TECHNISCHE UNIVERSITÄT CHEMNITZ

Sonderforschungsbereich 393

Parallele Numerische Simulation für Physik und Kontinuumsmechanik

Cornelia Pester

A residual a posteriori error estimator for the eigenvalue problem for the Laplace-Beltrami operator

Preprint SFB393/05-01

Abstract The Laplace-Beltrami operator corresponds to the Laplace operator on curved surfaces. In this paper, we consider an eigenvalue problem for the Laplace-Beltrami operator on subdomains of the unit sphere in $\mathbb{R}^3$. We develop a residual a posteriori error estimator for the eigenpairs and derive a reliable estimate for the eigenvalues. A global parametrization of the spherical domains and a carefully chosen finite element discretization allows us to use an approach similar to the one for the two-dimensional case. In order to assure results in the quality of those for plane domains, weighted norms and an adapted Clément-type interpolation operator have to be introduced.

Key Words Clément-type interpolation, spherical domains, Laplace-Beltrami operator, eigenvalue problem, a posteriori error estimation

AMS subject classification 65N15; 65N30, 65N50

Preprintreihe des Chemnitzer SFB 393

ISSN 1619-7178 (Print) ISSN 1619-7186 (Internet)

SFB393/05-01 **January 2005**

Contents

Author's address:

Cornelia Pester
Institut für Mathematik und Bauinformatik
Fakultät für Bauingenieur- und Vermessungswesen
Universität der Bundeswehr München
85577 Neubiberg, Germany

cornelia.pester@unibw-muenchen.de
http://www.bauv.unibw-muenchen.de/bauv1/oc/html/personen/pester/

1 Introduction

The mathematical modelling of many practical tasks leads to problems that are usually not solvable by analytic means. Numerical methods have to be applied; the given problem has to be discretized and the solution can be computed only approximately. Therefore, reliable and efficient a posteriori error estimates based on a finite element solution are of high interest to assess and to improve the discretization and therefore the quality of the solution. A posteriori estimates for eigenvalue problems were derived, for instance, by Larson (2000) and Heuveline and Rannacher (2001). An overview on error estimation for problems in two- or three-dimensional domains is given, for example, in the monographs by Verfürth (1996), Ainsworth and Oden (2000), Bangerth and Rannacher (2003).

The consideration of more complex domains, for instance two-dimensional manifolds in $\mathbb{R}^3$, requires a more careful analysis. The mixed boundary value problem for the Laplace-Beltrami operator is a popular model problem for the analysis of finite element methods for such domains, see, for example, Dziuk (1988), Mu (1996), Apel and Pester (2005) and references therein.

In this paper, we concentrate on subdomains of the unit sphere $\mathcal{S}^2$ in $\mathbb{R}^3$. Our interest in spherical domains arose from the computation of three-dimensional corner singularities for elliptic operators like the Laplace or the Lamé operator, see Apel, Mehrmann, and Watkins (2002a) for computational results. The quantitative knowledge of these singularities is of interest, for example, for engineers to predict the onset of cracks in brittle material, see work by Leguillon (1995, 2002); Leguillon and Sanchez-Palencia (1999), and Dimitrov, Buchholz, and Schnack (2002b,a); Schnack and Dimitrov (1999).

In the neighbourhood of polyhedral corners, the structure of the solutions to elliptic boundary value problems is known, see Kondrat'ev (1967). The idea is to consider a ball centered at the corner and to write the solution in terms of the form

$$r^\alpha u(\omega),$$

where r is the distance to the corner and ω is a point on the unit sphere. The regularity of such solutions was analyzed, for instance, by Kufner and Sändig (1987); Kozlov, Maz'ya, and Roßmann (2001). The singular exponent α and the function u are the solution to an eigenvalue problem that is related to the given boundary value problem. Usually, the eigenvalues with smallest magnitude are of interest. For the computation of α and u, the unit sphere around the corner has to be parametrized.

We choose spherical coordinates for this parametrization as it was done, for example, by Leguillon (1995); Kozlov, Maz'ya, and Roßmann (2001); Apel, Sändig, and Solov'ev (2002b); Apel and Pester (2005), but we are aware that the creation of artificial poles leads to difficulties in the further analysis. Other parametrizations were suggested, but they produce difficulties, too. For instance, the stereographic projection as proposed by Fichera (1975) or Steger (1983) possesses a singularity as well. It leads to a non-uniform parameter domain, which can become arbitrarily large. The projection of a refined icosahedron onto the sphere is a popular discretization method, see Baumgardner and Frederickson (1985); Mu (1996). Since the neighbourhood of the corner is intersected with the unit sphere, we have to consider subdomains of the sphere depending on the structure of the original domain. But it is not clear how the icosahedron has to look like to allow for arbitrary spherical subdomains.

The big advantage of spherical coordinates is that there is a global map which transforms the domain on the sphere to a bounded, connected two-dimensional parameter domain. This allows us to apply techniques similar to those for the two-dimensional case. No matter how the parametrization is chosen, the eigenvalue problem corresponding the homogeneous Laplace equation in a conical domain is given by

$$-\Delta_{\mathcal{S}} u = \lambda u,$$

where $\lambda = \alpha(\alpha+1)$ and $\Delta_{\mathcal{S}}$ denotes the Laplace-Beltrami operator, see, for example, Grisvard (1985, 1992); Kozlov, Maz'ya, and Roßmann (2001). A detailed deduction of this eigenvalue problem in a general parametrization is given by Meyer and Pester (2004). In this paper, we derive a posteriori error estimates for this eigenvalue problem, where we follow the theory of a posteriori error estimation for the eigenvalue problem for the Laplace operator in plane domains, as demonstrated, for instance, by Verfürth (1996). A priori error estimates for the eigenvalue problem for the Laplace-Beltrami operator on spherical domains have been derived, for example, by Steger (1983).

For the introduction of finite element spaces, we have to find proper triangulations so that a reliable and efficient error estimator can be obtained. Careful analysis of an a posteriori error estimator reveals that the elements produced by such triangulations must have approximately the same size on the sphere, which means that the corresponding elements in the parameter domain are the flatter the nearer they are placed to the pole, see Apel and Pester (2005) and references therein.

For technical reasons, we require that all elements in the parameter domain have straight-lined boundaries. An algorithm which produces a triangulation with all the desired properties was given by Apel and Pester (2005). The main difficulty in the numerical analysis of the problem and the main difference to the two-dimensional case is that the transformation of the parameter domain to the spherical domain influences the operators and norms that are used for the estimations. We have to introduce special norms and a weighted Clément-type interpolation operator; consequently, the definition of the Sobolev space on which the eigenvalue problem is defined has to be adapted to the weighted norms.

In the following, we denote the domain on the sphere by Ω and the corresponding parameter domain by G, which is spanned by the spherical angles φ and θ. Note that $\Omega \subset \mathcal{S}^2 \subset \mathbb{R}^3$ and $G \subset [0, 2\pi) \times [0, \pi] \subset \mathbb{R}^2$. All functions shall be functions of the two parameters φ and θ. We call Ω or G *regular* if Ω is an open, connected subset of the unit sphere, G is connected, polygonal and its boundary ∂G is piecewise parallel to the φ- or θ-axes. For simplicity, we consider only regular domains in this paper.

Throughout the paper, we write $\psi_1 \lesssim \psi_2$ if $\psi_1 \leq C\,\psi_2$ and $\psi_1 \sim \psi_2$ if $c\,\psi_2 \leq \psi_1 \leq C\,\psi_2$, where c and C are generic constants which vary with the context, but are independent of the triangulation and the functions under consideration.

Section 2 contains the introduction of the model problem and provides the necessary notation for a finite-element discretization. In Section 3, we introduce a weighted Clément-type operator and present important estimates which allow the derivation of a residual a posteriori error estimator for the eigenpairs in Section 4. We enrich our results by a separate a posteriori estimate for the eigenvalues using a well-known approximation result which yields an upper bound for the approximation error. We finish the paper with some numerical results in Section 5, where the residual error estimator is used for an adaptive mesh refinement based on the finite element solution of the problem.

2 Nomenclature

2.1 The model problem

For an open, connected subset Ω of the unit sphere $\mathcal{S}^2$ with boundary $\Gamma = \partial\Omega$, we consider the model problem

$$\begin{aligned} -\Delta_{\mathcal{S}} u &= \lambda u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{2.1}$$

where $\Delta_{\mathcal{S}}$ denotes the *Laplace-Beltrami operator*. If Γ is the empty set, i.e., $\Omega = \mathcal{S}^2$, then we omit the boundary condition.

Let $\nabla_{\mathcal{S}}$ be the *spherical gradient*, so that $\nabla_{\mathcal{S}} \cdot \nabla_{\mathcal{S}} = \Delta_{\mathcal{S}}$, and denote by $\mathrm{d}\sigma$ and $\mathrm{d}\omega$ the line and surface elements, respectively. For a subset $S \subset \Omega$, we denote by $\lVert\lceil \cdot \rceil\rVert_{k,S}$ and $\lceil \cdot \rceil_{k,S}$ the (weighted) *Sobolev norms* and *seminorms* of order k, $k = 0, 1$,

$$\lceil u \rceil_{0,S} = \lVert\lceil u \rceil\rVert_{0,S} := \left(\int\limits_S |u|^2 \,\mathrm{d}\omega \right)^{1/2}, \qquad \lceil u \rceil_{1,S} := \left(\int\limits_S |\nabla_{\mathcal{S}} u|^2 \,\mathrm{d}\omega \right)^{1/2},$$
$$\lVert\lceil u \rceil\rVert_{1,S}^2 := \lceil u \rceil_{0,S}^2 + \lceil u \rceil_{1,S}^2.$$

By analogy to the usual Sobolev spaces, we introduce Sobolev spaces over spherical domains, denoted by $\mathcal{H}^k(S)$, which consist of functions u with bounded norms $\lVert\lceil u \rceil\rVert_{\ell,S}$, $0 \le \ell \le k$, $k = 0, 1$.

