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The Integrated Density of States for Operators on Groups

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Abstract

This book is devoted to the study of operators on discrete structures. The operators are supposed to be self-adjoint and obey a certain translation invariance property. The discrete structures are given as Cayley graphs via finitely generated groups. Here, sofic groups and amenable groups are in the center of our considerations. Note that every finitely generated amenable group is sofic. We investigate the spectrum of a discrete self-adjoint operator by studying a sequence of finite dimensional analogues of these operators. In the setting of amenable groups we obtain these approximating operators by restricting the operator in question to finite subsets Q_n, $n \in \mathbb{N}$. These finite dimensional operators are self-adjoint and therefore admit a well-defined normalized eigenvalue counting function. The limit of the normalized eigenvalue counting functions when $|Q_n| \to \infty$ (if it exists) is called the integrated density of states (IDS). It is a distribution function of a probability measure encoding the distribution of the spectrum of the operator in question on the real axis.

We prove the existence of the IDS in various geometric settings and for different types of operators. The models we consider include deterministic as well as random situations. Depending on the specific setting, we prove existence of the IDS as a weak limit of distribution functions or even as a uniform limit. Moreover, in certain situations we are able to express the IDS via a semi-explicit formula using the trace of the spectral projection of the original operator. This is sometimes referred to as the validity of the Pastur-Shubin trace formula.

In the most general geometric setting we study, the operators are defined on Cayley graphs of sofic groups. Here we prove weak convergence of the eigenvalue counting functions and verify the validity of the Pastur-Shubin trace formula for random and non-random operators . These results apply to operators which not necessarily bounded or of finite hopping range. The methods are based on resolvent techniques. This theory is established without having an ergodic theorem for sofic groups at hand. Note that ergodic theory is the usual tool used in the proof of convergence results of this type.

Specifying to operators on amenable groups we are able to prove stronger results. In the discrete case, we show that the IDS exists

uniformly for a certain class of finite hopping range operators. This is obtained by using a Banach space-valued ergodic theorem. We show that this applies to eigenvalue counting functions, which implies their convergence with respect to the Banach space norm, in this case the supremum norm. Thus, the heart of this theory is the verification of the Banach space-valued ergodic theorem. Proceeding in two steps we first prove this result for so-called ST-amenable groups. Then, using results from the theory of ε-quasi tilings, we prove a version of the Banach space-valued ergodic theorem which is valid for all amenable groups.

Focusing on random operators on amenable groups, we prove uniform existence of the IDS without the assumption that the operator needs to be of finite hopping range or bounded. Moreover, we verify the Pastur-Shubin trace formula. Here we present different techniques. First we show uniform convergence of the normalized eigenvalue counting functions adapting the technique of the Banach space-valued ergodic theorem from the deterministic setting. In a second approach we use weak convergence of the eigenvalue counting functions and additionally obtain control over the convergence at the jumps of the IDS. These ingredients are applied to verify uniform existence of the IDS. In both situations we employ results from the theory of large deviations, in order to deal with long-range interactions.

Contents

1 Introduction

1.1 Motivation

The topic of this book is located at the interface of geometric group theory and mathematical physics. We study spectral properties of operators defined on discrete structures. The considered operators themselves are supposed to be self-adjoint on ℓ^2-spaces, of which the most prominent example we treat is the discrete Laplacian. Self-adjoint operators are of central interest in modeling physical systems, where they appear in differential equations, e.g. in the Schrödinger equation, the wave equation or the heat equation. In these cases the operators are used to describe the time-evolution of a physical process on a crystalline structure.

Besides the study of deterministic situations, it is of major importance to understand random systems too. This topic is of relevance in the description of physical processes in perturbed media, i.e. in situations where the underlying crystalline structure obeys impurities. An example is the description of the spreading of waves or the transport of electrons in (randomly) perturbed solids. The main reason, why perturbed systems are modeled as random systems is a lag of information. For instance, when describing an atom lattice with impurities it is virtually impossible to detect the exact positions of the missing or wrong atoms. Thus, one rather estimates the amount of impurities and inserts them via random variables. In practice, there are many ways to introduce randomness into a physical model. One of them is to perturb the potential of the associated Hamiltonian, leading to the so-called Anderson model [And58]. Another way is to randomly delete elements of the edge or vertex set of an underlying graph, which gives a random Laplacian. Such a model is known under the term percolation model [Gri99, Kes06].

From the physical point of view, it is usual to assume that the described system obeys a certain homogeneity or ergodicity property, which is to be reflected in the associated operator. Thus, it is natural to consider homogeneous or ergodic self-adjoint operators.

Much information about the solution of differential equations is encoded in the spectrum of the associated operator. For example, there is a strong connection between the type of the spectrum of the Schrödinger operator and the long-time behavior of solutions of the Schrödinger time evolution equation. The most prominent example for a result describing this relation is the RAGE-theorem, see for example [Tes09]. Important spectral properties are the spectral type (absolutely continuous, singular continuous, pure point), the distribution of the spectrum and the asymptotic behavior near the spectral edges.

1.1.1 The spectral distribution

When studying properties of the spectrum, it turns out that one of the relevant objects is a distribution function, associated to a certain probability measure, encoding the distribution of the spectrum. Note that the self-adjointness of the operators causes their spectrum to be a subset of the real axis. Hence, this distribution function is defined on $\mathbb{R}$ as well.

There are two generic ways to obtain such a function. Let us describe them in the exemplary situation of a self-adjoint operator A on $\ell^2(\mathbb{Z}^d)$. The first possibility is to consider the trace of the spectral projector E_λ of the whole operator A on the interval $(-\infty, \lambda]$. This trace can be considered as counting the eigenstates of the operator A, not exceeding the value λ. However, since we consider operators on infinite graphs, the number of such eigenstates is very likely to be infinite. Therefore, in order to make sense of this trace, one needs to project on some finite cube $Q \subseteq \mathbb{Z}^d$, and afterwards normalize by the number of elements in Q, which we denote by $|Q|$. The explicit formula reads as follows

$$\mathfrak{N}(\lambda) := |Q|^{-1} \operatorname{Tr}(\chi_Q E_\lambda). \tag{1.1}$$

This defines a function $\mathfrak{N} : \mathbb{R} \to [0,1]$, called *spectral distribution function* (SDF). Here it is important to mention that usually (under certain homogeneity assumptions) the spectral distribution function $\mathfrak{N}$ is independent of the choice of Q. Hence, in the special case where Q consists of only one element $x \in \mathbb{Z}^d$, we get $\mathfrak{N}(\lambda) = \langle \delta_x, E_\lambda \delta_x \rangle$.

Later in this work, we will see how this formula generalizes, when we consider less restricted geometries or random models, respectively.

Let us remark that the SDF is sometimes also referred to as *von Neumann trace*, see for instance [LPV07]. If the focus is rather on the geometry, the associated measure is known as the *Plancherel measure* or *Kesten spectral measure*, see [BW05]. In the language of physics, one says that the spectral distribution function $\mathfrak{N}$ at λ counts the states (per unit volume) of a physical system which do not exceed the value λ.

