the Fenchel duality for vector optimization discussed in Section 4.2.

It is clear that $\bar{x} \in K$ is a solution to (VEP) if and only if 0 is a weak minimal point of the set $\{f(\bar{x},y)|\ y \in K\}$. Let us consider for a fixed $x \in K$ the following vector optimization problem

$$(P^{VEP};x) \qquad \text{WInf}\Big\{f(x,y)|\ y \in K\Big\}.$$

Redefining

$$\widetilde{f}(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in K \times K; \\ +\infty, & \text{otherwise;} \end{cases}$$

and setting it in (D_c), the corresponding Fenchel dual turns out to be

$$\begin{aligned}(D^{VEP};x)\text{WSup} & \bigcup_{T \in \mathcal{L}(\mathrm{X},\mathrm{Y})} \text{WInf}\Big\{\{\widetilde{f}(x,y) - \langle T,y\rangle|\ y \in X\} + \{\langle T,y\rangle|\ y \in K\}\Big\} \\ = \quad \text{WSup} & \bigcup_{T \in \mathcal{L}(\mathrm{X},\mathrm{Y})} \text{WInf}\Big\{\{f(x,y) - \langle T,y\rangle|\ y \in K\} + \{\langle T,y\rangle|\ y \in K\}\Big\}.\end{aligned}$$

In view of Proposition 2.6 in [84] it also can be written as

$$(D^{VEP};x) \qquad \text{WSup} \bigcup_{T \in \mathcal{L}(\mathrm{X},\mathrm{Y})} \text{WInf}\Big\{-f_K^*(T;x) + \{\langle T,y\rangle|\ y \in K\}\Big\},$$

where $f_K^*(T;x)$ is defined by $f_K^*(T;x) = \text{WSup}\{\langle T,y\rangle - f(x,y)|\ y \in K\}$. For any $x \in K$, we introduce the following mapping

$$\gamma_p(x) := \bigcup_{T \in \mathcal{L}(\mathrm{X},\mathrm{Y})} \Big[-\Phi_p^*(0,T;x)\Big],$$

where $\Phi_p^*(0,T;x) = \mathrm{WSup}\Big\{f_K^*(T;x) + \{-\langle T,y\rangle |\ y \in K\}\Big\}$. Consequently, we obtain that

$$\begin{aligned}\gamma_p(x) &= \bigcup_{T\in\mathcal{L}(\mathrm{X},\mathrm{Y})} \Big[-\mathrm{WSup}\Big\{f_K^*(T;x) + \{-\langle T,y\rangle |\ y\in K\}\Big\}\Big] \\ &= \bigcup_{T\in\mathcal{L}(\mathrm{X},\mathrm{Y})} \mathrm{WInf}\Big\{-f_K^*(T;x) + \{\langle T,y\rangle |\ y\in K\}\Big\} \\ &= \bigcup_{T\in\mathcal{L}(\mathrm{X},\mathrm{Y})} \mathrm{WInf}\Big\{\{f(x,y) - \langle T,y\rangle |\ y\in K\} + \{\langle T,y\rangle |\ y\in K\}\Big\}.\end{aligned}$$

We consider the following optimization problem

$$(P_\gamma) \qquad \mathrm{WSup}\bigcup_{x\in K}\gamma_p(x).$$

Lemma 4.1 *For any $x \in K$, if $z \in \gamma_p(x)$, then $z \not> 0$.*

Proof: Let $x \in K$ be fixed and

$$z \in \gamma_p(x) = \bigcup_{T\in\mathcal{L}(\mathrm{X},\mathrm{Y})} \mathrm{WInf}\Big\{\{f(x,y) - \langle T,y\rangle |\ y\in K\} + \{\langle T,y\rangle |\ y\in K\}\Big\}.$$

Whence, $\exists \overline{T} \in \mathcal{L}(\mathrm{X},\mathrm{Y})$ such that

$$z \in \mathrm{WInf}\Big\{\{f(x,y) - \langle \overline{T},y\rangle |\ y\in K\} + \{\langle \overline{T},y\rangle |\ y\in K\}\Big\}.$$

We assume that $z > 0$. This relation can be rewritten as

$$z > f(x,x) - \langle \overline{T},x\rangle + \langle \overline{T},x\rangle,$$

and this leads to a contradiction. □

Theorem 4.2 *Let the problem $(P^{VEP};x)$ be stable for each $x \in K^p$. Then*

(i) $\bar{x} \in K$ is a solution to (VEP) if and only if $0 \in \gamma_p(\bar{x})$;

(ii) $K^p \subseteq K_\gamma^p$, where K_γ^p denotes the solution set of (P_γ).

Proof:

(i) If $\bar{x} \in K$ is a solution to (VEP), then by Proposition 4.4 it holds

$$0 \in \mathrm{WInf}(P^{VEP};\bar{x}) = \mathrm{WMax}(D^{VEP};\bar{x}).$$

Whence

$$0 \in \mathrm{WMax}\bigcup_{T\in\mathcal{L}(\mathrm{X},\mathrm{Y})} \mathrm{WInf}\Big\{-f_K^*(T,\bar{x}) + \{\langle T,y\rangle |\ y\in K\}\Big\}.$$

Consequently, $0 \in \gamma_p(\bar{x})$. Let us now assume that

$$\begin{aligned}0 \in \gamma_p(\bar{x}) &= \bigcup_{T\in\mathcal{L}(\mathrm{X},\mathrm{Y})} \mathrm{WInf}\Big\{-f_K^*(T,\bar{x}) + \langle T,y\rangle |\ y\in K\}\Big\} \\ &= \bigcup_{T\in\mathcal{L}(\mathrm{X},\mathrm{Y})} \mathrm{WInf}\Big\{\{f(\bar{x},y) - \langle T,y\rangle |\ y\in K\} \\ &+ \ \{\langle T,y\rangle |\ y\in K\}\Big\}.\end{aligned}$$

Therefore, $\exists \overline{T} \in \mathcal{L}(\mathrm{X},\mathrm{Y})$ such that

$$0 \in \mathrm{WInf}\Big\{\{f(\bar{x},y) - \langle \overline{T}, y\rangle |\ y \in K\} + \{\langle \overline{T}, y\rangle |\ y \in K\}\Big\}.$$

Assume that $0 \notin \mathrm{WInf}\{f(\bar{x},y)|\ y \in K\}$. Then it is clear that $0 \notin \mathrm{WMin}\{f(\bar{x},y)|\ y \in K\}$. Hence $\exists y' \in K$ such that $f(\bar{x},y') < 0$ or, equivalently $f(\bar{x},y') - \langle \overline{T}, y'\rangle + \langle \overline{T}, y'\rangle < 0$, which leads to a contradiction.

