TECHNISCHE UNIVERSITÄT CHEMNITZ

Martina Balg   Arnd Meyer

Fast simulation of (nearly) incompressible nonlinear elastic material at large strain via adaptive mixed FEM

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Postanschrift:
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Sitz:
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Abstract

The main focus of this work lies on the simulation of the deformation of mechanical components that consist of nonlinear elastic, incompressible material and that are subject to large deformations. Starting from a nonlinear formulation a discrete problem can be derived by using linearisation techniques and an adaptive mixed finite element method. It turns out to be a saddle point problem that can be solved via a Bramble-Pasciak conjugate gradient method. With some modifications the simulation can be improved.

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Authors' address:

Martina Balg, Arnd Meyer
Chemnitz University of Technology
Department of Mathematics
Chair of Numerical Mathematics
(Numerical Analysis)
Reichenhainer Strasse 41
D-09126 Chemnitz

http://www.tu-chemnitz.de/mathematik/num_analysis

1 Introduction

1.1 Content

The object of this work is the numerical simulation of mechanical components, that consist of nonlinear elastic and (nearly) incompressible material and which are subject to a large deformation. As a special case that also includes linear elastic material behaviour with small deformations. The special property of incompressible material, namely the constant volume during any shape changing deformation, needs special treatment in the mathematical formulation. Therefore a mixed ansatz plays an important role.

1.2 Notation

In order to describe the considered problem of deformation, we need several mechanical quantities and the corresponding operators. These operators mostly are defined as tensors of order n and the space of these tensors is denoted with $\mathbb{T}_n$. By choosing a fixed basis the *notation of Voigt* can be used. That allows the representation of the tensors as matrices or vectors. For distinction we use different typefaces for different types of values. This is shown in table 1.1, only few exceptions can occur.

Q, ϕ	scalar function, tensor of order zero
$\boldsymbol{V}$, $\boldsymbol{v}$	vector field, tensor of order one
$\boldsymbol{\mathcal{T}}, \boldsymbol{\sigma}$	tensor of order two
$\mathfrak{M}$	tensor of order four
$\underline{a}$, $\underline{R}$	n-vector of expanding coefficients
A	matrix

Table 1.1: types of notation

Throughout this paper, pairs of vectors, such as $\boldsymbol{UV}$, form a 2^{nd} order tensor and in general a 2^{nd} order tensor is any linear combination of such pairs. In the same way, a pair of 2^{nd} order tensors, such as $\boldsymbol{\mathcal{E}\mathcal{F}}$, defines a 4^{th} order tensor. Obviously any 2^{nd} order tensor $\boldsymbol{\mathcal{E}}$ is a linear operator with $\boldsymbol{\mathcal{E}} : \mathbb{T}_1 \to \mathbb{T}_1$, as well as any 4^{th} order tensor $\mathfrak{M}$ is a linear operator with $\mathfrak{M} : \mathbb{T}_2 \to \mathbb{T}_2$.

As usual we define a dot product $\boldsymbol{V} \cdot \boldsymbol{U} \in \mathbb{R}$ for all first order tensors $\boldsymbol{U}, \ \boldsymbol{V} \in \mathbb{T}_1$.

The double dot products are defined as follows.

$$\boldsymbol{CD} : \boldsymbol{UV} = \left(\boldsymbol{D} \cdot \boldsymbol{U}\right) \left(\boldsymbol{C} \cdot \boldsymbol{V}\right) \in \mathbb{R} \qquad \forall \boldsymbol{U}, \boldsymbol{V}, \boldsymbol{C}, \boldsymbol{D} \in \mathbb{T}_1 \tag{1.1}$$

$$\boldsymbol{ABCD} : \boldsymbol{UV} = \left(\boldsymbol{D} \cdot \boldsymbol{U}\right) \left(\boldsymbol{C} \cdot \boldsymbol{V}\right) \boldsymbol{AB} \in \mathbb{T}_2 \quad \forall \boldsymbol{U}, \boldsymbol{V}, \boldsymbol{C}, \boldsymbol{D}, \boldsymbol{A}, \boldsymbol{B} \in \mathbb{T}_1 \tag{1.2}$$

For later use we also define the symmetric part of a second order tensor.

$$2\,\texttt{Sym}\left(\boldsymbol{\mathcal{Y}}\right) = \boldsymbol{\mathcal{Y}} + \boldsymbol{\mathcal{Y}}^{\mathrm{T}} \tag{1.3}$$

With $[\,\cdot\,,\,\cdot\,,\,\cdot\,]$ we denote the usual *scalar triple product*. Furthermore we use the L^2-scalar products (1.4)-(1.6) and norms (1.7)-(1.10) for all $Q, P \in \mathbb{T}_0$, $\boldsymbol{V}, \boldsymbol{U} \in \mathbb{T}_1$ and $\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{F}} \in \mathbb{T}_2$.

$$\langle Q\,,\, P\rangle_{0,\Omega} := \int_\Omega Q\, P \,\mathrm{d}\Omega \tag{1.4}$$

$$\langle \boldsymbol{V}\,,\, \boldsymbol{U}\rangle_{0,\Omega} := \int_\Omega \boldsymbol{V} \cdot \boldsymbol{U} \,\mathrm{d}\Omega \tag{1.5}$$

$$\langle \boldsymbol{\mathcal{E}}\,,\, \boldsymbol{\mathcal{F}}\rangle_{0,\Omega} := \int_\Omega \boldsymbol{\mathcal{E}} : \boldsymbol{\mathcal{F}}^{\mathrm{T}} \,\mathrm{d}\Omega \tag{1.6}$$

$$\|Q\|^2_{0,\Omega} := \langle Q\,,\, Q\rangle_{0,\Omega} \tag{1.7}$$

$$\|\boldsymbol{V}\|^2_{0,\Omega} := \langle \boldsymbol{V}\,,\, \boldsymbol{V}\rangle_{0,\Omega} \tag{1.8}$$

$$|\boldsymbol{V}|^2_{1,\Omega} = \int_\Omega \mathrm{Grad}\,\boldsymbol{V} : \mathrm{Grad}\,\boldsymbol{U}^{\mathrm{T}} \,\mathrm{d}\Omega \tag{1.9}$$

$$\|\boldsymbol{V}\|^2_{1,\Omega} := \|\boldsymbol{V}\|^2_{L^2(\Omega)^3} + |\boldsymbol{V}|^2_{H^1(\Omega)^3} \tag{1.10}$$

Next to the tensors itself, its derivatives are also of importance, especially the derivatives of scalar tensors or tensors of second order. The derivative of a scalar tensor $\zeta(\boldsymbol{\mathcal{Y}}) : \mathbb{T}_2 \to \mathbb{R}$ is denoted with $\boldsymbol{\zeta}' = \frac{\partial \zeta(\boldsymbol{\mathcal{Y}})}{\partial \boldsymbol{\mathcal{Y}}}$. This is a tensor of second order that fulfils equation (1.11) for all applied directions $\delta\boldsymbol{\mathcal{Y}} \in \mathbb{T}_2$.

$$\zeta(\boldsymbol{\mathcal{Y}} + \delta\boldsymbol{\mathcal{Y}}) = \zeta(\boldsymbol{\mathcal{Y}}) + \frac{\partial \zeta(\boldsymbol{\mathcal{Y}})}{\partial \boldsymbol{\mathcal{Y}}} : \delta\boldsymbol{\mathcal{Y}} + \mathcal{O}\left(\|\delta\boldsymbol{\mathcal{Y}}\|^2\right) \tag{1.11}$$

The derivative $\boldsymbol{\mathcal{T}}' = \frac{\partial \boldsymbol{\mathcal{T}}(\boldsymbol{\mathcal{Y}})}{\partial \boldsymbol{\mathcal{Y}}}$ of a tensor $\boldsymbol{\mathcal{T}}(\boldsymbol{\mathcal{Y}}) \in \mathbb{T}_2$ with $\boldsymbol{\mathcal{Y}} \in \mathbb{T}_2$ is of order four and is defined via (1.12).

$$\boldsymbol{\mathcal{T}}(\boldsymbol{\mathcal{Y}} + \delta\boldsymbol{\mathcal{Y}}) = \boldsymbol{\mathcal{T}}(\boldsymbol{\mathcal{Y}}) + \frac{\partial \boldsymbol{\mathcal{T}}(\boldsymbol{\mathcal{Y}})}{\partial \boldsymbol{\mathcal{Y}}} : \delta\boldsymbol{\mathcal{Y}} + \mathcal{O}\left(\|\delta\boldsymbol{\mathcal{Y}}\|^2\right) \tag{1.12}$$

2 Basics

The following subsections deal with the basic terms, that are needed to describe an elastic deformation ([1] (chap. 6) and [2]). In doing so geometrical and kinematic terms are introduced first and the basic equations of motion are shown later on.

2.1 Geometry

We consider an arbitrary elastic body K as a representation of any given mechanical component. We assume that K has the initial configuration (2.1) prior to the deformation.

$$\Omega = \Omega_0 = \left\{ X(\eta) \in \mathbb{R}^3 : \ \eta \in \mathcal{P} \subset \mathbb{R}^3 \right\} \tag{2.1}$$

Via the influences of various external loads a deformation process occurs that transfers K into a current configuration.

$$\Omega_\tau = \left\{ x(\eta) \in \mathbb{R}^3 : \ \eta \in \mathcal{P} \subset \mathbb{R}^3 \right\} \text{ with } \tau \geq 0 \tag{2.2}$$

We note that for all τ the parametrising set $\mathcal{P}$ remains the same. Hence the material point $X(\eta)$ in Ω turns over to the spacial point $x(\eta)$ during the deformation.

Because of the fixed parametrisation it is possible to define a covariant and contravariant tensor basis in Ω that is denoted with $\boldsymbol{G}_i$ and $\boldsymbol{G}^i$ respectively.

$$\boldsymbol{G}_i = \frac{\partial}{\partial \eta^i} X(\eta), \tag{2.3}$$

$$\boldsymbol{G}^j \cdot \boldsymbol{G}_i = \delta_i^j \tag{2.4}$$

Analogously one can define these bases in Ω_τ.

$$\boldsymbol{g}_i = \frac{\partial}{\partial \eta^i} x(\eta), \tag{2.5}$$

$$\boldsymbol{g}^j \cdot \boldsymbol{g}_i = \delta_i^j \tag{2.6}$$

Note: With these tensors of order one and by using the *summation convention of Einstein* the unit tensor of second order $\boldsymbol{\mathcal{I}} \in \mathbb{T}_2$ can be written as shown in (2.7).

$$\sum_{i=1}^{3} \boldsymbol{G}_i \boldsymbol{G}^i = \boldsymbol{G}_i \boldsymbol{G}^i = \boldsymbol{\mathcal{I}} = \boldsymbol{g}_i \boldsymbol{g}^i \tag{2.7}$$

We use the *summation convention of Einstein* and introduce two differential operators in each configuration, i. e. the gradient or the divergence.

$$\mathrm{Grad} = \boldsymbol{G}^i \frac{\partial}{\partial \eta^i} \qquad \text{in } \Omega \tag{2.8}$$

$$\mathrm{grad} = \boldsymbol{g}^i \frac{\partial}{\partial \eta^i} \qquad \text{in } \Omega_\tau \tag{2.9}$$

$$\mathrm{Div} = \mathrm{Grad} \cdot = \left(\boldsymbol{G}^i \frac{\partial}{\partial \eta^i} \right) \cdot \qquad \text{in } \Omega \tag{2.10}$$

$$\mathrm{div} = \mathrm{grad} \cdot = \left(\boldsymbol{g}^i \frac{\partial}{\partial \eta^i} \right) \cdot \qquad \text{in } \Omega_\tau \tag{2.11}$$

Note: For vector fields $\boldsymbol{U}$ we have $\big(\mathrm{Grad}\,\boldsymbol{U}\big) = \boldsymbol{G}^i \frac{\partial}{\partial \eta^i} \boldsymbol{U} = \boldsymbol{G}^i \boldsymbol{U}_{,i}$ which implies a correct *Taylor expansion* $\boldsymbol{U}(X + \boldsymbol{V}) = \boldsymbol{U}(X) + \boldsymbol{V} \cdot \mathrm{Grad}\,\boldsymbol{U} + \mathcal{O}\big(\| \boldsymbol{V} \|^2\big)$.

2.2 Kinematics

The deformation of the body K shall be described by a sufficiently smooth function $\boldsymbol{\Phi}$ such that (2.12) holds.

$$\boldsymbol{\Phi} : \ \Omega \to \Omega_\tau, \quad X(\eta) \mapsto x(\eta) \tag{2.12}$$

Introducing the first order displacement tensor $\boldsymbol{U}$ one can rewrite $\boldsymbol{\Phi}$ as a sum.

$$\boldsymbol{\Phi}(X) = x = X + \boldsymbol{U}(X) \tag{2.13}$$

Using the derivatives of $\boldsymbol{U}$ the covariant tensor basis of Ω_τ decomposes with

$$\boldsymbol{g}_i = \frac{\partial}{\partial \eta^i}(X + \boldsymbol{U}) = \boldsymbol{G}_i + \frac{\partial}{\partial \eta^i} \boldsymbol{U} = \boldsymbol{G}_i + \boldsymbol{U}_{,i} \quad . \tag{2.14}$$

We need some important tensors to measure the deformation. First of all we get the deformation gradient $\boldsymbol{\mathcal{F}}$ that is defined via (2.15).

$$\boldsymbol{\mathcal{F}} \cdot \mathrm{d}X = \mathrm{d}x \tag{2.15}$$

With a few transformation steps (use the deformation, the displacement and the tensor basis) the explicit representation (2.16) follows.

$$\boldsymbol{\mathcal{F}} = (\mathrm{Grad}\,\boldsymbol{\Phi})^{\mathrm{T}} = \boldsymbol{\mathcal{I}} + (\mathrm{Grad}\,\boldsymbol{U})^{\mathrm{T}} = \boldsymbol{g}_i \boldsymbol{G}^i \tag{2.16}$$

To guarantee that $\boldsymbol{\Phi}(X)$ is a feasible deformation the determinant of $\boldsymbol{\mathcal{F}}$, denoted by J, has to fulfil condition (2.17).

$$\det \boldsymbol{\mathcal{F}} =: J > 0 \tag{2.17}$$

Note: There exists a unique *polar decomposition* of $\boldsymbol{\mathcal{F}}$ with $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{P}}\cdot\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{U}}\cdot\boldsymbol{\mathcal{P}}$, where $\boldsymbol{\mathcal{V}}$ (and $\boldsymbol{\mathcal{U}}$ resp.) is a unitary tensor of rotation and $\boldsymbol{\mathcal{P}}$ is a symmetric stretch tensor. Due to $\boldsymbol{\mathcal{F}}$ being invertible $\boldsymbol{\mathcal{P}}$ is even positive definite.