Since only those solutions of (2.1) are of interest which do not vanish identically on Ω, we require

$$\lVert\lceil u \rceil\rVert_{0,\Omega} = 1.$$

We define the spaces $X := Y := \{v \in \mathcal{H}^1(\Omega) \mid v = 0 \text{ on } \Gamma\}$. We use the divergence theorem to derive the variational formulation of problem (2.1): Find $[\lambda, u] \in \mathbb{R} \times X$, such that

$$\int\limits_\Omega \nabla_{\mathcal{S}} u \cdot \nabla_{\mathcal{S}} v \,\mathrm{d}\omega - \int\limits_\Omega \lambda u v \,\mathrm{d}\omega + \mu \Big\{ \int\limits_\Omega u^2 \,\mathrm{d}\omega - 1 \Big\} = 0 \qquad \forall [\mu, v] \in \mathbb{R} \times Y,$$

where the last term on the left hand side assures that $\lVert\lceil u \rceil\rVert_{0,\Omega} = 1$. For the validity of the divergence theorem, we refer to work on spherical calculus, for example, Malvern (1969) and Freeden, Gervens, and Schreiner (1998).

2.2 Parametrization of the sphere

For the introduction of a finite element space, we have to parametrize Ω. As outlined in Section 1, we choose spherical coordinates $x = \sin\varphi \sin\theta$, $y = \cos\varphi \sin\theta$, $z = \cos\theta$. The consideration of the corresponding parameter domain G allows approaches similar to those for two-dimensional domains. The variable transformation $(\varphi, \theta) \to (x, y, z)$ influences operators and norms on G. The line and surface elements in spherical coordinates are given by

$$\mathrm{d}\sigma = \sqrt{\dot{\varphi}(t)^2 \sin^2\theta(t) + \dot{\theta}(t)^2}\,\mathrm{d}t, \qquad \mathrm{d}\omega = \sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta.$$

Here, and in the following, curves on the sphere are given in the form $\gamma = \gamma(t) = \{(\varphi(t), \theta(t)) \mid t \in [0,1]\}$. For more details on the definition of operators in spherical coordinates, we refer to Malvern (1969); a short introduction of the notation that is most important for our purposes is given in Meyer and Pester (2004).

When we insert spherical coordinates explicitly into the integrals, also the integration domain transforms, so that we have to use G instead of Ω. For line integrals over a curve $\gamma \subset \Omega$, one has to use the parametrized form likewise. In order to keep the amount of notation at the minimum, we will not distinguish the integration domains for line integrals; their actual meaning is always clear from the context. We remark that $\partial\Omega$ and ∂G do not coincide. Indeed, if $\Omega = \mathcal{S}^2$, then we have that $\partial\Omega = \emptyset$, but ∂G is the boundary of the rectangle $[0, 2\pi] \times [0, \pi]$. We do not introduce new symbols for X and Y, either, although the functions are now defined over the parameter domain G instead of Ω. The weak formulation of problem (2.1) in spherical coordinates reads: Find $[\lambda, \mu] \in \mathbb{R} \times X$, such that for all $[\mu, v] \in \mathbb{R} \times Y$

$$\int_G \nabla_{\mathcal{S}} u \cdot \nabla_{\mathcal{S}} v \sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta - \int_G \lambda u v \sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta + \mu\Big\{\int_G u^2 \sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta - 1\Big\} = 0. \tag{2.2}$$

2.3 Isotropic triangulation of the sphere

For practical reasons, we choose a triangulation of Ω, so that it is the image of a triangulation of the parameter domain G with straight-lined elements. From the point of view of implementation and for the comparison to the two-dimensional analysis, an isotropic triangulation of the parameter domain would be preferable. This means, the elements in the parameter domain have a bounded aspect ratio (they are shape-regular). Such a triangulation results in an anisotropic triangulation of the sphere with mesh crowding near the poles, see Apel and Pester (2005). Careful analysis revealed that it is not possible to find an a posteriori error estimator which provides both, an upper and a lower bound for the error. This property is called a reliability-efficiency gap, see for example Kunert (1998, 2003); Apel and Pester (2005), and involves the danger of over- or underestimating the exact error, which we could also confirm by numerical tests.

Consequently, we have to use an alternative method to obtain an error estimator which is both reliable and efficient. The mesh distortion near the poles is a well-known problem.

Layton (2002) tried to avoid it by creating a quadrilateral grid, where the number of grid points along a latitude circle decreases towards the poles. With this *skipped mesh partition*, hanging nodes are produced. We use a similar idea, but, for technical reasons, we prefer a mesh consisting of triangles without hanging nodes. To this end, we consider an isotropic triangulation of the sphere, which means that the elements on the sphere are shape-regular and have approximately the same spatial dimensions, while the aspect ratios of corresponding elements in the parameter domain are not bounded uniformly in the discretization parameter.

We proceed as suggested in Apel and Pester (2005). Let $\mathcal{T}_h$ be a family of triangulations of Ω and denote by $\mathcal{E}_h$ and $\mathcal{N}_h$ the sets of all edges and vertices, respectively. With $\mathcal{E}(T)$ and $\mathcal{N}(T)$, we denote the sets of edges and nodes of an element $T \in \mathcal{T}_h$. Furthermore, let $\mathcal{E}_{h,D}$ and $\mathcal{N}_{h,D}$ contain the boundary edges and nodes, and $\mathcal{E}_{h,\Omega} := \mathcal{E}_h \setminus \mathcal{E}_{h,D}$ and $\mathcal{N}_{h,\Omega} := \mathcal{N}_h \setminus \mathcal{N}_{h,D}$ the inner edges and the inner nodes.

Without loss of generality, we assume that a pole belongs to the set $\mathcal{N}_h$, if Ω contains this pole. Moreover, we assume that all elements and edges are open, and that

$$\Omega = \bigcup_{T \in \mathcal{T}_h} \overline{T}.$$

This is a standard assumption which we extend, as customary, by the conformity condition that the closures of two elements of $\mathcal{T}_h$ are either disjoint or have exactly one common edge or one common vertex. In addition, let the number of elements with one common vertex be bounded.

Remark 2.1. *It is possible that there occur edges in the parameter domain G, in particular, edges at the poles, which belong to ∂G but do not exist on the sphere. In fact, they correspond to the north or south pole and therefore have the spatial length zero. For this reason, the sets $\mathcal{E}_h$, $\mathcal{E}_{h,D}$ etc. consist only of those edges which actually exist on the sphere. Likewise, the nodal sets are defined as the union of the poles (where applicable) and the vertices of the elements that are not placed at a pole.*

The nodes and edges at $\varphi = 2\pi$ are identified with those at $\varphi = 0$ and are not counted twice.

For a domain $\omega \subset \Omega$, we introduce the parameters

$$\vartheta_{-,\omega} = \inf_{(\varphi,\theta)\in\omega} \sin\theta, \qquad \vartheta_{+,\omega} = \sup_{(\varphi,\theta)\in\omega} \sin\theta.$$

The relation $\vartheta_{-,\omega} = 0$ characterizes domains ω which are placed at a pole. We define the horizontal and vertical dimensions of a domain $\omega \subset \Omega$ *in the parameter plane*,

$$h_{\varphi,\omega} = \sup_{(\varphi,\theta)\in\omega} \varphi - \inf_{(\varphi,\theta)\in\omega} \varphi, \qquad h_{\theta,\omega} = \sup_{(\varphi,\theta)\in\omega} \theta - \inf_{(\varphi,\theta)\in\omega} \theta.$$

The term $h_{\varphi,T}\vartheta_{+,T}$ stands for the actual, *spatial* horizontal extent of ω.

Let $T \in \mathcal{T}_h$, $E \in \mathcal{E}_h$, $x \in \mathcal{N}_h$. We define the patches

$$\omega_T := \bigcup_{\mathcal{E}(T)\cap\mathcal{E}(T')\neq\emptyset} T', \quad \omega_E := \bigcup_{E\in\mathcal{E}(T')} T', \qquad \omega_x := \bigcup_{x\in\mathcal{N}(T')} T',$$
$$\tilde\omega_T := \bigcup_{\mathcal{N}(T)\cap\mathcal{N}(T')\neq\emptyset} T', \quad \tilde\omega_E := \bigcup_{\mathcal{N}(E)\cap\mathcal{N}(T')\neq\emptyset} T',$$

In addition to the above mentioned assumptions on the mesh (see page 5), we require the following properties:

Axiparallel triangles. The nodes $\mathbf{x}_{i,T} = (\varphi_{i,T}, \theta_{i,T})$, $i = 1, 2, 3$, of an element $T \in \mathcal{T}_h$ satisfy

$$\varphi_{1,T} \le \varphi_{3,T} \le \varphi_{2,T} \quad \text{and} \quad \theta_{1,T} = \theta_{2,T}. \tag{2.3}$$

This means, in particular, that one edge of each element is parallel to the φ-axis in the parameter plane.

Isotropy. The isotropy of $\mathcal{T}_h$ is characterized by

$$h_{\varphi,T}\vartheta_{+,T} \sim h_{\theta,T} \qquad \forall T \in \mathcal{T}_h. \tag{2.4}$$

This implies that the φ-extent of pole elements is independent of h, i.e.,

$$h_{\varphi,T} \sim 1 \quad \text{for } T \in \mathcal{T}_h \text{ with } \vartheta_{-,T} = 0. \tag{2.5}$$

Comparable size of adjacent elements. We require

$$h_{\theta,T} \lesssim \vartheta_{-,T} \qquad \forall T \in \mathcal{T}_h \text{ with } \vartheta_{-,T} > 0. \tag{2.6}$$

This is true, for example, if adjacent elements T have approximately the same size.