(ii) Let $\bar{x} \in K^p$. In view of (i), we have $0 \in \gamma_p(\bar{x})$. On the other hand, by Lemma 4.1, for any $x \in K$, if $z \in \gamma_p(x)$, then $z \not> 0$. Therefore, from $z \in \bigcup\limits_{x \in K} \gamma_p(x)$ follows $z \not> 0$. This means that

$$0 \in \mathrm{WMax} \bigcup_{x \in K} \gamma_p(x) \subseteq \mathrm{WSup} \bigcup_{x \in K} \gamma_p(x).$$

Whence $\bar{x} \in K^p_\gamma$. □

Remark 4.3 Taking instead of f the bifunction $\widetilde{f} : X \times X \to Y \cup \{+\infty\}$, the mapping γ_p can be rewritten as

$$\widetilde{\gamma}_p(x) = \bigcup_{T \in \mathcal{L}(\mathrm{X},\mathrm{Y})} \mathrm{WInf}\Big\{\{\widetilde{f}(x,y) - \langle T, y\rangle |\ y \in X\} + \{\langle T, y\rangle |\ y \in K\}\Big\}.$$

One can easy verify that Lemma 4.1 and Theorem 4.2 remain true in this case. This results will be used later for applications.

Remark 4.4 Let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^p$. Then a linear continuous operator $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$ can be identified with a $p \times n$ matrix. Moreover, let us assume that $p = 1$. Then for a given set $Z \subseteq \mathbb{R}$, we have (cf. [83])

$\widehat{y} \in \mathrm{WSup}\, Z$ if and only if $\widehat{y} > y$, $\forall y \in Z$ and if $y' < \widehat{y}$, then $\exists y \in Z$ such that $y' < y$.

In other words, $\mathrm{WSup}\, Z$ is reduced to the usual concept of the supremum of a set Z in $\mathbb{R}$. Assume that $\varphi : X \times X \to \mathbb{R} \cup \{+\infty\}$ is a bifunction satisfying $\varphi(x,x) = 0$, $\forall x \in K$. We can consider the equilibrium problem which consists in finding $x \in K$ such that

$$(EP) \qquad \varphi(x,y) \geq 0,\ \forall y \in K,$$

which is special case of (VEP). Taking φ instead of $\widetilde{f}$ in $(D^{VEP}; x)$, the dual becomes

$$\begin{aligned}(D^{EP}; x) \qquad & \sup_{T \in \mathbb{R}^{1 \times n}} \inf \Big\{\{\varphi(x,y) - Ty|\ y \in X\} + \{Ty|\ y \in K\}\Big\} \\ = \quad & \sup_{T \in \mathbb{R}^{1 \times n}} \Big\{\inf_{y \in X}\{\varphi(x,y) - Ty\} + \inf_{y \in K} Ty\Big\} \\ = \quad & \sup_{T \in \mathbb{R}^{1 \times n}} \Big\{-\varphi^*_y(x,T) + \inf_{y \in K} Ty\Big\},\end{aligned}$$

where $\varphi^*_y(x,T) := \sup\limits_{y \in X}\{Ty - \varphi(x,y)\}$ is the conjugate function of f with respect to the variable y for a fixed x. In this case, we can define the gap function for (EP) as follows:

$$\gamma^{EP}_F(x) := -v(D^{EP}; x) = \inf_{T \in \mathbb{R}^{1 \times n}} \Big\{\varphi^*_y(x,T) + \sup_{y \in K}[-Ty]\Big\},$$

where $v(D^{EP};x)$ is the optimal objective value of $(D^{EP};x)$. This is nothing else that the gap function introduced in Section 2.2.

Example 4.1 Let $u : X \to Y \cup \{+\infty\}$ be a given function. Let us define the bifunction $\widetilde{f} : \text{dom } u \times X \to Y \cup \{+\infty\}$ as $\widetilde{f}(x,y) = u(y) - u(x)$, where $\text{dom } u = \{x \in X |\ u(x) \in Y\}$. We assume that $K \times K \subseteq \text{dom } \widetilde{f}$. Then (VEP) is reduced to the following vector optimization problem of finding $x \in K$ such that

$$(\widetilde{P}_u) \qquad \widetilde{f}(x,y) = u(y) - u(x) \nless 0,\ \forall y \in K.$$

For any $x \in K$, $\widetilde{\gamma}_p$ turns out to be

$$\widetilde{\gamma}_p(x) = -u(x) + \bigcup_{T \in \mathcal{L}(\mathrm{X},\mathrm{Y})} \text{WInf}\Big\{\{u(y) - \langle T, y\rangle |\ y \in X\} + \{\langle T, y\rangle |\ y \in K\}\Big\}.$$

Assuming the stability of $(\widetilde{P}_u)$, by Proposition 4.4 it holds

$$\text{WInf}(\widetilde{P}_u) = \text{WSup}(\widetilde{D}_u) = \text{WMax}(\widetilde{D}_u), \tag{4.2}$$

where $(\widetilde{D}_u)$ is the Fenchel dual problem to $(\widetilde{P}_u)$.

Let $\bar{x} \in K$ be a solution to $(\widetilde{P}_u)$. From (4.2) follows

$$u(\bar{x}) \in \bigcup_{T \in \mathcal{L}(\mathrm{X},\mathrm{Y})} \text{WInf}\Big\{\{u(y) - \langle T, y\rangle |\ y \in X\} + \{\langle T, y\rangle |\ y \in K\}\Big\}.$$

In other words $0 \in \widetilde{\gamma}_p(\bar{x})$. The inverse implication follows analogously (see the proof of Theorem 4.2). On the other hand, by Proposition 4.4 and Proposition 2.6 in [84], one has $\text{WSup} \bigcup\limits_{x \in K} \widetilde{\gamma}_p(x) = \{0\}$. If $\bar{x} \in K$ solves $(\widetilde{P}_u)$, then as shown before $0 \in \widetilde{\gamma}_p(\bar{x})$. This means that $K^p \subseteq K^p_\gamma$. In other words, the assertions of Theorem 4.2 are fulfilled.