Furthermore we can derive the (right) Cauchy-Green strain tensor $\boldsymbol{\mathcal{C}}$ which describes the local change of length and the Green-Lagrange strain tensor $\boldsymbol{\mathcal{E}}$.

$$\boldsymbol{\mathcal{C}} := \boldsymbol{\mathcal{F}}^{\mathrm{T}} \cdot \boldsymbol{\mathcal{F}} \tag{2.18}$$

$$\boldsymbol{\mathcal{E}} := \tfrac{1}{2}(\boldsymbol{\mathcal{C}} - \boldsymbol{\mathcal{I}}) \tag{2.19}$$

$$2\,\boldsymbol{\mathcal{E}} = \mathrm{Grad}\;\boldsymbol{U} + (\mathrm{Grad}\;\boldsymbol{U})^{\mathrm{T}} + \mathrm{Grad}\;\boldsymbol{U} \cdot (\mathrm{Grad}\;\boldsymbol{U})^{\mathrm{T}} \tag{2.20}$$

For later use we additionally need the directional derivative $\boldsymbol{\mathcal{E}}(\boldsymbol{U};\,\boldsymbol{V})$ of the strain.

$$\begin{aligned} 2\,\boldsymbol{\mathcal{E}}(\boldsymbol{U};\,\boldsymbol{V}) = {} & \mathrm{Grad}\;\boldsymbol{V} + (\mathrm{Grad}\;\boldsymbol{V})^{\mathrm{T}} \\ & + \mathrm{Grad}\;\boldsymbol{U} \cdot (\mathrm{Grad}\;\boldsymbol{V})^{\mathrm{T}} + \mathrm{Grad}\;\boldsymbol{V} \cdot (\mathrm{Grad}\;\boldsymbol{U})^{\mathrm{T}} \\ = {} & \mathrm{Grad}\;\boldsymbol{V} \cdot \boldsymbol{\mathcal{F}}(\boldsymbol{U}) + \boldsymbol{\mathcal{F}}(\boldsymbol{U})^{\mathrm{T}} \cdot (\mathrm{Grad}\;\boldsymbol{V})^{\mathrm{T}} \end{aligned} \tag{2.21}$$

2.3 Equilibrium of Forces

We assume that the deformation $\boldsymbol{\Phi}$ of the body K is caused by the influences of external loads. These loads can be of different types: a given displacement (2.22) on the *Dirichlet boundary*, a deformation independent force density per unit mass (2.24), i.e. an acceleration field, or a force density per unit surface (2.23) on the *Neumann boundary*, a so called surface force.

$$\boldsymbol{U}_0 \in \mathbb{T}_1 \qquad \text{on } \Gamma_{D,\tau} \subset \partial\Omega_\tau \tag{2.22}$$

$$g \in \mathbb{R}^3 \qquad \text{on } \Gamma_{N,\tau} \subset \partial\Omega_\tau \tag{2.23}$$

$$f \in \mathbb{R}^3 \qquad \text{with } f(X(\eta)) = f(x(\eta))\;\forall\,\eta \in \mathcal{P} \tag{2.24}$$

After the deformation, K shall be in a state of *equilibrium of forces*. With these assumptions we can derive integral equilibriums and by using the *theorem of Cauchy* (see [1], p. 275) we can prove the existence of a displacement $\boldsymbol{U}(X(\eta))$ in Ω and even the existence of a symmetric, second order stress tensor $\boldsymbol{\sigma}(\boldsymbol{U}, x(\eta)) \in C^1(\Omega_\tau)$ with the properties (2.25) - (2.27).

$$\mathrm{div}(\boldsymbol{\sigma}) + \rho_\tau\, f = 0 \qquad \text{in } \Omega_\tau \tag{2.25}$$

$$\boldsymbol{n}_\tau \cdot \boldsymbol{\sigma} = g_\tau \qquad \text{on } \Gamma_{\tau,N} \subset \partial\Omega_\tau \tag{2.26}$$

$$\boldsymbol{U} = \boldsymbol{U}_0 \qquad \text{on } \Gamma_D \text{ s.t. } \Gamma_{\tau,D} = \Gamma_D + \boldsymbol{U}_0 \tag{2.27}$$

For a certain parameter τ the scalar material density in Ω_τ is given by ρ_τ and $\boldsymbol{n}_\tau \in \mathbb{T}_1$ describes the outer normal vector in a boundary point $x \in \partial\Omega_\tau$. The boundary conditions can also be stated component wise.

Note: On parts of the boundary without any given loads it is $\boldsymbol{n}_\tau \cdot \boldsymbol{\sigma} = 0$.

Our goal is now to simulate the displacement $\boldsymbol{U}$.

2.4 Incompressibility

Incompressibility means that there is no change in volume by any deformation and change of shape. To ensure this J has to fulfil condition (2.28).

$$\det \boldsymbol{\mathcal{F}} = J \equiv 1. \tag{2.28}$$

Because J gives the ratio of the volume of K before and after the deformation (see [3]), the condition above yields a constant volume. In case of almost incompressiblity the condition (2.28) needs to be fulfilled only approximately.

Furthermore one has a restriction to the material parameter $\mathtt{K}$, which is called the bulk modulus. This number describes how much pressure needs to be applied to produce a relative change of volume and to compress a body K. Because we want to consider nearly or complete incompressible material we have to include the limit

$$\mathtt{K} \to \infty \tag{2.29}$$

in our calculations.

3 Mixed variational formulation

In this chapter we formulate the nonlinear problem of deformation for incompressible material. To that it is necessary to introduce a new process variable. Additionally the representation of $\overset{2}{\boldsymbol{\mathcal{T}}}$ as a derivative of the specific energy density function $\phi(\boldsymbol{\mathcal{C}})$ becomes important, which is discussed in a later subsection.

3.1 Variational Formulation of nonlinear Elasticity

To solve the problem (2.25) - (2.27) by means of the finite element method the week formulation is needed. As usual we multiply with test functions $\boldsymbol{V} \in \mathbb{V}_0$

$$\begin{aligned} \mathbb{V}_D &:= \left\{ \boldsymbol{V} \in (H^1(\Omega))^3 : \ \boldsymbol{V}\Big|_{\Gamma_D} = \boldsymbol{U}_0 \right\} && (3.1) \\ &= \left\{ \boldsymbol{v} \in (H^1(\Omega_\tau))^3 : \ \boldsymbol{v} \circ \boldsymbol{\Phi}\Big|_{\Gamma_D} = \boldsymbol{U}_0 \right\} \\ \mathbb{V}_0 &:= \left\{ \boldsymbol{V} \in (H^1(\Omega))^3 : \ \boldsymbol{V}\Big|_{\Gamma_D} = \boldsymbol{0} \right\} && (3.2) \\ &= \left\{ \boldsymbol{v} \in (H^1(\Omega_\tau))^3 : \ \boldsymbol{v} \circ \boldsymbol{\Phi}\Big|_{\Gamma_D} = \boldsymbol{0} \right\} \end{aligned}$$

and integrate over the domain Ω_τ. This yields the integral equation

$$\langle \operatorname{div}(\boldsymbol{\sigma}) \ , \ \boldsymbol{v} \rangle_{0,\Omega_\tau} + \langle \rho_\tau \, f \ , \ \boldsymbol{v} \rangle_{0,\Omega_\tau} = 0 \qquad \forall \boldsymbol{v} \in \mathbb{V}_0 \quad . \tag{3.3}$$

After the application of the *integral theorem of Gauss* we get (3.4).

$$\langle \boldsymbol{\sigma} \ , \ \operatorname{grad} \boldsymbol{v} \rangle_{0,\Omega_\tau} = \langle \rho_\tau \, f \ , \ \boldsymbol{v} \rangle_{0,\Omega_\tau} + \langle \boldsymbol{n}_\tau \cdot \boldsymbol{\sigma} \ , \ \boldsymbol{v} \rangle_{0,\Gamma_{\tau,N}} \qquad \forall \, \boldsymbol{v} \in \mathbb{V}_0 \tag{3.4}$$

Contrary to the case of small deformations we need to distinguish between the values on Ω and Ω_τ. But there are some quite handsome transformation rules available.

$$\begin{aligned} \mathrm{d}\Omega_\tau &= [\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3] \ \mathrm{d}\eta^1 \, \mathrm{d}\eta^2 \, \mathrm{d}\eta^3 \\ &= [\boldsymbol{\mathcal{F}} \cdot \boldsymbol{G}_1, \boldsymbol{\mathcal{F}} \cdot \boldsymbol{G}_2, \boldsymbol{\mathcal{F}} \cdot \boldsymbol{G}_3] \ \mathrm{d}\eta \\ &= \det \boldsymbol{\mathcal{F}} \ [\boldsymbol{G}_1, \boldsymbol{G}_2, \boldsymbol{G}_3] \ \mathrm{d}\eta \\ &= J \ \mathrm{d}\Omega && (3.5) \\ \rho_\tau \circ \boldsymbol{\Phi} &= J^{-1} \rho_0 && (3.6) \end{aligned}$$

With the introduction of two new second order tensors, namely the first and second Piola-Kirchhoff stress tensor $\overset{1}{\boldsymbol{\mathcal{T}}}$ and $\overset{2}{\boldsymbol{\mathcal{T}}}$ respectively we can also generate a pull back of the stress $\boldsymbol{\sigma}$ from Ω_τ to Ω.

$$\begin{aligned} \boldsymbol{\sigma} &= J^{-1} \ \boldsymbol{\mathcal{F}} \cdot \overset{1}{\boldsymbol{\mathcal{T}}} && (3.7) \\ &= J^{-1} \ \boldsymbol{\mathcal{F}} \cdot \overset{2}{\boldsymbol{\mathcal{T}}} \cdot \boldsymbol{\mathcal{F}}^{\mathrm{T}} && (3.8) \end{aligned}$$

Differently from $\boldsymbol{\sigma}$ these new terms are living on the initial configuration and at least $\overset{2}{\boldsymbol{\mathcal{T}}}$ keeps the symmetry property of $\boldsymbol{\sigma}$.

With (3.8) we transform the left hand side of (3.4) and get (3.9).

$$
\begin{aligned}
\langle\, \boldsymbol{\sigma} \,,\, \operatorname{grad} \boldsymbol{v} \,\rangle_{0,\Omega_\tau} &= \left\langle\, J \cdot J^{-1} \boldsymbol{\mathcal{F}} \cdot \overset{1}{\boldsymbol{\mathcal{T}}} \,,\, \operatorname{grad} \boldsymbol{V} \,\right\rangle_{0,\Omega_\tau} \\
&= \left\langle\, \overset{1}{\boldsymbol{\mathcal{T}}} \,,\, \operatorname{Grad} \boldsymbol{V} \,\right\rangle_{0,\Omega} \\
&= \int_\Omega \overset{2}{\boldsymbol{\mathcal{T}}} : \left(\boldsymbol{\mathcal{F}}^{\mathrm{T}} \cdot (\operatorname{Grad} \boldsymbol{V})^{\mathrm{T}} \right) \, \mathrm{d}\Omega \\
&= \left\langle\, \overset{2}{\boldsymbol{\mathcal{T}}} \,,\, \boldsymbol{\mathcal{E}}(\, \boldsymbol{U};\, \boldsymbol{V}\,) \,\right\rangle_{0,\Omega}
\end{aligned}
\tag{3.9}
$$

W.l.o.g. we assume $\rho_0 = 1$. Applying $\rho_\tau(x) = J^{-1} \rho_0(X)$ from (3.6) on $\langle\, \rho_\tau\, f \,,\, \boldsymbol{v} \,\rangle_{0,\Omega_\tau}$ from (3.4) we obtain

$$
\langle\, \rho_\tau\, f \,,\, \boldsymbol{v} \,\rangle_{0,\Omega_\tau} = \langle\, f \,,\, \boldsymbol{V} \,\rangle_{0,\Omega} \quad . \tag{3.10}
$$

Again w.l.o.g., we assume that

$$
\Gamma_{N,\tau} := \left\{ x(\eta^1, \eta^2, \eta^3) : \ \eta^3 = \eta_0 \text{ constant}, \ (\eta^1, \eta^2) \in \mathcal{P}_N \right\} \quad . \tag{3.11}
$$

Then it is $\mathrm{d}s_\tau = \| \boldsymbol{g}_1 \times \boldsymbol{g}_2 \| \ \mathrm{d}\eta^1 \, \mathrm{d}\eta^2$ and the boundary integral in (3.4) can be transformed.

$$
\begin{aligned}
\langle \boldsymbol{n}_\tau \cdot \boldsymbol{\sigma} \,,\, \boldsymbol{v} \rangle_{0,\Gamma_{N,\tau}} &= \int_{\Gamma_{N,\tau}} \boldsymbol{n}_\tau \cdot \boldsymbol{\sigma} \cdot \boldsymbol{v} \, \mathrm{d}s_\tau \\
&= \int_{\mathcal{P}_N} \frac{\boldsymbol{g}_1 \times \boldsymbol{g}_2}{\| \boldsymbol{g}_1 \times \boldsymbol{g}_2 \|} \cdot \left(\boldsymbol{\sigma} \cdot \boldsymbol{v} \right) \| \boldsymbol{g}_1 \times \boldsymbol{g}_2 \| \ \mathrm{d}\eta^1 \, \mathrm{d}\eta^2 \\
&= \int_{\mathcal{P}_N} \left[\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{\sigma} \cdot \boldsymbol{v} \right] \mathrm{d}\eta^1 \, \mathrm{d}\eta^2 \\
&= \int_{\mathcal{P}_N} \left[\boldsymbol{\mathcal{F}} \cdot \boldsymbol{G}_1, \boldsymbol{\mathcal{F}} \cdot \boldsymbol{G}_2, J^{-1} \boldsymbol{\mathcal{F}} \cdot \overset{1}{\boldsymbol{\mathcal{T}}} \cdot \boldsymbol{V} \right] \mathrm{d}\eta^1 \, \mathrm{d}\eta^2 \\
&= \int_{\mathcal{P}_N} \frac{\det \boldsymbol{\mathcal{F}}}{J} \left[\boldsymbol{G}_1, \boldsymbol{G}_2, \overset{1}{\boldsymbol{\mathcal{T}}} \cdot \boldsymbol{V} \right] \ \mathrm{d}\eta^1 \, \mathrm{d}\eta^2 \\
&= \int_{\mathcal{P}_N} \frac{\boldsymbol{G}_1 \times \boldsymbol{G}_2}{\| \boldsymbol{G}_1 \times \boldsymbol{G}_2 \|} \cdot \left(\overset{1}{\boldsymbol{\mathcal{T}}} \cdot \boldsymbol{V} \right) \| \boldsymbol{G}_1 \times \boldsymbol{G}_2 \| \, \mathrm{d}\eta^1 \, \mathrm{d}\eta^2 \\
&= \left\langle\, \boldsymbol{n}_0 \cdot \overset{1}{\boldsymbol{\mathcal{T}}} \,,\, \boldsymbol{V} \,\right\rangle_{0,\Omega}
\end{aligned}
\tag{3.12}
$$

Setting $g\big|_{\Gamma_N} := \boldsymbol{n}_0 \cdot \overset{1}{\boldsymbol{\mathcal{T}}}$ we finally get the weak formulation on Ω.

Find $\boldsymbol{U} \in \mathbb{V}_D$ such that (3.13) is fulfilled.

$$\left\langle \overset{2}{\boldsymbol{\mathcal{T}}}, \boldsymbol{\mathcal{E}}(\boldsymbol{U}; \boldsymbol{V}) \right\rangle_{0,\Omega} = \underbrace{\left\langle f, \boldsymbol{V} \right\rangle_{0,\Omega} + \left\langle g, \boldsymbol{V} \right\rangle_{0,\Gamma_N}}_{=: f_0(\boldsymbol{V})} \quad \forall \; \boldsymbol{V} \in \mathbb{V}_0 \tag{3.13}$$

Note: $\quad 0 = \left\langle f + \mathrm{Div}\, \overset{1}{\boldsymbol{\mathcal{T}}}(\boldsymbol{U}, P), \boldsymbol{V} \right\rangle_{0,\Omega} \quad \forall \; \boldsymbol{V} \in \mathbb{V}_0$

3.2 Stress Tensor of nonlinear Elasticity

The tensor $\overset{2}{\boldsymbol{\mathcal{T}}}$ from (3.8) can be derived from the *Clausius-Duhem inequality* (see [4] sec. 2-5 or [3] p. 5).

$$\frac{1}{2} \overset{2}{\boldsymbol{\mathcal{T}}} : \dot{\boldsymbol{\mathcal{C}}} - \rho_0 \, \dot{\widetilde{\phi}}(\boldsymbol{\mathcal{C}}) \geq 0 \tag{3.14}$$

Here $\widetilde{\phi}(\boldsymbol{\mathcal{C}})$ stands for the *free Helmholtz energy density per mass unit* and the dot symbolises the usual time derivative. For simplification we define the *specific strain energy density function per volume unit* $\phi(\boldsymbol{\mathcal{C}})$.

$$\phi(\boldsymbol{\mathcal{C}}) := \rho_0 \, \widetilde{\phi}(\boldsymbol{\mathcal{C}}) \tag{3.15}$$

As it is shown in [4], sec. 5 the inequality (3.14) yields the so called law of *hyperelasticity*.

$$\overset{2}{\boldsymbol{\mathcal{T}}} = 2 \, \frac{\partial \, \phi(\boldsymbol{\mathcal{C}})}{\partial \boldsymbol{\mathcal{C}}} \tag{3.16}$$

3.3 Decomposition ansatz of Flory

We consider the so called *Flory split* of the deformation gradient $\boldsymbol{\mathcal{F}}$ (and later on of the Cauchy-Green strain tensor $\boldsymbol{\mathcal{C}}$) into a deviatoric and a volumetric (isochoric) part (see [3], [5] or [6]).

