TECHNISCHE UNIVERSITÄT CHEMNITZ

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The Koiter shell equation in a coordinate free description - extended

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Abstract

We give an alternate description of Koiter's shell equation that does not depend on the special mid surface coordinates, but uses differential operators defined on the mid surface. This is the continuation of the preprint [4] due to new additional simplifications of these operators.

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Contents

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1 Introduction

We consider the deformation of a thin shell of constant thickness h under mechanical loads.

If a usual linear elastic material behavior is proposed, then consequently a linearized strain tensor has to be considered. In this case, the well established Koiter shell equation is obtained after some additional simplifications.

We consider these simplifications from the initial large strain equation to the Koiter shell equation in an easier form due to Ciarlet [3]. Based on this, we are able to find a coordinate free description, which means that differential operators (defined on the mid surface of the shell) are used instead of derivatives with respect to the surface parameters (coordinates) (η^1, η^2).

2 Basic differential geometry

2.1 The initial mid surface

We start with the description of the basic differential geometry on both the undeformed shell (initial domain) and the shell after deformation. All vectors and matrices belonging to the initial configuration (mainly the co- and contravariant basis vectors and the matrices of first and second fundamental forms) are written as capital letters. All these quantities belonging to the deformed structure are the same lower case letters. Let

$$\mathcal{S}_0 = \left\{ \boldsymbol{Y}\left(\eta^1, \eta^2\right) : \left(\eta^1, \eta^2\right) \in \Omega \subset \mathbb{R}^2 \right\}$$

be the mid surface of the undeformed shell, where $\boldsymbol{Y}$ denote the points of the surface in the 3-dimensional space and (η^1, η^2) run through a parameter domain Ω. Then we have

$$\begin{aligned} \boldsymbol{A}_i &= \frac{\partial}{\partial \eta^i} \boldsymbol{Y} \quad \text{the tangential vectors } i = 1, 2 \\ \boldsymbol{A}_3 = \boldsymbol{A}^3 &= (\boldsymbol{A}_1 \times \boldsymbol{A}_2) \,\big/\, |\boldsymbol{A}_1 \times \boldsymbol{A}_2| \quad \text{surface normal vector.} \end{aligned}$$

This defines the first metrical fundamental forms $A_{ij} = \boldsymbol{A}_i \cdot \boldsymbol{A}_j$ written as the (2×2)-matrix

$$\underline{A} = \left(A_{ij}\right)_{ij=1}^2 .$$

The surface element is

$$d\mathcal{S} = |\boldsymbol{A}_1 \times \boldsymbol{A}_2| \; d\eta^1 d\eta^2 = (det\underline{A})^{1/2} \; d\eta^1 d\eta^2$$

and the contravariant basis is
$\boldsymbol{A}^j = A^{jk}\boldsymbol{A}_k$ with $\boldsymbol{A}^j \cdot \boldsymbol{A}_k = \delta_k^j$ and A^{jk} the entries of $\underline{A}^{-1}$.

The second fundamental forms are

$$B_{ij} = \left(\frac{\partial^2}{\partial\eta^i\partial\eta^j}\boldsymbol{Y}\right)\cdot\boldsymbol{A}_3 = \boldsymbol{A}_{i,j}\cdot\boldsymbol{A}_3 = -\boldsymbol{A}_i\cdot\boldsymbol{A}_{3,j}$$

forming the matrix $\underline{B} = (B_{ij})_{ij=1}^2$.

We recall the Gauss- and Weingarten-equations

$$\begin{aligned}
\boldsymbol{A}_{3,i} &= -B_{ij}A^{jk}\boldsymbol{A}_k = -B_{ij}\boldsymbol{A}^j,\\
\boldsymbol{A}_{i,j} &= \Gamma_{ij}^k\boldsymbol{A}_k + B_{ij}\boldsymbol{A}_3,\\
\boldsymbol{A}^i_{,j} &= -\Gamma_{jk}^i\boldsymbol{A}^k + B_j^i\boldsymbol{A}_3, \quad B_j^i = B_{jk}A^{ki}
\end{aligned}$$

with the Christoffel symbols $2\Gamma_{ij}^k = A^{kl}(A_{il,j} + A_{jl,i} - A_{ij,l})$.

Throughout this paper we use Einstein's summation convention, where consequently all indices run from 1 to 2 only.

Later on, we will need the two second order tensors $\mathcal{A} = A_{ij}\boldsymbol{A}^i\boldsymbol{A}^j$ and $\mathcal{B} = B_{ij}\boldsymbol{A}^i\boldsymbol{A}^j$ often referred as metric tensor and curvature tensor of the surface $\mathcal{S}_0$.

Throughout this paper a pair of vectors (first order tensors) as $\boldsymbol{A}^1\boldsymbol{A}^2$ (or $\boldsymbol{A}_1\boldsymbol{A}_2$ or similar) is understood as second order tensor. A second order tensor in general is any linear combination of such pairs. The main meaning of a second order tensor is its action as a map of the (3-dimensional) vector functions onto itself via the dot product:

$$\begin{aligned}
(\boldsymbol{A}^1\boldsymbol{A}^2)\cdot\boldsymbol{U} &= \boldsymbol{A}^1\ (\boldsymbol{A}^2\cdot\boldsymbol{U})\\
\boldsymbol{U}\cdot(\boldsymbol{A}^1\boldsymbol{A}^2) &= \boldsymbol{A}^2\ (\boldsymbol{U}\cdot\boldsymbol{A}^1)
\end{aligned}$$

consequently the second order tensor $\boldsymbol{A}^1\boldsymbol{A}^2$ has a trace $tr(\boldsymbol{A}^1\boldsymbol{A}^2) = \boldsymbol{A}^1\cdot\boldsymbol{A}^2$ and the transposed tensor of $\boldsymbol{A}^1\boldsymbol{A}^2$ is $(\boldsymbol{A}^1\boldsymbol{A}^2)^T = \boldsymbol{A}^2\boldsymbol{A}^1$.