Example 4.2 (see [79]) Let $X = \mathbb{R},\ Y = \mathbb{R}^2$ and $C = \mathbb{R}^2_+$. Let the vector-valued function $\varphi_1 : \mathbb{R} \to \mathbb{R}^2 \cup \{+\infty\}$ be given by

$$\varphi_1(x) = \begin{cases} (x, 0), & \text{if } x \in [0,1], \\ +\infty, & \text{otherwise.} \end{cases}$$

Introducing the bifunction $f_1 : \mathbb{R}^2 \to \mathbb{R}^2 \cup \{+\infty\}$ as

$$f_1(x,y) = \begin{cases} \varphi_1(y) - \varphi_1(x), & \text{if } (x,y)^T \in [0,1] \times [0,1], \\ +\infty, & \text{otherwise,} \end{cases}$$

we consider the vector equilibrium problem of finding $x \in K = [0,1]$ such that

$$(VEP_1) \qquad f_1(x,y) = \varphi_1(y) - \varphi_1(x) \nless 0,\ \forall y \in K.$$

According to γ_p, we have

$$\gamma_{p_1}(x) = \bigcup_{T \in \mathcal{L}(\mathbb{R},\mathbb{R}^2)} \text{WInf}\Big\{\{\varphi_1(y) - \varphi_1(x) - \langle T, y\rangle |\ y \in K\} + \{\langle T, y\rangle |\ y \in K\}\Big\}.$$

This can be written as (see Remark 4.2)

$$\begin{aligned} \gamma_{p_1}(x) &= -\varphi_1(x) - \bigcup_{T \in \mathcal{L}(\mathbb{R},\mathbb{R}^2)} \text{WSup}\Big\{\{\langle T, y\rangle - \varphi_1(y) |\ y \in K\} + \{-\langle T, y\rangle |\ y \in K\}\Big\} \\ &= -\varphi_1(x) - \bigcup_{T \in \mathcal{L}(\mathbb{R},\mathbb{R}^2)} \text{WSup}\Big\{ \text{WSup}\{\langle T, y\rangle - \varphi_1(y) |\ y \in K\} \\ &+ \quad \text{WSup}\{-\langle T, y\rangle |\ y \in K\}\Big\}. \end{aligned}$$

Notice that the linear continuous operator $T \in \mathcal{L}(\mathbb{R}, \mathbb{R}^2)$ has the form $T = (\alpha, \beta) \in \mathbb{R}^2$. Using the notations

$$\begin{aligned}
\psi_1(T) : &= \mathrm{WSup}\{\langle T, y\rangle - \varphi_1(y) |\ y \in K\} = \mathrm{WSup}\{(\alpha - 1, \beta)y |\ y \in [0,1]\}, \\
\psi_2(T) : &= \mathrm{WSup}\{-\langle T, y\rangle |\ y \in K\} = \mathrm{WSup}\{(-\alpha, -\beta)y |\ y \in [0,1]\},
\end{aligned}$$

let us calculate for any $T = (\alpha, \beta) \in \mathcal{L}(\mathbb{R}, \mathbb{R}^2)$ the sets $\psi_1(T), \psi_2(T)$ and $\mathrm{WSup}\{\psi_1(T)+\psi_2(T)\}$.

(i) If $\alpha \geq 1$ and $\beta \geq 0$, then

$$\psi_1(T) = \{(x,y)^T \in \mathbb{R}^2 |\ (x = \alpha - 1,\ y \leq \beta) \vee (y = \beta,\ x \leq \alpha - 1)\}$$

and

$$\psi_2(T) = \{(x,y)^T \in \mathbb{R}^2 |\ (x = 0,\ y \leq 0) \vee (y = 0,\ x \leq 0)\}.$$

Whence $\mathrm{WSup}\{\psi_1(T) + \psi_2(T)\} = \psi_1(T)$.

(ii) If $\alpha > 1$ and $\beta < 0$, then

$$\begin{aligned}
\psi_1(T) &= \{(x,y)^T \in \mathbb{R}^2 |\ (x = \alpha - 1,\ y \leq \beta) \vee (y = 0,\ x \leq 0) \\
&\vee\ (y = \frac{\beta}{\alpha - 1}x,\ 0 \leq x \leq \alpha - 1)\}, \\
\psi_2(T) &= \{(x,y)^T \in \mathbb{R}^2 |\ (x = 0,\ y \leq 0) \vee (y = -\beta,\ x \leq -\alpha) \\
&\vee\ (y = \frac{\beta}{\alpha}x,\ -\alpha \leq x \leq 0)\}.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\mathrm{WSup}\{\psi_1(T) &+ \psi_2(T)\} = \{(x,y)^T \in \mathbb{R}^2 |\ (x = \alpha - 1,\ y \leq \beta) \\
&\vee\ (y = -\beta,\ x \leq -\alpha) \vee (y = \frac{\beta}{\alpha}x,\ -\alpha \leq x \leq 0) \\
&\vee\ (y = \frac{\beta}{\alpha - 1}x,\ 0 \leq x \leq \alpha - 1)\}.
\end{aligned}$$

If $\alpha = 1$ and $\beta < 0$, then we can easy see that

$$\begin{aligned}
\mathrm{WSup}\{\psi_1(T) &+ \psi_2(T)\} = \{(x,y)^T \in \mathbb{R}^2 |\ (x = 0,\ y \leq 0) \\
&\vee\ (y = -\beta,\ x \leq -\alpha) \vee (y = \frac{\beta}{\alpha}x,\ -\alpha \leq x \leq 0)\}.
\end{aligned}$$