The second way of defining a distribution function for the spectrum of a self-adjoint operator is more constructive. Focusing again on the situation on $\mathbb{Z}^d$, one considers a sequence of cubes (Q_n) of increasing side length. For each n we obtain a finite dimensional operator A_n by restricting A to the set Q_n. In the canonical basis, the operator A_n can be represented as a symmetric matrix with real-valued entries, having at most $|Q_n|$ real eigenvalues. The eigenvalue counting function $\mathfrak{e}(A_n) : \mathbb{R} \to [0, 1]$ of A_n at the point λ is defined as the number of eigenvalues of A_n not exceeding λ. Here one counts eigenvalues according to their multiplicities. If there exists a function $\mathfrak{I} : \mathbb{R} \to \mathbb{N}_0$ such that

$$\mathfrak{I}(\lambda) = \lim_{n\to\infty} |Q_n|^{-1} \mathfrak{e}(A_n)(\lambda), \tag{1.2}$$

at least for all continuity points λ of $\mathfrak{I}$, then this function is called the *integrated density of states* (IDS). Note that, depending on the specific situation (operator and geometry), it is a priori not clear whether the functions $\mathfrak{N}$ and $\mathfrak{I}$ coincide. If one can show that they are equal, this equality is referred to as the *Pastur-Shubin trace formula*, see [Pas71, Shu79]. For recent literature on this formula we refer to [Ves08] and references therein. Let us emphasize the fact that there exist examples where the Pastur-Shubin trace formula does not hold, cf. [AS93].

If the operator under consideration is random, one has to deal with a whole family of operators $(A^{(\omega)})_{\omega\in\Omega}$. In this situation, the SDF is defined as the expectation of the expression in (1.1). In order to show the existence of the IDS, one has to prove that the limit function in (1.2) is the same for almost all realizations. This is usually a consequence of ergodicity. The Pastur-Shubin trace formula for random operators reads as follows:

$$\mathfrak{N}(\lambda) := |Q|^{-1}\mathbb{E}\big(\mathrm{Tr}(\chi_Q E_\lambda^{(\omega)})\big) = \lim_{n\to\infty}|Q_n|^{-1}\mathfrak{e}(A_n^{(\omega)})(\lambda) =: \mathfrak{I}^{(\omega)}(\lambda), \tag{1.3}$$

where one needs to show that the right-hand side of this formula is the same for almost all $\omega \in \Omega$. Here, $E_\lambda^{(\omega)}$ is the spectral projection of $A^{(\omega)}$ on the interval $(-\infty, \lambda]$ and $A_n^{(\omega)}$ is the restriction of $A^{(\omega)}$ to the set Q_n.

1.1.2 Physical relevance of the IDS and the SDF

Having introduced the most important objects for this work, let us proceed with the discussion of their relevance for the investigation of self-adjoint operators. Fundamental quantities, which are often in the center of the interest, are

(a) the behavior of $\mathfrak{N}$ (or $\mathfrak{I}$) at the spectral edges,

(b) continuity properties of $\mathfrak{N}$ (or $\mathfrak{I}$),

(c) the approximability of $\mathfrak{N}$ with finite volume analogues.

Let us remark that item (c) covers two aspects. Firstly, the validity of the Pastur-Shubin trace formula, and secondly, the topology in which the convergence of the eigenvalue counting functions holds.

We already explained that in the context of perturbed solids, it is convenient to describe the physical system via a random model. Thus, one does not investigate *one* geometrical setting, but rather studies a whole *family* of (similar) geometric settings. Each realization in this family describes one specific physical system, as it may appear in reality. A priori it is not clear whether different elements of this family exhibit similar properties. However, under appropriate homogeneity assumptions, one can show that certain (spectral) properties coincide for almost all realizations. A well-known example for a result in this direction is the non-randomness of the spectrum (as a set) for ergodic operators, cf. [PF92]. In the present work, ergodicity (Definition 2.17) and translation invariance in distribution (formula (2.16)) are the central homogeneity assumptions for random operators. Considering a random operator $A = (A^{(\omega)})$ which is ergodic or translation invariant in distribution, we prove for instance the existence of the integrated density of states or the validity of the Pastur-Shubin trace formula (1.3). These results are closely related to the non-randomness of the spectrum: they show that the distribution of the spectrum of a realization of this random operator is almost surely given by *one* (non-random) distribution function.

As described before, by knowing the spectral type of an operator one can draw conclusions about the long-time behavior of solutions of differential equations. Precise information about the properties (a) and/or (b) is a basic ingredient in many proofs determining the spectral type. However, for the investigation of the quantities (a) and (b), it is crucial to have information about (c). The reason is that known methods allowing to understand continuity properties or the low energy asymptotics of $\mathfrak{N}$ (or $\mathfrak{I}$), relies at some point on an approximation of the original operator by a finite dimensional one. This shows the centered role of the Pastur-Shubin trace formula.

Besides this, it is of relevance in which topology the convergence of the eigenvalue counting functions holds. The weakest (and usually obtained) type of convergence of the eigenvalue counting functions is weak convergence. This is by definition pointwise convergence of these distribution functions at each continuity point of the limit function. Knowing that there are situations where the IDS may exhibit jumps, sometimes even at each point of a dense subset of the spectrum, shows that weak convergence does often contain only little information. For results in this direction we refer to [CCF$^+$86, Ves05] and [KLS03], where the authors studied these discontinuities for percolation operators and operators on quasi-crystal graphs, respectively. Hence, there is a growing interest in stronger forms of convergence, namely convergence at any point in $\mathbb{R}$ or even uniform convergence.

1.2 Geometry and operators

In this book we will rather be concerned with points (b) and (c). Thus, we are interested in verifying a Pastur-Shubin trace formula and studying the type of convergence of the eigenvalue counting functions. Besides this, we present situations where we are in the position to estimate the speed of convergence by giving precise bounds on the approximation error. Moreover, we will prove results concerning the (dis-)continuity of the IDS. Large part of this book can be interpreted as an investigation of issue (c) under different conditions, resulting from the interplay of

- the generality of the geometric setting, and
- the generality of the operator in question.

1.2.1 Geometric setting

In the following, we discuss the variety of geometric settings and operators.

From the geometrical point of view we cover a wide range of settings since the graphs we consider are given in a very general manner, via finitely generated groups. Fixing a finite set of generators, each such group gives rise to a translation invariant graph in a natural way, a so-called Cayley graph. To be precise, given a group G and a finite and symmetric generating system $S \subseteq G$, the associated Cayley graph Γ is the graph with vertex set G, where two vertices x and y are adjacent if and only if $xy^{-1} \in S$. This graph is regular and G acts on Γ via graph automorphisms. Hence, each finite generating system gives rise to an induced metric on the group, the graph metric of the Cayley graph. For instance, if we consider $G = \mathbb{Z}^d$ with the canonical generating system, we obtain the $\mathbb{Z}^d$-lattice. The induced metric is in this case the ℓ^1-metric on $\mathbb{Z}^d$. As a second example, consider the free group $G = F_2$ with generator set $S = \{a, b, a^{-1}, b^{-1}\}$, which consists of all finite products of the elements a, b, a^{-1} and b^{-1}. The associated Cayley graph is a 4-regular tree. The branch of mathematics which investigates groups as geometric objects is known under the term *geometric group theory*. We refer the interested reader to [dlH00] and the references therein.