The double dot product between two second order tensors such as

$$(\boldsymbol{A}^1\boldsymbol{A}^2) : (\boldsymbol{A}^3\boldsymbol{A}^4) = (\boldsymbol{A}^2\cdot\boldsymbol{A}^3)(\boldsymbol{A}^1\cdot\boldsymbol{A}^4)$$

is a scalar function on (η^1, η^2). Later on, we use 4th order tensors in the same manner, as a 4-tuple of vectors ($\boldsymbol{A}^1\boldsymbol{A}^2\boldsymbol{A}^3\boldsymbol{A}^4$ and an arbitrary linear combination of those) or as a pair of second order tensors. Here, the main operation is the double dot product as a map of second order tensors onto second order tensors.

From this definition both tensors $\mathcal{A}$ and $\mathcal{B}$ are rank-2 tensors mapping each vector into the tangential space $span(\boldsymbol{A}_1, \boldsymbol{A}_2) = span(\boldsymbol{A}^1, \boldsymbol{A}^2)$. Especially $\mathcal{A}$ is the orthogonal projector onto this 2-dimensional space, due to:

$$\mathcal{A} = A_{ij}\boldsymbol{A}^i\boldsymbol{A}^j = \boldsymbol{A}_j\boldsymbol{A}^j = I - \boldsymbol{A}_3\boldsymbol{A}_3.$$

(Here, I denotes the identity tensor mapping each vector $\boldsymbol{U}$ onto itself).
It should be stressed that the two vectors $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are dependent on the parametrization (η^1, η^2) chosen to define $\mathcal{S}_0$ but $\boldsymbol{A}_3$ not, hence $\mathcal{A}$ and $\mathcal{B}$ are independent on the special coordinates (η^1, η^2) but functions on the given point $\boldsymbol{Y}$ of $\mathcal{S}_0$ only. So, $(\boldsymbol{Y} \mapsto \boldsymbol{A}_3)$ is called the Gaussian map and $\mathcal{B}$ the Weingarten map. Furthermore the surface gradient as gradient operator on the tangential space also is independent on the special parametrization (η^1, η^2), obviously

$$Grad_{\mathcal{S}} = \boldsymbol{A}^i \frac{\partial}{\partial \eta^i}.$$

The matrix $\underline{A}^{-1}\underline{B}$ has two eigenvalues λ_1 and λ_2 as main curvatures at $\boldsymbol{Y}(\eta^1, \eta^2)$, as well as the tensor $\mathcal{B}$ has these eigenvalues (together with a 0 as rank-2 tensor), so

$H = (\lambda_1 + \lambda_2)/2 = tr\mathcal{B}/2 = tr(\underline{A}^{-1}\underline{B})/2$ is the mean curvature and
$K = \lambda_1 \cdot \lambda_2 = det(\underline{A}^{-1}\underline{B})$ the Gaussian curvature at $\boldsymbol{Y}$.

2.2 The initial shell

The initial shell is the 3-dimensional manifold

$$\mathcal{H}_0 = \left\{ \boldsymbol{X}\left(\eta^1, \eta^2, \tau = \eta^3\right) = \boldsymbol{Y}\left(\eta^1, \eta^2\right) + \tau h \boldsymbol{A}_3,\ \left(\eta^1, \eta^2\right) \in \Omega,\ |\tau| \leq \frac{1}{2} \right\} \quad (1)$$

with the constant thickness h and $\boldsymbol{A}_3$ from 2.1. For an easy description of the following let $\tau = \eta^3$ be a synonym for the (dimensionless) thickness coordinate. We may use η^1 and η^2 dimensionless as well, then $\boldsymbol{A}_i$ have length dimension (in m) and $\boldsymbol{A}^i$ in $1/m$ while $\boldsymbol{A}_3 = \boldsymbol{A}^3$ is dimensionless in any case. In 3D we have to consider the covariant basis

$$\boldsymbol{G}_i = \frac{\partial}{\partial \eta^i} \boldsymbol{X} = \boldsymbol{A}_i + \tau h \boldsymbol{A}_{3,i}\ , \quad i = 1, 2$$

and $\boldsymbol{G}_3 = h\boldsymbol{A}_3$ as well as the contravariant tensor basis $\boldsymbol{G}^i$ $(i = 1, 2)$ and $\boldsymbol{G}^3 = h^{-1}\boldsymbol{A}_3$.

The volume element of the shell is

$$dV = [\boldsymbol{G}_1\ ,\ \boldsymbol{G}_2\ ,\ \boldsymbol{G}_3]\ d\eta^1 d\eta^2 d\tau = h\ det(\underline{G})^{1/2}\ d\eta^1 d\eta^2 d\tau$$

with the (2×2)-matrix $\underline{G} = (G_{ij})^2_{i,j=1}$, $G_{ij} = \boldsymbol{G}_i \cdot \boldsymbol{G}_j$, which is simply calculated as

$$\underline{G} = \underline{A}\left(\underline{I} - \tau h \underline{A}^{-1}\underline{B}\right)^2 = \left(\underline{A} - \tau h \underline{B}\right) \underline{A}^{-1} \left(\underline{A} - \tau h \underline{B}\right). \quad (2)$$

From this, the volume element is well-known as

$$\begin{aligned} dV &= h\ (1 - 2\tau h H + (\tau h)^2 K)\ d\tau\ d\mathcal{S} \\ &= (1 - \tau h \lambda_1)(1 - \tau h \lambda_2)\ h\, d\tau\ d\mathcal{S}. \end{aligned}$$

Here, the necessary condition

$$\epsilon_{\mathcal{H}} := (h/2) \max_{(\eta^1,\eta^2)\in\Omega} (\max(\lambda_1, \lambda_2)) \ < \ 1$$

guarantees the admissibility of the parametrization of the initial shell. Consistent with the historic literature, we strengthen this inequality in the following considerations to the case of **thin shells** as

$$\epsilon_{\mathcal{H}} << 1. \tag{3}$$

This allows the approximation of the volume element by $h\ d\tau\ d\mathcal{S}$ as well as the approximation of the matrix $(\underline{I} - \tau h \underline{A}^{-1}\underline{B})$ by $\underline{I}$ without significant errors.