(iii) If $0 < \alpha < 1$ and $\beta \geq 0$, then

$$\begin{aligned}
\psi_1(T) &= \{(x,y)^T \in \mathbb{R}^2 |\ (x = 0,\ y \leq 0) \vee (y = \beta,\ x \leq \alpha - 1) \\
&\vee\ (y = \frac{\beta}{\alpha - 1}x,\ \alpha - 1 \leq x \leq 0)\}, \\
\psi_2(T) &= \{(x,y)^T \in \mathbb{R}^2 |\ (x = 0,\ y \leq 0) \vee (y = 0,\ x \leq 0)\}.
\end{aligned}$$

As a consequence, one has $\mathrm{WSup}\{\psi_1(T) + \psi_2(T)\} = \psi_1(T)$. In additional, if $\alpha = 0$ and $\beta \geq 0$, then it holds

$$\begin{aligned}
\mathrm{WSup}\{\psi_1(T) &+ \psi_2(T)\} = \{(x,y)^T \in \mathbb{R}^2 |\ (x = 0,\ y \leq 0) \\
&\vee\ (y = \beta,\ x \leq \alpha - 1) \vee (y = \frac{\beta}{\alpha - 1}x,\ \alpha - 1 \leq x \leq 0)\}.
\end{aligned}$$

(iv) If $0 < \alpha < 1$ and $\beta < 0$, then

$$\begin{aligned}\psi_1(T) &= \{(x,y)^T \in \mathbb{R}^2 | \ (x=0,\ y \leq 0) \vee (y=0,\ x \leq 0)\},\\ \psi_2(T) &= \{(x,y)^T \in \mathbb{R}^2 | \ (x=0,\ y \leq 0) \vee (y=-\beta,\ x \leq -\alpha)\\ &\vee \quad (y=\frac{\beta}{\alpha}x,\ \ -\alpha \leq x \leq 0)\}.\end{aligned}$$

Thus $\mathrm{WSup}\{\psi_1(T)+\psi_2(T)\} = \psi_2(T)$. Moreover, if $\alpha = 0$ and $\beta < 0$, then it holds

$$\mathrm{WSup}\{\psi_1(T)+\psi_2(T)\} = \{(x,y)^T \in \mathbb{R}^2 | \ (x=0,\ y \leq \beta) \vee (y=-\beta,\ x \leq 0)\}.$$

(v) If $\alpha < 0$ and $\beta \geq 0$, then

$$\begin{aligned}\psi_1(T) &= \{(x,y)^T \in \mathbb{R}^2 | \ (x=0,\ y \leq 0) \vee (y=\beta,\ x \leq \alpha-1)\\ &\vee \quad (y=\frac{\beta}{\alpha-1}x,\ \alpha-1 \leq x \leq 0)\},\\ \psi_2(T) &= \{(x,y)^T \in \mathbb{R}^2 | \ (x=-\alpha,\ y \leq -\beta) \vee (y=0,\ x \leq 0)\\ &\vee \quad (y=\frac{\beta}{\alpha}x,\ 0 \leq x \leq -\alpha)\}.\end{aligned}$$

Consequently, we get

$$\begin{aligned}\mathrm{WSup}\{\psi_1(T) &+ \psi_2(T)\} = \{(x,y)^T \in \mathbb{R}^2 | \ (x=-\alpha,\ y \leq -\beta)\\ &\vee \quad (y=\beta,\ x \leq \alpha-1) \vee (y=\frac{\beta}{\alpha-1}x,\ \alpha-1 \leq x \leq 0)\\ &\vee \quad (y=\frac{\beta}{\alpha}x,\ 0 \leq x \leq -\alpha)\}.\end{aligned}$$

(vi) If $\alpha < 0$ and $\beta < 0$, then

$$\begin{aligned}\psi_1(T) &= \{(x,y)^T \in \mathbb{R}^2 | \ (x=0,\ y \leq 0) \vee (y=0,\ x \leq 0),\\ \psi_2(T) &= \{(x,y)^T \in \mathbb{R}^2 | \ (x=-\alpha,\ y \leq -\beta) \vee (y=-\beta,\ x \leq -\alpha)\}.\end{aligned}$$

In conclusion, we have $\mathrm{WSup}\{\psi_1(T)+\psi_2(T)\} = \psi_2(T)$.

Summarizing all above cases, we obtain the complete description of γ_{p_1}.

4.4 Variational principles for $(DVEP)$

It is well known that (VEP) is closely related to the so-called dual vector equilibrium problem of finding $x \in K$ such that

$$(DVEP) \qquad f(y,x) \not> 0,\ \forall y \in K.$$

In the same way as before, we can obtain similar results for $(DVEP)$. Indeed, let us denote by K^d the solution set of $(DVEP)$. We mention that $\widehat{x} \in K$ is a solution to $(DVEP)$ if and only if 0 is a weak maximal point of the set $\{f(y,\widehat{x}) | \ y \in K\}$. For any $x \in K$ we consider the vector optimization problem

$$\begin{aligned}(P^{DVEP};x) \qquad &\mathrm{WSup}\{f(y,x) | \ y \in K\}\\ = \quad &-\mathrm{WInf}\{-f(y,x) | \ y \in K\}.\end{aligned}$$

In other words, we can reduce $(P^{DVEP};x)$ to the following vector optimization problem

$$(\widetilde{P}^{DVEP};x) \qquad \mathrm{WInf}\{-f(y,x) | \ y \in K\}.$$

By using the extended function

$$\widehat{f}(x,y) = \begin{cases} -f(y,x), & \text{if } (x,y) \in K \times K; \\ +\infty, & \text{otherwise,} \end{cases}$$

the Fenchel dual to $(\widetilde{P}^{DVEP}; x)$ turns out to be

$$\begin{aligned} (\widetilde{D}^{DVEP}; x) \;\; & \mathrm{WSup} \bigcup_{\Lambda \in \mathcal{L}(\mathrm{X},\mathrm{Y})} \mathrm{WInf} \Big\{ \{ \widehat{f}(x,y) - \langle \Lambda, y \rangle | \; y \in X \} + \{ \langle \Lambda, y \rangle | \; y \in K \} \Big\} \\ = \;\; & \mathrm{WSup} \bigcup_{\Lambda \in \mathcal{L}(\mathrm{X},\mathrm{Y})} \mathrm{WInf} \Big\{ \{ -f(y,x) - \langle \Lambda, y \rangle | \; y \in K \} + \{ \langle \Lambda, y \rangle | \; y \in K \} \Big\}. \end{aligned}$$

Whence for $x \in K$ we can define the following mapping

$$\gamma_d(x) := \bigcup_{\Lambda \in \mathcal{L}(\mathrm{X},\mathrm{Y})} \Phi_d^*(0, \Lambda; x),$$

where $\Phi_d^*(0, \Lambda; x) = \mathrm{WSup} \Big\{ \{ f(y,x) + \langle \Lambda, y \rangle | \; y \in K \} + \{ -\langle \Lambda, y \rangle | \; y \in K \} \Big\}$.