As the class of finitely generated groups is very rich, studying Cayley graphs, one covers a huge range of geometries. For example, one may study non-abelian groups, groups of different growth regimes (polynomial, intermediate or exponential), residually finite groups, amenable groups or sofic groups. The latter two (amenable and sofic groups) are in the focus of our interest. In particular the class of sofic groups is very large. It contains all amenable groups, all residually finite groups, all groups of intermediate and polynomial growth. One specific example of a sofic group is the above mentioned free group F_k, $k \geq 2$, whose Cayley graph is a regular tree of exponential growth. This shows that we deal with hyperbolic geometry as well. Note that trees are discrete analogues of hyperbolic manifolds, while grids like $\mathbb{Z}^d$ correspond to euclidean manifolds.

1.2.2 Amenable versus sofic

Here we discuss the classes of groups which are in the center of our investigations: amenable groups and sofic groups. We already mentioned that the class of sofic groups contains all amenable group. However, there exist substantially more sofic than amenable groups.

Let us first introduce amenable groups. A finitely generated group G is *amenable*, if an only if there exists a Følner sequence in G. A Følner sequence (Q_n) is a sequence of finite subsets in G such that for any finite $K \subseteq G$:

$$\lim_{n\to\infty} \frac{|KQ_n \triangle Q_n|}{|Q_n|} = 0.$$

Here $KQ_n \triangle Q_n$ denotes the symmetric difference of the sets KQ_n and Q_n. This can be interpreted as follows: the volume of the boundary of Q_n, divided by the volume of the sets Q_n tends to zero. Thus, amenable groups are by definition those groups which contain sets for which the proportion of the boundary with respect to the volume can be made arbitrarily small.

Amenable groups have firstly been studied in 1929 by von Neumann [vN29] in connection with the Banach-Tarski paradox. His definition can be formulated as follows: a group G is called amenable if and only if there exists a left-invariant mean on G. For finitely generated groups this is equivalent to the existence of a Følner sequence. Though von Neumann already studied this class of groups, the term "amenable" firstly appeared in 1949, cf. [Day49]. From there on amenable groups examined an increase in relevance. For our purposes a milestone in the theory of amenable groups is [Lin01], where Lindenstrauss proved a pointwise ergodic theorem for amenable groups, cf. Theorem 2.12.

Let us now discuss sofic groups. A finitely generated group is called *sofic*, if we have for each $r \in N$ a finite graph Γ_r, such that the r-balls around the all elements of Γ_r, up to a portion of $1/r$, are isomorphic to the r-ball of the Cayley graph of the group.

The importance of sofic groups relies on two aspects: on the one hand this class of groups is very rich, it contains not only the amenable groups and the residually finite groups. In fact, there is - up to now - no example of a non-sofic group known. On the other hand a sofic group has very useful approximation properties. Roughly speaking, the above definition says that the Cayley graph of a sofic group can be approximated on arbitrary good scales by finite graphs.

The concept of sofic groups is rather new. It has been introduced in 1999 by Gromov [Gro99]. The term "sofic" firstly appeared in [Wei00b] where the author studied these groups in connection with dynamical systems. For a survey on sofic groups we refer to [Pes08, KP09]. Besides this, let us mention the papers [ES04, ES06, Cor11] for research on the class of sofic groups concerning closedness properties.

Let us compare the approximability properties of amenable and sofic groups. In the toy example from Subsection 1.1.1 we considered an operator on $\mathbb{Z}^d$, which is an amenable group. There we already made use of the fact that a sequence of cubes (Q_n) in $\mathbb{Z}^d$ with increasing side length is a Følner sequence. Let us describe this in detail. We restricted the operator in question to these cubes and obtained a sequence (A_n) of approximating operators. The IDS in (1.2) is defined as the pointwise limit of the associated normalized eigenvalue counting functions. The hope that this limit exists, relies on the idea that boundary effects, which are caused by the restriction of the operator to the cube, vanish when n tends to infinity. In order to justify of this "hope" rigorously, one shows that the error, which appears with the restriction, can be estimated using the size of the boundary. Since we divide by the volume of Q_n in formula (1.2), the Følner property implies that this error term vanishes for n to infinity.

1.2.3 Operators

Depending on the given geometry and on the aimed type of convergence, we present our results for preferably large classes of operators. Moreover, we establish many results for deterministic as well as for random operators.

Let us briefly discuss some assumptions on the operators which frequently appear in our investigations. The most important property which our operators need to fulfill is *self-adjointness*. Another central assumption is that the operator under consideration A needs to be *translation invariant*. In deterministic situations this means that for all $x, y, z \in G$ the matrix element $\langle \delta_x, A\delta_y \rangle$ equals $\langle \delta_{xz}, A\delta_{yz} \rangle$, where δ_x is the Kronecker delta. If A is a random operator, the matrix elements are random variables. In this case translation invariance means that these random variables are identically distributed. A stricter assumption which is sometimes needed is *ergodicity* of the operator. In some cases this condition can be weakened using *colorings* of graphs. Here a coloring of a graph is a mapping from the vertices into some finite set (of colors).

Besides this, a relevant quantity is the *hopping range* of an operator. We say that an operator A is of finite hopping range, if there exists $r \in \mathbb{N}$ such that for an arbitrary finitely supported ϕ we have $A\phi(x) = 0$, whenever the distance between x and the support of ϕ is larger then r. As we will see, a translation invariant operator which is of finite hopping range (and in the random case admits uniform bound on the matrix elements) is automatically bounded. In many situations we deal with operators which are not of finite hopping range and which can therefore be unbounded.

In Section 1.4 we discuss in further detail, in which geometric setting which properties of the operator are necessary to obtain convergence of the eigenvalue counting functions, cf. Table 1.1. Remark that, due to our physical motivation, it is natural that in any geometric setting, we ensure that the discrete Laplacian or the adjacency operator fits in our framework.

1.3 Historical remarks

Let us give a short overview on previous results related to the present work. Convergence of eigenvalue counting functions has firstly been proven in the seminal papers [Pas71] by Pastur and [Shu79] by Shubin, where they established weak convergence in the euclidean setting for almost periodic and random ergodic operators. Starting from this, there have been many results in the topology of pointwise convergence for discrete operators [MY02, MSY03, DLM$^+$03, Ves05] as well as for continuous operators on manifolds [Szn89, Szn90, AS93, PV02, LPV04]. We refer to [Ves08] for a survey on results up to the year 2007, but also recommend the book [PF92].

A related topic is the approximation of L^2-invariants. These quantities can be interpreted as the evaluation of the IDS at one single point. Results concerning the approximation of L^2-invariants can be found in [Lüc94, MSY03, DM97, DM98, Eck99, Sch01] and in references therein. In a closely related geometric setting, namely on sofic groups, these questions have been studied in [Tho08, ES05]. Moreover, let us mention the monograph [Lüc02] as a survey on L^2-invariants.