2.3 Special case: the plate

Here we have a simplification on $\mathcal{S}_0$ such as $\boldsymbol{Y} = L_1\boldsymbol{e}_1 \cdot \eta^1 + L_2\boldsymbol{e}_2 \cdot \eta^2$, yielding $\boldsymbol{A}_3 = \boldsymbol{e}_3$ independent on (η^1, η^2). From this a lot of simplifications arise:

$$\boldsymbol{G}_i = \boldsymbol{A}_i, \qquad \underline{B} = \mathbb{O}, \qquad \mathcal{B} = 0.$$

2.4 The deformed shell

The basic assumption of the simple shell models consists in keeping a straight line of the points

$$\left\{ \boldsymbol{Y}(\eta^1, \eta^2) + \tau h \boldsymbol{A}_3(\eta^1, \eta^2) : |\tau| \le \frac{1}{2} \right\}$$

after the deformation also, i. e. the mid surface is deformed as

$$\mathcal{S}_t = \left\{ \boldsymbol{y}(\eta^1, \eta^2) = \boldsymbol{Y}(\eta^1, \eta^2) + \boldsymbol{U}(\eta^1, \eta^2) \ : \ (\eta^1, \eta^2) \in \Omega \right\}$$

with an unknown displacement vector $\boldsymbol{U}$ (a function of (η^1, η^2) as well as of $\boldsymbol{Y}$). The weaker assumption defines the deformed shell as

$$\mathcal{H}_t = \left\{ \boldsymbol{x}(\eta^1, \eta^2, \tau) = \boldsymbol{y}(\eta^1, \eta^2) + \tau h\ \boldsymbol{d}(\eta^1, \eta^2) \right\} \tag{4}$$

with an additional vector field $\boldsymbol{d}(\eta^1, \eta^2)$ (the so called director vector).

Here, we concentrate on the stronger Kirchhoff assumption, where $\boldsymbol{d}$ is the new surface normal vector $\boldsymbol{a}_3$ of the deformed surface $\mathcal{S}_t$ following its differential geometry:
Let

$$\boldsymbol{a}_i = \frac{\partial}{\partial \eta^i}\boldsymbol{y} = \boldsymbol{A}_i + \boldsymbol{U}_{,i}$$

the tangential vectors after deformation, then analogously we have

$$\boldsymbol{a}_3 = (\boldsymbol{a}_1 \times \boldsymbol{a}_2) / |\boldsymbol{a}_1 \times \boldsymbol{a}_2|$$

as surface normal vector of $\mathcal{S}_t$. Again we have

$$\begin{aligned} \underline{a} &= (a_{ij})_{i,j=1}^{2} \quad \text{with} \quad a_{ij} = \boldsymbol{a}_i \cdot \boldsymbol{a}_j \\ \underline{b} &= (b_{ij})_{i,j=1}^{2} \quad \text{with} \quad b_{ij} = \boldsymbol{a}_{i,j} \cdot \boldsymbol{a}_3 \end{aligned}$$

as new first and second fundamental forms.

With

$$\boldsymbol{d} = \boldsymbol{a}_3, \tag{5}$$

the 3D covariant basis is now

$$\boldsymbol{g}_i = \frac{\partial}{\partial \eta^i}\boldsymbol{x} = \boldsymbol{a}_i + \tau h \boldsymbol{a}_{3,i} = \boldsymbol{A}_i + \boldsymbol{U}_{,i} + \tau h \boldsymbol{a}_{3,i}$$

and $\boldsymbol{g}_3 = h\boldsymbol{a}_3$. Hence, we can define the (2×2)-matrix $\underline{g} = (g_{ij})$ from $g_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j$, which is analogously to $\underline{G}$:

$$\underline{g} = \underline{a}\left(\underline{I} - \tau h\ \underline{a}^{-1}\underline{b}\right)^2 = \left(\underline{a} - \tau h\ \underline{b}\right)\underline{a}^{-1}\left(\underline{a} - \tau h\ \underline{b}\right). \tag{6}$$

3 The strain tensor and its simplifications

From the definition of the deformed shell as (4,5), we may deduce the 3D-deformation gradient

$$\mathcal{F} = \boldsymbol{g}_i \boldsymbol{G}^i + \boldsymbol{g}_3 \boldsymbol{G}^3 = \boldsymbol{g}_i \boldsymbol{G}^i + \boldsymbol{a}_3 \boldsymbol{A}^3,$$

the right Cauchy Green tensor $\mathcal{C} = \mathcal{F}^T \cdot \mathcal{F}$ and the strain tensor

$$\mathcal{E} = \frac{1}{2}(\mathcal{C} - I) = \frac{1}{2}(g_{ij} - G_{ij})\,\boldsymbol{G}^i\boldsymbol{G}^j = \frac{1}{2}(g_{ij}\boldsymbol{G}^i\boldsymbol{G}^j - \mathcal{A}), \tag{7}$$

which is a rank-2 tensor only without components of $\boldsymbol{G}^i\boldsymbol{G}^3$ or $\boldsymbol{G}^3\boldsymbol{G}^3$.