Sufficient fineness. The mesh generated by $\mathcal{T}_h$ is fine enough that $h_{\theta,\tilde\omega_T} \le \pi/4$ at least for elements near the pole (i.e. for T with $\vartheta_{-,\tilde\omega_T} = 0$). Moreover, each element $T \in \mathcal{T}_h$ touches maximum one boundary corner. For technical reasons, elements with $\vartheta_{-,\tilde\omega_T} = 0$ must not touch a crack tip.

Remark 2.2. *Pole elements T (with $\vartheta_{-,T} = 0$) appear as rectangles in the parameter plane with the nodes $\mathbf{x}_{i,T} = (\varphi_i, \theta_i)$, $i = 1, \ldots, 4$, where $\varphi_1 = \varphi_4$, $\varphi_2 = \varphi_3$, $\theta_1 = \theta_2$, $\theta_3 = \theta_4 \in \{0, \pi\}$. The nodes $\mathbf{x}_{3,T}$ and $\mathbf{x}_{4,T}$ are identified in the parameter plane; they both correspond to one of the poles.*

An algorithm for the construction of meshes which satisfy the above assumptions is given in Apel and Pester (2005, Section 3). Figure 1 shows an example of an isotropic refinement of the sphere with the corresponding (anisotropic) triangulation of the parameter domain. Due to the singularity of the transformation, one cannot achieve meshes, which are shape-regular both over G and Ω.

To each edge $E \in \mathcal{E}_h$ and each $\mathbf{x} \in E$, we assign a unit vector $\mathbf{n}_E(\mathbf{x})$ so that $\mathbf{n}_E(\mathbf{x})$ is orthogonal to the tangential vector on the curve $E \subset \mathcal{S}^2$ at $\mathbf{x}$ and so that $\mathbf{n}_E(\mathbf{x})$ lies in

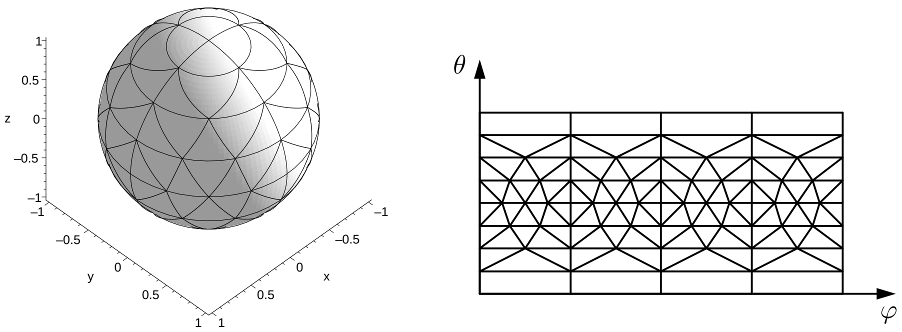

Figure 1: Isotropic triangulation of the sphere with $n_\varphi = 4$ divisions in the φ-direction and $n_\theta = 8$ divisions in the θ-direction and the corresponding anisotropic triangulation in the parameter domain.

the tangential plane at $\mathbf{x}$. For boundary edges $E \subset \partial\Omega$, this vector will equal the exterior normal vector to $\partial\Omega$. For any interior edge $E \in \mathcal{E}_{h,\Omega}$, the jump of a function ψ with $\psi\,|_{T'} \in C(T')$ for all $T' \subset \omega_E$ across E in direction $\mathbf{n}_E$ is defined by

$$[\psi]_E(\mathbf{x}) := \lim_{t\to 0+} \psi(\mathbf{x} + t\mathbf{n}_E(\mathbf{x})) - \lim_{t\to 0+} \psi(\mathbf{x} - t\mathbf{n}_E(\mathbf{x})).$$

We finish this section stating some important properties of the proposed triangulation. They are consequences of (2.6) and (2.4).

Lemma 2.3. *The relation*

$$\vartheta_{+,T} \sim \vartheta_{-,T} \tag{2.7}$$

holds true for all elements $T \in \mathcal{T}_h$ with $\vartheta_{-,T} > 0$.

Corollary 2.4. *Let $T \in \mathcal{T}_h$ with $\vartheta_{-,T} > 0$. Then*

$$\sin\theta \sim \frac{h_{\theta,T}}{h_{\varphi,T}} \quad \forall(\varphi,\theta) \in T. \tag{2.8}$$

Remark 2.5. *The properties (2.4), (2.5) and (2.8) yield that*

$$\lceil T \rceil \sim h_{\theta,T}^2 \quad \forall T \in \mathcal{T}_h \quad \textit{and} \quad \lceil E \rceil \sim h_{\theta,T} \quad \forall E \in \mathcal{E}_h,\ \forall T \subset \omega_E, \tag{2.9}$$

where $\lceil E \rceil := \|\lceil 1 \rceil\|_{1,E}^2$ and $\lceil T \rceil := \|\lceil 1 \rceil\|_{1,T}^2$ denote the spatial length of the edge E and the spatial size of the T, respectively. This implies, in particular, that

$$h_{\theta,T_1} \sim h_{\theta,T_2} \quad \textit{for any two adjacent elements } T_1, T_2 \in \mathcal{T}_h. \tag{2.10}$$

3 Interpolation error estimates on the sphere

3.1 Finite element discretization

Based on the triangulation $\mathcal{T}_h$ of Ω defined in Section 2.3, we introduce the finite element spaces X_h, Y_h which consist of continuous functions v with $v|_T \in \operatorname{span}\{1, \varphi, \theta\}$ if $\vartheta_{-,T} > 0$, and with $v|_T \in \operatorname{span}\{1, \theta, \varphi\theta\}$ if $\vartheta_{-,T} = 0$. Moreover, the functions of X_h and Y_h shall vanish on Γ.

Remark 3.1. *The functions of X_h or Y_h are elementwise affine linear or affine bilinear with respect to φ or θ, depending on whether the element is a triangle ($\vartheta_{-,T} > 0$) or a rectangle (pole element) in the parameter plane. They are linear combinations of nodal basis functions ϕ_x which have the value 1 in the node $x \in \mathcal{N}_h$ and the value 0 in all other nodes and whose support is ω_x. These basis functions can be chosen so that*

$$\sum_{x \in \mathcal{N}(T)} \phi_x = 1 \quad \textit{for all} \quad T \in \mathcal{T}_h$$

and $\phi_x \in \mathcal{H}^1(G)$ for all $x \in \mathcal{N}_h$. Hence $X_h \subset X$, $Y_h \subset Y$.

Moreover, the nodal basis functions for elements $T \in \mathcal{T}_h$ with $\vartheta_{-,T} > 0$ equal the barycentric coordinates corresponding to T (considered as a planar triangle in the parameter domain), and the nodal basis functions for elements $T \in \mathcal{T}_h$ with $\vartheta_{-,T} = 0$ have the same properties as barycentric coordinates, especially $\phi_{x_{1,T}}\phi_{x_{2,T}}\phi_{x_{3,T}} \le 1/27$ with maximum value 1/27 and $\phi_{x_{i,T}}\phi_{x_{j,T}} \le 1/4$ for $i \neq j$ with maximum value 1/4.

For details and a specific choice of such functions, we refer to Apel and Pester (2005).

The finite element discretization of problem (2.1) is given by: Find $([\lambda_h, u_h] \in \mathbb{R} \times X_h$, such that for all $[\mu_h, v_h] \in \mathbb{R} \times Y_h$

$$\int_G \nabla_{\mathcal{S}} u_h \cdot \nabla_{\mathcal{S}} v_h \sin\theta \,\mathrm{d}\varphi \,\mathrm{d}\theta - \int_G \lambda_h u_h v_h \sin\theta \,\mathrm{d}\varphi \,\mathrm{d}\theta + \mu_h \Big\{ \int_G u_h^2 \sin\theta \,\mathrm{d}\varphi \,\mathrm{d}\theta - 1 \Big\} = 0. \quad (3.1)$$

3.2 Clément-type interpolation

Let $\mathcal{P}_0$ be the space of all functions which are constant over Ω and let $\omega \subset \Omega$. We denote by $\pi_{0,\omega} : \mathcal{H}^0(\omega) \to \mathcal{P}_0$ the weighted L^2-projection of a function $v \in \mathcal{H}^0(\omega)$ onto $\mathcal{P}_0$,

$$\pi_{0,\omega} v = \frac{1}{\lceil \omega \rceil} \int_\omega v(\varphi, \theta) \sin\theta \,\mathrm{d}\varphi \,\mathrm{d}\theta = \|\lceil 1 \rceil\|_{0,\omega}^{-2} \int_\omega v \,\mathrm{d}\omega.$$

We define the (weighted) Clément-type interpolation operator $I_h : \mathcal{H}^0(\Omega) \mapsto X_h$ by

$$I_h v(\varphi, \theta) = \sum_{x \in \mathcal{N}_h \setminus \mathcal{N}_{h,D}} (\pi_{0,\omega_x} v)\ \phi_x(\varphi, \theta),$$

where the nodal basis functions ϕ_x are elementwise affine linear (or bilinear) with respect to φ and θ, i.e. piecewise polynomials of first degree over the parameter domain, see Remark 3.1.