$$\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_D \cdot \boldsymbol{\mathcal{F}}_V \tag{3.17}$$

The part $\boldsymbol{\mathcal{F}}_D$ shall describe the change of shape whereas $\boldsymbol{\mathcal{F}}_V$ shall describe the change of volume during the deformation $\boldsymbol{\Phi}$. That is why one can postulate

$$\det\Big(\boldsymbol{\mathcal{F}}_D\Big) = 1 \quad , \tag{3.18}$$

$$\det\Big(\boldsymbol{\mathcal{F}}_V\Big) = J \overset{\kappa = 0}{=} 1 \quad . \tag{3.19}$$

These conditions can be fulfilled easily by setting $\boldsymbol{\mathcal{F}}_D$ to $J^{-1/3}\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{F}}_V$ to $J^{1/3}\boldsymbol{\mathcal{I}}$. Analogously one can multiplicatively decompose $\boldsymbol{\mathcal{C}}$ with $\boldsymbol{\mathcal{C}} = \boldsymbol{\mathcal{C}}_D \cdot \boldsymbol{\mathcal{C}}_V$. Both quantities have to fulfil (3.20) and (3.21).

$$\boldsymbol{\mathcal{C}}_D = {\boldsymbol{\mathcal{F}}_D}^{\mathrm{T}} \cdot \boldsymbol{\mathcal{F}}_D = \left(J^{-1/3}\boldsymbol{\mathcal{F}}\right)^{\mathrm{T}} \cdot \left(J^{-1/3}\boldsymbol{\mathcal{F}}\right) = J^{-2/3}\boldsymbol{\mathcal{C}} \tag{3.20}$$

$$\boldsymbol{\mathcal{C}}_V = {\boldsymbol{\mathcal{F}}_V}^{\mathrm{T}} \cdot \boldsymbol{\mathcal{F}}_V = \left(J^{1/3}\boldsymbol{\mathcal{I}}\right)^{\mathrm{T}} \cdot \left(J^{1/3}\boldsymbol{\mathcal{I}}\right) = J^{2/3}\boldsymbol{\mathcal{I}} \tag{3.21}$$

This decomposition also affects the specific strain energy density, in such a way that $\phi(\boldsymbol{\mathcal{C}})$ splits into two additive parts. Again we get a deviatoric part $\phi_D(\boldsymbol{\mathcal{C}})$ which is dedicated to the energy of the changing shape and a volumetric part $\phi_V(\boldsymbol{\mathcal{C}})$ which only describes the energy of the changing volume during a deformation.

$$\phi(\boldsymbol{\mathcal{C}}) = \phi_D(\boldsymbol{\mathcal{C}}_D) + U(\boldsymbol{\mathcal{C}}_V) = \phi_D(\boldsymbol{\mathcal{C}}_D) + \frac{\mathtt{K}}{2}\phi_V(\boldsymbol{\mathcal{C}}_V)^2 \tag{3.22}$$

As we consider isotropic and objective material we can choose an energy density that only depends on the three tensor invariants (3.25) instead of the tensor itself.

$$\begin{aligned}
\phi(\boldsymbol{\mathcal{C}}) = \quad & \phi_D(I_1(\boldsymbol{\mathcal{C}}_D), I_2(\boldsymbol{\mathcal{C}}_D), I_3(\boldsymbol{\mathcal{C}}_D)) \\
& + \frac{\mathtt{K}}{2}\phi_V(I_1(\boldsymbol{\mathcal{C}}_V), I_2(\boldsymbol{\mathcal{C}}_V), I_3(\boldsymbol{\mathcal{C}}_V))^2 && (3.23)\\
\text{with} \quad & I_1(\boldsymbol{\mathcal{Y}}) = \operatorname{tr}(\boldsymbol{\mathcal{Y}}) && (3.24)\\
& I_2(\boldsymbol{\mathcal{Y}}) = \frac{1}{2}\left(\operatorname{tr}(\boldsymbol{\mathcal{Y}})^2 - \operatorname{tr}\left(\boldsymbol{\mathcal{Y}}^2\right)\right) && (3.25)\\
& I_3(\boldsymbol{\mathcal{Y}}) = \det(\boldsymbol{\mathcal{Y}}) \qquad \forall\ \boldsymbol{\mathcal{Y}} \in \mathbb{T}_2
\end{aligned}$$

With (3.20) and (3.21) the identity $I_3(\boldsymbol{\mathcal{C}}_D) = 1$ follows. In addition we can show that $I_k(\boldsymbol{\mathcal{C}}_V)$ depends only on J for all $k = 1, 2, 3$. Therefore we can omit the dependencies on $I_3(\boldsymbol{\mathcal{C}}_D)$ and replace $I_k(\boldsymbol{\mathcal{C}}_V)$ by J^2. Additionally we also decide to omit all terms depending on $I_2(\boldsymbol{\mathcal{C}}_D)$.

$$\phi(\boldsymbol{\mathcal{C}}) = \phi_D(I_1(\boldsymbol{\mathcal{C}}_D)) + \frac{\mathtt{K}}{2}\phi_V(J^2)^2 \tag{3.26}$$

We also want to achieve that $\phi_V(J^2)$ is convex and is zero in $J = 1$. In this case a suitable material function is the *Neo-Hooke material* (see [5]). Plugging in our values this yields (3.27).

$$\begin{aligned}
\phi(\boldsymbol{\mathcal{C}}) &= c_{10}(I_1(\boldsymbol{\mathcal{C}}_D) - 3) + \frac{\mathtt{K}}{2}\left(\ln(J)\right)^2 \\
&= \underbrace{c_{10}\left(I_1(\boldsymbol{\mathcal{C}})I_3(\boldsymbol{\mathcal{C}})^{-1/3} - 3\right)}_{=:\,\phi_D(I_1(\boldsymbol{\mathcal{C}}), I_3(\boldsymbol{\mathcal{C}}))} + \underbrace{\frac{\mathtt{K}}{2}\left(\tfrac{1}{2}\ln(I_3(\boldsymbol{\mathcal{C}}))\right)^2}_{=:\,\frac{\mathtt{K}}{2}\phi_V(I_3(\boldsymbol{\mathcal{C}}))^2} && (3.27)
\end{aligned}$$

Note: As an abbreviation we use $I_k := I_k(\boldsymbol{\mathcal{C}})$ from now on.

3.4 Hydrostatic Pressure

With (3.16) and (3.27) we can define the second Piola-Kirchhoff stress tensor more precisely.

$$\overset{2}{\boldsymbol{\mathcal{T}}}(\boldsymbol{\mathcal{C}}) = 2\,\frac{\partial\,\phi(\boldsymbol{\mathcal{C}})}{\partial\,\boldsymbol{\mathcal{C}}} = \underbrace{2\,\frac{\partial\,\phi_D}{\partial\,\boldsymbol{\mathcal{C}}}}_{=:\boldsymbol{\mathcal{T}}_D} + \mathtt{K}\,\phi_V\;\underbrace{2\,\frac{\partial\,\phi_V}{\partial\,\boldsymbol{\mathcal{C}}}}_{=:\boldsymbol{\mathcal{S}}_V} \tag{3.28}$$

However the product term $\left(\mathtt{K}\,\phi_V\right)$ cannot be determined explicitly, because $\mathtt{K}$ diverges according to (2.29) and ϕ_V vanishes for $J = 1$. One way out is given by the substitution (3.29), which eliminates the problematic term.

$$P := \mathtt{K}\,\phi_V(I_3) = \frac{1}{\kappa}\;\phi_V(I_3) \tag{3.29}$$

The new material parameter κ in (3.29) is defined as the reciprocal of the bulk modulus and it is called the compressibility of the material. Obviously, we have $\kappa = 0$ for incompressible material and $\kappa \gtrsim 0$ for nearly incompressible material.

After plugging in the substitution (3.29) in (3.28) we obtain a new formulation of the stress

$$\overset{2}{\boldsymbol{\mathcal{T}}} = \overset{2}{\boldsymbol{\mathcal{T}}}(\boldsymbol{U}, P) = \overset{2}{\boldsymbol{\mathcal{T}}}(\boldsymbol{\mathcal{C}}(\boldsymbol{U}), P) = \boldsymbol{\mathcal{T}}_D(\boldsymbol{\mathcal{C}}) + P\cdot\boldsymbol{\mathcal{S}}_V(\boldsymbol{\mathcal{C}}) \tag{3.30}$$

together with the side condition (3.31), which is later used in a weak sense.

$$\phi_V(I_3) - \kappa\,P = 0 \tag{3.31}$$

3.5 Piola-Kirchhoff stress

The derivative $\frac{\partial\,\phi(\boldsymbol{\mathcal{C}})}{\partial\,\boldsymbol{\mathcal{C}}}$ in (3.28) can be build with the help of the pseudo invariants a_k (see [7], [8]).

$$a_i(\boldsymbol{\mathcal{Y}}) := \frac{1}{i}\,\mathrm{tr}\left(\boldsymbol{\mathcal{Y}}^i\right) \qquad i = 1, 2, 3,\ \forall\,\boldsymbol{\mathcal{Y}} \in \mathbb{T}_2 \tag{3.32}$$

$$\underline{a} = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}^T \qquad \text{with } a_i := a_i(\boldsymbol{\mathcal{C}}) \tag{3.33}$$

They permit the reformulation of the tensor invariants

$$\begin{aligned} I_1(\boldsymbol{\mathcal{Y}}) &= a_1(\boldsymbol{\mathcal{Y}}) \\ I_2(\boldsymbol{\mathcal{Y}}) &= \frac{1}{2}\left(a_1(\boldsymbol{\mathcal{Y}})^2 - 2a_2(\boldsymbol{\mathcal{Y}})\right) \\ I_3(\boldsymbol{\mathcal{Y}}) &= \frac{1}{6}\left(a_1(\boldsymbol{\mathcal{Y}})^3 - 6a_1(\boldsymbol{\mathcal{Y}})a_2(\boldsymbol{\mathcal{Y}}) + 6a_3(\boldsymbol{\mathcal{Y}})\right) \end{aligned} \tag{3.34}$$

and they have a simple derivative w. r. t. any second order tensor $\boldsymbol{\mathcal{Y}}$ (see [8], use *Taylor expansion*).

$$\frac{\partial\, a_i(\boldsymbol{\mathcal{Y}})}{\partial\, \boldsymbol{\mathcal{Y}}} = \boldsymbol{\mathcal{Y}}^{i-1} \tag{3.35}$$

With (3.34) the specific energy density function can be reformulated in terms of $\underline{a}$, instead of (3.27).

$$\phi(\boldsymbol{\mathcal{C}}) = \phi_V(a_1, I_3(\underline{a})) + \frac{\mathtt{K}}{2}\phi_V(I_3(\underline{a}))^2 \tag{3.36}$$

$$= \phi_V(\underline{a}) + \frac{\mathtt{K}}{2}\phi_V(\underline{a})^2 \tag{3.37}$$

Note: We keep the notation ϕ_D and ϕ_V although the dependencies of the functions has been changed.

By applying the chain rule to (3.37) the derivatives of $\phi_D(\boldsymbol{\mathcal{C}})$ and $\phi_V(\boldsymbol{\mathcal{C}})$ with respect to $\boldsymbol{\mathcal{C}}$ can be determined as the second order tensors

$$\boldsymbol{\mathcal{T}}_D = 2\,\frac{\partial\, \phi_D(\boldsymbol{\mathcal{C}})}{\partial\, \boldsymbol{\mathcal{C}}} = 2\sum_{i=1}^{3}\left(\frac{\partial\, \phi_D(\underline{a})}{\partial\, a_i}\frac{\partial\, a_i}{\partial\, \boldsymbol{\mathcal{C}}}\right) \quad , \tag{3.38}$$

$$\boldsymbol{\mathcal{S}}_V = 2\,\frac{\partial\, \phi_V(\boldsymbol{\mathcal{C}})}{\partial\, \boldsymbol{\mathcal{C}}} = 2\sum_{i=1}^{3}\left(\frac{\partial\, \phi_V(\underline{a})}{\partial\, a_i}\frac{\partial\, a_i}{\partial\, \boldsymbol{\mathcal{C}}}\right) \quad . \tag{3.39}$$

In general we get

$$\boldsymbol{\mathcal{T}}_D = 2\,\frac{\partial\, \phi_D}{\partial\, a_1}\boldsymbol{\mathcal{I}} + 2\,\frac{\partial\, \phi_D}{\partial\, a_2}\boldsymbol{\mathcal{C}} + 2\,\frac{\partial\, \phi_D}{\partial\, a_3}\boldsymbol{\mathcal{C}}^2 \quad , \tag{3.40}$$

$$\boldsymbol{\mathcal{S}}_V = 2\,\frac{\partial\, \phi_V}{\partial\, a_1}\boldsymbol{\mathcal{I}} + 2\,\frac{\partial\, \phi_V}{\partial\, a_2}\boldsymbol{\mathcal{C}} + 2\,\frac{\partial\, \phi_V}{\partial\, a_3}\boldsymbol{\mathcal{C}}^2 \quad . \tag{3.41}$$

If the dependency on I_1 and I_3 (see (3.27)) is used directly we obtain in the same way

$$\boldsymbol{\mathcal{T}}_D = 2\, c_{10} I_3^{-1/3}\left(\boldsymbol{\mathcal{I}} - \frac{a_1}{3}\boldsymbol{\mathcal{C}}^{-1}\right) \quad , \tag{3.42}$$

$$\boldsymbol{\mathcal{S}}_V = \boldsymbol{\mathcal{C}}^{-1} \quad . \tag{3.43}$$

The equivalence of (3.42), (3.43) with (3.40), (3.41) would follow from the *theorem of Cayley-Hamilton* as well.

3.6 Mixed Formulation

With the decomposition (3.30) of $\overset{2}{\boldsymbol{\mathcal{T}}}$ we get the equation (3.44) instead of (3.13).

$$\langle\, \boldsymbol{\mathcal{T}}_D(\boldsymbol{U})\,,\, \boldsymbol{\mathcal{E}}(\boldsymbol{U};\, \boldsymbol{V})\,\rangle_{0,\Omega} + \langle\, P\cdot \boldsymbol{\mathcal{S}}_V(\boldsymbol{U})\,,\, \boldsymbol{\mathcal{E}}(\boldsymbol{U};\, \boldsymbol{V})\,\rangle_{0,\Omega} = f_0(\boldsymbol{V}) \qquad \forall\ \boldsymbol{V} \in \mathbb{V}_0 \tag{3.44}$$

Additionally we have to guarantee condition (3.31) in a weak sense. Therefor we choose suitable test functions $Q \in \mathbb{Q} := L^2(\Omega)$ and get

$$\langle \phi_V(I_3(\boldsymbol{U})) \, , \, Q \, \rangle_{0,\Omega} - \langle \, \kappa \cdot P \, , \, Q \, \rangle_{0,\Omega} = 0 \qquad \forall \, Q \in \mathbb{Q} \quad . \tag{3.45}$$

We introduce a new notation

$$a_D(\boldsymbol{U}; \, \boldsymbol{V}) = \langle \, \boldsymbol{\mathcal{T}}_D(\boldsymbol{U}) \, , \, \boldsymbol{\mathcal{E}}(\boldsymbol{U}; \, \boldsymbol{V}) \, \rangle_{0,\Omega} \tag{3.46}$$

$$a_V(\boldsymbol{U}, P; \, \boldsymbol{V}) = \langle \, P \cdot \boldsymbol{\mathcal{S}}_V(\boldsymbol{U}) \, , \, \boldsymbol{\mathcal{E}}(\boldsymbol{U}; \, \boldsymbol{V}) \, \rangle_{0,\Omega} \tag{3.47}$$

$$b_0(\boldsymbol{U}; Q) = \langle \, \phi_V(I_3(\boldsymbol{U})) \, , \, Q \, \rangle_{0,\Omega} \tag{3.48}$$

$$c(P; Q) = \langle \, \kappa \cdot P \, , \, Q \, \rangle_{0,\Omega} \tag{3.49}$$

$$f_0(\boldsymbol{V}) = \langle \, f \, , \, \boldsymbol{V} \, \rangle_{0,\Omega} + \langle g \, , \, \boldsymbol{V} \rangle_{0,\Gamma_N} \tag{3.50}$$

that allows us to formulate the mixed boundary value problem of (nearly) incompressible nonlinear elasticity with large deformations via an nonlinear system of equations. This yields the nonlinear saddle-point like problem.