To the problem $(DVEP)$ can be associated the following set-valued vector optimization problem

$$(D_\gamma) \qquad \mathrm{WInf} \bigcup_{x \in K} \gamma_d(x).$$

Lemma 4.2 *For any $x \in K$, if $z \in \gamma_d(x)$, then $z \not< 0$.*

Proof: Let $x \in K$ be fixed and

$$z \in \gamma_d(x) = \bigcup_{\Lambda \in \mathcal{L}(\mathrm{X},\mathrm{Y})} \mathrm{WSup} \Big\{ \{ f(y,x) + \langle \Lambda, y \rangle | \; y \in K \} + \{ -\langle \Lambda, y \rangle | \; y \in K \} \Big\}.$$

Consequently, $\exists \widetilde{\Lambda} \in \mathcal{L}(\mathrm{X},\mathrm{Y})$ such that

$$z \in \mathrm{WSup} \Big\{ \{ f(y,x) + \langle \widetilde{\Lambda}, y \rangle | \; y \in K \} + \{ -\langle \widetilde{\Lambda}, y \rangle | \; y \in K \} \Big\}.$$

Let $z < 0$. In other words

$$z < f(x,x) + \langle \widetilde{\Lambda}, x \rangle - \langle \widetilde{\Lambda}, x \rangle.$$

This contradicts the fact that z is a weak supremal point of the set $\Big\{ \{ f(y,x) + \langle \widetilde{\Lambda}, y \rangle | \; y \in K \} + \{ -\langle \widetilde{\Lambda}, y \rangle | \; y \in K \} \Big\}$. □

Theorem 4.3 *Let the problem $(\widetilde{P}^{DVEP}; x)$ be stable for each $x \in K^d$. Then*

(i) $\widetilde{x} \in K$ is a solution to $(DVEP)$ if and only if $0 \in \gamma_d(\widetilde{x})$;

(ii) $K^d \subseteq K_\gamma^d$, where K_γ^d denotes the solution set of (D_γ).

Proof:

(i) Let $\widetilde{x} \in K$ be a solution to $(DVEP)$. Then by Proposition 4.4, it follows that

$$0 \in \mathrm{WSup}(P^{DVEP}; \widetilde{x}) = -\,\mathrm{WInf}(\widetilde{P}^{DVEP}; \widetilde{x}) = -\,\mathrm{WMax}(\widetilde{D}^{DVEP}; \widetilde{x}).$$

Thus

$$0 \in \text{WMin} \bigcup_{\Lambda \in \mathcal{L}(\mathrm{X},\mathrm{Y})} \text{WSup} \Big\{ \{ f(y, \widetilde{x}) + \langle \Lambda, y \rangle |\ y \in K \} + \{ -\langle \Lambda, y \rangle |\ y \in K \} \Big\}.$$

In other words, we have $0 \in \gamma_d(\widetilde{x})$. Let now $0 \in \gamma_d(\widetilde{x})$. Then, $\exists \widetilde{\Lambda} \in \mathcal{L}(\mathrm{X},\mathrm{Y})$ such that

$$0 \in \text{WSup} \Big\{ \{ f(y, \widetilde{x}) + \langle \widetilde{\Lambda}, y \rangle |\ y \in K \} + \{ -\langle \widetilde{\Lambda}, y \rangle |\ y \in K \} \Big\}.$$

If $0 \notin \text{WSup}(P^{DVEP}; \widetilde{x})$, then $0 \notin \text{WMax}(P^{DVEP}; \widetilde{x})$. Whence $\exists \widetilde{y} \in K$ such that $f(\widetilde{y}, \widetilde{x}) > 0$, i.e. $f(\widetilde{y}, \widetilde{x}) + \langle \widetilde{\Lambda}, \widetilde{y} \rangle - \langle \widetilde{\Lambda}, \widetilde{y} \rangle > 0$, which leads to a contradiction.

(ii) Let $\widetilde{x} \in K^d$. Taking into account (i), one has $0 \in \gamma_d(\widetilde{x})$. By Lemma 4.2 we obtain that

$$0 \in \text{WMin} \bigcup_{x \in K} \gamma_d(x) \subseteq \text{WInf} \bigcup_{x \in K} \gamma_d(x).$$

This means $\widetilde{x} \in K^d_\gamma$. □

Remark 4.5 As mentioned in Remark 4.3, choosing instead of γ_d the bifunction $\widehat{f} : X \times X \to Y \cup \{+\infty\}$, one can define the following mapping

$$\widetilde{\gamma}_d(x) = \text{WSup} \Big\{ \{ \widehat{f}(y, x) + \langle \Lambda, y \rangle |\ y \in X \} + \{ -\langle \Lambda, y \rangle |\ y \in K \} \Big\}.$$

Under (generalized) convexity and monotonicity assumptions, the relationships between the solution sets of (VEP) and $(DVEP)$ have been investigated in [6] and [59]. Whence, under the assumptions considered in these papers, the mapping γ_d can be related to the problem (VEP). Before doing this, let us recall some definitions and results.