Having discussed pointwise convergence, let us remark that the history of uniform convergence of eigenvalue counting functions is considerably shorter. The first approach in this direction was established in [LS05] for operators on Delone sets. In [LMV08] these ideas were applied in order to show uniform existence of the IDS for operators on $\mathbb{Z}^d$. This result was used in [GLV07] to verify uniform convergence of the approximating functions for operators on metric graphs over $\mathbb{Z}^d$. Besides this, in [LV09] the authors presented a method which applies to a large class of discrete models. They are able to treat, for instance, percolation models on groups and quasi-crystal Hamiltonians on Delone sets. In the chronological order of results concerning uniform convergence of eigenvalue counting functions, this is the point, where the topic of the present work is located. Let us mention that based on papers constituting this book, in [PSS13] the authors established uniform existence of the IDS and a Pastur-Shubin trace formula for metric graphs over amenable groups.

Large part of the present work applies to (long-range) percolation models and associated operators, where the graphs are given via finitely generated groups. For such models we prove for example the existence of the IDS, the validity of the Pastur-Shubin trace formula and (dis-)continuity properties of the IDS, cf. aspects (b) and (c) in the list in Subsection 1.1.2. Closely related work has been done in [AV09b, AV09a]. The authors of these papers studied in the same geometric setting the asymptotic behavior of the IDS at the spectral edges, cf. aspect (a). Other spectral properties for a similar model have been investigated in [Aya09] and [Aya10]. There it was shown that scaled versions of the approximating operators lead to a semicircle law of the limiting distribution. The last two mentioned

	deterministic	**random**
sofic	Chapter 3 • unbounded hopping range • weak convergence • approximation of the free group	Chapter 4 • unbounded hopping range • almost sure weak convergence • long-range percolation
amenable	Chapter 5 • finite hopping range • uniform convergence • coloring of graphs • tilings and ergodic theorems	Chapter 6 • unbounded hopping range • almost sure uniform convergence • long-range percolation

Table 1.1: Outline of the book

papers show that the long-range percolation model can be interpreted as an interpolation between the theory of random operators and random matrix theory. For further investigation of the connection between these topics we refer to [Pas12].

1.4 Models and main results

Here we give a short discussion of the main results of this work. Chapter 2 is devoted to basic facts about finitely generated groups and operators on groups. In the following Chapters 3 to 6 we study properties of the integrated density of states for deterministic and random operators on sofic and amenable groups, see Table 1.1. Let us in the following list describe the content of the single chapters in detail. We will formulate our main results and explain the applied techniques.

- In Chapter 2 we present basic definitions and facts concerning finitely generated groups and deterministic and random operators on groups. We prove essential self-adjointness for certain random operators and give (for later purposes) useful results on the measurability of certain sets or functions. Furthermore, we give precise definitions of eigenvalue counting functions and prove some elementary properties of them. We also introduce different notions of convergence for distribution functions and discuss necessary and sufficient conditions to verify them.

The next four items can be regarded as a discussion of the four cells in Table 1.1. Here we present the content of the main part of this work.

- In Chapter 3 we consider deterministic operators on sofic groups. The operators are supposed to be translation invariant and self-adjoint. In particular the theory we present here applies to unbounded operators and operators which are not of finite hopping range.

  A major step to prove existence of the IDS and to verify a Pastur-Shubin trace formula is an appropriate choice of the finite dimensional approximation operators. Here we use the fact that Cayley graphs of sofic groups can be approximated (on arbitrary good scales) by finite graphs and present a procedure to transfer the operator under consideration to these graphs. This leads to an appropriate sequence of approximating operators.

  Having the right definition at hand, we use a method, known as the *resolvent method*, to obtain weak convergence of the eigenvalue counting functions. The idea is to integrate certain test functions against the approximating distribution functions and against the limiting distribution function. The verification that the difference between these integrals tends to zero implies the desired convergence. By the specific choice of these test functions, one must deal with resolvents of the original operator and of the approximating operators. This is the reason why it is called resolvent method. The theorem we prove here is closely related to [Lüc94], where the author obtains pointwise convergence of eigenvalue counting functions for bounded operators in a more restricted geometry (given by residually finite groups). For related results on sofic groups see [ES04].

  As a specific example of a non-amenable sofic group, we consider the free group F_k with $2k$ generators. We construct a sequence of approximation graphs for F_k. This is especially important as the Cayley graph of the free group is a tree and the approximation of trees via finite graphs is an intensively studied problem.

In [AW06] the authors show that choosing the balls of the free group does not lead to a (spectral) approximation of the Cayley graph of the whole group. They rather prove that this procedure leads to an approach of the canopy tree, i.e. the horoball of the free group. In [FHS11] different approximations are suggested. Here one also considers balls, but rewires the vertices at the boundary with weighted edges. The idea we follow in this chapter is that a regular tree should be approximated using regular graphs with increasing girth, cf. [McK81]. The construction of [Big88] leads us to the presented approach.

- The generalization of these deterministic results to random operators is presented in Chapter 4. We consider random operators, which are almost surely self-adjoint and translation invariant in distribution. These conditions allow unbounded operators and unbounded hopping range. In particular, the developed theory of this chapter applies to the graph Laplacian of a long-range percolation graph and the Anderson model.

  We proceed in several steps. First, we give again an appropriate definition of a sequence of approximating operators. Then we prove weak convergence of the normalized eigenvalue counting functions in expectation. Afterwards, using a large deviation estimate by McDiarmid [McD98], this result is improved to obtain weak convergence for almost all realizations.

  The reason why we study the convergence in expectation in an intermediate step, is that our operators are only translation invariant in distribution, but not for each single realization. Hence, taking the expectation makes them translation invariant. This is crucial for our methods as we do not have an ergodic theorem at hand. The results of Chapters 3 and 4 are published in [SS12].

- The relevant objects of Chapter 5 are deterministic operators on amenable groups. The operators are assumed to be of finite hopping range and translation invariant with respect to a given graph coloring.

  As explained before, in the setting of amenable groups it is a reasonable choice to restrict the operator under consideration to elements of a Følner sequence (Q_n). This gives rise to a sequence of finite dimensional operators and hence a sequence of eigenvalue counting functions. For a moment let us consider these functions as mappings, which take a finite subset of G (in this case a set Q_n) to the Banach space $B(\mathbb{R})$ consisting of all bounded, right-continuous functions, equipped with supremum norm. Eigenvalue interlacing shows that these functions are almost-additive. For such functions we prove and apply an ergodic theorem. This gives convergence of the eigenvalue counting functions as elements in the Banach space $B(\mathbb{R})$. Thus, we obtain uniform convergence of the approximating distribution functions.