In the following we expand this tensor with respect to the surface basis vectors $\boldsymbol{A}^i\boldsymbol{A}^j$ instead of $\boldsymbol{G}^i\boldsymbol{G}^j$ which is possible due to $span(\boldsymbol{G}^1, \boldsymbol{G}^2) = span(\boldsymbol{A}^1, \boldsymbol{A}^2)$:

$$\begin{aligned}\boldsymbol{G}_i &= \boldsymbol{A}_i + \tau h \boldsymbol{A}_{3,i} \\ &= \boldsymbol{A}_i - \tau h B_{ij} A^{jk} \boldsymbol{A}_k \\ &= (A_{ij} - \tau h B_{ij}) \boldsymbol{A}^j\end{aligned}$$

yields

$$\boldsymbol{G}^k = G^{ki}\boldsymbol{G}_i = G^{ki}(A_{ij} - \tau h B_{ij})\boldsymbol{A}^j. \tag{8}$$

Hence, $(g_{ij} - G_{ij})\boldsymbol{G}^i\boldsymbol{G}^j$ is written as

$$\left[(A_{ks} - \tau h B_{ks})G^{si}\right](g_{ij} - G_{ij})\left[G^{jm}(A_{ml} - \tau h B_{ml})\right]\boldsymbol{A}^k\boldsymbol{A}^l$$

which means that

$$2\mathcal{E} = \epsilon_{ij}\boldsymbol{A}^i\boldsymbol{A}^j \tag{9}$$

and the matrix of the coefficients ϵ_{ij} is the following matrix product

$$(\epsilon_{ij})_{i,j=1}^2 = \underline{A}(\underline{A} - \tau h\underline{B})^{-1}\underline{g}(\underline{A} - \tau h\underline{B})^{-1}\underline{A} - \underline{A} \tag{10}$$

$$= (\underline{I} - \tau h\underline{B}\,\underline{A}^{-1})^{-1}(\underline{g} - \underline{G})(\underline{I} - \tau h\underline{A}^{-1}\underline{B})^{-1}. \tag{11}$$

Note, that the expression (9) describes the correct strain tensor without any simplifications belonging to the Kirchhoff-assumption (5). In contrast to the formula

$$2\mathcal{E} = (g_{ij} - G_{ij})\,\boldsymbol{G}^i\,\boldsymbol{G}^j$$

all dependencies on the thickness coordinate τ are contained in ϵ_{ij} only, the tensor basis is constant w.r.t. τ. From this formula the simplifications towards Koiter's shell equation can be deduced.

This is done in 3 steps:

1. Due to the thin shell assumption (3) we approximate ϵ_{ij} by $\epsilon_{ij}^{(1)} = (g_{ij} - G_{ij})$, which are now quadratic functions w.r.t. (τh).
2. We neglect the quadratic terms of order $(\tau h)^2$, yielding $\mathcal{E}^{(2)}$.
3. Then $\mathcal{E}^{(2)}$ still contains nonlinear expressions similar to large strain, which are products of derivatives of $\boldsymbol{U}$, such as $(\boldsymbol{U}_{,i} \cdot \boldsymbol{U}_{,j})$. As usually for "small strain" equations we neglect all such products, yielding $\mathcal{E}^{(3)}$ being a linear differential operator applied to $\boldsymbol{U}$.

These 3 steps result in an approximate strain tensor, which is typically a linear differential operator acting on $\boldsymbol{U}$ (as small strain) and is a sum of "change of metric" and "change of curvature" as

$$\begin{aligned}\mathcal{E}^{(3)} &= \mathcal{E}^a - \tau h \mathcal{E}^b \\ \mathcal{E}^a &= \gamma_{ij}(\boldsymbol{U})\boldsymbol{A}^i\boldsymbol{A}^j \\ \mathcal{E}^b &= \varrho_{ij}(\boldsymbol{U})\boldsymbol{A}^i\boldsymbol{A}^j.\end{aligned}$$

The calculations within these 3 steps are easily done:

1. First we consider the matrix (11) and neglect $(\underline{I} - \tau h \underline{A}^{-1}\underline{B})$, yielding

$$\begin{aligned}(\epsilon_{ij}^{(1)})_{i,j=1}^2 &= \underline{g} - \underline{G} \\ &= \underline{a} - 2\tau h\ \underline{b} + (\tau h)^2\ \underline{b}\underline{a}^{-1}\underline{b} \\ &\quad - (\underline{A} - 2\tau h\ \underline{B} + (\tau h)^2\ \underline{B}\underline{A}^{-1}\underline{B}).\end{aligned}$$

2. Hence

$$2\mathcal{E}^{(2)} = \epsilon_{ij}^{(2)}\boldsymbol{A}^i\boldsymbol{A}^j$$

   with

$$\epsilon_{ij}^{(2)} = (a_{ij} - A_{ij}) - 2\tau h\ (b_{ij} - B_{ij}).$$

3. We linearize $\mathcal{E}^{(2)}$ to

$$\mathcal{E}^{(3)} = \mathcal{E}^a - \tau h\ \mathcal{E}^b$$

   which can be seen as Koiter-strain-tensor because the coefficients $\gamma_{ij}(\boldsymbol{U})$ and $\varrho_{ij}(\boldsymbol{U})$ are linear differential operators applied to $\boldsymbol{U}$ and are the same expressions as in Koiter's shell model.

4 Linearization of $\mathcal{E}^{(2)}$ to small strain and coordinate free description

We recall

$$\mathcal{E}^{(2)} = [\tfrac{1}{2}(a_{ij} - A_{ij}) - \tau h\,(b_{ij} - B_{ij})]\boldsymbol{A}^i\boldsymbol{A}^j$$

and linearize both parts separately.

4.1 Change of metric tensor

From

$$a_{ij} = A_{ij} + \boldsymbol{A}_i \cdot \boldsymbol{U}_{,j} + \boldsymbol{A}_j \cdot \boldsymbol{U}_{,i} + \boldsymbol{U}_{,i} \cdot \boldsymbol{U}_{,j}$$

simply follows

$$\gamma_{ij} = \tfrac{1}{2}(\boldsymbol{A}_i \cdot \boldsymbol{U}_{,j} + \boldsymbol{A}_j \cdot \boldsymbol{U}_{,i}) \tag{12}$$

as found in the Koiter shell equation as well.