Remark 3.2. *It was proven in Apel and Pester (2005, Lemma 4.5) that the interpolation operator I_h is bounded. Since $\sum_{x\in\mathcal{N}(T)} \phi_x = 1$, the structure of I_h assures that constant functions are interpolated exactly over elements $T \in \mathcal{T}_h$ which have no nodes on the Dirichlet boundary, that is, $I_h v = v$ for $v \in \mathcal{P}_{0|T}$, $T \in \mathcal{T}_h$, $\partial T \cap \Gamma_D = \emptyset$. The definition of the interpolation operator implies that $I_h v = 0$ on Γ_D. This allows us to prove the interpolation error estimates in the following theorem.*

Theorem 3.3 (Interpolation Error Estimates). *Let $\mathcal{T}_h$ be an isotropic triangulation of a regular spherical domain, $T \in \mathcal{T}_h$, $E \in \mathcal{E}(T)$. Then the following interpolation error estimates hold true:*

$$\begin{aligned} \lceil\!\lceil v - I_h v \rceil\!\rceil_{0,T} &\lesssim h_{\theta,T} \lceil v \rceil_{1,\tilde{\omega}_T} \quad \forall v \in \mathcal{H}^1(\tilde{\omega}_T), \\ \lceil\!\lceil v - I_h v \rceil\!\rceil_{0,E} &\lesssim h_{\theta,T}^{1/2} \lceil v \rceil_{1,\tilde{\omega}_E} \quad \forall v \in \mathcal{H}^1(\tilde{\omega}_E). \end{aligned}$$

The interpolation error estimates in Theorem 3.3 are important ingredients in the derivation of a reliable error estimator. We refer to Apel and Pester (2005) for a proof of the validity of these estimates.

4 A residual error estimator

4.1 An a posteriori error estimate for the eigenpairs

For the deduction of a residual a posteriori error estimator, we trace the route of the planar case, see for example Verfürth (1996), and adapt the ideas for two-dimensional domains to our specific problem with weighted norms and operators.

Let V and W be two Banach spaces, and let $F : D \subset V \to W^\star$ be a given linear functional that is differentiable on a subset of D. If $F(x_0) = 0$ for a fixed element $x_0 \in V$, that is, if $\langle F(x_0), y \rangle = 0$ for all $y \in W$, then x_0 is called a *solution* to

$$F(x) = 0; \tag{4.1}$$

x_0 is called a *regular solution* to (4.1), if the Fréchet derivative $\mathrm{D}F(x_0)$ exists and is a linear homeomorphism, that is, $\mathrm{D}F(x_0) : V \to W^\star$ is bijective and continuous in both directions.

Lemma 4.1 (Verfürth (1996, Proposition 2.1)). *Let $x_0 \in D$ be a regular solution to problem (4.1). We assume that $\mathrm{D}F(u_0)$ is Lipschitz continuous with a constant $\gamma > 0$. Then, there is a constant $R > 0$ depending on γ and x_0, so that*

$$\frac{1}{2}\|\mathrm{D}F(x_0)\|^{-1}_{\mathcal{L}(V,W^\star)}\|F(x)\|_{W^\star} \le \|x - x_0\|_V \le 2\|\mathrm{D}F(x_0)^{-1}\|_{\mathcal{L}(W^\star,V)}\|F(x)\|_{W^\star}$$

for all $x \in V$ with $\|x - x_0\|_V < R$.

The terms $\|\mathrm{D}F(x_0)\|^{-1}_{\mathcal{L}(V,W^\star)}$ and $\|\mathrm{D}F(x_0)^{-1}\|_{\mathcal{L}(W^\star,V)}$ in the estimate of Lemma 4.1 depend only on V, W, F and x_0; they can therefore be treated as constants and we can write

$$\|x - x_0\|_V \sim \|F(x)\|_{W^\star} \quad \text{for all } x \text{ with } \|x - x_0\|_V < R. \tag{4.2}$$

Lemma 4.2. *Let V_h and W_h be finite dimensional subspaces of V and W, and let $F_h \in \mathcal{C}(V_h, W_h^\star)$ be an approximation of F. We consider an approximate solution $x_h \in V_h$ to $F_h(x) = 0$ and a restriction operator $R_h \in \mathcal{L}(W, W_h)$. Then, the following estimate holds:*

$$\begin{aligned}\|F(x_h)\|_{W^\star} \;\leq\; & \|(\mathrm{Id}_W - R_h)^\star F(x_h)\|_{W^\star} + \|R_h^\star\|_{\mathcal{L}(W,W_h)}\|F(x_h) - F_h(x_h)\|_{W_h^\star} \\ & + \|R_h^\star\|_{\mathcal{L}(W,W_h)}\|F_h(x_h)\|_{W_h^\star},\end{aligned}$$

where Id_W *denotes the identity operator on* W.

Proof. The assertion follows from

$$\begin{aligned}\langle F(x_h), y\rangle &= \langle F(x_h), y - R_h y\rangle + \langle F(x_h) - F_h(x_h), R_h y\rangle + \langle F_h(x_h), R_h y\rangle \\ &\leq \Big(\|(\mathrm{Id}_W - R_h)^\star F(x_h)\|_{W^\star} + \|R_h^\star(F(x_h) - F_h(x_h))\|_{W^\star} \\ &\qquad + \|R_h^\star F_h(x_h)\|_{W^\star}\Big)\,\|y\|_W \\ &\leq \Big(\|(\mathrm{Id}_W - R_h)^\star F(x_h)\|_{W^\star} + \|R_h^\star\|_{\mathcal{L}(W,W_h)}\|F(x_h) - F_h(x_h)\|_{W_h^\star} \\ &\qquad + \|R_h^\star\|_{\mathcal{L}(W,W_h)}\|F_h(x_h)\|_{W_h^\star}\Big)\,\|y\|_W\end{aligned}$$

for all $y \in W$. □

We set, in particular, $V := \mathbb{R} \times X$, $W := \mathbb{R} \times Y$, $V_h := \mathbb{R} \times X_h$ and $W_h := \mathbb{R} \times Y_h$. The norm on V is given by

$$\|[\lambda, u]\|_{\mathbb{R}\times X} := \Big(|\lambda|^2 + \|\ulcorner u\rrbracket^2_{1,G}\Big)^{1/2}.$$

The norm on W is defined analogously. We define the linear functional $F : \mathbb{R} \times X \to (\mathbb{R} \times Y)^\star$ by

$$\langle F([\lambda, u]), [\mu, v]\rangle := \int_G \big(\nabla_{\mathcal{S}} u \cdot \nabla_{\mathcal{S}} v - \lambda u v\big) \sin\theta \,\mathrm{d}\varphi\,\mathrm{d}\theta + \mu\Big\{\int_G u^2 \sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta - 1\Big\}.$$

for $[\lambda, u] \in \mathbb{R} \times X$, $[\mu, v] \in \mathbb{R} \times Y$.

Lemma 4.3. *Let $[\lambda_0, u_0] \in \mathbb{R} \times X$ be a solution to $F([\lambda_0, u_0]) = 0$. Then,* $\mathrm{D}F([\lambda_0, u_0])$ *is a linear homeomorphism, if and only if $\lambda_0 \in \mathbb{R}$ is a simple eigenvalue.*

Proof. For abbreviation, let $\mathcal{A} : X \to Y^\star$ be the differential operator associated with problem (2.1) and let $\mathcal{I} : X \to Y^\star$ be an embedding operator (it exists since X is compactly embedded into $Y^\star$). The Fréchet derivative of $F : \mathbb{R} \times X \to (\mathbb{R} \times Y)^\star$,

$$\langle F([\lambda, u]), [\mu, v]\rangle = \int_\Omega (\mathcal{A} - \lambda\mathcal{I}) u v \,\mathrm{d}\omega - \mu\Big\{\int_\Omega u^2\,\mathrm{d}\omega - 1\Big\},$$

in $[\lambda_0, u_0]$ is given by

$$\langle \mathrm{D}F([\lambda_0, u_0])[\lambda, u], [\mu, v] \rangle = \int_\Omega \Big[(\mathcal{A} - \lambda_0 \mathcal{I}) u - \lambda u_0 \Big] v \, \mathrm{d}\omega + 2\mu \int_\Omega u_0 u \, \mathrm{d}\omega$$

for $[\lambda, u] \in \mathbb{R} \times X$ and $[\mu, v] \in \mathbb{R} \times Y$.

Employing spectral theory for linear operators (a short introduction is given in the book by Kozlov, Maz'ya, and Roßmann (2001), see also Karma (1996a)), one can prove that λ_0 is a simple eigenvalue of problem (2.1), if and only if $\ker \mathrm{D}F([\lambda_0, u_0]) = \{[0,0]\}$. The latter condition is equivalent to the injectivity of $\mathrm{D}F([\lambda_0, u_0])$.

It remains to show that $\mathrm{D}F([\lambda_0, u_0])$ is bijective; then $\mathrm{D}F([\lambda_0, u_0])^{-1}$ exists. To this end, compact embedding theory is needed; the space $\mathcal{H}^1$ is compactly embedded into the space $\mathcal{H}^0$. The bijectivity follows then from the Fredholm alternative, see, for example Cotlar and Cignoli (1974); Werner (1997); Alt (1999). It follows from Banach's theorem (theorem on inverse operators) that the inverse operator is continuous, see for example Alt (1999) or Berezanskij, Sheftel, and Us (1996). □

Let the discretization of F be given by $F_h : \mathbb{R} \times X_h \to (\mathbb{R} \times Y_h)^\star$,

$$\langle F_h([\lambda_h, u_h]), [\mu_h, v_h] \rangle := \langle F([\lambda_h, u_h]), [\mu_h, v_h] \rangle \quad \forall\, [\lambda_h, u_h] \in \mathbb{R} \times X_h,\ \forall\, [\mu_h, v_h] \in \mathbb{R} \times Y_h.$$

Problem (3.1) can be rewritten then: Find $[\lambda_h, u_h] \in \mathbb{R} \times X_h$, such that $F_h([\lambda_h, u_h]) = 0$, that is, $\langle F_h([\lambda_h, u_h]), [\mu_h, v_h] \rangle = 0$ for all $[\mu_h, v_h] \in \mathbb{R} \times Y_h$.