Definition 4.6 *[6, Definition 2.1]*
A function $f : K \times K \to Y$ is called

(i) monotone if, for all $x, y \in K$, we have

$$f(x, y) + f(y, x) \leq 0;$$

(ii) pseudomonotone if, for all $x, y \in K$, we have

$$f(x, y) \not< 0 \quad \textit{implies} \quad f(y, x) \not> 0,$$

or, equivalently,

$$f(x, y) > 0 \quad \textit{implies} \quad f(y, x) < 0.$$

Definition 4.7 *[6, cf. Definition 2.2]*
A function $h : K \to Y$ is called:

(i) quasiconvex if, for all $\alpha \in Y$, the set $L(\alpha) = \{x \in K |\ h(x) \leq \alpha\}$ is convex;

(ii) explicitly quasiconvex if h is quasiconvex and, for all $x, y \in K$ such that $h(x) < h(y)$, we have

$$h(z_t) < h(y), \quad \textit{for all } z_t = tx + (1-t)y \textit{ and } t \in (0, 1);$$

(iii) *hemicontinuous if, for any* $x, y \in K$ *and* $t \in [0,1]$, *the mapping* $t \mapsto h(tx + (1-t)y)$ *is continuous at* 0^+.

Proposition 4.6 *[6, Proposition 2.1]*
Let K *be a nonempty convex subset of a Hausdorff topological vector space* X *and let* $f : K \times K \to Y$ *be a bifunction such that* $f(x,x) = 0,\ \forall x \in K$.

(i) *If* f *is pseudomonotone, then* $K^p \subseteq K^d$;

(ii) *If* $f(x,\cdot)$ *is explicitly quasiconvex and* $f(\cdot,y)$ *is hemicontinuous for all* $x, y \in K$, *then* $K^d \subseteq K^p$.

By Theorem 4.3 and Proposition 4.6 we can easy verify the following assertion.

Proposition 4.7 *Let all the assumptions of Proposition 4.6 and Theorem 4.3 be fulfilled. Then*

(i) $\widetilde{x} \in K$ *is a solution to* (VEP) *if and only if* $0 \in \gamma_d(\widetilde{x})$;

(ii) $K^p \subseteq K^d_\gamma$.

4.5 Gap functions for weak vector variational inequalities

This section deals with the construction of gap functions for the so-called weak vector variational inequalities. Therefore we apply the results for vector equilibrium problems in the previous section. As before, let X and Y be real topological spaces. Assume that K is a closed and convex subset of X and $F : X \to \mathcal{L}(\mathrm{X},\mathrm{Y})$ is a given mapping. The weak vector variational inequality consists in finding $x \in K$ such that

$$(WVVI) \qquad \langle F(x), y - x\rangle \not< 0,\ \forall y \in K.$$

Definition 4.8 *[23, Definition 5(ii)]*
A set-valued mapping $\psi : X \rightrightarrows Y$ *is said to be a gap function for the problem* $(WVVI)$ *if it satisfies the following conditions*

(i) $0 \in \psi(x)$ *if and only if* $x \in K$ *solves* $(WVVI)$;

(ii) $0 \not> \psi(y),\ \forall y \in K$.

It is clear that $\bar{x} \in K$ is a solution to $(WVVI)$ if and only if 0 is a weak minimal point of the set $\{\langle F(\bar{x}), y - \bar{x}\rangle \mid y \in K\}$. Let us consider the vector optimization problem:

$$(P^{WVVI}; x) \qquad \mathrm{WInf}\{\langle F(x), y - x\rangle \mid y \in K\}.$$

Taking for any $x \in K$, $\widetilde{f}(x,y) := \langle F(x), y - x\rangle$ in $\widetilde{\gamma}_p$, we suggest the following map for $(WVVI)$

$$\begin{aligned}\psi_p(x) :=& \bigcup_{T \in \mathcal{L}(\mathrm{X},\mathrm{Y})} \mathrm{WSup}\Big\{\{\langle T, y\rangle - \langle F(x), y - x\rangle \mid y \in X\} + \{-\langle T, y\rangle \mid y \in K\}\Big\} \\ =& \bigcup_{T \in \mathcal{L}(\mathrm{X},\mathrm{Y})} \mathrm{WSup}\Big\{\{\langle T - F(x), y\rangle \mid y \in X\} + \{-\langle T, y\rangle \mid y \in K\}\Big\} + \langle F(x), x\rangle.\end{aligned}$$

Theorem 4.4 ψ_p *is a gap function for the problem* $(WVVI)$.

Proof:

(i) Since $\langle F(x), y-x\rangle$ is a linear mapping with respect to y, one can apply Proposition 4.5. Consequently, for any $x \in K$ the problem $(P^{WVVI}; x)$ is stable. For $\widetilde{f}(x,y) = \langle F(x), y-x\rangle$, the first condition in the definition of a gap function follows from Theorem 4.2(i).

(ii) By Lemma 4.1, for any $y \in K$ and $z \in -\psi_p(y)$ implies $z \not> 0$. Consequently, we have $0 \not> \psi_p(y),\ \forall y \in K$. □

The relations between $(WVVI)$ and the so-called Minty vector variational inequality have been investigated by several authors (see [40], [59], [94] and [96]). Here we consider the Minty weak vector variational inequality consists in finding $x \in K$ such that

$$(MWVVI) \qquad \langle F(y), x-y\rangle \not> 0,\ \forall y \in K.$$

Likewise in Section 4, $(MWVVI)$ can be related to the following vector optimization problem:

$$(P^{MWVVI}; x) \qquad \mathrm{WInf}\{\langle F(y), y-x\rangle |\ y \in K\}$$

in the sense that $\bar{x} \in K$ is a solution to $(MWVVI)$ if and only if 0 is a weak minimal point of the set $\{\langle F(y), y-\bar{x}\rangle |\ y \in K\}$. Taking $\widehat{f}(x,y) := \langle F(x), y-x\rangle$ in $\widetilde{\gamma}_d$, we can introduce the following mapping

$$\psi_d(x) = \bigcup_{\Lambda \in \mathcal{L}(\mathrm{X},\mathrm{Y})} \mathrm{WSup}\Big\{\{\langle F(y), x-y\rangle + \langle \Lambda, y\rangle |\ y \in X\} + \{-\langle \Lambda, y\rangle |\ y \in K\}\Big\}.$$

From Theorem 4.3(i) and Lemma 4.2 follows the following assertion.

Theorem 4.5 *Let the problem $(P^{MWVVI}; x)$ be stable for any solution $x \in K$ to $(MWVVI)$. Then ψ_d is a gap function for the problem $(MWVVI)$.*

Under certain assumptions the mapping ψ_d is also a gap function for $(WVVI)$. Let us recall first the following definitions.