Find the displacement $\boldsymbol{U} \in \mathbb{V}_D$ and the hydrostatic pressure $P \in \mathbb{Q}$ such that

$$\begin{array}{rcrcc} a_D(\boldsymbol{U}; \, \boldsymbol{V}) & + & a_V(\boldsymbol{U}, P; \, \boldsymbol{V}) & = & f_0(\boldsymbol{V}) \\ b_0(\boldsymbol{U}; Q) & - & c(P; Q) & = & 0 \end{array} \tag{3.51}$$

holds for all test functions $(\boldsymbol{V}, Q) \in \mathbb{V}_0 \times \mathbb{Q}$.

4 Solution method

To solve problem (3.51) we use *Newton's method* to linearise the equations and mixed finite elements to discretise them afterwards. The resulting system can be solved with a *method of conjugate gradients* due to *Bramble and Pasciak* (see [9] or [10]). Altogether this leads to a nested iteration method.

4.1 Newton's method

First of all we transform (3.51) into a homogeneous problem.

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_D(\boldsymbol{U};\, \boldsymbol{V}) + a_V(\boldsymbol{U}, P;\, \boldsymbol{V}) - f_0(\boldsymbol{V}) \\ b_0(\boldsymbol{U}; Q) - c(P; Q) \end{pmatrix} =: S(\boldsymbol{U}, P;\, \boldsymbol{V}, Q) \tag{4.1}$$

Even though (4.1) seems to be nonlinear in $\boldsymbol{U}$ only, we have to perform a linearisation in both $\boldsymbol{U}$ and P. This can be done with *Newton's method*. We choose a initial solution $(\boldsymbol{U}_0, P_0)$, solve equation (4.2) several times for $i \geq 0$

$$S'(\boldsymbol{U}_i, P_i;\, \boldsymbol{V}, Q, \delta\boldsymbol{U}, \delta P) = -S(\boldsymbol{U}_i, P_i;\, \boldsymbol{V}, Q) \qquad \forall\ (\boldsymbol{V}, Q) \in \mathbb{V}_0 \times \mathbb{Q} \tag{4.2}$$

and update the current initial solution $(\boldsymbol{U}_i, P_i)$ with the increment $(\delta\boldsymbol{U}, \delta P)$ in each step.

$$\begin{pmatrix} \boldsymbol{U}_{i+1} \\ P_{i+1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{U}_i \\ P_i \end{pmatrix} + \begin{pmatrix} \delta\boldsymbol{U} \\ \delta P \end{pmatrix} \tag{4.3}$$

Because this method has only local convergence we additionally use incremental load steps. Instead of (4.2) and (4.1) we consider the operator (4.4) with $t \in (0, 1] : t \to 1$ and equation (4.5) for all test functions $\boldsymbol{V} \in \mathbb{V}_0$ und $Q \in \mathbb{Q}$.

$$\begin{pmatrix} a_D(\boldsymbol{U};\, \boldsymbol{V}) + a_V(\boldsymbol{U}, P;\, \boldsymbol{V}) - t\, f_0(\boldsymbol{V}) \\ b_0(\boldsymbol{U}; Q) - c(P; Q) \end{pmatrix} = S(\boldsymbol{U}, P, t;\, \boldsymbol{V}, Q) \tag{4.4}$$

$$S'(\boldsymbol{U}, P;\, \boldsymbol{V}, Q, \delta\boldsymbol{U}, \delta P) = -S(\boldsymbol{U}, P, t;\, \boldsymbol{V}, Q) \tag{4.5}$$

For any fixed t we choose $\frac{\|\delta\boldsymbol{U}\|}{\|\boldsymbol{U}\|} < \varepsilon_{\text{tol}}$ as the stopping criteria. For $t = 1$, this yields the approximate solution of the problem (3.51).

Note: We abbreviate $S := S(\boldsymbol{U}, P, t;\, \boldsymbol{V}, Q)$ and $S' := S'(\boldsymbol{U}, P;\, \boldsymbol{V}, Q, \delta\boldsymbol{U}, \delta P)$.

4.2 Newton's equation

We need the representation of the linear operator S' as the first derivative of S applyied to $(\delta\boldsymbol{U}, \delta P)$. This follows from the *Taylor expansion* (4.6) of S with omitting the sufficient small nonlinear terms $\mathcal{O}(\|\delta P\|^2)$, $\mathcal{O}\left(\|\delta\boldsymbol{U}\|^2\right)$ and $\mathcal{O}(\|\delta\boldsymbol{U}\| \cdot \|\delta P\|)$

as well as the reordering of the remaining terms w. r. t. S' and S.

$$\begin{aligned}
&S(\boldsymbol{U}+\delta\boldsymbol{U}, P+\delta P, t;\, \boldsymbol{V}, Q)\\
&\quad= \begin{pmatrix} a_D(\boldsymbol{U}+\delta\boldsymbol{U};\, \boldsymbol{V}) + a_V(\boldsymbol{U}+\delta\boldsymbol{U}, P+\delta P;\, \boldsymbol{V}) - t\, f_0(\boldsymbol{V}) \\ b_0(\boldsymbol{U}+\delta\boldsymbol{U}; Q) - c(P+\delta P, Q) \end{pmatrix} && (4.6)\\
&\quad= \quad S(\boldsymbol{U}, P, t;\, \boldsymbol{V}, Q)\\
&\qquad + S'(\boldsymbol{U}, P;\, \boldsymbol{V}, Q, \delta\boldsymbol{U}, \delta P)\\
&\qquad + \mathcal{O}\Big(\|\delta\boldsymbol{U}\|^2\Big) + \mathcal{O}\Big(\|\delta P\|^2\Big) + \mathcal{O}\Big(\|\delta\boldsymbol{U}\| \cdot \|\delta P\|\Big) && (4.7)
\end{aligned}$$

Therefor we consider the *Taylor expansions* of the integral forms (3.46) - (3.49) and of the integrands $\boldsymbol{\mathcal{E}}(\boldsymbol{U})$, $\boldsymbol{\mathcal{E}}(\boldsymbol{U};\, \boldsymbol{V})$, $\boldsymbol{\mathcal{T}}_D(\boldsymbol{U})$, $\boldsymbol{\mathcal{S}}_V(\boldsymbol{U})$ and $\phi_V(\boldsymbol{U})$. Althogether this yields (4.8) and (4.9).

$$\begin{aligned}
&a_D(\boldsymbol{U}+\delta\boldsymbol{U};\, \boldsymbol{V}) + a_V(\boldsymbol{U}+\delta\boldsymbol{U}, P+\delta P;\, \boldsymbol{V}) - t\, f_0(\boldsymbol{V})\\
&\qquad= \quad a_D(\boldsymbol{U};\, \boldsymbol{V}) + a_V(\boldsymbol{U}, P;\, \boldsymbol{V}) - t\, f_0(\boldsymbol{V})\\
&\qquad\quad + a_V(\boldsymbol{U}, \delta P;\, \boldsymbol{V})\\
&\qquad\quad + \Big\langle\, \boldsymbol{\mathcal{E}}(\boldsymbol{U};\, \boldsymbol{V})\,,\, \Big(P \cdot 2\frac{\partial\, \boldsymbol{\mathcal{S}}_V(\boldsymbol{\mathcal{C}})}{\partial\, \boldsymbol{\mathcal{C}}} + 2\frac{\partial\, \boldsymbol{\mathcal{T}}_D(\boldsymbol{\mathcal{C}})}{\partial\, \boldsymbol{\mathcal{C}}}\Big) : \boldsymbol{\mathcal{E}}(\boldsymbol{U}; \delta\boldsymbol{U})\, \Big\rangle_{0,\Omega}\\
&\qquad\quad + \Big\langle\, \mathrm{Grad}\, \boldsymbol{V}\,,\, \Big(P{\cdot}\boldsymbol{\mathcal{S}}_V(\boldsymbol{U}) + \boldsymbol{\mathcal{T}}_D(\boldsymbol{U})\Big) \cdot \mathrm{Grad}\, \delta\boldsymbol{U}\, \Big\rangle_{0,\Omega}\\
&\qquad\quad + \mathcal{O}\Big(\|\delta\boldsymbol{U}\|^2\Big) + \mathcal{O}\Big(\|\delta P\|^2\Big) + \mathcal{O}\Big(\|\delta\boldsymbol{U}\| \cdot \|\delta P\|\Big) && (4.8)\\
&b_0(\boldsymbol{U}+\delta\boldsymbol{U}; Q) - c(P+\delta P, Q)\\
&\qquad= \quad b_0(\boldsymbol{U}; Q) - c(P, Q)\\
&\qquad\quad + \langle\, Q{\cdot}\boldsymbol{\mathcal{S}}_V(\boldsymbol{U})\,,\, \boldsymbol{\mathcal{E}}(\boldsymbol{U}; \delta\boldsymbol{U})\, \rangle_{0,\Omega} - c(\delta P, Q)\\
&\qquad\quad + \mathcal{O}\Big(\|\delta\boldsymbol{U}\|^2\Big) && (4.9)
\end{aligned}$$

Using the material tensor

$$\mathfrak{M}(\boldsymbol{U}, P) := \underbrace{2\frac{\partial\, \boldsymbol{\mathcal{T}}_D(\boldsymbol{\mathcal{C}})}{\partial\, \boldsymbol{\mathcal{C}}}}_{=:\, \mathfrak{M}_D(\boldsymbol{U})} + P \cdot \underbrace{2\frac{\partial\, \boldsymbol{\mathcal{S}}_V(\boldsymbol{\mathcal{C}})}{\partial\, \boldsymbol{\mathcal{C}}}}_{=:\, \mathfrak{M}_V(\boldsymbol{U})} \qquad (4.10)$$

we introduce a new notation.

$$a(\boldsymbol{U}, P; \delta\boldsymbol{U}, \boldsymbol{V}) = \Big\langle \boldsymbol{\mathcal{E}}(\boldsymbol{U}; \boldsymbol{V}), \mathfrak{M}(\boldsymbol{U}, P) : \boldsymbol{\mathcal{E}}(\boldsymbol{U}; \delta\boldsymbol{U}) \Big\rangle_{0,\Omega} + \left\langle \text{Grad}\,\boldsymbol{V}, \overset{2}{\boldsymbol{\mathcal{T}}}(\boldsymbol{U}, P) \cdot \text{Grad}\,\delta\boldsymbol{U} \right\rangle_{0,\Omega} \tag{4.11}$$

$$b(\boldsymbol{U}; \delta P, \delta\boldsymbol{U}) = \langle \delta P \cdot \boldsymbol{\mathcal{S}}_V(\boldsymbol{U}), \boldsymbol{\mathcal{E}}(\boldsymbol{U}; \delta\boldsymbol{U}) \rangle_{0,\Omega} \tag{4.12}$$

$$c(\delta P, Q) = \langle \kappa \cdot \delta P, Q \rangle_{0,\Omega} \tag{4.13}$$

$$f(t, \boldsymbol{U}, P; \boldsymbol{V}) = t\, f_0(\boldsymbol{V}) - a_D(\boldsymbol{U}; \boldsymbol{V}) - a_V(\boldsymbol{U}, P; \boldsymbol{V}) \tag{4.14}$$

$$g(\boldsymbol{U}, P; Q) = c(P, Q) - b_0(\boldsymbol{U}; Q) \tag{4.15}$$

Finally by omitting the nonlinear terms in (4.8) and (4.9) resp. the operators S' und S are built.

$$S'(\boldsymbol{U}, P; \boldsymbol{V}, Q, \delta\boldsymbol{U}, \delta P) = \begin{pmatrix} a(\boldsymbol{U}, P; \delta\boldsymbol{U}, \boldsymbol{V}) + b(\boldsymbol{U}; \delta P, \boldsymbol{V}) \\ b(\boldsymbol{U}; Q, \delta\boldsymbol{U}) - \quad c(\delta P, Q) \end{pmatrix}, \tag{4.16}$$

$$-S(\boldsymbol{U}, P, t; \boldsymbol{V}, Q) = \begin{pmatrix} f(t, \boldsymbol{U}, P; \boldsymbol{V}) \\ g(\boldsymbol{U}, P; Q) \end{pmatrix}. \tag{4.17}$$

This formulation together with the linear *Newton's equation* leads to the *linear saddle-point problem* in each *Newton's step.*

Let $(t, \boldsymbol{U}, P)$ be a given tripel in $(0,1] \times \mathbb{V}_D \times \mathbb{Q}$. Find the solution pair $(\delta\boldsymbol{U}, \delta P) \in \mathbb{V}_0 \times \mathbb{Q}$ that fulfils the system (4.18)

$$\begin{aligned} a(\boldsymbol{U}, P; \delta\boldsymbol{U}, \boldsymbol{V}) + b(\boldsymbol{U}; \delta P, \boldsymbol{V}) &= f(t, \boldsymbol{U}, P; \boldsymbol{V}) \\ b(\boldsymbol{U}; Q, \delta\boldsymbol{U}) - \quad c(\delta P, Q) &= \quad g(\boldsymbol{U}, P; Q) \end{aligned} \tag{4.18}$$

for all test functions $(\boldsymbol{V}, Q) \in \mathbb{V}_0 \times \mathbb{Q}$.

4.3 Material tensor

We want to take a closer look on the material tensor (4.10). If we plug in the representations (3.38) and (3.39) of the stress tensors we can give a more detailed description of $\mathfrak{M}$.

$$\mathfrak{M}_* = 2\frac{\partial}{\partial \boldsymbol{\mathcal{C}}}\left(2\sum_{i=1}^{3}\left(\frac{\partial\, \phi_*(\underline{a})}{\partial\, a_i}\boldsymbol{\mathcal{C}}^{i-1}\right)\right) \qquad \text{for } * = D/V \tag{4.19}$$

$$
\begin{aligned}
\mathfrak{M}_* &= 4\sum_{i=1}^{3}\left(\boldsymbol{\mathcal{C}}^{i-1}\frac{\partial}{\partial\boldsymbol{\mathcal{C}}}\left(\frac{\partial\,\phi_*(\underline{a})}{\partial\,a_i}\right)+\frac{\partial\,\phi_*(\underline{a})}{\partial\,a_i}\frac{\partial\,\boldsymbol{\mathcal{C}}^{i-1}}{\partial\boldsymbol{\mathcal{C}}}\right)\\
&= 4\sum_{i,j=1}^{3}\left(\boldsymbol{\mathcal{C}}^{i-1}\frac{\partial}{\partial\,a_j}\left(\frac{\partial\,\phi_*}{\partial\,a_i}\right)\frac{\partial\,a_j}{\partial\boldsymbol{\mathcal{C}}}\right)+4\sum_{i=1}^{3}\left(\frac{\partial\,\phi_*}{\partial\,a_i}\frac{\partial\,\boldsymbol{\mathcal{C}}^{i-1}}{\partial\boldsymbol{\mathcal{C}}}\right)\\
&= 4\sum_{i,j=1}^{3}\left(\frac{\partial^2\,\phi_*}{\partial\,a_i\,a_j}\boldsymbol{\mathcal{C}}^{i-1}\boldsymbol{\mathcal{C}}^{j-1}\right)+4\sum_{i=1}^{3}\left(\frac{\partial\,\phi_*}{\partial\,a_i}\frac{\partial\,\boldsymbol{\mathcal{C}}^{i-1}}{\partial\boldsymbol{\mathcal{C}}}\right)
\end{aligned}
\tag{4.20}
$$

The new tensors $\boldsymbol{\mathcal{C}}^{i-1}\boldsymbol{\mathcal{C}}^{j-1}$ and $\dfrac{\partial\,\boldsymbol{\mathcal{C}}^{i-1}}{\partial\boldsymbol{\mathcal{C}}}$ are of order four with

$$
\frac{\partial\,\boldsymbol{\mathcal{C}}^{i-1}}{\partial\boldsymbol{\mathcal{C}}}=\begin{cases}\dfrac{\partial\boldsymbol{\mathcal{I}}}{\partial\boldsymbol{\mathcal{C}}}=\mathfrak{o} & \text{for } i=1\\ \dfrac{\partial\boldsymbol{\mathcal{C}}}{\partial\boldsymbol{\mathcal{C}}}=\mathfrak{I} & \text{for } i=2\\ \mathfrak{C} & \text{for } i=3\end{cases}
\tag{4.21}
$$