The consistency error $\|F([\lambda_h, u_h]) - F_h([\lambda_h, u_h])\|_{(\mathbb{R} \times Y_h)^\star}$ vanishes. Moreover, the pair $[\lambda_h, u_h]$ is a solution to $F_h([\lambda_h, u_h]) = 0$, if and only if $[\lambda_h, u_h]$ solves the system

$$\begin{aligned} \int_G \nabla_{\mathcal{S}} u_h \cdot \nabla_{\mathcal{S}} v_h \sin\theta \, \mathrm{d}\varphi \, \mathrm{d}\theta &= \lambda_h \int_G u_h v_h \sin\theta \, \mathrm{d}\varphi \, \mathrm{d}\theta, \\ \int_G u_h^2 \sin\theta \, \mathrm{d}\varphi \, \mathrm{d}\theta &= 1. \end{aligned} \tag{4.3}$$

With this, we transformed problem (3.1) to a finite-dimensional standard eigenvalue problem. In the following, let $[\lambda_h, u_h] \in \mathbb{R} \times X_h$ denote a solution to problem (4.3). The estimate in Lemma 4.2 reduces then to

$$\|F([\lambda_h, u_h])\|_{(\mathbb{R} \times Y)^\star} \le \|(\mathrm{Id}_{\mathbb{R} \times Y} - R_h)^\star F([\lambda_h, u_h])\|_{(\mathbb{R} \times Y)^\star}, \tag{4.4}$$

where $R_h := [0, I_h]$ with the Clément-type interpolation operator I_h from Section 3.2.

Employing Green's formula, one readily verifies that

$$\begin{aligned} \langle F([\lambda_h, u_h]), [\mu, v] \rangle &= \sum_{T \in \mathcal{T}_h} \int_T \big(-\Delta_{\mathcal{S}} u_h - \lambda_h u_h \big) v \sin\theta \, \mathrm{d}\varphi \, \mathrm{d}\theta \\ &\quad - \sum_{E \in \mathcal{E}_{h,\Omega}} \int_E [\mathbf{n}_E \cdot \nabla u_h]_E v \, \mathrm{d}\sigma \qquad \forall [\mu, v] \in \mathbb{R} \times Y. \end{aligned} \tag{4.5}$$

We define the residual error estimator

$$\eta_T := \Big\{ h_{\theta,T}^2 \|\!\lceil -\Delta_{\mathcal{S}} u_h - \lambda_h u_h \rceil\!\|_{0,T}^2 + \sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_{h,\Omega}} h_{\theta,T} \|\!\lceil [\mathbf{n}_E \cdot \nabla_{\mathcal{S}} u_h]_E \rceil\!\|_{0,E}^2 \Big\}^{1/2} \tag{4.6}$$

and obtain the following upper bound for the error.

Theorem 4.4. *Let $\lambda_0 \in \mathbb{R}$ be a simple eigenvalue of the eigenvalue problem (2.1) with the corresponding eigenfunction $u_0 \in X$, $\|\!\lceil u_0 \rceil\!\|_{0,G} = 1$. Let $[\lambda_h, u_h] \in \mathbb{R}\times X_h$ be a solution to the discretized problem (4.3), so that λ_h and u_h are sufficiently close to λ_0 and u_0, respectively, in the sense of Lemma 4.1. Then, an upper bound for the error is given by*

$$|\lambda_0 - \lambda_h| + \|\!\lceil u_0 - u_h \rceil\!\|_{1,G} \lesssim \Big\{ \sum_{T\in\mathcal{T}_h} \eta_T^2 \Big\}^{1/2}.$$

Proof. We know from the estimates (4.2) and (4.4) that

$$\begin{aligned} |\lambda_0 - \lambda_h| + \|\!\lceil u_0 - u_h \rceil\!\|_{1,G} &\le \sqrt{2}\|[\lambda_0, u_0] - [\lambda_h, u_h]\|_{\mathbb{R}\times X} \sim \|F([\lambda_h, u_h])\|_{(\mathbb{R}\times Y)^\star} \\ &\le \|(\mathrm{Id}_{\mathbb{R}\times Y} - R_h)^\star F([\lambda_h, u_h])\|_{(\mathbb{R}\times Y)^\star}. \end{aligned}$$

It remains to show that $\|(\mathrm{Id}_{\mathbb{R}\times Y} - R_h)^\star F([\lambda_h, u_h])\|_{(\mathbb{R}\times Y)^\star} \lesssim \big\{ \sum_{T\in\mathcal{T}_h} \eta_T^2 \big\}^{1/2}$. By definition, we have that $(\mathrm{Id}_{\mathbb{R}\times Y} - R_h)[\mu, v] = [\mu, v - I_h v]$ for all $\mu \in \mathbb{R}$, $v \in Y$. Hence, we get from (4.5) and Lemma 3.3 that

$$\begin{aligned} |\langle (\mathrm{Id}_{\mathbb{R}\times Y} - R_h)^\star F([\lambda_h, u_h]), [\mu, v]\rangle| &= |\langle F([\lambda_h, u_h]), (\mathrm{Id}_{\mathbb{R}\times Y} - R_h)[\mu, v]\rangle| \\ &= \Big| \sum_{T\in\mathcal{T}_h} \int_T (-\Delta_{\mathcal{S}} u_h - \lambda_h u_h)(v - I_h v)\,\mathrm{d}\omega - \sum_{E\in\mathcal{E}_{h,\Omega}} \int_E [\mathbf{n}_E \cdot \nabla_{\mathcal{S}} u_h]_E (v - I_h v)\,\mathrm{d}\sigma \Big| \\ &\le \Big(\sum_{T\in\mathcal{T}_h} \|\!\lceil -\Delta_{\mathcal{S}} u_h - \lambda_h u_h \rceil\!\|_{0,T} \cdot \|\!\lceil v - I_h v \rceil\!\|_{0,T} + \sum_{E\in\mathcal{E}_{h,\Omega}} \|\!\lceil [\mathbf{n}_E \cdot \nabla_{\mathcal{S}} u_h] \rceil\!\|_E \cdot \|\!\lceil v - I_h v \rceil\!\|_{0,E} \Big) \\ &\lesssim \sum_{T\in\mathcal{T}_h} \Big(h_{\theta,T} \|\!\lceil -\Delta_{\mathcal{S}} u_h - \lambda_h u_h \rceil\!\|_{0,T} \cdot \|\!\lceil v \rceil\!\|_{1,\tilde\omega_T} + \sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_{h,\Omega}} h_{\theta,T}^{1/2} \|\!\lceil [\mathbf{n}_E \cdot \nabla_{\mathcal{S}} u_h]_E \rceil\!\|_{0,E} \|\!\lceil v \rceil\!\|_{1,\tilde\omega_E} \Big) \end{aligned}$$

for all $[\mu, v] \in \mathbb{R}\times Y$.

Since the number of elements with one common node is bounded by a fixed constant independent of the specific triangulation, the number of patches $\tilde\omega_T$ or $\tilde\omega_E$ that contain the element T or the edge $E \in \mathcal{E}(T)$ is bounded, too. Therefore, $\sum_{T\in\mathcal{T}_h} \|\!\lceil v \rceil\!\|_{1,\tilde\omega_T}^2 \lesssim \|\!\lceil v \rceil\!\|_{1,G}$ and $\sum_{T\in\mathcal{T}_h} \sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_{h,\Omega}} \|\!\lceil v \rceil\!\|_{1,\tilde\omega_E}^2 \lesssim \|\!\lceil v \rceil\!\|_{1,G}$. The Cauchy-Schwarz inequality yields

$$|\langle (\mathrm{Id}_{\mathbb{R}\times Y} - R_h)^\star F([\lambda_h, u_h]), [\mu, v]\rangle| \lesssim \Big\{ \sum_{T\in\mathcal{T}_h} \eta_T^2 \Big\}^{1/2} \|\!\lceil v \rceil\!\|_{1,G}.$$

The assertion follows from $\|\!\lceil v \rceil\!\|_{1,G} \le \|[\mu, v]\|_{\mathbb{R}\times Y}$. □

Remark 4.5. *Due to Lemma 4.3, the requirement that λ_0 is a simple eigenvalue assures that $[\lambda_0, u_0]$ is a regular solution of (4.1), so that the assumptions of Lemma 4.1 are satisfied.*

The condition that λ_h is sufficiently close to λ_0 means, in particular, that λ_h is closer to λ_0 than to any other eigenvalue of problem (2.1). With the condition that u_h is sufficiently close to u_0, we exclude that u_h approximates $-u_0$ instead of u_0.

The constants in the estimate of Theorem 4.4 depend on the constants in the standard trace theorem and the Bramble-Hilbert lemma as well as the constants in (4.2), especially $\|DF([\lambda_0, u_0])^{-1}\|$; they are also influenced by the spatial aspect ratios of the spherical elements, the relation of the sizes of adjacent elements and the maximum number of elements with one common vertex. Due to the assumptions on the mesh, these constants are independent of h and $\mathcal{T}_h$. We did not trace their exact magnitude, but conclude from numerical experience that they are about 2.

The tricky part is to prove an estimate in converse direction. This requires special care, because of our weighted norms. In fact, the proof of the efficiency is similar to the planar case, except for the adaption of norms and separate consideration of pole and non-pole elements, once we have chosen the right discretization of Ω. In our initial tests, we observed that the condition $h_{\varphi,T}\vartheta_{+,T} \sim h_{\theta,T}$ is essential and that the efficiency of the error estimator is not a trivial consequence of the proofs in the planar case. Nevertheless, the steps are mainly the same, so that we present here only the basic ideas and refer, for instance, to Verfürth (1996) for details.