Definition 4.9 *[96] Let $F : K \to \mathcal{L}(\mathrm{X},\mathrm{Y})$ be a given function.*

(i) F is weakly C-pseudomonotone on K if for each $x, y \in K$, we have

$$\langle F(x), y-x\rangle \not< 0 \ \ \textit{implies} \ \ \langle F(y), x-y\rangle \not> 0;$$

(ii) F is v-hemicontinuous if for each $x, y \in K$ and $t \in [0,1]$, the mapping $t \mapsto \langle F(x+t(y-x)), y-x\rangle$ is continuous at 0^+.

Proposition 4.8 *[96, Lemma 2.1]*
Let X, Y be Banach spaces and let K be a nonempty convex subset of X. Assume that a function $F : K \to \mathcal{L}(\mathrm{X},\mathrm{Y})$ is weakly C-pseudomonotone on K and v-hemicontinuous. Then $x \in K$ is a solution to $(WVVI)$ if and only if it is also a solution to $(MWVVI)$.

As a consequence, we can easy verify the following assertion.

Proposition 4.9 *Let the assumptions of Theorem 4.5 and Proposition 4.8 be fulfilled. Then ψ_d is a gap function for $(WVVI)$.*

Index of notation

$\mathbb{R}$ – the set of real numbers

$\overline{\mathbb{R}}$ – the extended set of real numbers

$\mathbb{R}^n$ – n-dimensional Euclidean space

$\mathbb{R}^{p\times n}$ – the set of $p \times n$ matrices with real entries

$\mathbb{R}^m_+$ – the non-negative orthant of $\mathbb{R}^m$

$\text{int } C$ – the interior of the set C

$\text{ri}(C)$ – the relative interior of the set C

$\text{dom } h$ – the effective domain of the function h

$\text{epi } h$ – the epigraph of the function h

$h_1 \Box h_2$ – the infimal convolution of the functions h_1 and h_2

h^* – the conjugate function of the function h

h^*_C – the conjugate function of the function h relative to the set C

G^* – the conjugate mapping of the set-valued map G

h^*_p – the conjugate of the vector-valued function h
with p-dimensional vector variable

δ_C – the indicator function of the set C

σ_C – the support function of the set C

$N_C(x)$ – the normal cone operator to the set C at $x \in C$

$\partial h(x)$ – the subdifferential of the function h at x

$\partial h(x; y)$ – the subdifferential of the set-valued map h at $(x; y)$

$\underset{\mathbb{R}^m_+}{\geq}$ – the partial ordering induced by $\mathbb{R}^m_+$

$\underset{\text{int } \mathbb{R}^m_+}{\geq}$ – the weak partial ordering induced by int $\mathbb{R}^m_+$

$\underset{C}{\geq}$ – the partial ordering induced by cone C

$\underset{C\setminus\{0\}}{\geq}$ – the partial ordering induced by $C\setminus\{0\}$

$>$ – the weak partial ordering induced by int C

$\xi \underset{C\setminus\{0\}}{\ngeq} \mu$	– the partial ordering which means that $\xi - \mu \notin C\setminus\{0\}$
$\xi \not> \mu$	– the weak partial ordering which means that $\xi - \mu \notin \text{int } C$
X^*	– the topological dual of a topological vector space X
$x^T y$	– the inner product of the vectors x and y
$\langle \cdot, \cdot \rangle$	– the linear pairing between a topological vector space and its topological dual
$\mathcal{L}(\mathrm{X}, \mathrm{Y})$	– the space of all linear continuous operators from X to Y
$w(X^*, X)$	– the weak*-topology of X^*
$\max\limits_{\mathbb{R}^p_+\setminus\{0\}} Z$	– the set of all maximal points of the set Z
$\min\limits_{\mathbb{R}^p_+\setminus\{0\}} Z$	– the set of all minimal points of the set Z
$\mathrm{WSup}\, Z$	– the set of all weak supremal points of the set Z
$\mathrm{WMax}\, Z$	– the set of all weak maximal points of the set Z
$\mathrm{WInf}\, Z$	– the set of all weak infimal points of the set Z
$\mathrm{WMin}\, Z$	– the set of all weak minimal points of the set Z
$v(P)$	– the optimal objective value of the primal optimization problem (P)
$v(D)$	– the optimal objective value of the dual optimization problem (D)

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Theses

of the dissertation

A duality approach to gap functions for variational inequalities and equilibrium problems by Lkhamsuren Altangerel

1. This thesis deals with some applications of the conjugate duality theory for optimization problems to the construction of gap functions for variational inequalities and equilibrium problems. It concerns both scalar and vector cases. The proposed approach is considered first for variational inequalities, afterwards this is applied to more general cases including variational inequalities, the equilibrium problems. We observe that the proposed gap functions for equilibrium problems provide a convenient way of explaining as special cases the conjugate duality results for convex optimization problems and some gap functions for variational inequalities.

2. We consider the optimization problem

$$(P) \qquad \inf_{x \in X \cap G} f(x), \quad G = \{x \in \mathbb{R}^n |\ g(x) \leq_{\mathbb{R}^m_+} 0\},$$

where $X \subseteq \mathbb{R}^n$ is a nonempty set and $f : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\},\ g = (g_1, ..., g_m)^T : \mathbb{R}^n \to \mathbb{R}^m$ are given functions. For $x, y \in \mathbb{R}^m,\ x \leq_{\mathbb{R}^m_+} y$ means $y - x \in \mathbb{R}^m_+$. In associated to some perturbations, the so-called Fenchel-type and Fenchel-Lagrange-type dual problems to (P) are discussed. Closely related to this study, we reformulate the strong duality theorem in [16].