The tensor

$$\mathcal{E}^a = \gamma_{ij}(\boldsymbol{U})\boldsymbol{A}^i\boldsymbol{A}^j$$

has a simple coordinate free representation from:

$$\begin{aligned} 2\mathcal{E}^a &= \mathcal{E}_1 + \mathcal{E}_1^T, \\ \mathcal{E}_1 &= (\boldsymbol{A}_j \cdot \boldsymbol{U}_{,i})\boldsymbol{A}^i\boldsymbol{A}^j \\ &= \boldsymbol{A}^i\boldsymbol{U}_{,i} \cdot \boldsymbol{A}_j\boldsymbol{A}^j \\ &= Grad_{\mathcal{S}}\boldsymbol{U} \cdot \mathcal{A}. \end{aligned}$$

Hence, we have

$$2\mathcal{E}^a = \mathcal{A} \cdot (Grad_{\mathcal{S}}\boldsymbol{U})^T + (Grad_{\mathcal{S}}\boldsymbol{U}) \cdot \mathcal{A} \tag{13}$$

with the orthogonal projector $\mathcal{A}$ onto the tangential plane at $\boldsymbol{Y}$.

4.2 Change of curvature tensor

Here, the linearization of $(b_{ij} - B_{ij})\boldsymbol{A}^i\boldsymbol{A}^j$ to

$$\mathcal{E}^b = \varrho_{ij}(\boldsymbol{U})\boldsymbol{A}^i\boldsymbol{A}^j$$

is a longer calculation. We start with abbreviations for

$$B_{ij} = \boldsymbol{A}_3 \cdot \boldsymbol{A}_{i,j} = \frac{1}{\alpha}[\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_{i,j}]$$

with the spatial inner product $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$ and $\alpha = |\boldsymbol{A}_1 \times \boldsymbol{A}_2|$.

In the same way we have

$$\begin{aligned} b_{ij} &= \zeta_{ij}\beta \qquad \text{with} \\ \zeta_{ij} &= \frac{1}{\alpha}[\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_{i,j}] \qquad \text{and} \\ \beta &= \frac{\alpha}{|\boldsymbol{a}_1 \times \boldsymbol{a}_2|} \end{aligned}$$

and we linearize both parts ζ_{ij} and β separately. In ζ_{ij} we delete all quadratic terms in $\boldsymbol{U}$ to:

$$\zeta_{ij} = B_{ij} + \frac{1}{\alpha}[\boldsymbol{U}_{,1}\ ,\ \boldsymbol{A}_2\ ,\ \boldsymbol{A}_{i,j}] + \frac{1}{\alpha}[\boldsymbol{A}_1\ ,\ \boldsymbol{U}_{,2}\ ,\ \boldsymbol{A}_{i,j}] + \frac{1}{\alpha}[\boldsymbol{A}_1\ ,\ \boldsymbol{A}_2\ ,\ \boldsymbol{U}_{,ij}]\,. \tag{14}$$

(The last term is $\boldsymbol{A}_3 \cdot \boldsymbol{U}_{,ij}$). Now, let us start with the first two terms. Using Gauss–Weingarten equations we get

$$\frac{1}{\alpha}\left[\boldsymbol{A}_2, \boldsymbol{A}_{i,j}, \boldsymbol{U}_{,1}\right] = \frac{1}{\alpha}\left[\boldsymbol{A}_2\ ,\ \Gamma^k_{ij}\boldsymbol{A}_k + B_{ij}\boldsymbol{A}_3\ ,\ \boldsymbol{U}_{,1}\right] =$$

$$= -\Gamma^1_{ij}(\boldsymbol{A}_3 \cdot \boldsymbol{U}_{,1}) + \frac{1}{\alpha}B_{ij}\left[\boldsymbol{A}_2, \boldsymbol{A}_3, \boldsymbol{U}_{,1}\right].$$

So,

$$\begin{aligned}
\zeta_{ij} &= B_{ij} + \varrho_{ij} + \zeta^{(2)}_{ij} \\
\varrho_{ij} &= \boldsymbol{A}_3 \cdot (\boldsymbol{U}_{,ij} - \Gamma^k_{ij}\boldsymbol{U}_{,k}) \\
\zeta^{(2)}_{ij} &= \frac{1}{\alpha}B_{ij}\left([\boldsymbol{A}_2, \boldsymbol{A}_3, \boldsymbol{U}_{,1}] + [\boldsymbol{A}_3, \boldsymbol{A}_1, \boldsymbol{U}_{,2}]\right).
\end{aligned}$$

The first term $\varrho_{ij}(\boldsymbol{U})$ is exactly found in the Koiter's shell equation (compare [3]) and will be transformed into a coordinate free description at the end of this chapter. The other part $\zeta^{(2)}_{ij}$ has a simple coordinate free meaning and will vanish after linearizing β.

$$\begin{aligned}
&\frac{1}{\alpha}(\boldsymbol{A}_2 \times \boldsymbol{A}_3) \cdot \boldsymbol{U}_{,1} + \frac{1}{\alpha}(\boldsymbol{A}_3 \times \boldsymbol{A}_1) \cdot \boldsymbol{U}_{,2} \\
&= \boldsymbol{A}^1 \cdot \boldsymbol{U}_{,1} + \boldsymbol{A}^2 \cdot \boldsymbol{U}_{,2} \\
&= Div_{\mathcal{S}}\boldsymbol{U} = Grad_{\mathcal{S}} \cdot \boldsymbol{U}
\end{aligned}$$

So,

$$\zeta^{(2)}_{ij} = (Div_{\mathcal{S}}\boldsymbol{U})\ B_{ij}.$$

The term

$$\beta^{-1} = \frac{|\boldsymbol{a}_1 \times \boldsymbol{a}_2|}{\alpha}$$

is

$$|\boldsymbol{A}_3 + \frac{1}{\alpha}(\boldsymbol{U}_{,1} \times \boldsymbol{A}_2) + \frac{1}{\alpha}(\boldsymbol{A}_1 \times \boldsymbol{U}_{,2})| + \text{ h.o.t. },$$

so

$$\begin{aligned}
\beta^{-2} &= 1 + 2\frac{1}{\alpha}(\boldsymbol{U}_{,1} \times \boldsymbol{A}_2) \cdot \boldsymbol{A}_3 + 2\frac{1}{\alpha}(\boldsymbol{A}_1 \times \boldsymbol{U}_{,2}) \cdot \boldsymbol{A}_3 + \text{ h.o.t.} \\
&= 1 + 2Div_{\mathcal{S}}\boldsymbol{U} + \text{ h.o.t.}
\end{aligned}$$