Lemma 4.6. *The following estimate holds:*

$$\Big\{ \sum_{T\in\mathcal{T}_h} \eta_T^2 \Big\}^{1/2} \lesssim \|F([\lambda_h, u_h])\|_{\mathbb{R}\times Y^\star}.$$

Proof. For $T \in \mathcal{T}_h$ and $E \in \mathcal{E}(T)$, let $\mathfrak{b}_T$ and $\mathfrak{b}_E$ denote the element- and edge-bubble functions defined over the parameter domain $G \subset \mathbb{R}^2$. They are given by

$$\mathfrak{b}_T := \begin{cases} 27\phi_{x_{1,T}}\phi_{x_{2,T}}\phi_{x_{2,T}} & \text{on } T, \\ 0 & \text{on } G\setminus T, \end{cases} \qquad \text{and} \quad \mathfrak{b}_E = \begin{cases} 4\phi_{x_{1,T'}}\phi_{x_{2,T'}} & \text{on each } T'\subset\omega_E, \\ 0 & \text{on } G\setminus\omega_E, \end{cases}$$

where $\phi_{x_{i,T'}}$ are the nodal basis functions over the element T' corresponding to the nodes $x_{i,T'}$, see Remark 3.1, and where the vertices of E are counted first. If $\vartheta_{-,T} > 0$, the functions $\mathfrak{b}_T$ and $\mathfrak{b}_E$ are the common triangle- and edge-bubble functions.

Let F_{ext} be an extension operator that extends the domain of definition of a function ψ_E smoothly from an edge $E \in \mathcal{E}_h$ to the domain ω_E (see, for example, Verfürth (1996, Section 3.1)). The functions $w_T := \{-\Delta_{\mathcal{S}} u_h - \lambda_h u_h\}\mathfrak{b}_T$ and $w_E := \{F_{\text{ext}}([\mathbf{n}_E \cdot \nabla_{\mathcal{S}} u_h]_E)\}\mathfrak{b}_E$ vanish on $G\setminus T$ and $G\setminus\omega_E$, respectively. They are elements of a finite-dimensional space $\tilde{Y}_h \subset Y$ that consists of combinations of the functions $\mathfrak{b}_T$ and $\mathfrak{b}_E$:

$$\begin{aligned}\tilde{Y}_h := \operatorname{span}\{&\mathfrak{b}_T v,\, \mathfrak{b}_T\Delta_{\mathcal{S}} v,\, \mathfrak{b}_E F_{\text{ext}}([\mathbf{n}_E\cdot\nabla_{\mathcal{S}}\sigma]_E) \mid \\ &v\in\mathcal{P}_{1|T},\, \sigma\in\mathcal{P}_{1|E},\, T\in\mathcal{T}_h,\, E\in\mathcal{E}_{h,\Omega}\}.\end{aligned}$$

Consider $T \in \mathcal{T}_h$ and $E \in \mathcal{E}(T)$. We transform T to a reference element $\hat{T}$, so that E is mapped to a reference edge $\hat{E}$. The customary way is to apply known estimates on the reference domains, where the functions usually live in finite dimensional spaces, and to obtain the desired estimates by backwards transformation.

In our case, however, the operators $\nabla_{\mathcal{S}}$ and $\Delta_{\mathcal{S}}$ appear in the definition of the space $\tilde{Y}_h$ and comprise terms such as $\sin\theta$ or $\cos\theta$ so that the transformed space is not necessarily finite dimensional. The trigonometric terms have to be dispelled (for example with (2.8)) before further estimates. Hence, we have to distinguish the specific transformation maps for pole and non-pole elements as the term $\sin\theta$ tends to zero and estimate (2.8) does not hold if $\vartheta_{-,T} = 0$. For a specification of these maps and appropriate techniques for the estimation on the reference elements, we refer to Apel and Pester (2005, Appendix).

After backwards transformation, we can conclude that

$$\begin{aligned}
\lceil\!\lceil -\Delta_{\mathcal{S}} u_h - \lambda_h u_h \rceil\!\rceil_{0,T}^2 &\lesssim \int_T (-\Delta_{\mathcal{S}} u_h - \lambda_h u_h) w_T \,\mathrm{d}\omega \;=\; \langle F([\lambda_h, u_h]), [0, w_T]\rangle, \\
\lceil\!\lceil [\mathbf{n}_E \cdot \nabla_{\mathcal{S}} u_h]_E \rceil\!\rceil_{0,E}^2 &\lesssim \int_T [\mathbf{n}_E \cdot \nabla_{\mathcal{S}} u_h]_E w_E \,\mathrm{d}\sigma \\
&= -\langle F([\lambda_h, u_h]), [0, w_E]\rangle + \sum_{T' \subset \omega_E} \int_{T'} (-\Delta_{\mathcal{S}} u_h - \lambda_h u_h) w_E \,\mathrm{d}\omega, \\
\lceil\!\lceil w_T \rceil\!\rceil_{1,T} &\lesssim h_{\theta,T}^{-1} \lceil\!\lceil -\Delta_{\mathcal{S}} u_h - \lambda_h u_h \rceil\!\rceil_{0,T}, \\
\lceil\!\lceil w_E \rceil\!\rceil_{k,T} &\lesssim h_{\theta,T}^{1/2-k} \lceil\!\lceil [\mathbf{n}_E \cdot \nabla_{\mathcal{S}} u_h]_E \rceil\!\rceil_{0,E}, \qquad k \in \{0, 1\}.
\end{aligned}$$

Thus, we have that

$$\begin{aligned}
\lceil\!\lceil -\Delta_{\mathcal{S}} u_h - \lambda_h u_h \rceil\!\rceil_{0,T}^2 &\lesssim \lceil\!\lceil w_T \rceil\!\rceil_{1,T}^{-1} h_{\theta,T}^{-1} \lceil\!\lceil -\Delta_{\mathcal{S}} u_h - \lambda_h u_h \rceil\!\rceil_{0,T} \langle F([\lambda_h, u_h]), [0, w_T]\rangle, \\
\lceil\!\lceil [\mathbf{n}_E \cdot \nabla_{\mathcal{S}} u_h]_E \rceil\!\rceil_{0,E}^2 &\lesssim \lceil\!\lceil w_E \rceil\!\rceil_{1,\omega_E}^{-1} h_{\theta,T}^{-1/2} \lceil\!\lceil [\mathbf{n}_E \cdot \nabla_{\mathcal{S}} u_h]_E \rceil\!\rceil_{0,E} \cdot |\langle F([\lambda_h, u_h]), [0, w_E]\rangle| \\
&\quad + \sum_{T' \subset \omega_E} \lceil\!\lceil (-\Delta_{\mathcal{S}} u_h - \lambda_h u_h) \rceil\!\rceil_{0,T'} \lceil\!\lceil w_E \rceil\!\rceil_{0,T'} \\
&\lesssim \lceil\!\lceil [\mathbf{n}_E \cdot \nabla_{\mathcal{S}} u_h]_E \rceil\!\rceil_{0,E} \Big\{ h_{\theta,T}^{-1/2} \lceil\!\lceil w_E \rceil\!\rceil_{1,\omega_E}^{-1} |\langle F([\lambda_h, u_h]), [0, w_E]\rangle| \\
&\qquad + \sum_{T' \subset \omega_E} h_{\theta,T'}^{1/2} \lceil\!\lceil (-\Delta_{\mathcal{S}} u_h - \lambda_h u_h) \rceil\!\rceil_{0,T'} \Big\}.
\end{aligned}$$

Since the size of adjacent elements does not change rapidly, the terms $h_{\theta,T}$ and $h_{\theta,T'}$ differ only in a constant for $\omega_E = T \cup T'$. In both estimates, we divide by the norm $\lceil\!\lceil \cdot \rceil\!\rceil_{0,T}$ or $\lceil\!\lceil \cdot \rceil\!\rceil_{0,E}$. Inserting the estimate for the $\lceil\!\lceil \cdot \rceil\!\rceil_{0,T}$-norm into the second estimate, we get that

$$\eta_T \lesssim \sup_{\substack{v \in \tilde{Y}_h \\ \operatorname{supp} v \subset \omega_T}} \lceil\!\lceil v \rceil\!\rceil_{1,T}^{-1} \langle F([\lambda_h, u_h]), [0, v]\rangle.$$

The summation over all $T \in \mathcal{T}_h$ and the relation $\|\cdot\|_{\tilde{Y}_h^\star} \le \|\cdot\|_{Y^\star}$ due to $\tilde{Y}_h \subset Y$ yield the assertion. $\square$

The following theorem is a simple consequence of Lemma 4.6 and inequality (4.2).

Theorem 4.7. *Let $[\lambda_0, u_0] \in \mathbb{R}\times X$ and $[\lambda_h, u_h] \in \mathbb{R}\times X_h$ be solutions to the problems (2.1) and (4.3), respectively, satisfying the assumptions of Theorem 4.4. Then, a lower bound for the error is given by*

$$\Big\{ \sum_{T\in\mathcal{T}_h} \eta_T^2 \Big\}^{1/2} \lesssim |\lambda_0 - \lambda_h| + \|\!\lceil u_0 - u_h \rceil\!\|_{1,G}.$$

Remark 4.8. *Theorems 4.4 and 4.7 yield a global estimate for the exact error,*

$$|\lambda_0 - \lambda_h| + \|\!\lceil u_0 - u_h \rceil\!\|_{1,G} \sim \Big\{ \sum_{T\in\mathcal{T}_h} \eta_T^2 \Big\}^{1/2}.$$

It is known that the eigenvalues converge faster than the eigenfunctions, which means that the above estimator is sub-optimal for the eigenvalues; the estimate is dominated by the approximation error of the eigenfunctions. From Theorem 4.7, we can extract the estimate

$$\|\!\lceil u_0 - u_h \rceil\!\|_{1,G} \lesssim \Big\{ \sum_{T\in\mathcal{T}_h} \eta_T^2 \Big\}^{1/2}. \tag{4.7}$$

A similar estimate holds for $|\lambda_0 - \lambda_h|$, but it can be improved. In the following subsection, we deduce a separate estimate for the eigenvalues.