Thereby it is $\mathfrak{o}$ the forth order null tensor and $\mathfrak{I}$ the forth order unit tensor. $\mathfrak{C}$ is the tensor that realises definition (4.23). For later use we introduce another forth order tensor $\hat{\mathfrak{C}}$ with (4.24).

$$
\mathfrak{I}:\boldsymbol{\mathcal{Y}}=\boldsymbol{\mathcal{Y}} \qquad \forall\,\boldsymbol{\mathcal{Y}}\in\mathbb{T}_2 \tag{4.22}
$$

$$
\mathfrak{C}:\boldsymbol{\mathcal{Y}}=\boldsymbol{\mathcal{Y}}\cdot\boldsymbol{\mathcal{C}}+\boldsymbol{\mathcal{C}}\cdot\boldsymbol{\mathcal{Y}} \qquad \forall\,\boldsymbol{\mathcal{Y}}\in\mathbb{T}_2 \tag{4.23}
$$

$$
\hat{\mathfrak{C}}:\boldsymbol{\mathcal{Y}}=\boldsymbol{\mathcal{C}}^{-1}\cdot\boldsymbol{\mathcal{Y}}\cdot\boldsymbol{\mathcal{C}}^{-1} \qquad \forall\,\boldsymbol{\mathcal{Y}}\in\mathbb{T}_2 \tag{4.24}
$$

Reformulating both parts of the material tensor from (4.20)

$$
\mathfrak{M}_*=4\sum_{i,j=1}^{3}\left(\frac{\partial^2\phi_*}{\partial a_i\,\partial a_j}\boldsymbol{\mathcal{C}}^{j-1}\boldsymbol{\mathcal{C}}^{i-1}\right)+4\,\frac{\partial\,\phi_*}{\partial\,a_2}\mathfrak{I}+4\,\frac{\partial\,\phi_*}{\partial\,a_3}\mathfrak{C}
\tag{4.25}
$$

yields (4.26).

$$
\begin{aligned}
\mathfrak{M}(\boldsymbol{U},P)= \quad & 4\sum_{i,j=1}^{3}\left(\left\{\frac{\partial^2\phi_D}{\partial a_i\,\partial a_j}+P\cdot\frac{\partial^2\phi_V}{\partial a_i\,\partial a_j}\right\}\boldsymbol{\mathcal{C}}^{j-1}\boldsymbol{\mathcal{C}}^{i-1}\right)\\
&+4\left(\frac{\partial\,\phi_D}{\partial\,a_2}+P\cdot\frac{\partial\,\phi_V}{\partial\,a_2}\right)\mathfrak{I}+4\left(\frac{\partial\,\phi_D}{\partial\,a_3}+P\cdot\frac{\partial\,\phi_V}{\partial\,a_3}\right)\mathfrak{C}
\end{aligned}
\tag{4.26}
$$

By means of the material function (3.27) that is under consideration we can find a more detailed representation.

$$\begin{aligned}\mathfrak{M}(\boldsymbol{U},P) = &\quad 4{I_3}^2\left(\frac{\partial^2\phi_D}{\partial {I_3}^2}+P\frac{\partial^2\phi_V}{\partial {I_3}^2}\right)\boldsymbol{\mathcal{C}}^{-1}\boldsymbol{\mathcal{C}}^{-1}\\ &+4\left(\frac{\partial\phi_D}{\partial I_3}+P\frac{\partial\phi_V}{\partial I_3}\right)\left(a_1\boldsymbol{\mathcal{I}}\boldsymbol{\mathcal{I}}-\boldsymbol{\mathcal{I}}\boldsymbol{\mathcal{C}}-\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{I}}-a_1\mathfrak{J}+\mathfrak{C}\right)\\ &+4I_3\frac{\partial^2\phi_D}{\partial a_1\,\partial I_3}\left(\boldsymbol{\mathcal{I}}\boldsymbol{\mathcal{C}}^{-1}+\boldsymbol{\mathcal{C}}^{-1}\boldsymbol{\mathcal{I}}\right)\end{aligned} \tag{4.27}$$

$$\begin{aligned}\mathfrak{M}(\boldsymbol{U},P) = &-4I_3\left(\frac{\partial\phi_D}{\partial I_3}+P\frac{\partial\phi_V}{\partial I_3}\right)\hat{\mathfrak{C}}\\ &+4I_3\left(\frac{\partial\phi_D}{\partial I_3}+I_3\frac{\partial^2\phi_D}{\partial {I_3}^2}\right)\boldsymbol{\mathcal{C}}^{-1}\boldsymbol{\mathcal{C}}^{-1}\\ &+4I_3\frac{\partial^2\phi_D}{\partial a_1\,\partial I_3}\left(\boldsymbol{\mathcal{C}}^{-1}\boldsymbol{\mathcal{I}}+\boldsymbol{\mathcal{I}}\boldsymbol{\mathcal{C}}^{-1}\right)\end{aligned} \tag{4.28}$$

4.4 Mixed finite element method

The given *linear saddle-point problem* (4.18) shall be solved by means of a mixed FE method. Hence we assume that there is a regular triangulation Υ_h of the domain Ω available with n_T hexahedral elements T_i and n_N nodal points. We choose the stable *Taylor-Hood element* which implies the ansatz of an element wise triquadratic displacement and an element wise trilinear pressure. The computations can be made with the help of the reference element $\widehat{T} = [-1,1]^3$ and the associated transformation map (4.29).

$$B_i: \quad \widehat{T} \to T_i, \qquad \hat{X} \mapsto X \tag{4.29}$$

As suitable function spaces we take (4.30).

$$\begin{aligned}\mathbb{V}_h &= \left\{\boldsymbol{V} \in C(\overline{\Omega})^3 \cap H^1(\Omega)^3 : \boldsymbol{V}|_{T_i}\circ B_i \in \mathcal{Q}_2(\widehat{T})^3 \;\; \forall\, T_i \in \Upsilon_h\right\}\\ \mathbb{V}_{h,0} &= \left\{\boldsymbol{V} \in \mathbb{V}_h : \boldsymbol{V}|_{\Gamma_D} = \boldsymbol{0}\right\}\\ \mathbb{V}_{h,D} &= \left\{\boldsymbol{V} \in \mathbb{V}_h : \boldsymbol{V}|_{\Gamma_D} = \boldsymbol{U}_0\right\}\\ \mathbb{Q}_h &= \left\{Q \;\in C(\Omega) \cap L^2(\Omega) : Q|_{T_i}\circ B_i \in \mathcal{Q}_1(\widehat{T}) \;\; \forall\, T_i \in \Upsilon_h\right\}\end{aligned} \tag{4.30}$$

We choose suitable triquadratic and trilinear nodal ansatz functions $\Phi^{(i)}$ and $\Psi^{(j)}$ resp. to represent the functions of $\mathbb{V}_{h,*}$ and $\mathbb{Q}_h$ as a linear combination of these ansatz functions with certain coefficients $\delta U^{(j)}$ and $\delta P^{(j)}$. That leads to a *discrete saddle point problem* which can be rewritten in matrix-vector form.

Find $\begin{pmatrix} \delta\underline{U} \\ \delta\underline{P} \end{pmatrix}$ such that (4.31) holds.

$$S_h \cdot \begin{pmatrix} \delta\underline{U} \\ \delta\underline{P} \end{pmatrix} := \begin{pmatrix} A & B \\ B^T & -C \end{pmatrix} \cdot \begin{pmatrix} \delta\underline{U} \\ \delta\underline{P} \end{pmatrix} = \begin{pmatrix} \underline{R}_f \\ \underline{R}_g \end{pmatrix} \tag{4.31}$$

At first glance this system (4.31) is indefinite, however it can be solved with the *conjugate gradient method by Bramble and Pasciak*. The idea is to use the matrix ${S_{h,0}}^{-1}$ in (4.32) as a preconditioner on the system and to define a new suitable duality product (4.33).

$${S_{h,0}}^{-1} := \begin{bmatrix} A_0^{-1} & 0 \\ \delta C_0^{-1}\, B^T A_0^{-1} & -\gamma\; \delta C_0^{-1} \end{bmatrix} \tag{4.32}$$

$$\left\langle \underline{x} \, , \, \underline{y} \right\rangle := \left\langle \underline{x}_o \, , \, \underline{y}_o \right\rangle_o + \left\langle \underline{x}_u \, , \, \underline{y}_u \right\rangle_u \tag{4.33}$$

$$:= ((A - \gamma A_0) \cdot \underline{x}_o, \underline{y}_o) + ((\delta^{-1} C_0) \cdot \underline{x}_u, \underline{y}_u) \tag{4.34}$$

Here we choose A_0 to be a preconditioner of A (e. g. with BPX) and C_0 to be a preconditioner of the *Schur complement* $K := B^{\mathrm{T}} {\cdot} {A_0}^{-1} {\cdot} B + \gamma C$ (e. g. by taking the diagonal of C). The positive parameters δ and γ shall guarantee $(A - \gamma A_0) \succ 0$ and decrease the condition number of the given system (see next subsection). To the resulting system we apply the usual *conjugate gradient method*. But we use a *matrix free* method, i. e. the stiffness matrix is never assembled completely but used element wise. According to that the occurring matrix-vector products are determined on an element wise basis.

4.5 Optimal parameter

The described *conjugate gradient method* by *Bramble and Pasciak* is especially good if the "right" parameters γ and δ are used.

To determine γ we use a rather simple method. We consider the value of $\left(\underline{w}^{\mathrm{T}} \cdot A \cdot \underline{w} - \gamma\, \underline{w}^{\mathrm{T}} \cdot A_0 \cdot \underline{w}\right)$. If it is negative we reduce γ by a fixed factor until the condition is fulfilled again.

To determine δ we can use the eigenvalues of the matrices (see [11], section 4.2). First we consider the diagonal of $\left({S_{h,0}}^{-1} \cdot S_h\right)$ that is denoted by M.

$$M = \begin{bmatrix} {A_0}^{-1} {\cdot} A & 0 \\ 0 & \delta {C_0}^{-1} {\cdot} K \end{bmatrix}$$

The corresponding condition number fulfils (4.36).

$$\kappa({S_{h,0}}^{-1} {\cdot} S_h) \le c_A(\gamma, A, A_0)\, \kappa(M) \tag{4.35}$$

$$= c_A \frac{\max \left\{ \lambda_{\max}({A_0}^{-1} {\cdot} A), \delta \lambda_{\max}({C_0}^{-1} {\cdot} K) \right\}}{\min \left\{ \lambda_{\min}({A_0}^{-1} {\cdot} A), \delta \lambda_{\min}({C_0}^{-1} {\cdot} K) \right\}} \tag{4.36}$$

Since the matices are self adjoint

$$\left\langle {A_0}^{-1}{\cdot}A{\cdot}x_o\,,\,y_o\right\rangle_o = \left\langle x_o\,,\,{A_0}^{-1}{\cdot}A{\cdot}y_o\right\rangle_o \tag{4.37}$$

$$\left\langle {C_0}^{-1}{\cdot}K{\cdot}x_u\,,\,y_u\right\rangle_u = \left\langle x_u\,,\,{C_0}^{-1}{\cdot}K{\cdot}y_u\right\rangle_u \tag{4.38}$$

we can estimate the eigenvalues in (4.36) with the *Rayleigh quotient* $\Theta_*(x)$.

$$\Theta_{o,{A_0}^{-1}\cdot A}(x_o) := \frac{\left\langle {A_0}^{-1}{\cdot}A{\cdot}x_o\,,\,x_o\right\rangle_o}{\langle x_o\,,\,x_o\rangle_o} \le \lambda_{\max}({A_0}^{-1}{\cdot}A) \tag{4.39}$$

$$\lambda_{\min}({C_0}^{-1}{\cdot}K) \le \frac{\left\langle {C_0}^{-1}{\cdot}K{\cdot}x_u\,,\,x_u\right\rangle_u}{\langle x_u\,,\,x_u\rangle_u} =: \Theta_{u,{C_0}^{-1}\cdot K}(x_u) \tag{4.40}$$

We choose

$$\delta = \frac{\Theta_{o,{A_0}^{-1}\cdot A}(x_o)}{\Theta_{u,{C_0}^{-1}\cdot K}(x_u)} \le \frac{\lambda_{\max}({A_0}^{-1}{\cdot}A)}{\lambda_{\min}({C_0}^{-1}{\cdot}K)} \tag{4.41}$$

which yields the estimate (4.42) after applying this choice to (4.36).

$$\kappa({S_{h,0}}^{-1}{\cdot}S_h) \le c_A\,\kappa(M) \le c_A\,\kappa({A_0}^{-1}{\cdot}A)\cdot\kappa({C_0}^{-1}{\cdot}K) \tag{4.42}$$

Note: With the exact choice of δ we would get

$$\kappa(M) = \max\left\{\kappa({A_0}^{-1}{\cdot}A), \kappa({C_0}^{-1}{\cdot}K)\right\} \qquad \text{for} \quad \delta = \frac{\lambda_{\max}({A_0}^{-1}{\cdot}A)}{\lambda_{\max}({C_0}^{-1}{\cdot}K)},$$

$$\kappa(M) = \kappa({A_0}^{-1}{\cdot}A)\cdot\kappa({C_0}^{-1}{\cdot}K) \qquad \text{for} \quad \delta = \frac{\lambda_{\max}({A_0}^{-1}{\cdot}A)}{\lambda_{\min}({C_0}^{-1}{\cdot}K)}.$$

5 Error estimation

In order to control the adaptive mesh refinement we need an appropriate error estimator of the approximated solution $(\boldsymbol{U}_h, P_h)$ of (5.1).

$$\begin{aligned} a^*(\boldsymbol{U}_h, P_h;\, \boldsymbol{V}) &= f_0(\boldsymbol{V}) \quad \forall\, \boldsymbol{V} \in \mathbb{V}_{h,0} \\ b_0(\boldsymbol{U}_h; Q) - c(P_h; Q) &= 0 \qquad\quad \forall Q \in \mathbb{Q}_h \end{aligned} \tag{5.1}$$

$$\begin{aligned} \text{s.t.}\quad a^*(\boldsymbol{U}, P;\, \boldsymbol{V}) &:= a_D(\boldsymbol{U};\, \boldsymbol{V}) + a_V(\boldsymbol{U}, P;\, \boldsymbol{V}) \\ &= \left\langle \overset{1}{\boldsymbol{\mathcal{T}}}(\boldsymbol{U}, P)\,,\, \mathrm{Grad}\, \boldsymbol{V} \right\rangle_{0,\Omega} \end{aligned} \tag{5.2}$$

5.1 Reliability

We use $\boldsymbol{e}_{\boldsymbol{U}} \in \mathbb{V}_0$ with $\boldsymbol{e}_{\boldsymbol{U}} := \boldsymbol{U} - \boldsymbol{U}_h$ to denote the error and we define a test function $\boldsymbol{V} \in \mathbb{V}_0$ with $\boldsymbol{V} = I\,\boldsymbol{e}_{\boldsymbol{U}} - I_h \boldsymbol{e}_{\boldsymbol{U}}$ and the projection $I_h : \mathbb{V}_0 \to \mathbb{V}_{h,0}$.