This leads to

$$\beta = 1 - (Div_{\mathcal{S}}\boldsymbol{U}) + \text{ h.o.t.}$$

and

$$\begin{aligned}
b_{ij} &= \zeta_{ij}\beta = (B_{ij} + \varrho_{ij} + B_{ij}(Div_{\mathcal{S}}\boldsymbol{U}))(1 - (Div_{\mathcal{S}}\boldsymbol{U})) + \text{ h.o.t.} \\
&= B_{ij} + \varrho_{ij} + \text{ h.o.t.}
\end{aligned}$$

Hence, the linearization of

$$(b_{ij} - B_{ij})\boldsymbol{A}^i\boldsymbol{A}^j$$

leads to

$$\mathcal{E}^b = \varrho_{ij}\boldsymbol{A}^i\boldsymbol{A}^j$$

with

$$\varrho_{ij} = \boldsymbol{A}_3 \cdot (\boldsymbol{U}_{,ij} - \Gamma^k_{ij}\boldsymbol{U}_{,k}). \tag{15}$$

This tensor can be written in a coordinate free manner due to the following manipulations. First we note that

$$\varrho_{ij}\boldsymbol{A}^i\boldsymbol{A}^j = (\varrho_{ij}\boldsymbol{A}^i\boldsymbol{A}^j) \cdot \mathcal{A}.$$

This leads to

$$\begin{aligned}
\varrho_{ij}\boldsymbol{A}^i\boldsymbol{A}^j &= (\ \boldsymbol{A}^i \left[\boldsymbol{A}^j(\boldsymbol{U}_{,ij} \cdot \boldsymbol{A}_3) - \Gamma^k_{ij}\boldsymbol{A}^j(\boldsymbol{U}_{,k} \cdot \boldsymbol{A}_3)\right]\ ) \cdot \mathcal{A} \\
&= (\ \boldsymbol{A}^i \left[\boldsymbol{A}^j(\boldsymbol{U}_{,ij} \cdot \boldsymbol{A}_3) - \Gamma^k_{ij}\boldsymbol{A}^j(\boldsymbol{U}_{,k} \cdot \boldsymbol{A}_3) + B^k_i\boldsymbol{A}_3(\boldsymbol{U}_{,k} \cdot \boldsymbol{A}_3)\right]\ ) \cdot \mathcal{A} \\
&= (\ \boldsymbol{A}^i \left[\frac{\partial}{\partial\eta^i}\left(\boldsymbol{A}^k\boldsymbol{U}_{,k}\right)\right] \cdot \boldsymbol{A}_3\ ) \cdot \mathcal{A} \\
&= (\ \boldsymbol{A}^i \left[\frac{\partial}{\partial\eta^i} Grad_{\mathcal{S}}\boldsymbol{U}\right] \cdot \boldsymbol{A}_3\ ) \cdot \mathcal{A} \\
&= (\ [Grad_{\mathcal{S}}\ Grad_{\mathcal{S}}\ \boldsymbol{U}] \cdot \boldsymbol{A}_3\ ) \cdot \mathcal{A}
\end{aligned}$$

This is the 3rd order tensor $[Grad_{\mathcal{S}}\ Grad_{\mathcal{S}}\ \boldsymbol{U}]$ applied to $\boldsymbol{A}_3$, yielding a second order tensor, which is multiplied with the orthogonal projector $\mathcal{A}$.

So, the final result is a coordinate free Koiter strain as

$$\begin{aligned}
\mathcal{E}^{linK} &= \mathcal{E}^a\ - \tau h\ \mathcal{E}^b, \\
2\mathcal{E}^a &= \mathcal{A} \cdot (Grad_{\mathcal{S}}\boldsymbol{U})^T + (Grad_{\mathcal{S}}\boldsymbol{U}) \cdot \mathcal{A}, \qquad (16) \\
\mathcal{E}^b &= ([Grad_{\mathcal{S}}\ Grad_{\mathcal{S}}\ \boldsymbol{U}] \cdot \boldsymbol{A}_3) \cdot \mathcal{A}. \qquad (17)
\end{aligned}$$

5 The resulting Koiter energy

We complete the resulting deformation energy of the shell by inserting $\mathcal{E}^{(3)}$ into the energy functional. Due to the desired small strain assumption in $\mathcal{E}^{(3)}$, we use a linear material law, such as

$$W(\boldsymbol{U}) = \frac{1}{2}\int_{\mathcal{H}_0} \mathcal{E} : \mathfrak{C} : \mathcal{E}\ dV \tag{18}$$

with a (possibly space dependent) 4th order material tensor $\mathfrak{C}$. The most simple case, the St.Venant–Kirchhoff material, is considered to be

$$\mathfrak{C} = 2\mu\mathfrak{I} + \lambda(I\ I)$$

with the Lamé constants

$$2\mu = \frac{E}{1+\nu} \qquad \text{and} \qquad \lambda = 2\mu\frac{\nu}{1-\nu}$$

(for the plane stress assumption). Here, $\mathfrak{I}$ is the 4th order identity map ($\mathfrak{I} : \mathcal{X} = \mathcal{X}$ for each 2nd order tensor $\mathcal{X}$) and $(I\ I) : \mathcal{X} = I\ (I : \mathcal{X}) = I\ tr\mathcal{X}$.

Now, we end up with three different representations of the Koiter shell energy, depending on which strain formulation is inserted into (18). If the material tensor $\mathfrak{C}$ is constant over the thickness (independent on τ), we integrate over $\tau \in [-1/2, +1/2]$ and end up with the 2 parts:

$$W(\boldsymbol{U}) = h\ W^a(\boldsymbol{U}) + \frac{h^3}{12}W^b(\boldsymbol{U})$$

with

$$W^a(\boldsymbol{U}) = \frac{1}{2}\int\limits_{\mathcal{S}_0} \mathcal{E}^a : \mathfrak{C} : \mathcal{E}^a\ d\mathcal{S}$$

$$W^b(\boldsymbol{U}) = \frac{1}{2}\int\limits_{\mathcal{S}_0} \mathcal{E}^b : \mathfrak{C} : \mathcal{E}^b\ d\mathcal{S}.$$

With the formulas (16) and (17) this is a coordinate free form of Koiter's shell energy.