3. By using the conjugate duality for scalar optimization we extend the investigations of the duality for the convex partially separable optimization problem

$$(P^{cps}) \quad \inf_{u \in W} \sum_{i=1}^{n} F_i(A_i u),$$

where

$$W = \left\{ u = (u_0, ..., u_n)^T \in \mathbb{R}^{n+1} \middle|\ \sum_{i=1}^{n} G_i(A_i u) \leq_{\mathbb{R}^m_+} 0,\ A_i u \in W_i,\ i = \overline{1, n} \right\}.$$

Here $F_i : \mathbb{R}^{l_i} \to \mathbb{R},\ G_i : \mathbb{R}^{l_i} \to \mathbb{R}^m,\ i = \overline{1, n}$, are convex functions and $W_i \subseteq \mathbb{R}^{l_i},\ \ i = \overline{1, n}$, are convex sets. Moreover, $A_i \in \mathbb{R}^{l_i \times (n+1)},\ l_i \in \mathbb{N}$ are given matrices. Optimality conditions for (P^{cps}) and some of its particular cases are derived (see also L. Altangerel, R. I. Boţ and G. Wanka [1]).

4. The second chapter of the work is devoted to the construction of some new gap functions for variational inequalities and equilibrium problems on the basis of the conjugate duality for scalar optimization. The approach is based on the reformulation of the variational inequality problem which consists in finding $x \in K$ such that

$$(VI) \qquad F(x)^T(y - x) \geq 0,\ \forall y \in K,$$

where $K \subseteq \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ is a vector-valued function, into the optimization problem

$$(P^{VI}; x) \qquad \inf_{y \in K} F(x)^T(y - x).$$

For a fixed $x \in \mathbb{R}^n$ we define the gap functions as the negative optimal objective values of the corresponding dual problems. In order to prove that the introduced functions satisfy the properties in the definition of a gap function for (VI) we use weak and strong duality results for convex optimization. Moreover, gap functions for the mixed variational inequality problem and the so-called dual gap functions for (VI) are also studied (see also L. Altangerel, R. I. Boţ and G. Wanka [2]).

5. Since the construction of a gap function based on the Fenchel duality does not depend on the ground set K, the approach is extended to the more general case including variational inequalities, namely to the equilibrium problem of finding $x \in K$ such that

$$(EP) \qquad f(x,y) \geq 0, \ \forall y \in K,$$

where $f : X \times X \to \mathbb{R} \cup \{+\infty\}$ is a bifunction such that $f(x,x) = 0, \ \forall x \in K$, and K is a closed, convex subset of a real topological vector space X satisfying $K \times K \subseteq \text{dom } f$. To verify the properties of a gap function for (EP), the duality results in the settings of locally convex spaces by Boţ and Wanka [18] are used. Gap functions for (EP) based on the Fenchel duality are applied to convex optimization problems and variational inequalities in a real Banach space (see also L. Altangerel, R. I. Boţ and G. Wanka [3]).

6. The remainder of this thesis deals with the extension of this approach to vector variational inequalities and vector equilibrium problems. As tools here we use duality results for vector optimization by Tanino and Sawaragi [82] and Tanino [84]. Introducing vector-valued perturbation functions in analogy to the scalar case (see [16] and [90]), at the beginning of the third chapter we obtain for the following vector optimization problem

$$(VO) \qquad \min_{\mathbb{R}^p_+ \setminus \{0\}} \Big\{ f(x) | \ x \in G \Big\}, \ G = \Big\{ x \in X | \ g(x) \underset{\mathbb{R}^m_+}{\leqq} 0 \Big\}$$

different dual problems having set-valued objective maps, where $f : \mathbb{R}^n \to \mathbb{R}^p, \ g : \mathbb{R}^n \to \mathbb{R}^m$ are vector-valued functions and $X \subseteq \mathbb{R}^n$.

7. Additionally, considering the conjugate maps with vector variables we investigate further dual problems to (VO). Remark that these can be seen as special cases of the dual problems investigated before. As shown in this work, there are some advantages of investigating such dual problems.

8. Related to the different dual problems and duality results discussed in the third chapter, we define some new gap functions for the vector variational inequality of finding $x \in K$ such that

$$(VVI) \qquad F(x)^T(y-x) \underset{\mathbb{R}^p_+ \setminus \{0\}}{\nleq} 0, \ \forall y \in K,$$

where $F : \mathbb{R}^n \to \mathbb{R}^{n \times p}$ is a matrix-valued function and $K \subseteq \mathbb{R}^n$.

9. Finally, we consider the vector equilibrium problem which consists in finding $x \in K$ such that

$$(VEP) \qquad f(x,y) \nless 0, \ \forall y \in K,$$

where $f : K \times K \to Y$ is a bifunction such that $f(x,x) = 0, \ \forall x \in K$, and K is a subset of a real topological vector space X. In analogy to the scalar case, we introduce the set-valued mapping on the basis of the Fenchel duality. The proposed set-valued mapping and the duality results developed by Tanino [84] give us the possibility to investigate variational principles for (VEP). Similar discussions are applied to the so-called variational principles for the dual vector equilibrium problem and to constructing gap functions for weak vector variational inequalities.

Lebenslauf

Persönliche Daten

Name:	Lkhamsuren Altangerel
Adresse:	Vettersstrasse 52/320 09126 Chemnitz
Geburtsdatum:	22.06.1974
Geburtsort:	Khovd, Mongolei

Schulausbildung

Sept.1982 – Jun.1992	10 Jahrige Oberschule N37 in Ulaanbaatar, Mongolei Abschluss: Staatsprüfung

Studium

Sept.1993 – Jun.1997	Mongolische Nationale Universität Fakultät für Mathematik Fachbereich: Mathematik Abschluss: Bakkalaureus Diplom
Sept.1997 – Jan.1999	Mongolische Nationale Universität Fakultät für Mathematik und Informatik Fachbereich: Numerische Mathematik Abschluss: Masterdiplom
seit Okt.2001	Technische Universität Chemnitz Fakultät für Mathematik Promotionstudent des internationalen Master-und Promotionsstudiengangs Mathematik

Berufstätigkeit

Sept.1997 – Aug.1998	Informatik Fachhochshule “Tsahim“, Lehrer
Sept.1998 – Jun.2000	Mongolische Nationale Universität Fakultät für Mathematik und Informatik Assistent Lehrer

Erklärung gemäß §6 der Promotionsordnung

Hiermit erkläre ich an Eides Statt, dass ich die von mir eingereichte Arbeit "A duality approach to gap functions for variational inequalities and equilibrium problems" selbstständig und nur unter Benutzung der in der Arbeit angegebenen Hilfsmittel angefertigt habe.

Chemnitz, den 12.04.2006 Lkhamsuren Altangerel