As it is shown in [2] the form $a^*(\boldsymbol{U}, P;\, \boldsymbol{V})$ does not lead to a proper energy norm (in a sense of $\|(\boldsymbol{U}, P)\|_{A^*}$) and the *Galerkin orthogonality* becomes like (5.3).

$$\begin{aligned} \Delta a^*(\boldsymbol{V}) &:= a^*(\boldsymbol{U}, P;\, \boldsymbol{V}) - a^*(\boldsymbol{U}_h, P_h;\, \boldsymbol{V}) \\ &= a^*(\boldsymbol{U}, P;\, \boldsymbol{V}) - \qquad f_0(\boldsymbol{V}) = 0 \qquad \forall\, \boldsymbol{V} \in \mathbb{V}_0 \end{aligned} \tag{5.3}$$

Now we consider (5.4) as a functional to meassure the error, whereas $d_{\boldsymbol{U}}(\boldsymbol{V}, \boldsymbol{V})$ denots some kind of norm.

$$\mathcal{J}\big(\boldsymbol{e}_{\boldsymbol{U}}\big) = \Big(a^*(\boldsymbol{U}, P; \boldsymbol{e}_{\boldsymbol{U}}) - a^*(\boldsymbol{U}_h, P_h; \boldsymbol{e}_{\boldsymbol{U}})\Big) \cdot d_{\boldsymbol{U}}(\boldsymbol{e}_{\boldsymbol{U}}, \boldsymbol{e}_{\boldsymbol{U}})^{-1/2} \tag{5.4}$$

After exploiting the *Galerkin orthogonality* the denominator $\mathcal{D} := \Delta a^*(\boldsymbol{e}_{\boldsymbol{U}})$ yields (5.5).

$$\begin{aligned} \mathcal{D} &= a^*(\boldsymbol{U}, P;\, \boldsymbol{V}) - a^*(\boldsymbol{U}_h, P_h;\, \boldsymbol{V}) \\ &= \left\langle \overset{2}{\boldsymbol{\mathcal{T}}}(\boldsymbol{U}, P)\,,\, \boldsymbol{\mathcal{E}}(\boldsymbol{U};\, \boldsymbol{V}) \right\rangle_{0,\Omega} - \left\langle \overset{2}{\boldsymbol{\mathcal{T}}}(\boldsymbol{U}_h, P_h)\,,\, \boldsymbol{\mathcal{E}}(\boldsymbol{U}_h;\, \boldsymbol{V}) \right\rangle_{0,\Omega} \\ &= \langle\, \rho_0\, f\,,\; \boldsymbol{V}\,\rangle_{0,\Omega} + \langle g\,,\; \boldsymbol{V}\rangle_{0,\Gamma_N} - \left\langle \overset{1}{\boldsymbol{\mathcal{T}}}(\boldsymbol{U}_h, P_h)\,,\, \mathrm{Grad}\, \boldsymbol{V} \right\rangle_{0,\Omega} \end{aligned} \tag{5.5}$$

Using the discretisation Υ_h we get (5.6).

$$\begin{aligned} \mathcal{D} = &\sum_{T \in \Upsilon_h} \langle\, \rho_0\, f\,,\; \boldsymbol{V}\,\rangle_{0,T} - \left\langle \overset{1}{\boldsymbol{\mathcal{T}}}(\boldsymbol{U}_h, P_h)\,,\, \mathrm{Grad}\, \boldsymbol{V} \right\rangle_{0,T} \\ &+ \sum_{F \subset \Gamma_N} \langle\, g\,,\; \boldsymbol{V}\,\rangle_{0,F} \end{aligned} \tag{5.6}$$

We introduce the abbreviation $\overset{1}{\boldsymbol{\mathcal{T}}}_h = \overset{1}{\boldsymbol{\mathcal{T}}}(\boldsymbol{U}_h, P_h)$ and the operator $\mathbb{1}_{F,\Gamma_N}$ that is the identity if $F \in \Gamma_N$ and zero elsewhere. Again we set ρ_0 equal to one, w.l.o.g. The *integral theorem of Gauss* yields (5.7).

$$\begin{aligned}\mathcal{D} &= \sum_{T\in\Upsilon_h} \langle f \,,\, \boldsymbol{V}\rangle_{0,T} + \sum_{F\subset\Gamma_N} \langle g \,,\, \boldsymbol{V}\rangle_{0,F} \\ &\quad + \sum_{T\in\Upsilon_h} \left(\left\langle \operatorname{Div}(\overset{1}{\boldsymbol{\mathcal{T}}}_h) \,,\, \boldsymbol{V} \right\rangle_{0,T} - \sum_{F\subset\partial T} \left\langle \boldsymbol{n}\cdot\overset{1}{\boldsymbol{\mathcal{T}}}_h \,,\, \boldsymbol{V} \right\rangle_{0,F} \right) \\ &= \sum_{T\in\Upsilon_h} \left(\left\langle f + \operatorname{Div}(\overset{1}{\boldsymbol{\mathcal{T}}}_h) \,,\, \boldsymbol{V} \right\rangle_{0,T} + \sum_{F\subset\partial T} \left\langle \mathbb{1}_{F,\Gamma_N}\cdot g - \boldsymbol{n}\cdot\overset{1}{\boldsymbol{\mathcal{T}}}_h \,,\, \boldsymbol{V} \right\rangle_{0,F} \right) \end{aligned} \tag{5.7}$$

We define the residuals $\boldsymbol{R}_T$ and $\boldsymbol{R}_{F(T)}$.

$$\boldsymbol{R}_T := f + \operatorname{Div}(\overset{1}{\boldsymbol{\mathcal{T}}}_h) \tag{5.8}$$

$$\boldsymbol{R}_{F(T)} := \begin{cases} \frac{1}{2}\left(-\left(\boldsymbol{n}\cdot\overset{1}{\boldsymbol{\mathcal{T}}}_h\right)\Big|_{T_2} - \left(\boldsymbol{n}\cdot\overset{1}{\boldsymbol{\mathcal{T}}}_h\right)\Big|_{T} \right) & \text{if } F = T\cap T_2 \\ g - \boldsymbol{n}\cdot\overset{1}{\boldsymbol{\mathcal{T}}}_h & \text{if } F\subset\Gamma_N \\ 0 & \text{else} \end{cases} \tag{5.9}$$

Note: $\boldsymbol{n}\Big|_{T_2} = -\boldsymbol{n}\Big|_{T} \;\Rightarrow\; \boldsymbol{R}_{F(T)} = \frac{1}{2}\boldsymbol{n}\Big|_{T}\cdot\left(\overset{1}{\boldsymbol{\mathcal{T}}}_h\Big|_{T_2} - \overset{1}{\boldsymbol{\mathcal{T}}}_h\Big|_{T}\right)$

With the new notation, (5.7) can be reformulated as follows.

$$\mathcal{D} = \sum_{T\in\Upsilon_h} \left(\langle \boldsymbol{R}_T \,,\, \boldsymbol{V}\rangle_{0,T} + \sum_{F\subset\partial T} \left\langle \boldsymbol{R}_{F(T)} \,,\, \boldsymbol{V} \right\rangle_{0,F} \right) \tag{5.10}$$

This can be estimated from above by using *Cauchy-Schwarz inequalities*, interpolation estimates and the *Clément interpolation operator* as well as the patches $\hat{T}$ and $\hat{F}$ around T and F respectively and a scalar material function $\gamma(T) = \gamma_T|_T$ that is constant on every element T. Finally this approach yields (5.11).

$$\mathcal{D} \le C \left\{ \sum_{T\in\Upsilon_h} \frac{{h_T}^2}{{\gamma_T}^2} \|\boldsymbol{R}_T\|^2_{0,T} + \sum_{T,F\subset\partial T} \frac{h_T}{{\gamma_T}^2} \left\|\boldsymbol{R}_{F(T)}\right\|^2_{0,F} \right\}^{1/2} \cdot |\gamma(T)\, \boldsymbol{e}_{\boldsymbol{U}}|_{1,\Omega} \tag{5.11}$$

From (5.11) we can define the element wise error indicator η_T as shown in (5.12) and $d_{\boldsymbol{U}}(\boldsymbol{e}_{\boldsymbol{U}}, \boldsymbol{e}_{\boldsymbol{U}})$ with $d_{\boldsymbol{U}}(\boldsymbol{e}_{\boldsymbol{U}}, \boldsymbol{e}_{\boldsymbol{U}})^{1/2} := |\gamma(T)\, \boldsymbol{e}_{\boldsymbol{U}}|_{1,\Omega}$.

$${\eta_T}^2 := \frac{{h_T}^2}{{\gamma_T}^2} \|\boldsymbol{R}_T\|^2_{0,T} + \sum_{F\subset\partial T} \frac{h_T}{{\gamma_T}^2} \left\|\boldsymbol{R}_{F(T)}\right\|^2_{0,F} \tag{5.12}$$

The sum of all error indicators gives an upper bound on our error functional.

$$\mathcal{J}\Big(\boldsymbol{e_U}\Big) = \frac{1}{|\gamma(T)\,\boldsymbol{e_U}|_{1,\Omega}} \Big(a^*(\boldsymbol{U}, P; \boldsymbol{e_U}) - a^*(\boldsymbol{U}_h, P_h; \boldsymbol{e_U})\Big) \tag{5.13}$$

$$\leq c_3 \left\{ \sum_{T\in\Upsilon_h} {\eta_T}^2 \right\}^{1/2} \tag{5.14}$$

Following the derivation for the case of linear elasticity, where we usually choose $\gamma(T)^2$ to represent the smallest eigenvalue of $\mathfrak{M}$, we take $\gamma(T)^2 = c_{10}$.

Note: Analogously one can treat the second equation of (5.1). We denote the error with $e_P := P - P_h$ in $0, \Omega$ and define a test function $Q := \Big(I - I_h\Big)\, e_P$ in $0, \Omega$. Subsequent we consider the error functional (5.17) and the corresponding denominator $\mathcal{D}$.

$$\mathcal{J}\Big(e_P\Big) = \frac{b_0(\boldsymbol{U}; e_P) - c(P; e_P) - b_0(\boldsymbol{U}_h; e_P) + c(P_h; e_P)}{d_P(e_P, e_P)^{1/2}} \tag{5.15}$$

$$\begin{aligned} \mathcal{D} &= b_0(\boldsymbol{U}; e_P) - c(P; e_P) - b_0(\boldsymbol{U}_h; e_P) + c(P_h; e_P) \\ &= -b_0(\boldsymbol{U}_h; Q) + c(P_h; Q) \\ &= \langle\, \kappa\, P_h - \phi_V(\boldsymbol{U}_h)\,,\, Q\,\rangle_{0,\Omega} \end{aligned} \tag{5.16}$$

With $\Big(\kappa\, P_h - \phi_V(\boldsymbol{U}_h)\Big) =: r_P \in 0, T$ we get the following estimate.

$$\mathcal{D} \leq C \left(\sum_T \|r_P\|^2_{0,T} \right)^{1/2} \cdot \|e_P\|_{0,\Omega} \tag{5.17}$$

We should set (5.18).

$$d_P(e_P, e_P)^{1/2} = \|e_P\|_{0,\Omega} \tag{5.18}$$

If we combine these two results, we would get the following.

$$\begin{aligned} & d(\boldsymbol{e_U}, e_P)^{1/2} \cdot \mathcal{J}\Big(\boldsymbol{e_U}, e_P\Big) \\ &= \quad a^*(\boldsymbol{U}, P; \boldsymbol{e_U}) - a^*(\boldsymbol{U}_h, P_h; \boldsymbol{e_U}) \\ &\quad - b_0(\boldsymbol{U}; e_P) \qquad + b_0(\boldsymbol{U}_h; e_P) \\ &\quad + c(P; e_P) \qquad - c(P_h; e_P) \\ &\leq c_2 \left\{ \sum_T \frac{{h_T}^2}{{\gamma_T}^2} \|\boldsymbol{R}_T\|^2_{0,T} + \sum_{T, F\subset\partial T} \frac{h_T}{{\gamma_T}^2} \Big\|\boldsymbol{R}_{F(T)}\Big\|^2_{0,F} + \sum_T \|r_P\|^2_{0,T} \right\}^{1/2} \\ &\qquad \cdot \Big\{ |\gamma(T)\boldsymbol{e_U}|^2_{1,\Omega} + \|e_P\|^2_{0,\Omega} \Big\}^{1/2} \end{aligned} \tag{5.19}$$

With $d(\boldsymbol{e_U}, e_P) := |\gamma(T)\boldsymbol{e_U}|^2_{1,\Omega} + \|e_P\|^2_{0,\Omega}$ we get the element wise error indicator ${\eta_{T,\mathrm{comb}}}^2$ with (5.21).

$$ {\eta_{T,\mathrm{comb}}}^2 := \|r_P\|^2_{0,T} + \frac{{h_T}^2}{{\gamma_T}^2}\,\|\boldsymbol{R}_T\|^2_{0,T} + \frac{h_T}{{\gamma_T}^2}\sum_{F\subset\partial T}\left\|\boldsymbol{R}_{F(T)}\right\|^2_{0,F} \tag{5.20} $$

$$ \mathcal{J}\Big(\boldsymbol{e_U}, e_P\Big) \le c_2\left\{\sum_{T\in\Upsilon_h}{\eta_{T,\mathrm{comb}}}^2\right\}^{1/2} \tag{5.21} $$

However, numerical tests indicate that the r_P part can be left out because it is dominated by the other ones. There are also reasonable arguments why it is possible to leave out the element residual too.

To prevent the case that there is no mesh refinement if the estimated error is zero (e. g. if $\partial\Omega = \Gamma_D$ without any inner faces) we set the refinement condition as follows.

$$ \texttt{refine element } T_i \texttt{ if } \eta_{T_i} \ge p\cdot\max_{T_i}\Big\{\eta_{T_i}\Big\},\quad p\in\Big(0,1\Big) \tag{5.22} $$

6 LBB conditions

It is known that a saddle point problem is uniquely solvable, if the three bilinear forms $a(\boldsymbol{U}, P; \delta\boldsymbol{U}, \boldsymbol{V})$, $b(\boldsymbol{U}; Q, \boldsymbol{V})$ and $c(P, Q)$ are continuous and fulfil the conditions (6.4) - (6.3) (see [1] sec. III.4 theorem 4.11 or [5] sec. 3.2).

$$a(\boldsymbol{U}, P; \boldsymbol{V}, \boldsymbol{V}) \geq \alpha \, \|\boldsymbol{V}\|^2_{\mathbb{V}_0} \qquad \forall\, \boldsymbol{V} \in \mathcal{N}_0(B);\ \alpha > 0 \tag{6.1}$$

$$\sup_{\boldsymbol{V}\in\mathbb{V}_0} \frac{b(\boldsymbol{U}; Q, \boldsymbol{V})}{\|\boldsymbol{V}\|_{\mathbb{V}_0}} \geq \beta \, \|Q\|_{\mathbb{Q}} \qquad \forall\, Q \in \mathbb{Q};\ \beta > 0 \tag{6.2}$$

$$c(Q, Q) \geq 0 \qquad \forall\, Q \in \mathbb{Q} \tag{6.3}$$

Here we use the kernel of B, which is an associate operator to $b(\ldots)$.

$$\mathcal{N}_0(B) = \Big\{ \boldsymbol{V} \in \mathbb{V}_0 :\ b(\boldsymbol{U}; Q, \boldsymbol{V}) = \langle Q \,,\, B(\boldsymbol{U}) \cdot \boldsymbol{V} \rangle = 0 \quad \forall\, Q \in \mathbb{Q} \Big\}$$

Instead of showing the coercivity (6.1) it suffices to show the stability on the kernel.

$$\sup_{\boldsymbol{V}\in\mathcal{N}_0(B)} \frac{a(\boldsymbol{U}, P; \delta\boldsymbol{U}, \boldsymbol{V})}{\|\boldsymbol{V}\|_{\mathbb{V}_0}} \geq \alpha \, \|\delta\boldsymbol{U}\|_{\mathbb{V}_0} \qquad \forall\, \delta\boldsymbol{U} \in \mathcal{N}_0(B);\ \alpha > 0 \tag{6.4}$$

$$\sup_{\delta\boldsymbol{U}\in\mathcal{N}_0(B)} \frac{a(\boldsymbol{U}, P; \delta\boldsymbol{U}, \boldsymbol{V})}{\|\delta\boldsymbol{U}\|_{\mathbb{V}_0}} \geq \alpha \, \|\boldsymbol{V}\|_{\mathbb{V}_0} \qquad \forall\, \boldsymbol{V} \in \mathcal{N}_0(B);\ \alpha > 0 \tag{6.5}$$

6.1 First steps

Condition (6.3) can be shown easily with (4.13).

$$c(Q, Q) = \int_\Omega \kappa \cdot Q^2 \, \mathrm{d}\Omega \overset{\kappa \cdot Q^2 \geq 0}{\geq} 0 \tag{6.6}$$

$$c(Q, Q) = \int_\Omega \kappa \cdot Q^2 \, \mathrm{d}\Omega \leq \underbrace{\max \kappa}_{=:\gamma<\infty} \cdot \underbrace{\|Q\|^2_{0,\Omega}}_{<\infty} < \infty \tag{6.7}$$