The second equivalent formula (comparable to [3]) is obtained from the coordinate dependent strain equation

$$\mathcal{E}^a = \gamma_{ij}(\boldsymbol{U})\boldsymbol{A}^i\boldsymbol{A}^j \qquad \text{and} \qquad \mathcal{E}^b = \varrho_{ij}(\boldsymbol{U})\boldsymbol{A}^i\boldsymbol{A}^j$$

inserted into the energy functional.

Here, we end up with

$$\begin{aligned} c^{ijkl} &= (\boldsymbol{A}^i\boldsymbol{A}^j) : \mathfrak{C} : (\boldsymbol{A}^k\boldsymbol{A}^l) \\ &= 2\mu A^{il}A^{jk} + \lambda A^{ij}A^{kl} \\ &= \frac{E}{1+\nu}(A^{il}A^{jk} + \frac{\nu}{1-\nu}A^{ij}A^{kl}) \end{aligned}$$

and

$$W^a(\boldsymbol{U}) = \frac{1}{2}\int\limits_{\mathcal{S}_0} \gamma_{ij}(\boldsymbol{U})\ c^{ijkl}\ \gamma_{kl}(\boldsymbol{U})d\mathcal{S}$$
$$W^b(\boldsymbol{U}) = \frac{1}{2}\int\limits_{\mathcal{S}_0} \varrho_{ij}(\boldsymbol{U})\ c^{ijkl}\ \varrho_{kl}(\boldsymbol{U})d\mathcal{S}$$

(Note that the volume element was simplified to $hd\tau d\mathcal{S}$).

The third (original) Koiter energy is obtained from a special ansatz for the unknown vector $\boldsymbol{U}$ as an expansion w.r.t. the contravariant surface basis:

$$\boldsymbol{U} = U_i\boldsymbol{A}^i + U_3\boldsymbol{A}_3$$

with the 3 unknown functions (U_1, U_2, U_3) depending on (η^1, η^2).

Now, the derivatives of $\boldsymbol{U}$ lead to longer expressions, such as

$$\begin{aligned}\boldsymbol{U}_{,j} &= U_{i,j}\boldsymbol{A}^i + U_i\boldsymbol{A}^i_{,j} + U_{3,j}\boldsymbol{A}_3 + U_3\boldsymbol{A}_{3,j} \\ &= (U_{k,j} - U_i\Gamma^i_{jk})\boldsymbol{A}^k + (U_{3,j} - U_iB^i_j)\boldsymbol{A}_3 - U_3B_{jk}\boldsymbol{A}^k\end{aligned}$$

and the coefficients $\gamma_{ij}(\boldsymbol{U})$ and $\varrho_{ij}(\boldsymbol{U})$ are changed into the well-known complicate functions on derivatives of (U_1, U_2, U_3), compare [2].

If a more general material law is considered (still constant w.r.t. τ), we have a change in the expressions of the c^{ijkl} only. For instance, a transversely isotropic material (compare [6, 7]) uses a spatial dependent normalized direction vector $\boldsymbol{a}(\eta^1, \eta^2)$ of given "fiber directions" and 5 material parameters $(\mu, \mu_a, \lambda, \alpha, \beta)$ to define the material tensor as

$$\mathfrak{C} = 2\mu\mathfrak{I} + \lambda(I\ I) + \alpha(\boldsymbol{aa}I + I\boldsymbol{aa}) + 2(\mu_a - \mu)\hat{\mathfrak{C}} + \beta(\boldsymbol{aaaa})$$

(Here, $\hat{\mathfrak{C}} : \mathcal{X} = (\boldsymbol{aa}) \cdot \mathcal{X} + \mathcal{X} \cdot (\boldsymbol{aa})$ for each second order tensor $\mathcal{X}$.)

Usually, the fiber direction $\boldsymbol{a}$ is in the tangential plane of $\mathcal{S}_0$ with $\boldsymbol{a} = a^1\boldsymbol{A}_1 + a^2\boldsymbol{A}_2$ then the material coefficients are generalized to

$$\begin{aligned}c^{ijkl} &= (\boldsymbol{A}^i\boldsymbol{A}^j) : \mathfrak{C} : (\boldsymbol{A}^k\boldsymbol{A}^l) \\ &= 2\mu A^{il}A^{jk} + \lambda A^{ij}A^{kl} + \alpha(a^ia^jA^{kl} + A^{ij}a^ka^l) \\ &\quad + 2(\mu_a - \mu)(a^ja^kA^{il} + A^{jk}a^la^i) + \beta a^ia^ja^ka^l\end{aligned}$$

without any change in the differential operators $\gamma_{ij}(\boldsymbol{U})$ and $\varrho_{ij}(\boldsymbol{U})$.

6 Remarks on the differential operators

6.1 The plate equation

The special case of the well–known plate equation is obtained from the simplifications of Section 2.3. Note that $x_1 = L_1\eta^1$ and $x_2 = L_2\eta^2$, so

$$Grad_{\mathcal{S}} = \boldsymbol{A}^i \frac{\partial}{\partial \eta^i} = \boldsymbol{e}_i \frac{\partial}{\partial x_i}.$$

Here, we may split $\boldsymbol{U} = \boldsymbol{u} + w\boldsymbol{e}_3$ into an in–plane part $\boldsymbol{u} = u_i\boldsymbol{e}_i$ and the plate deflection $w(x_1, x_2)\boldsymbol{e}_3$. Then we have from (16) and (17)

$$2\mathcal{E}^a = \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)\boldsymbol{e}_i\boldsymbol{e}_j,$$

the 2D-strain tensor in the plane and

$$\mathcal{E}^b = ([\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{U}_{,ij}] \cdot \boldsymbol{e}_3) \cdot \boldsymbol{e}_k\boldsymbol{e}_k = \boldsymbol{e}_i\boldsymbol{e}_j\frac{\partial^2 w}{\partial x_i x_j},$$

the strain in the Kirchhoff plate bending equation.