In order to verify the ergodic theorem, we follow the ideas of [LS05, LMV08], where this procedure has been established in a different geometric setting. The results of Chapter 5 can be seen as a direct generalization of [LMV08] where the authors considered $\mathbb{Z}^d$. The two basic problems one faces, when replacing $\mathbb{Z}^d$ by an arbitrary amenable group (and a sequence of cubes by a Følner sequence), are the following: first one needs to deal with non-abelian group structures; second one needs to substitute the property that each large cube can be decomposed into smaller cubes. Both facts are intensively used in previous versions of the Banach space-valued ergodic theorem.

We address these problems one at a time. First we show, how to handle non-commutativity and assume a certain tiling condition, replacing the decomposition property of cubes. This tiling condition allows to (symmetrically) tile the group with each element of a certain Følner sequence. We refer to groups fulfilling this property as *ST-amenable* groups. The class of ST-amenable groups is a large subclass of the amenable groups, cf. Remark 5.5. In the first part of Chapter 5 we already obtain the ergodic theorem and uniform convergence of the eigenvalue counting functions for ST-amenable groups. In order to overcome the second difficulty, we apply results from the theory of *ε-quasi tilings*, cf. [OW87]. Let us explain the two basic ideas of this theory: first, one tiles the group *not with one* element of a Følner sequence, but with *finitely many* elements of a Følner sequence. Second, one is not interested in an exact tiling of the group, but rather allows *small intersections* of the tiles. Here we generalize and improve results from the seminal work [OW87], leading to a version of the Banach space-valued ergodic theorem which is valid for *all* amenable groups.

After having established uniform existence of the IDS for amenable groups, we discuss further properties of the IDS. For instance, we prove a characterization of its discontinuity points via finitely supported eigenfunctions. Besides this, we specify our results to operators on $\mathbb{Z}^d$ and the Heisenberg group, respectively. Let us remark that the relevant papers where the results of this chapter are published are [LSV11] and [PS12].

- In Chapter 6 we study random operators on amenable groups. We assume these operators to be ergodic and almost surely self-adjoint. In particular, we allow unbounded hopping range and unbounded operators. Again, we firstly concentrate on ST-amenable groups and afterwards present results for general amenable groups. However, the methods we use in these two cases are rather different.

  Let us describe the part dealing with ST-amenable groups. Here we present a random model given by a (long-range) percolation process. We consider the Laplacian of a percolated sub-graph of the complete graph on the vertices of an ST-amenable group. This operator is due to long-range interactions almost surely unbounded and not of finite hopping range. Applying results from the theory of large deviation, namely a Bernstein inequality, we prove again an almost-additivity property for the eigenvalue counting functions. This enables us to verify an adapted version of the ergodic theorems from Chapter 5, leading to the proof of uniform convergence of the associated eigenvalue counting functions. Studying a rather specific model, we are in the position to give detailed information about the set of points of discontinuity of the IDS.

  In the second part of this chapter we pursue a different technique to verify uniform convergence. We firstly show that the eigenvalue counting functions converge weakly to some limiting function and then improve this result by obtaining detailed information about the convergence at the jumps. Note that this procedure was firstly suggested in [LV09] in a similar, but different setting. Again, we apply a Bernstein inequality in order to obtain control over the number of edges exceeding a certain length. This is used to prove estimates for error terms, caused by long-range interactions. The results here generalize the previous ones on ST-amenable groups, not just in terms of the geometry, but also in terms of the operator. Here we allow weighted edges where the weights are taken randomly from a possibly uncountable and unbounded subset of $\mathbb{R}$. Additionally we prove a Pastur-Shubin trace formula.

Let us emphasize the fact that in all previous works studying uniform existence of the IDS a central assumption is the finite hopping range of the operator, cf. [LS05, LMV08, LV09]. The results in this chapter are the first ones, where uniform existence of the IDS is established for operators which are not of finite hopping range. Large part of the content of Chapter 6 is published in the papers [Sch12] and [ASV13].

2 Preliminaries

2.1 Finitely generated groups

This section is devoted to the introduction of the objects which determine the geometric setting of this work. We are interested in operators defined on discrete structures, in particular on graphs. The graphs are given via finitely generated groups, as so-called Cayley graphs. The reason why we concentrate on groups which are finitely generated is that these are exactly the groups, where the associated Cayley graph has finite vertex degree. First, we present some notion for general graphs. Then, we give definitions closely related to groups and Cayley graphs. Finally, and divided into three subsections, we present special cases of finitely generated groups, which will play an important role in this book.

We begin with general graphs. Given a countable set V and a set $\vec{E} \subseteq V \times V$ we call the pair $\vec{\Gamma} = (V, \vec{E})$ *directed graph* with *vertex set* V and *edge set* $\vec{E}$. The notion "directed" refers to the fact that the elements $(x, y), (y, x) \in V \times V$ are not equal. Similarly we define an *(undirected) graph* $\Gamma = (V, E)$. Here again the set V is some countable set, which we call the *vertex set* of Γ. The set E is in this case a subset of the power set of V, containing only sets with at most two elements. As before, the set E is called the *edge set*. In particular, this allows loops but no multiple edges. In the special case where E contains all subsets of V with at most two elements, we say that $\Gamma = (V, E)$ is the *complete (undirected) graph* over V. Equivalently, $\vec{\Gamma} = (V, \vec{E})$ is called the *complete directed graph* over V, if $\vec{E} = V \times V$. When speaking about undirected graphs, we will often drop the notion "undirected". Moreover, sometimes we omit the arrow for directed graphs. A directed or (undirected graph) is called *finite*, if the vertex set is finite.

A *path* of length $n \in \mathbb{N}$ in an undirected graph Γ is a sequence $(e_1, \dots, e_n)$ of elements where $e_i = \{z_{i-1}, z_i\} \in E$, $i = 1, \dots, n$ for some $z_0, \dots, z_n \in V$. In this situation we say that the path $(e_1, \dots, e_n)$

connects the elements z_0 and z_n. A graph is called *connected* if for each pair of vertices there exists a path connecting them. Using this notion, there is a natural way to define a metric on the vertex set of a connected undirected graph. If $x, y \in V$ are distinct elements of the vertex set of an undirected graph Γ, we define the distance $d^\Gamma(x, y)$ to be the length of the shortest path, connecting the elements x and y in Γ. If $x = y \in V$, we set $d^\Gamma(x, y) = 0$.

In the same manner we define paths and distances in directed graphs. Let $\vec{\Gamma} = (V, \vec{E})$ be a directed graph. A sequence $(e_1, \ldots, e_n)$ is called a *directed path* of length n in $\vec{\Gamma}$, if there exist $z_0, \ldots, z_n \in V$ such that $e_i = (z_{i-1}, z_i) \in \vec{E}$ for all $i = 1, \ldots, n$. Furthermore we call $(e_1, \ldots, e_n)$ an *undirected path* of length n in $\vec{\Gamma}$ if there exist $z_0, \ldots, z_n \in V$ such that $e_i \in \{(z_{i-1}, z_i), (z_i, z_{i-1})\} \cap \vec{E}$ for all $i = 1, \ldots, n$. Hence, an undirected path ignores the direction of the edges. With this notion, each directed path in $\vec{\Gamma}$ is an undirected path. Again we say that the (un-)directed path connects x and y if $z_0 = x$ and $z_n = y$. A directed graph $\vec{\Gamma}$ is called connected, if for each pair of vertices, there exists an undirected path connecting them. Let $\vec{\Gamma} = (V, \vec{E})$ be a connected directed graph. The distance $d^{\vec{\Gamma}} : V \times V \to [0, \infty)$ is given in the following way: if $x, y \in V$ are distinct and the length of the shortest undirected path connecting x and y equals n, then we set $d^{\vec{\Gamma}}(x, y) = n$. If $x = y$, we set $d^{\vec{\Gamma}}(x, y) = 0$.