6.2 Inf-sup condition

Due to the symmetry of $\boldsymbol{\mathcal{S}}_V(\boldsymbol{U}) = \boldsymbol{\mathcal{C}}^{-1}(\boldsymbol{U})$ (6.8) holds.

$$\begin{aligned} &\sup_{\boldsymbol{V}\in\mathbb{V}_0} \frac{b(\boldsymbol{U}; Q, \boldsymbol{V})}{\|\boldsymbol{V}\|_{\mathbb{V}_0}} \\ &\quad = \sup_{\boldsymbol{V}\in H^1(\Omega)^3} \frac{1}{\|\boldsymbol{V}\|_{\mathbb{V}_0}} \langle Q \,,\, \boldsymbol{\mathcal{S}}_V(\boldsymbol{U}) : \boldsymbol{\mathcal{E}}(\boldsymbol{U}; \boldsymbol{V}) \rangle_{0,\Omega} \\ &\quad = \sup_{\boldsymbol{V}\in H^1(\Omega)^3} \frac{1}{\|\boldsymbol{V}\|_{\mathbb{V}_0}} \Big\langle Q \,,\, \boldsymbol{\mathcal{C}}^{-1} : \Big(\boldsymbol{\mathcal{F}}^{\mathrm{T}} \cdot (\mathrm{Grad}\, \boldsymbol{V})^{\mathrm{T}} \Big) \Big\rangle_{0,\Omega} \end{aligned} \tag{6.8}$$

The scalar product gives

$$\begin{aligned}
&\left\langle Q\,,\, \boldsymbol{\mathcal{C}}^{-1} : \left(\boldsymbol{\mathcal{F}}^{\mathrm{T}} \cdot (\mathrm{Grad}\ \boldsymbol{V})^{\mathrm{T}}\right)\right\rangle_{0,\Omega} \\
&= \left\langle Q\,,\, \boldsymbol{\mathcal{F}}^{-1} : (\mathrm{Grad}\ \boldsymbol{V})^{\mathrm{T}}\right\rangle_{0,\Omega} = \left\langle Q\,,\, \boldsymbol{\mathcal{F}}^{-1} : \left((\mathrm{grad}\ \boldsymbol{V})^{\mathrm{T}} \cdot \boldsymbol{\mathcal{F}}\right)\right\rangle_{0,\Omega} \\
&= \langle\, Q \cdot \boldsymbol{\mathcal{I}}\,,\, \mathrm{grad}\ \boldsymbol{V}\,\rangle_{0,\Omega} \\
&= \int_{\Omega} Q(X) \cdot \mathrm{div}(\,\boldsymbol{V}(X))\,\mathrm{d}X = \int_{\Omega_\tau} q(x) \cdot \mathrm{div}(\boldsymbol{v}(x))\,\frac{1}{J(x)}\,\mathrm{d}x \quad . \qquad (6.9)
\end{aligned}$$

We can apply the theorem of the surjectivity of the divergence on L^2: For all $q \in L^2(\Omega_\tau)$ there is a $\boldsymbol{v} \in H^1(\Omega_\tau)^3$ with the properties

$$q = \mathrm{div}\ \boldsymbol{v} \quad , \qquad (6.10)$$

$$\|\boldsymbol{v}\|_{1,\Omega_\tau} \leq c_1\, \|q\|_{0,\Omega_\tau} \quad . \qquad (6.11)$$

We can choose such a pair $\left(q, \boldsymbol{v}\right)$ and use it in (6.9).

$$\begin{aligned}
\left\langle Q \cdot \boldsymbol{\mathcal{C}}^{-1}\,,\, \mathrm{Grad}\ \boldsymbol{V} \cdot \boldsymbol{\mathcal{F}}\right\rangle_{0,\Omega} &= \left\langle J^{-1} \cdot q\,,\, \mathrm{div}(\boldsymbol{v})\right\rangle_{0,\Omega_\tau} \\
&= \left\langle J^{-1} \cdot q\,,\, q\right\rangle_{0,\Omega_\tau} \\
&= \langle\, Q\,,\, Q\,\rangle_{0,\Omega} = \|Q\|^2_{0,\Omega} \qquad (6.12)
\end{aligned}$$

With (6.11) we estimate the norms.

$$c_0\, \|\,\boldsymbol{V}\|_{\mathbb{V}} \leq \|\boldsymbol{v}\|_{1,\Omega_\tau} \leq c_1\, \|q\|_{0,\Omega_\tau} \leq c_2\, \|Q\|_{0,\Omega} \qquad (6.13)$$

Hence the inf-sup condition follows.

$$\begin{aligned}
\sup_{\boldsymbol{V}\in\mathbb{V}_0} \frac{b(\,\boldsymbol{U}; Q, \,\boldsymbol{V})}{\|\,\boldsymbol{V}\|_{\mathbb{V}_0}} &= \sup_{\boldsymbol{V}\in\mathbb{V}_0} \frac{\langle Q\,,\, \mathrm{div}\ \boldsymbol{V}\rangle_{0,\Omega}}{\|\,\boldsymbol{V}\|_{\mathbb{V}_0}} \\
&\geq \frac{\|Q\|^2_{0,\Omega}}{\|\,\boldsymbol{V}\|_{\mathbb{V}_0}} \geq \frac{\|Q\|^2_{0,\Omega}}{c_2 c_0{}^{-1}\, \|Q\|_{0,\Omega}} = \frac{c_0}{c_2}\, \|Q\|_{0,\Omega} \qquad (6.14)
\end{aligned}$$

6.3 Coercivity

We plug in $\mathfrak{M}(\boldsymbol{U},P)$ from (4.28) and the stress tensor from (3.43).

$$
\begin{aligned}
&\Big\langle \boldsymbol{\mathcal{E}}(\boldsymbol{U};\,\boldsymbol{V})\,,\,\mathfrak{M}(\boldsymbol{U},P):\boldsymbol{\mathcal{E}}(\boldsymbol{U};\delta\boldsymbol{U})\Big\rangle_{0,\Omega}\\
&= -\Big\langle \boldsymbol{\mathcal{E}}(\boldsymbol{U};\,\boldsymbol{V})\,,\,4I_3\left(\frac{\partial\,\phi_D}{\partial\,I_3}+P\frac{\partial\,\phi_V}{\partial\,I_3}\right)\hat{\mathfrak{C}}:\boldsymbol{\mathcal{E}}(\boldsymbol{U};\delta\boldsymbol{U})\Big\rangle_{0,\Omega}\\
&\quad+\left\langle \boldsymbol{\mathcal{E}}(\boldsymbol{U};\,\boldsymbol{V})\,,\,4I_3\left(\frac{\partial\,\phi_D}{\partial\,I_3}+I_3\frac{\partial^2\,\phi_D}{\partial {I_3}^2}\right)\left(\boldsymbol{\mathcal{C}}^{-1}\boldsymbol{\mathcal{C}}^{-1}\right):\boldsymbol{\mathcal{E}}(\boldsymbol{U};\delta\boldsymbol{U})\right\rangle_{0,\Omega}\\
&\quad+\left\langle \boldsymbol{\mathcal{E}}(\boldsymbol{U};\,\boldsymbol{V})\,,\,4I_3\frac{\partial^2\phi_D}{\partial a_1\,\partial I_3}\left(\boldsymbol{\mathcal{C}}^{-1}\boldsymbol{\mathcal{I}}+\boldsymbol{\mathcal{I}}\boldsymbol{\mathcal{C}}^{-1}\right):\boldsymbol{\mathcal{E}}(\boldsymbol{U};\delta\boldsymbol{U})\right\rangle_{0,\Omega}
\end{aligned}
\tag{6.15}
$$

Since $\delta\boldsymbol{U}$ and $\boldsymbol{V}$ should be in the kernel of B we are allowed to omit the two last terms because it is $\big\langle Q\,,\,\boldsymbol{\mathcal{C}}^{-1}:\boldsymbol{\mathcal{E}}(\boldsymbol{U};\,\boldsymbol{V})\big\rangle_{0,\Omega}=0$ for $Q\in L^2(\Omega)$. Furthermore it is (6.16).

$$
\begin{aligned}
&\left\langle \mathrm{Grad}\,\boldsymbol{V}\,,\,\overset{2}{\boldsymbol{\mathcal{T}}}(\boldsymbol{U},P)\cdot\mathrm{Grad}\,\delta\boldsymbol{U}\right\rangle_{0,\Omega}\\
&=\quad\left\langle \mathrm{Grad}\,\boldsymbol{V}\,,\,2\frac{\partial\,\phi_D}{\partial\,a_1}\mathrm{Grad}\,\delta\boldsymbol{U}\right\rangle_{0,\Omega}\\
&\quad+\Big\langle \boldsymbol{\mathcal{F}}^{-\mathrm{T}}\cdot\mathrm{Grad}\,\boldsymbol{V}\,,\,2I_3\left(\frac{\partial\,\phi_D}{\partial\,I_3}+P\frac{\partial\,\phi_V}{\partial\,I_3}\right)\boldsymbol{\mathcal{F}}^{-\mathrm{T}}\cdot\mathrm{Grad}\,\delta\boldsymbol{U}\Big\rangle_{0,\Omega}
\end{aligned}
\tag{6.16}
$$

After direct computing from (3.27) we define $\pi(I_3,P)$.

$$
\pi(I_3,P):=I_3\left(\frac{\partial\,\phi_D}{\partial\,I_3}+P\frac{\partial\,\phi_V}{\partial\,I_3}\right)=-\;c_{10}\frac{a_1}{3}{I_3}^{-1/3}+\frac{1}{2}P \tag{6.17}
$$

$$
2\frac{\partial\,\phi_D}{\partial\,a_1}=2c_{10}{I_3}^{-1/3}>0 \tag{6.18}
$$

Note: In $-\pi(I_3,P)$ it clearly is $c_{10}\frac{a_1}{3}{I_3}^{-1/3}>0$, but the second addend depends on the problem.

By a short computation with interchanging tensor factors it is

$$
\begin{aligned}
&4\,\boldsymbol{\mathcal{E}}(\boldsymbol{U};\delta\boldsymbol{U}):\hat{\mathfrak{C}}:\boldsymbol{\mathcal{E}}(\boldsymbol{U};\,\boldsymbol{V})\\
&\qquad=\;4\,\boldsymbol{\mathcal{E}}(\boldsymbol{U};\delta\boldsymbol{U})\cdot\boldsymbol{\mathcal{C}}^{-1}:\boldsymbol{\mathcal{E}}(\boldsymbol{U};\,\boldsymbol{V})\cdot\boldsymbol{\mathcal{C}}^{-1}\\
&\qquad=\;2\,\mathtt{Sym}\Big((\mathrm{Grad}\,\delta\boldsymbol{U})^{\mathrm{T}}\cdot\boldsymbol{\mathcal{F}}^{-1}\Big):2\,\mathtt{Sym}\Big((\mathrm{Grad}\,\boldsymbol{V})^{\mathrm{T}}\cdot\boldsymbol{\mathcal{F}}^{-1}\Big)\\
&\qquad=\;\Big(\mathrm{grad}\,\delta\boldsymbol{U}+(\mathrm{grad}\,\delta\boldsymbol{U})^{\mathrm{T}}\Big):\Big(\mathrm{grad}\,\boldsymbol{V}+(\mathrm{grad}\,\boldsymbol{V})^{\mathrm{T}}\Big)\\
&\qquad=:\;4\,\boldsymbol{\varepsilon}(\delta\boldsymbol{U}):\boldsymbol{\varepsilon}(\boldsymbol{V})
\end{aligned}
\tag{6.19}
$$

and in general it follows (6.20) or (6.21) as a pull back onto Ω_τ.

$$
\begin{aligned}
& a(\boldsymbol{U}, P; \delta\boldsymbol{U}, \boldsymbol{V}) \\
& = -4\Big\langle \pi(I_3, P)\, \mathtt{Sym}\Big((\mathrm{Grad}\ \boldsymbol{V})^{\mathrm{T}} \cdot \boldsymbol{\mathcal{F}}^{-1}\Big)\ ,\ \mathtt{Sym}\Big((\mathrm{Grad}\,\delta\boldsymbol{U})^{\mathrm{T}} \cdot \boldsymbol{\mathcal{F}}^{-1}\Big)\Big\rangle_{0,\Omega} \\
& \quad + 2\Big\langle \pi(I_3, P)\ (\mathrm{Grad}\ \boldsymbol{V})^{\mathrm{T}} \cdot \boldsymbol{\mathcal{F}}^{-1}\ ,\ (\mathrm{Grad}\,\delta\boldsymbol{U})^{\mathrm{T}} \cdot \boldsymbol{\mathcal{F}}^{-1}\Big\rangle_{0,\Omega} \\
& \quad + 2\Big\langle \mathrm{Grad}\ \boldsymbol{V}\ ,\ \frac{\partial\,\phi_D}{\partial\,a_1}\mathrm{Grad}\,\delta\boldsymbol{U}\Big\rangle_{0,\Omega} \qquad (6.20) \\
& = -\Big\langle J^{-1}\,\pi(i_3, p)\, 2\,\mathtt{Sym}\big(\mathrm{grad}\,\boldsymbol{v}\big)\ ,\ 2\,\mathtt{Sym}\big(\mathrm{grad}\,\delta\boldsymbol{u}\big)\Big\rangle_{0,\Omega_\tau} \\
& \quad + 2\Big\langle J^{-1}\,\pi(i_3, p)\ \mathrm{grad}\,\boldsymbol{v}\,,\ \mathrm{grad}\,\delta\boldsymbol{u}\Big\rangle_{0,\Omega_\tau} \\
& \quad + 2\Big\langle \mathrm{Grad}\ \boldsymbol{V}\ ,\ \frac{\partial\,\phi_D}{\partial\,a_1}\mathrm{Grad}\,\delta\boldsymbol{U}\Big\rangle_{0,\Omega} \qquad (6.21)
\end{aligned}
$$

We could sum up.

$$
\begin{aligned}
& a(\boldsymbol{U}, P; \delta\boldsymbol{U}, \boldsymbol{V}) \\
& = \quad 2\int_\Omega \frac{\partial\,\phi_D}{\partial\,a_1}\,(\mathrm{Grad}\ \boldsymbol{V})^{\mathrm{T}} : \mathrm{Grad}\,\delta\boldsymbol{U}\ \mathrm{d}\Omega \\
& \qquad - 2\int_\Omega \pi(I_3, P)\,\Big(\boldsymbol{\mathcal{F}}^{-\mathrm{T}} \cdot \mathrm{Grad}\ \boldsymbol{V}\Big) : \Big(\boldsymbol{\mathcal{F}}^{-\mathrm{T}} \cdot \mathrm{Grad}\,\delta\boldsymbol{U}\Big)\ \mathrm{d}\Omega \qquad (6.22)
\end{aligned}
$$

To show (6.1) we use (6.20) with the argument $(\boldsymbol{V}, \boldsymbol{V})$. Then, the term $\Big\langle \mathrm{Grad}\ \boldsymbol{V}\ ,\ \frac{\partial\phi_D}{\partial a_1}\ \mathrm{Grad}\ \boldsymbol{V}\Big\rangle_{0,\Omega}$ would give us the needed norm estimation if the rest vanishes. We get the feeling that *Korn's inequality* (see [12], [13] thm. 3 or 4) is applicable with (6.23) for $\boldsymbol{v}\Big|_{\Gamma_{\tau,D}} = \boldsymbol{0}$.

$$
\begin{aligned}
& \Big\|2\,\mathtt{Sym}\big(\mathrm{grad}\,\boldsymbol{v}\big)\Big\|^2_{0,\Omega_\tau} \\
& = \Big\|\mathrm{grad}\,\boldsymbol{v} + (\mathrm{grad}\,\boldsymbol{v})^{\mathrm{T}}\Big\|^2_{0,\Omega_\tau} \geq c^+\ \left\|\mathrm{grad}\,\boldsymbol{v}\right\|^2_{0,\Omega_\tau} \qquad (6.23)
\end{aligned}
$$

Note: It is $c^+ = 2$ if $\boldsymbol{v}\Big|_{\partial\Omega_\tau} = 0$.

A positive coefficient $-J\pi(i_3, p)$ would not destroy this inequality, but yet we have no knowledge about its sign in (6.21). Furthermore if c^+ is smaller than 2 then a remainder is left.