6.2 The behavior of the differential operators when applied to the initial midsurface $\boldsymbol{Y}$

A second remark is targeted at the behavior of both operators when applied to the surface definition $\boldsymbol{Y}$:
While

$$Grad_{\mathcal{S}}\boldsymbol{Y} = \mathcal{A}$$

is a simple consequence of the initial definition or may serve as the definition of the surface gradient operator, we have an analogous result for the other operator $[Grad_{\mathcal{S}}\, Grad_{\mathcal{S}}\, .] \cdot \boldsymbol{A}_3$:

$$\begin{aligned}
[Grad_{\mathcal{S}}\, Grad_{\mathcal{S}}\, \boldsymbol{Y}] \cdot \boldsymbol{A}_3 &= [Grad_{\mathcal{S}}\mathcal{A}] \cdot \boldsymbol{A}_3 \\
&= [Grad_{\mathcal{S}}(I - \boldsymbol{A}_3\boldsymbol{A}_3)] \cdot \boldsymbol{A}_3 \\
&= -\boldsymbol{A}^i(\boldsymbol{A}_{3,i}\boldsymbol{A}_3 + \boldsymbol{A}_3\boldsymbol{A}_{3,i}) \cdot \boldsymbol{A}_3 \\
&= \boldsymbol{A}^i(B_{ij}\boldsymbol{A}^j\boldsymbol{A}_3) \cdot \boldsymbol{A}_3 \\
&= \mathcal{B}.
\end{aligned} \tag{19}$$

Hence, if the linearized strain tensor is applied to $\boldsymbol{Y}$ instead of $\boldsymbol{U}$, we obtain

$$\mathcal{E}^a = \mathcal{A} \qquad \text{and} \qquad \mathcal{E}^b = \mathcal{B}\,.$$

6.3 The relation to the Laplace–Beltrami operator

As in the plane plate case, the surface Laplace–Beltrami operator $\triangle_{\mathcal{S}} = Div_{\mathcal{S}} Grad_{\mathcal{S}}$ arises from the *trace* of the $[Grad_{\mathcal{S}}\, Grad_{\mathcal{S}}\, .] \cdot \boldsymbol{A}_3$ operator. Note that this is correctly done for the second order tensor $[Grad_{\mathcal{S}}\, Grad_{\mathcal{S}}\, \boldsymbol{U}] \cdot \boldsymbol{A}_3$, not for the 3rd order $[Grad_{\mathcal{S}}\, Grad_{\mathcal{S}}\, \boldsymbol{U}]$. Obviously,

$$\begin{aligned} tr([Grad_{\mathcal{S}}\, Grad_{\mathcal{S}}\, \boldsymbol{U}] \cdot \boldsymbol{A}_3) &= tr([\boldsymbol{A}^i (Grad_{\mathcal{S}} \boldsymbol{U})_{,i}] \cdot \boldsymbol{A}_3) \\ &= tr(\boldsymbol{A}^i \boldsymbol{V}_i) \end{aligned}$$

with some vector function $\boldsymbol{V}_i = (Grad_{\mathcal{S}} \boldsymbol{U})_{,i} \cdot \boldsymbol{A}_3$. This simply yields

$$\begin{aligned} tr([Grad_{\mathcal{S}}\, Grad_{\mathcal{S}}\, \boldsymbol{U}] \cdot \boldsymbol{A}_3) &= tr(\boldsymbol{A}^i \boldsymbol{V}_i) = \boldsymbol{A}^i \cdot \boldsymbol{V}_i \\ &= \boldsymbol{A}^i \cdot (Grad_{\mathcal{S}} \boldsymbol{U})_{,i} \cdot \boldsymbol{A}_3 \\ &= (Div_{\mathcal{S}} Grad_{\mathcal{S}} \boldsymbol{U}) \cdot \boldsymbol{A}_3 \\ &= (\triangle_{\mathcal{S}}\, \boldsymbol{U}) \cdot \boldsymbol{A}_3. \end{aligned} \tag{20}$$

Note that $tr(\ ([Grad_{\mathcal{S}}\, Grad_{\mathcal{S}}\, \boldsymbol{U}] \cdot \boldsymbol{A}_3) \cdot (\boldsymbol{A}_3 \boldsymbol{A}_3)\ ) = tr(\ (\boldsymbol{A}_3 \boldsymbol{A}_3) \cdot (\boldsymbol{A}^i \boldsymbol{V}_i)\ ) = 0$. Hence, for an arbitrary displacement vector $\boldsymbol{U}$ we have the interesting relations

$$tr\mathcal{E}^a = Div_{\mathcal{S}} \boldsymbol{U} \qquad \text{and} \qquad tr\mathcal{E}^b = (\triangle_{\mathcal{S}}\, \boldsymbol{U}) \cdot \boldsymbol{A}_3\,.$$

References

[1] D.Chapelle, K.-J.Bathe, *The Finite Element Analysis of Shells - Fundamentals*, 2nd ed.,410p.,Comp.Fluid and Solid Mech., Springer (2011).

[2] P.G.Ciarlet,*The Finite Element Method for Elliptic Problems*, 530p., North-Holland, Amsterdam (1978).

[3] P.G.Ciarlet, L.Gratie, *A new approach to linear shell theory*, Preprint Ser.No.17, Liu Bie Ju Centre for Math.Sc. (2005).

[4] A.Meyer,*The Koiter shell equation in a coordinate free description*, CSC12-02, TU Chemnitz (2012).

[5] P.M.Naghdi, *Foundations of elastic shell theory*, *Progress in Solid Mechanics* vol.4, 3-90, North-Holland Publ.Amsterdam (1963).

[6] A.F.Bower, *Applied mechanics of solids*, CRC (2009).

[7] J.P.Boehler, *Applications of tensor functions in solid mechanics*, CISM Courses no 292, Springer (1987).

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