In the situation where the (un-)directed graph is not connected, one defines the metrics in the almost same way. Here the only difference is, that one sets the distance between elements which are not connected by a path, to be equal to infinity. We will in the following refer to the metric d^Γ or $d^{\vec{\Gamma}}$ as *graph metric* of Γ or $\vec{\Gamma}$, respectively.

Applying the above defined metrics, we define balls in graphs. To this end let $\Gamma = (V, E)$ be and undirected graph and $\vec{\Gamma} = (V, \vec{E})$ a directed graph. For $x \in V$ and $r \geq 0$ we define

$$B_r^\Gamma(x) := \{y \in V \mid d^\Gamma(x, y) \leq r\}$$

and

$$B_r^{\vec{\Gamma}}(x) := \{y \in V \mid d^{\vec{\Gamma}}(x, y) \leq r\}.$$

In order to compare graphs, we use the language of graph isomorphism. Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be two undirected graphs. A function $\Psi : V_1 \to V_2$ is called a *graph isomorphism*, if Ψ is bijective and for $x, y \in V_1$ we have $\{\Psi(x), \Psi(y)\} \in E_2$ if and only if $\{x, y\} \in E_1$. If such Ψ exists, the graphs Γ_1 and Γ_2 are called *isomorphic* and we write $\Gamma_1 \simeq \Gamma_2$. A graph isomorphism for directed graphs is defined in the same way, with the only difference, that here we also require, that the direction of the edge is preserved by the function Ψ.

For directed graphs, we will also need the notion of being (edge) labeled. Let $\vec{\Gamma} = (V, \vec{E})$ be a directed graph and L some set. We say that $\vec{\Gamma}$ is *edge labeled* by L with ϑ, if ϑ is a function $\vartheta : \vec{E} \to L$.

In the following we assume that the directed graphs $\vec{\Gamma}_1 = (V_1, \vec{E}_1)$ and $\vec{\Gamma}_2 = (V_2, \vec{E}_2)$ are both edge labeled by L, with ϑ_1 and ϑ_2, respectively. Then we say that they are *isomorphic as labeled graphs* if there exists a graph isomorphism $\Psi : V_1 \to V_2$, which satisfies for all $(x, y) \in \vec{E}_1$:

$$\vartheta_1((x, y)) = \vartheta_2((\Psi(x), \Psi(y))).$$

If $\vec{\Gamma}_1$ and $\vec{\Gamma}_2$ are edge labeled by L and isomorphic as labeled graphs, we write $\Gamma_1 \simeq_L \Gamma_2$.

Let $\Gamma = (V, E)$ be an undirected graph and $U \subseteq V$, then we define the *induced subgraph* $\Gamma|_U$ as the graph with vertex set U and edge set

$$E|_U = \{\{x, y\} \in E \mid x, y \in U\}.$$

Similarly for a directed graph $\vec{\Gamma} = (V, \vec{E})$ and $U \subseteq V$ the *induced subgraph* $\vec{\Gamma}|_U$ is the graph with vertex set U and edge set $\vec{E}|_U = \{(x, y) \in \vec{E} \mid x, y \in U\}$. If $\vec{\Gamma}$ is edge labeled by L with ϑ, then $\vec{\Gamma}|_U$ is edge labeled by L with $\vartheta|_U : \vec{E}|_U \to L$, where $\vartheta|_U(e) = \vartheta(e)$ for all e in $\vec{E}|_U$.

In the following we introduce notion, which is related to finitely generated groups. Let G be a group, then $S \subseteq G$ is called a *generating set* or *set of generators*, if each $g \in G$ can be expressed as a finite product of elements in S. A group G is called *finitely generated*, if there exists a finite set of generators in G. Note that each finitely generated group contains at most countably many elements. The unit element of the group will always be denoted by id. Given a set $Q \subseteq G$, we denote by Q^{-1} the set of the inverse elements, i.e.

$$Q^{-1} = \{g \in G \mid g^{-1} \in Q\}.$$

A set $Q \subseteq G$ is called *symmetric*, if $Q = Q^{-1}$.

Let G be a finitely generated group and $S \subseteq G$ a finite set of generators. Then directed graph $\vec{\Gamma}(G,S) = (V, \vec{E})$ with vertex set $V = G$ and edge set

$$\vec{E} := \{(x,y) \in V \times V \mid xy^{-1} \in S\}$$

is called *directed Cayley graph* of G and S. Furthermore, each such graph can be interpreted as being edge labeled with S, where we define the function ϑ by setting for $(x,y) \in \vec{E}$: $\vartheta((x,y)) := xy^{-1}$. We will refer to this as the *canonical labeling* of $\vec{\Gamma}(G,S)$. In order to define an undirected Cayley graph, we need a symmetric generating set S, i.e. $S = S^{-1}$. Note that this implies: $(x,y) \in \vec{E}$ if and only if $(y,x) \in \vec{E}$. Let G be a group and $S \subseteq G$ a finite and symmetric set of generators. Then the *(undirected) Cayley graph* $\Gamma(G,S) = (V,E)$ is defined via the vertex set $V := G$ and the edge set

$$E := \{\{x,y\} \subseteq G \mid xy^{-1} \in S\}.$$

If $\Gamma = \Gamma(G,S)$ is an undirected Cayley graph, we will use the notation $d_S := d^\Gamma : G \times G \to [0,\infty)$ for the graph metric on Γ. Note that this metric is sometimes called *word metric* and can be also defined by setting for distinct $x, y \in G$:

$$d_S(x,y) = \min\{k \in \mathbb{N} \mid \exists s_1, \dots, s_k \in S \text{ with } s_1 \cdots s_k y = x\}$$

and $d_S(x,x) = 0$. As indicated in the index, the word metric depends on the specific choice of the generating set S. For a given group G and finite generating set $S \subseteq G$ we write for $r \geq 0$ and $x \in G$:

$$B_r^G(x) := B_r^{\Gamma(G,S)}(x) = \{y \in G \mid d_S(x,y) \leq r\}$$

and

$$B_r^G := B_r^G(\mathrm{id}).$$

We will drop the superscript G in this notation whenever it is clear to which group the balls are associated. We use the notations (Q_n) and $(Q_n)_{n\in\mathbb{N}}$ for a sequence of finite subsets of G, where the index n takes values in $\mathbb{N}$. The set of all finite subsets of G is denoted by $\mathcal{F}(G)$. Given a set $Q \in \mathcal{F}(G)$, we define the *diameter* of Q by