7 Improvement suggestions

Especially in cases of very large deformations we notice that the numerical realisation sometimes exhibits a bad behaviour. One problem is the decreasing minimum of $J(\boldsymbol{U}_h)$. It tents to zero because sometimes given rectangular angles are deformed into a nearly straight line. On the other hand after a certain adaptive refinement we witness a jump in the estimated error and a negative residual $\underline{w}_o^T \cdot A \cdot \underline{w}_o = (A_0 \cdot \underline{r}_o)^T \cdot A \cdot (A_0 \cdot \underline{r}_o)$ or even a negative determinant J occurs. May be the problem is that we use all terms of $a(\ldots)$ to build the system matrix A although $a(\ldots)$ is only coercive on the kernel of $b(\ldots)$. Or maybe in that case the initial value $(\boldsymbol{U}_h, P_h)$ contains a high error and that influences A. The next sections show some improvements.

7.1 Specific energy density function

Instead of (3.27) we can use (7.1) to formulate the deviatoric part of $\phi(\boldsymbol{\mathcal{C}})$.

$$\phi_D(\boldsymbol{\mathcal{C}}) = \phi_D(a_1, I_3) = c_{10}\left(a_1 - 3 - \ln(I_3)\right) \tag{7.1}$$

This leads to an easier formulation of $\boldsymbol{\mathcal{T}}_D$ and $\mathfrak{M}_D$.

$$\begin{aligned}
\boldsymbol{\mathcal{T}}_D(\boldsymbol{\mathcal{C}}(\boldsymbol{U})) &= 2c_{10}\boldsymbol{\mathcal{I}} - 2c_{10}\boldsymbol{\mathcal{C}}^{-1}(\boldsymbol{U}) &(7.2)\\
\frac{\partial \boldsymbol{\mathcal{T}}_D(\boldsymbol{\mathcal{C}})}{\partial \boldsymbol{\mathcal{C}}} &= 2c_{10}\left\{\left(\boldsymbol{\mathcal{C}}^{-1}\boldsymbol{\mathcal{C}}^{-1}\right) - I_3{}^{-1}\left(a_1\boldsymbol{\mathcal{I}}\boldsymbol{\mathcal{I}} - (\boldsymbol{\mathcal{I}}\boldsymbol{\mathcal{C}} + \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{I}}) - a_1\mathfrak{I} + \mathfrak{C}\right)\right\} &(7.3)\\
&= 2c_{10}\,\hat{\mathfrak{C}} &(7.4)
\end{aligned}$$

7.2 Simple split of the bulk modulus

The bulk modulus can be decomposed into two parts.

$$\mathtt{K}_D + \mathtt{K}_\infty := \mathtt{K} \leq \infty \quad \text{such that} \quad 0 \leq \mathtt{K}_D \ll \mathtt{K}_\infty \leq \infty \tag{7.5}$$

Thereby we predefine a new structure of $\phi(\boldsymbol{\mathcal{C}})$.

$$\phi(\boldsymbol{\mathcal{C}}) = \underbrace{\phi_D((I_1),(I_3)) + \frac{1}{2}\mathtt{K}_D(\phi_V(I_3))^2}_{=:\ \phi_1((I_1),(I_3))} + \frac{1}{2}\mathtt{K}_\infty(\phi_V(I_3))^2 \tag{7.6}$$

The hydrostatic pressure comes in with the substitution (7.7) but in the case of incompressibility with $\mathtt{K} = \mathtt{K}_\infty = \infty$ the new variable P_∞ equals P.

$$P_\infty = \mathtt{K}_\infty \phi_V(\boldsymbol{\mathcal{C}}) \tag{7.7}$$

With this ansatz we receive a new formulation of the stress tensor and the material tensor in the initial configuration.

$$\overset{2}{\boldsymbol{\mathcal{T}}} = \underbrace{\boldsymbol{\mathcal{T}}_D(\boldsymbol{U}) + \mathtt{K}_D \phi_V(I_3(\boldsymbol{U})) \cdot \boldsymbol{\mathcal{S}}_V}_{=:\ \boldsymbol{\mathcal{T}}_1(\boldsymbol{U})} + P_\infty \cdot \boldsymbol{\mathcal{S}}_V \tag{7.8}$$

7.3 Scaling function

We change the volumetric part of the stress tensor $\overset{2}{\boldsymbol{\mathcal{T}}}$ by introducing a suitable scaling function $\zeta(I_3) \in \mathbb{R}$ for all $I_3 = I_3(\boldsymbol{\mathcal{C}}(\boldsymbol{U}))$ which leads to (7.10).

$$P \cdot \boldsymbol{\mathcal{S}}_V = \mathtt{K}\, \phi_V(I_3) \cdot 2 \frac{\partial\, \phi_V(I_3(\boldsymbol{\mathcal{C}}))}{\partial\, \boldsymbol{\mathcal{C}}} \tag{7.9}$$

$$\Rightarrow \quad P_\zeta \cdot \boldsymbol{\mathcal{S}}_{V,\zeta} = \frac{\mathtt{K}\, \phi_V(I_3)}{\zeta(I_3)} \cdot 2\zeta(I_3) \frac{\partial\, \phi_V(I_3(\boldsymbol{\mathcal{C}}))}{\partial\, \boldsymbol{\mathcal{C}}} \tag{7.10}$$

Together with the decomposition of the bulk modulus we also could think of a change like (7.11).

$$P_{\zeta,\infty} \cdot \boldsymbol{\mathcal{S}}_{V,\zeta} = \frac{\mathtt{K}_\infty\, \phi_V(I_3)}{\zeta(I_3)} \cdot 2\zeta(I_3) \frac{\partial\, \phi_V(I_3(\boldsymbol{\mathcal{C}}))}{\partial\, \boldsymbol{\mathcal{C}}} \tag{7.11}$$

A reasonable property of $\zeta(I_3)$ seems to be given by $\zeta(1) = 1$ and that is why (7.12) could be a good choice.

$$\zeta(I_3) = (I_3)^y, \qquad y \neq 0 \tag{7.12}$$

Now the substitution with the hydrostatic pressure P_ζ requires a slightly changed side condition (7.13).

$$\kappa P_\zeta = \zeta(I_3)^{-1} \phi_V(I_3) \tag{7.13}$$

$$\text{or} \quad \kappa_\infty P_{\zeta,\infty} = \zeta(I_3)^{-1} \phi_V(I_3) \tag{7.14}$$

Note: Formula (7.10) yields $\boldsymbol{\mathcal{S}}_{V,\zeta} = \zeta(I_3)\, \boldsymbol{\mathcal{C}}^{-1}$.

Instead of (3.51) we now have to consider a modified system of equations.

$$\begin{pmatrix} t\, f_0(\boldsymbol{V}) \\ 0 \end{pmatrix} = \begin{pmatrix} a_D(\boldsymbol{U};\, \boldsymbol{V}) + a_V(\boldsymbol{U}, P_{\zeta,\infty};\, \boldsymbol{V}) \\ b_0(\boldsymbol{U}; Q) \quad - c(P_{\zeta,\infty}, Q) \end{pmatrix} \tag{7.15}$$

$$= \begin{pmatrix} \langle \boldsymbol{\mathcal{T}}_1(\boldsymbol{U})\,,\, \boldsymbol{\mathcal{E}}(\boldsymbol{U};\, \boldsymbol{V}) \rangle_{0,\Omega} \quad + \langle P_{\zeta,\infty} \cdot \boldsymbol{\mathcal{S}}_{V,\zeta}(\boldsymbol{U})\,,\, \boldsymbol{\mathcal{E}}(\boldsymbol{U};\, \boldsymbol{V}) \rangle_{0,\Omega} \\ \left\langle \zeta(\boldsymbol{U})^{-1} \cdot \phi_V(\boldsymbol{U})\,,\, Q \right\rangle_{0,\Omega} - \langle \kappa \cdot P_{\zeta,\infty}\,,\, Q \rangle_{0,\Omega} \end{pmatrix} \tag{7.16}$$

We adapt our notation from (3.46) - (3.49) and define a new bilinear form.

$$b_\zeta(\boldsymbol{U};\delta\boldsymbol{U},Q) := \langle\, \mathcal{Z}(\boldsymbol{U})\cdot \boldsymbol{\mathcal{S}}_{V,\zeta}(\boldsymbol{U}) : \boldsymbol{\mathcal{E}}(\boldsymbol{U};\delta\boldsymbol{U})\,,\, Q\,\rangle_{L^2(\Omega)} \tag{7.17}$$

$$\begin{aligned}\mathcal{Z}(\boldsymbol{U}) &:= \frac{1}{\zeta(I_3(\boldsymbol{U}))^2}\left(1-\frac{\phi_V(I_3(\boldsymbol{U}))}{\zeta(I_3(\boldsymbol{U}))}\left(\frac{\partial\,\phi_V(I_3)}{\partial\, I_3}\right)^{-1}\frac{\partial\,\zeta(I_3)}{\partial\, I_3}\right) \\ &= \Big(1-y\,\ln(I_3)\Big)\end{aligned} \tag{7.18}$$

Then we can recompute the linearisation, which results in a modified linear problem for all $Q\in\mathbb{Q}$ and $\boldsymbol{V}\in\mathbb{V}_0$

$$\begin{aligned} t\, f_0(\boldsymbol{V}) - a_D(\boldsymbol{U};\,\boldsymbol{V}) - a_V(\boldsymbol{U},P_{\zeta,\infty};\,\boldsymbol{V}) \qquad\qquad & \\ = a(\boldsymbol{U},P_{\zeta,\infty};\delta\boldsymbol{U},\,\boldsymbol{V}) + a_V(\boldsymbol{U},\delta P_{\zeta,\infty};\,\boldsymbol{V}) &\end{aligned} \tag{7.19}$$

$$c(P_{\zeta,\infty},Q) - b_0(\boldsymbol{U};Q) = b_\zeta(\boldsymbol{U};\delta\boldsymbol{U},Q) - c(\delta P_{\zeta,\infty},Q) \tag{7.20}$$

with a slightly changed stress and material tenor in $a(\ldots)$.

$$\overset{2}{\boldsymbol{\mathcal{T}}}(\boldsymbol{U},P_{\zeta,\infty}) = \Big(\boldsymbol{\mathcal{T}}_D(\boldsymbol{U}) + \mathtt{K}_D\cdot\phi_V(I_3)\cdot\boldsymbol{\mathcal{S}}_V(\boldsymbol{U})\Big) + P_{\zeta,\infty}\cdot(\zeta(I_3)\cdot\boldsymbol{\mathcal{S}}_V(\boldsymbol{U})) \tag{7.21}$$

$$\begin{aligned}\mathfrak{M}(\boldsymbol{U},P_{\zeta,\infty}) = 2\frac{\partial\,\boldsymbol{\mathcal{T}}_D}{\partial\,\boldsymbol{\mathcal{C}}} + \mathtt{K}_D\boldsymbol{\mathcal{S}}_V\boldsymbol{\mathcal{S}}_V + \mathtt{K}_D\phi_V(I_3)\,2\frac{\partial\,\boldsymbol{\mathcal{S}}_V}{\partial\,\boldsymbol{\mathcal{C}}} & \\ + P_{\zeta,\infty}\left(2\frac{\partial\,\zeta(\boldsymbol{\mathcal{C}})}{\partial\,\boldsymbol{\mathcal{C}}}\boldsymbol{\mathcal{S}}_V(\boldsymbol{\mathcal{C}}) + \zeta(I_3)\cdot 2\frac{\partial\,\boldsymbol{\mathcal{S}}_V(\boldsymbol{\mathcal{C}})}{\partial\,\boldsymbol{\mathcal{C}}}\right) &\end{aligned} \tag{7.22}$$

In order to achieve a saddle point problem again we have to match the bilinear forms $b_\zeta(\boldsymbol{U};\delta\boldsymbol{U},Q)$ and $a_V(\boldsymbol{U},\delta P_{\zeta,\infty};\,\boldsymbol{V})$. In (7.20) we rewrite Q as $\mathcal{Z}^{-1}\cdot Q_\zeta$ with $Q_\zeta\in\mathbb{Q}_\zeta := \Big\{Q_\zeta = \mathcal{Z}\cdot Q:\ Q\in\mathbb{Q}\Big\}$ (in brief it is $\mathcal{Z}:=\mathcal{Z}(\boldsymbol{U})$) and reformulate the scalar products with equivalent equations (7.23)-(7.25). In the last step (7.26) we define new bilinear forms.

$$c(P_{\zeta,\infty},\mathcal{Z}^{-1}\cdot Q_\zeta) - b_0(\boldsymbol{U};\mathcal{Z}^{-1}\cdot Q_\zeta) = b_\zeta(\boldsymbol{U};\delta\boldsymbol{U},\mathcal{Z}^{-1}\cdot Q_\zeta) - c(\delta P_{\zeta,\infty},\mathcal{Z}^{-1}\cdot Q_\zeta) \tag{7.23}$$

$$\begin{aligned}\Big\langle\,\kappa_\infty\cdot P_{\zeta,\infty}\,,\,\mathcal{Z}^{-1}\cdot Q_\zeta\,\Big\rangle_{L^2(\Omega)} - \Big\langle\,\phi_V\,,\,\mathcal{Z}^{-1}\cdot Q_\zeta\,\Big\rangle_{L^2(\Omega)} & \\ = \Big\langle\,\mathcal{Z}\cdot\boldsymbol{\mathcal{S}}_{V,\zeta}:\boldsymbol{\mathcal{E}}(\boldsymbol{U};\delta\boldsymbol{U})\,,\,\mathcal{Z}^{-1}\cdot Q_\zeta\,\Big\rangle_{L^2(\Omega)} & \\ - \Big\langle\,\kappa_\infty\cdot\delta P_{\zeta,\infty}\,,\,\mathcal{Z}^{-1}\cdot Q_\zeta\,\Big\rangle_{L^2(\Omega)} &\end{aligned} \tag{7.24}$$

$$\begin{aligned}\Leftrightarrow\quad \Big\langle\kappa_\infty\cdot\mathcal{Z}^{-1}\cdot P_{\zeta,\infty}\,,\,Q_\zeta\Big\rangle_{0,\Omega} - \Big\langle\mathcal{Z}^{-1}\cdot\phi_V\,,\,Q_\zeta\Big\rangle_{0,\Omega} & \\ = \langle\boldsymbol{\mathcal{S}}_{V,\zeta}:\boldsymbol{\mathcal{E}}(\boldsymbol{U};\delta\boldsymbol{U})\,,\,Q_\zeta\rangle_{0,\Omega} & \\ - \Big\langle\kappa_\infty\cdot\mathcal{Z}^{-1}\cdot\delta P_{\zeta,\infty}\,,\,Q_\zeta\Big\rangle_{0,\Omega} &\end{aligned} \tag{7.25}$$

$$c_\zeta\Big(\boldsymbol{U};P_{\zeta,\infty},Q_\zeta\Big) - b_\zeta\Big(\boldsymbol{U};Q_\zeta\Big) = a_V(\boldsymbol{U};\delta\boldsymbol{U},Q_\zeta) - c_\zeta(\boldsymbol{U};\delta P_{\zeta,\infty},Q_\zeta) \tag{7.26}$$

Because of $\mathbb{Q} = \mathbb{Q}_\zeta$ one can replace Q_ζ by Q and this leads to the desired saddle point problem for all $Q \in \mathbb{Q}$ and $\boldsymbol{V} \in \mathbb{V}_0$.

$$\begin{aligned} t\, f_0(\boldsymbol{V}) - a_D(\boldsymbol{U}; \boldsymbol{V}) - a_V(\boldsymbol{U}, P_{\zeta,\infty}; \boldsymbol{V}) & \\ = a(\boldsymbol{U}, P_{\zeta,\infty}; \delta\boldsymbol{U}, \boldsymbol{V}) + a_V(\boldsymbol{U}, \delta P_{\zeta,\infty}; \boldsymbol{V}) & \end{aligned} \tag{7.27}$$

$$c_\zeta\Big(\boldsymbol{U}; P_{\zeta,\infty}, Q\Big) - b_\zeta\Big(\boldsymbol{U}; Q\Big) = a_V(\boldsymbol{U}; \delta\boldsymbol{U}, Q) - c_\zeta(\boldsymbol{U}; \delta P_{\zeta,\infty}, Q) \tag{7.28}$$

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