4.2 An a posteriori error estimate for the eigenvalues

A posteriori error estimates for the eigenvalues were derived for the planar case, for example, by Larson (2000) or Heuveline and Rannacher (2001). Defining the cell-residuals $\rho_T = \rho_T(u_h, \lambda_h)$ for the primal problem and $\rho_T^\star = \rho_T^\star(u_h^\star, \lambda_h^\star)$ for the dual problem (see Heuveline and Rannacher (2001)), they proved estimates of the form

$$|\lambda_0 - \lambda_h| \lesssim \Big(\sum_{T\in\mathcal{T}_h} h_T^4 \rho_T(u_h, \lambda_h)^2 \Big)^{1/2}$$

for a symmetric eigenvalue problem and

$$|\lambda_0 - \lambda_h| \lesssim \sum_{T\in\mathcal{T}_h} h_T^2 \{\rho_T(u_h, \lambda_h)^2 + \rho_T^\star(u_h^\star, \lambda_h^\star)^2\} \tag{4.8}$$

for a non-symmetric eigenvalue problem. We conjecture that it is possible to verify these estimates for spherical domains with similar techniques as proposed by Larson (2000) or Heuveline and Rannacher (2001) and that it is sufficient to replace h_T by $h_{\theta,T}$ and to define ρ_T by

$$\rho_T := \|\!\lceil -\Delta_{\mathcal{S}} u_h - \lambda_h u_h \rceil\!\|_{0,T} + \frac{1}{2} \sum_{E\in\mathcal{E}(T)} \|\!\lceil [\mathbf{n}_E \cdot \nabla_{\mathcal{S}} u_h]_E \rceil\!\|_{0,E}.$$

As the Laplace operator is self-adjoint, the dual and the primal problem coincide in our case, so that $\rho_T^\star = \rho_T$.

In this paper, however, we prove an estimate similar to (4.8) by considering a general estimate for $|\lambda_0 - \lambda_h|$ which was proven by Karma (1996b, Theorem 3). It states that $|\lambda_0 - \lambda_h| \lesssim (d_h d_h^\star)^{1/\kappa}$, where κ is the dimension of the generalized eigenspace corresponding to λ_0 and where d_h is the maximum distance of the generalized eigenfunctions to the interpolation of the space which is used for the discretization of the eigenvalue problem; $d_h^\star$ is defined similarly to d_h for the dual problem. As the primal and the dual problems coincide in our case, we have that $d_h^\star = d_h$.

Since we required in Section 4.1 that λ_0 is a simple eigenvalue of (2.1), the dimension κ of the eigenspace corresponding to λ_0 equals 1. Hence,

$$|\lambda_0 - \lambda_h| \lesssim d_h^2. \tag{4.9}$$

Furthermore, we can estimate

$$d_h \lesssim \inf_{w \in X_h} \|\tilde{u}_0 - w\|_X = \inf_{w \in X_h} [\![\tilde{u}_0 - w]\!]_{1,G},$$

provided that $\tilde{u}_0$ is an eigenfunction of (2.1) corresponding to λ_0 with $\|\tilde{u}_0\|_X = [\![\tilde{u}_0]\!]_{1,G} = 1$.

In the calculations of the previous subsection, however, we considered eigenfunctions u_0 with $[\![u_0]\!]_{0,G} = 1$. Therefore, $\tilde{u}_0 = u_0 / [\![u_0]\!]_{1,G} = u_0 \, [\![\tilde{u}_0]\!]_{0,G} = \alpha \, u_0$ with $0 < \alpha \leq 1$. We conclude that

$$d_h \lesssim \inf_{w \in X_h} \alpha [\![u_0 - w/\alpha]\!]_{1,G}.$$

Choosing $w = \alpha u_h$ and using (4.9) and (4.7), we obtain the following theorem.

Theorem 4.9. *Let $[\lambda_0, u_0] \in \mathbb{R} \times X$ and $[\lambda_h, u_h] \in \mathbb{R} \times X_h$ be solutions to the problems (2.1) and (4.3), respectively, satisfying the assumptions of Theorem 4.4. Then,*

$$|\lambda_0 - \lambda_h| \lesssim [\![u_0 - u_h]\!]_{1,G}^2$$

and consequently

$$|\lambda_0 - \lambda_h| \lesssim \sum_{T \in \mathcal{T}_h} \eta_T^2.$$

The constants in Theorem 4.9 depend on the constants in Theorem 4.7, see also Remark 4.5.

5 Numerical results

We consider the model problem of a zone Ω which is bounded by two geodesic lines that span an arbitrary but constant angle ξ. We choose $\xi = 300^\circ = \frac{5}{3}\pi$. The associated parameter domain is the rectangle $G = (0, \frac{5}{3}\pi) \times [0, \pi]$, see Figure 2.

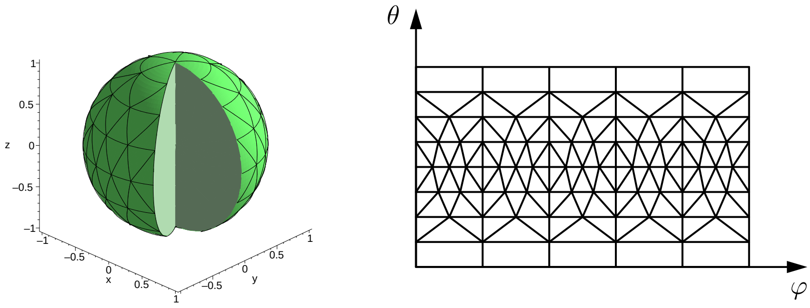

Figure 2: The domains Ω and G for $\xi = \frac{5}{3}\pi$ with isotropic triangulation.

The smallest eigenvalue of the problem

$$-\Delta_{\mathcal{S}} u = \lambda u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

is given by $\lambda_0 = \alpha(\alpha+1)$ with $\alpha = \frac{\pi}{\xi} = 0.6$. A corresponding eigenfunction reads

$$u_0 = (\sin\theta)^\alpha \sin(\alpha\varphi).$$

The left diagram of Figure 3 shows the development of the relative errors $\wp^{-1}\big\{|\lambda_0 - \lambda_h| + [\![u_0 - u_h]\!]_{1,G}\big\}$ and $\wp^{-1}\big\{\sum_{T\in\mathcal{T}} \eta_T^2\big\}^{1/2}$ with $\wp = |\lambda_0| + \lceil u_0 \rceil_{1,G}$ with respect to the corresponding problem sizes in the course of an adaptive refinement. The dotted line has the slope of the function $N^{-1/2}$, where N is the problem size. By analogy, the right diagram of Figure 3 shows the development of the errors $|\lambda_0 - \lambda_h|$ and $\sum_T \eta_T^2$ in the course of the adaptive refinement, where the dotted line has the slope of the function N^{-1}.

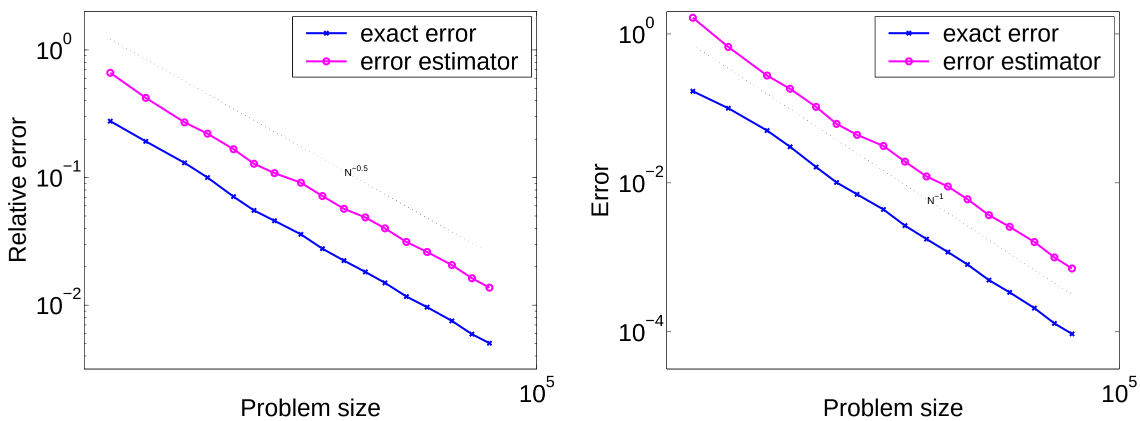

Figure 3: Left: Relative exact error and relative estimated error (adaptive refinement); Right: Exact error of the eigenvalues ($|\lambda_0 - \lambda_h|$) and squared error estimator ($\sum_T \eta_T^2$)

The numerical tests were performed with our program package *CoCoS* (computation of corner singularities) which comprises some libraries that were provided by the Department of Mathematics of Chemnitz University of Technology (see the documentation by Pester (1996)).

6 Conclusion

We studied the eigenvalue problem for the Laplace-Beltrami operator on subdomains of the unit sphere and derived a residual a posteriori error estimator for the eigenpairs. In order to apply techniques which are known for the planar case, we parametrized the sphere with spherical coordinates and chose a finite element discretization that yields straight-lines elements in the corresponding parameter domain.