$\operatorname{diam}(Q) := \max\{d_S(g,h) \mid g,h \in Q\}$ and use $|Q|$ for the *cardinality* of Q. For a set $Q \subseteq G$ and $K \in \mathcal{F}(G)$ we define the *K-boundary of Q* by

$$\partial_K(Q) := \{g \in G \mid Kg \cap Q \neq \emptyset \text{ and } Kg \cap (G \setminus Q) \neq \emptyset\}.$$

For $r > 0$ we set $\partial^r(Q) := \partial_{B_r}(Q)$. Furthermore, we will sometimes use the *inner* or *outer boundary*, which we define by $\partial^r_{\text{int}}(Q) := \partial^r(Q) \cap Q$ and $\partial^r_{\text{ext}}(Q) := \partial^r(Q) \setminus Q$, respectively. It is easy to see that

$$\partial^R_{\text{int}}(Q) = \{x \in Q \,|\, d_S(x, G \setminus Q) \leq R\}$$

and

$$\partial^R_{\text{ext}}(Q) = \{x \in G \setminus Q \,|\, d_S(x, Q) \leq R\}.$$

Furthermore, we introduce the following notation: $Q^{(r)} := Q \setminus \partial^r_{\text{int}}(Q)$. For $\delta > 0$ and $K \in \mathcal{F}(G)$ we will say that a set $Q \in \mathcal{F}(G)$ is *(K,δ)-invariant* if

$$|\partial_K(Q)| \leq \delta |Q|. \tag{2.1}$$

The next lemma contains useful properties of the K-boundary. Since the other types of boundaries which we defined above are based on this notion, one can easily deduce similar properties for them.

Lemma 2.1. *Let $Q, U \subseteq G$, $K \in \mathcal{F}(G)$ be non-empty and assume that $g \in G$. Then:*

(i) $\partial_K(Q) = \partial_K(G \setminus Q)$,

(ii) $\partial_K(U \cup Q) \subseteq \partial_K(U) \cup \partial_K(Q)$,

(iii) $\partial_K(U \setminus Q) \subseteq \partial_K(U) \cup \partial_K(Q)$,

(iv) $\partial_K(Q) \subseteq \partial_L(Q)$ *if* $K \subseteq L \subseteq G$,

(v) $\partial_K(Qg) = \partial_K(Q)g$,

(vi) $\partial_K(QU) \subseteq \partial_K(Q)U$ *and*

(vii) $\partial_K(U \setminus Q) \subseteq \partial_K(U) \cup (\partial_K(Q) \cap U)$ *if* $\mathrm{id} \in K$.

Proof. The statements (i) to (v) follow directly from the definition of the K-boundary. Let us prove (vi). In order to do so, choose some $g \in \partial_K(QU)$. Then there exists some $u \in U$ such that $Kg \cap Qu$ is non-empty. Furthermore, we have

$$\emptyset \neq Kg \cap (G \setminus QU) \subseteq Kg \cap (G \setminus Qu).$$

Therefore, we obtain $g \in \partial_K(Qu) = \partial_K(Q)u \subseteq \partial_K(Q)U$, where we used (v).

In order to show (vii) let $g \in \partial_K(U \setminus Q)$ be given and let $\mathrm{id} \in K$. We assume that $g \notin \partial_K(U)$. Then, by definition of the K-boundary, we have $Kg \cap (U \setminus Q) \neq \emptyset$. Since $U \setminus Q$ is a subset of U, we have $Kg \cap U \neq \emptyset$. This together with $g \notin \partial_K(U)$ and $\mathrm{id} \in K$ yields $g \in Kg \subseteq U$. Now it remains to show that $g \in \partial_K(Q)$. To see this, we use statement (iii) and

$$g \in \partial_K(U \setminus Q) \subseteq \partial_K(U) \cup \partial_K(Q).$$

This, $g \notin \partial_K(U)$ and $g \in U$, implies $g \in \partial_K(Q) \cap U$. ■

The *(volume) growth* of a group is defined using the cardinality of balls. We say that a group G is of *polynomial (volume) growth*, if there exists $a > 0$ and $d \in \mathbb{N}$ such that $|B_r| \leq ar^d$ for all $r \in \mathbb{N}$. Furthermore a group is said to be of *exponential (volume) growth*, if there exists $c > 0$ such that $|B_r| \geq ce^{cr}$ for all $r \in \mathbb{N}$. If a group is neither of polynomial growth nor of exponential growth, one says that it obeys *intermediate (volume) growth*.

We say that a group G is *abelian*, if for any $g, h \in G$ we have $gh = hg$. All finitely generated abelian groups, in particular $\mathbb{Z}^d$, are of polynomial growth. The Heisenberg group is an example for a non-abelian group of polynomial growth. The first group which was shown to be of intermediate growth is the Grigorchuk group. Examples for exponentially growing groups are the Lamplighter group as well as the free group. We refer to [dlH00], as they give many examples and study a variety of properties which are related to the volume growth of a group.

In order to approximate infinite Cayley graphs of a finitely generated group by finite graphs, there are several classes of groups, where one has canonical candidates. The next three subsections discuss such classes of groups.

2.1.1 Sofic groups

The notion of sofic groups was introduced by Gromov in [Gro99] and the specific definition we will use, goes back to Weiss [Wei00b].

Definition 2.2. Let G be a finitely generated group, S a finite and symmetric set of generators and $\vec{\Gamma} = \vec{\Gamma}(G, S)$ the canonically labeled directed Cayley graph. Then G is called *sofic*, if for all $\varepsilon > 0$ and $r \in \mathbb{N}$ there is a finite directed graph $\Gamma_{r,\varepsilon} = (V_{r,\varepsilon}, E_{r,\varepsilon})$, edge labeled by S with $\vartheta : E_{r,\varepsilon} \to S$ and a subset $V_{r,\varepsilon}^{(0)} \subseteq V_{r,\varepsilon}$, such that

(S1) $|V_{r,\varepsilon}^{(0)}| \geq (1 - \varepsilon)|V_{r,\varepsilon}|$, and

(S2) for all $v \in V_{r,\varepsilon}^{(0)}$ the graph $\Gamma_{r,\varepsilon}$ restricted to the r-ball around v is isomorphic as a labeled graph to $\vec{\Gamma}|_{B_r^G}$, i.e.

$$\Gamma_{r,\varepsilon}|_{B_r^{\Gamma_{r,\varepsilon}}(v)} \simeq_S \vec{\Gamma}|_{B_r^G}.$$

Note that the property of being sofic is independent of the specific choice of the symmetric generating set S, cf. [Wei00b]. Though the notion of being sofic was already introduced in 1999, there is up to now no group which is known to fail being sofic. Besides this, it is easy to show that each finitely generated amenable group is sofic, cf. Lemma 2.11. Furthermore, each finitely generated residually finite group is sofic, cf. Lemma 2.5. Both facts are well-known and emphasize the importance of the investigation of sofic groups.