As a consequence of the consideration of spherical domains, all operators or norms had to be adapted and provided with certain weights. Therefore, it was not a simple consequence of the two-dimensional case that the error estimator defined in (4.6) provides an upper an a lower bound for the error. The specific triangulation had to be chosen with care and certain parts of the proof had to be adapted as well, since, for example, the term $\Delta_{\mathcal{S}} u_h$ does not vanish for linear functions u_h although it does in the usual two-dimensional theory.

The derived error estimator can be used to estimate the eigenvalues and eigenfunctions at once; the appropriate result is summarized in Remark 4.8. Since the eigenvalues actually converge faster and the error estimator estimates mainly the error in the eigenfunctions, we proved a separate estimate for the eigenvalues.

Acknowledgement

Special thanks go to Thomas Apel (Universität der Bundeswehr München) and Bernd Hofmann (Technische Universität Chemnitz) for valuable advice. We thank the DFG (Deutsche Forschungsgemeinschaft) for financial support.

References

Ainsworth, M., Oden, J. T., 2000. A Posteriori Error Estimation in Finite Element Analysis. Wiley, New York.

Alt, H. W., 1999. Lineare Funktionalanalysis. Springer, Berlin, Heidelberg, New York, 3. Auflage.

Apel, T., Mehrmann, V., Watkins, D., 2002a. Structured eigenvalue methods for the computation of corner singularities in 3D anisotropic elastic structures. Comput. Methods Appl. Mech. Engrg. 191, 4459–4473.

Apel, T., Pester, C., 2005. Clément-type interpolation on spherical domains — interpolation error estimates and application to a posteriori error estimation. To appear in IMAJNA. .

Apel, T., Sändig, A.-M., Solov'ev, S. I., 2002b. Computation of 3D vertex singularities for linear elasticity: Error estimates for a finite element method on graded meshes. Math. Model. Numer. Anal. 36, 1043–1070.

Bangerth, W., Rannacher, R., 2003. Adaptive Finite Element Methods for Differential Equations. Birkhäuser, Basel.

Baumgardner, J. R., Frederickson, P. O., 1985. Icosahedral discretization of the two-sphere. SIAM J. Numer. Anal. 22 (6), 1107–1115.

Berezanskij, J. M., Sheftel, Z. G., Us, G. F., 1996. Functional Analysis Vol. I. Birkhäuser.

Cotlar, M., Cignoli, R., 1974. An Introduction to Functional Analysis. North-Holland Publishing Company, Amsterdam.

Dimitrov, A., Buchholz, F.-G., Schnack, E., 2002a. 3D-Corner effects in crack propagation. In: On-line Proc. of the 5th World Congress in Comp. Mech. (WCCM V). H. A. Mang, F.G. Rammerstorfer, J. Eberhardsteiner, Vienna, Austria, July 7–12, 2002.

Dimitrov, A., Buchholz, F.-G., Schnack, E., 2002b. Free surface effects in crack propagation: A theoretical-numerical-experimental correlation. Tech. Rep. Preprint 2002-1, Universität Karlsruhe.

Dziuk, G., 1988. Finite elements for the Beltrami operator on arbitrary surfaces. In: Partial differential equations and calculus of variations. Vol. 1357 of Lecture Notes in Math. Springer, Berlin, pp. 142–155.

Fichera, G., 1975. Asymptotic behaviour of the electric field and density of the electric charge in the neighbourhood of singular points of a conducting surface. Russian Math. Surveys 30 (3), 107–127.

Freeden, W., Gervens, T., Schreiner, M., 1998. Constuctive Approximation on the Sphere. With Application to Geomathematics. Numerical Mathematics and Scientific Computation. The Clarendon Press Oxford University Press, New York.

Grisvard, P., 1985. Elliptic problems in nonsmooth domains. Monographs and Studies in Mathematics, 24. Pitman Advanced Publishing Program. Boston-London-Melbourne: Pitman Publishing Inc. XIV, 410 p.

Grisvard, P., 1992. Singularities in boundary value problems. Recherches en Mathmatiques Appliques. 22. Paris: Masson. Berlin: Springer-Verlag. xiv, 198 p.

Heuveline, V., Rannacher, R., 2001. A posteriori error control for finite element aproximations of elliptic eigenvalue problems. Journal on Advances in Computational Mathematics. Special issue “A posteriori Error Estimation and Adaptive Computational Methods” 15 (1–4).

Karma, O. O., 1996a. Approximation in eigenvalue problems for holomorphic Fredholm operator functions. I. Numer. Funct. Anal. Optimization 17, 365–387.

Karma, O. O., 1996b. Approximation in eigenvalue problems for holomorphic Fredholm operator functions. II: Convergence rate. Numer. Funct. Anal. Optimization 17, 389–408.

Kondrat’ev, V. A., 1967. Boundary problems for elliptic equations in domains with conical or angular points. Trans. Moscow Math. Soc. 16, 227–313, english translation.

Kozlov, V. A., Maz’ya, V. G., Roßmann, J., 2001. Spectral problems associated with corner singularities of solutions to elliptic equations. Mathematical Surveys and Monographs, Volume 85, American Mathematical Society.

Kufner, A., Sändig, A.-M., 1987. Some Applications of Weighted Sobolev Spaces. Teubner, Leipzig.

Kunert, G., 1998. A posteriori error estimation for anisotropic tetrahedral and triangular finite element meshes. Ph.D. thesis, TU Chemnitz, Logos, Berlin.

Kunert, G., 2003. Advances in a posteriori error estimation on anisotropic finite element discretizations. Logos, Berlin.

Larson, M. G., 2000. A posteriori and a priori error analysis for finite element approximations of self-adjoint elliptic eigenvalue problems. SIAM J. Numer. Anal. 38 (2), 608–625, society for Industrial and Applied Mathematics.

Layton, A. T., 2002. Cubic spline collocation method for the shallow water equations on the sphere. J. Comput. Phys. 179 (2), 578–592.

Leguillon, D., 1995. Computation of 3d singularities in elasticity. In: Costabel, M., Dauge, M., Nicaise, S. (Eds.), Boundary value problems and integral equations in nonsmooth domains. Vol. 167 of Lecture notes in pure and applied mathematics. Marcel Dekker, New York, pp. 161–170, proceedings of a conference at CIRM, Luminy, France, May 3-7, 1993.

Leguillon, D., 2002. Strength or toughness? A criterion for crack onset at a notch. Eur. J. of Mechanics – A/Solids 21, 61–72.

Leguillon, D., Sanchez-Palencia, E., 1999. On 3D cracks intersecting a free surface in laminated composites. Int. J. Fracture 99, 25–40.

Malvern, L., 1969. Introduction to the Mechanics of a Continuous Medium. Prentice Hall, Inc. Englewood Cliff, New Jersey.

Meyer, A., Pester, C., 2004. The Laplace and the linear elasticity problems near polyhedral corners and associated eigenvalue problems. Preprint SFB393/04-12, Preprint-Reihe des SFB393 der Technischen Universität Chemnitz.

Mu, J., 1996. Solving the Laplace-Beltrami equation on S^2 using spherical triangles. Numer. Methods Partial Differential Equations 12 (5), 627–641.

Pester, M., 1996. Parallelrechner-Bibliotheken, internet publication, `http://www-user.tu-chemnitz.de/~pester/par_lib.html`.

Schnack, E., Dimitrov, A., 1999. Modelling of crack propagation under consideration of corner and edge singularities and local plastifications. Forschungsbericht, Universität Karlsruhe.

Steger, L. K., 1983. Sphärische finite Elemente und ihre Anwendung auf Eigenwertprobleme des Laplace-Beltrami-Operators. Ph.D. thesis, Ludwig-Maximilian-Universität München.

Verfürth, R., 1996. A review of a posteriori error estimation and adaptive mesh-refinement techniques. Wiley and Teubner, Chichester and Stuttgart.

Werner, D., 1997. Funktionalanalysis. Springer, Berlin, Heidelberg, New York, 2. Auflage.

Other titles in the SFB393 series:

04-01 A. Meyer, F. Rabold, M. Scherzer. Efficient Finite Element Simulation of Crack Propagation. February 2004.

04-02 S. Grosman. The robustness of the hierarchical a posteriori error estimator for reaction-diffusion equation on anisotropic meshes. March 2004.

04-03 A. Bucher, A. Meyer, U.-J. Görke, R. Kreißig. Entwicklung von adaptiven Algorithmen für nichtlineare FEM. April 2004.

04-04 A. Meyer, R. Unger. Projection methods for contact problems in elasticity. April 2004.

04-05 T. Eibner, J. M. Melenk. A local error analysis of the boundary concentrated FEM. May 2004.

04-06 H. Harbrecht, U. Kähler, R. Schneider. Wavelet Galerkin BEM on unstructured meshes. May 2004.

04-07 M. Randrianarivony, G. Brunnett. Necessary and sufficient conditions for the regularity of a planar Coons map. May 2004.

04-08 P. Benner, E. S. Quintana-Ortí, G. Quintana-Ortí. Solving Linear Matrix Equations via Rational Iterative Schemes. October 2004.

04-09 C. Pester. Hamiltonian eigenvalue symmetry for quadratic operator eigenvalue problems. October 2004.

04-10 T. Eibner, J. M. Melenk. An adaptive strategy for hp-FEM based on testing for analyticity. November 2004.

04-11 B. Heinrich, B. Jung. The Fourier-finite-element method with Nitsche-mortaring. November 2004.

04-12 A. Meyer, C. Pester. The Laplace and the linear elasticity problems near polyhedral corners and associated eigenvalue problems. December 2004.

04-13 M. Jung, T. D. Todorov. On the Convergence Factor in Multilevel Methods for Solving 3D Elasticity Problems. December 2004.

The complete list of current and former preprints is available via
`http://www.tu-chemnitz.de/sfb393/preprints.html`.