In this subsection we always assume that G is sofic and generated by a finite and symmetric set S. In order to simplify notation of the approximations, we choose some function $\varepsilon\colon \mathbb{N} \to (0, \infty)$ with $\lim_{r\to\infty} \varepsilon(r) = 0$ and write

$$\Gamma_r := (V_r, E_r) := (V_{r,\varepsilon(r)}, E_{r,\varepsilon(r)}), \quad V_r^{(0)} := V_{r,\varepsilon(r)}^{(0)}. \tag{2.2}$$

Hence, for each $r \in \mathbb{N}$ we obtain a finite approximating graph of the Cayley graph $\Gamma = \Gamma(G, S)$. We denote the graph metric in Γ_r by d_r.

We fix $r \in \mathbb{N}$. For each $v \in V_r^{(0)}$ we have by definition a graph isomorphism

$$\Psi_{r,v}\colon B_r^{\Gamma_r}(v) \to B_r^G, \tag{2.3}$$

which preserves the labels. It is immediately clear that $\Psi_{r,v}(v) = \mathrm{id}$. This implies for any choice $v, w \in V_r^{(0)}$ with $d_r(v,w) \le r$, that

$$\Psi_{r,v}(w) = (\Psi_{r,w}(v))^{-1}. \tag{2.4}$$

Before we verify (2.4) let us briefly discuss some elementary properties of edge labeled Cayley graphs and sofic approximations. Recall that the symmetry of S implies that if (x,y) is an edge of the directed Cayley graph $\vec{\Gamma} = \vec{\Gamma}(G,S)$, then the inverse edge (y,x) is an edge of $\vec{\Gamma}$ as well. By property (S2) the same holds true for edges in the r-balls in Γ_r around elements of $V_r^{(0)}$. Thus, for each $v \in V_r^{(0)}$ every undirected path in $\Gamma_r|_{B_r^{\Gamma_r}(v)}$ can be transferred in an directed path, by an appropriate change of the directions of the involved edges.

Moreover, let $(e_1, \dots, e_k)$ be a directed path in $\vec{\Gamma} = \vec{\Gamma}(G,S)$ from x to y and let $s_1, \dots, s_k \in S$ be the canonical labels of these edges, i.e. $\vartheta(e_i) = s_i$ for all $i = 1, \dots, k$. Then by definition of these labels we have $s_1 \cdots s_k = xy^{-1}$. Again using (S2) this property transfers to the r-balls around elements in $V_r^{(0)}$ in the approximating graph Γ_r.

With these considerations we easily conclude (2.4): let $v, w \in V_r^{(0)}$ with $d_r(v,w) \le r$ be given and let $s_1, \dots, s_k$ be the labels along a directed path contained in $B_r^{\Gamma_r}(v) \cap B_r^{\Gamma_r}(w)$ from v to w. As the labels are preserved by $\Psi_{r,v}$ and $\Psi_{r,w}$ we have

$$\begin{aligned}(\Psi_{r,v}(w))^{-1} = \Psi_{r,v}(v)(\Psi_{r,v}(w))^{-1} = s_1 \cdots s_k \\ = \Psi_{r,w}(v)(\Psi_{r,w}(w))^{-1} = \Psi_{r,w}(v),\end{aligned}$$

which shows (2.4). We generalize these ideas in the next Lemma.

Lemma 2.3. *Let $r \in \mathbb{N}$. If $x, y \in V_r$ and $v, w \in V_r^{(0)}$ satisfy $x, y \in B_{r/2}^{\Gamma_r}(v) \cap B_{r/2}^{\Gamma_r}(w)$, then we have*

$$\Psi_{r,v}(x)(\Psi_{r,v}(y))^{-1} = \Psi_{r,w}(x)(\Psi_{r,w}(y))^{-1}.$$

Proof. Let $x, y \in V_r$ and $v, w \in V_r^{(0)}$ be such that $x, y \in B_{r/2}^{\Gamma_r}(v) \cap B_{r/2}^{\Gamma_r}(w)$. Then $k := d_r(x,y) \le r$ and hence all shortest paths in Γ_r connecting x and y are completely contained in $B_r^{\Gamma_r}(v)$ as well as in $B_r^{\Gamma_r}(w)$. We consider one of these shortest (directed) paths from x

to y. Let $s_1, \ldots, s_k$ be the labels of this path. By the choice of $\Psi_{r,v}$ we have that

$$\Psi_{r,v}(x)(\Psi_{r,v}(y))^{-1} = s_1 \cdots s_k(\Psi_{r,v}(y))(\Psi_{r,v}(y))^{-1} = s_1 \cdots s_k.$$

As we also have $\Psi_{r,w}(x) = s_1 \cdots s_k(\Psi_{r,w}(y))$, the claim follows. ■

2.1.2 Residually finite groups

In this section we define residually finite groups. Roughly speaking, this is the class of groups, where for each element of the group (except the identity), one can find a normal, finite index subgroup, which does not contain this element. Quotients of these groups will lead to approximating graphs. Before stating the definition of a residually finite group, let us explain the notion of quotients of groups. If G is a group and U is a subgroup of G we call

$$G/U := \{uG \mid u \in U\}$$

the *quotient* of the groups G and U. The *index* of U in G is the number of elements in G/U. We write $[G : U] := |G/U|$. If $[G : U]$ is finite, we say that U is a subgroup of finite index or a finite index subgroup of G. A subgroup U of G is called *normal* in G if for any $g \in G$ and $u \in U$ we have $gug^{-1} \in U$.

If U is a normal subgroup of G, the quotient G/U is a group itself. The multiplication in G/U is given by induced by the multiplication in G, i.e. for $u, v \in U$ we have

$$(uG)(vG) = (uv)G.$$

In this situation the group G/U is called *quotient group*.

Definition 2.4. Let G be a finitely generated group. We call G *residually finite* if there exists a sequence $(G_n)_{n\in\mathbb{N}}$ of subgroups of G such that

(R1) $\forall\, n \in \mathbb{N} : G_n$ is normal in G,

(R2) $\forall\, n \in \mathbb{N} : [G : G_n]$ is finite,

(R3) $\forall\, n \in \mathbb{N} : G_{n+1} \subseteq G_n$,

(R4) $\bigcap_{n\in\mathbb{N}} G_n = \{\mathrm{id}\}$.

Note that conditions (R1),(R2) and (R4) are already sufficient to construct a sequence of subgroups $(\tilde{G}_n)$ for which (R1) to (R4) hold true. In order to see this, let (G_n) be a sequence satisfying (R1),(R2) and (R4). For each $n \in \mathbb{N}$ we set

$$\tilde{G}_n := \bigcap\nolimits_{j \leq n} G_j.$$

Then $\tilde{G}_n$ is normal in G. Condition (R2) holds true since for each $n \in \mathbb{N}$

$$[G : \tilde{G}_n] \leq \prod\nolimits_{j=1}^{n} [G : G_j] < \infty.$$

Besides this (R3) and (R4) are satisfied by construction. This shows in particular, that a finitely generated group G is residually finite if and only if for any $x \in G \setminus \{\mathrm{id}\}$ there is a normal subgroup G_x of G which is of finite index and does not contain x.

Let us remark that among the finitely generated groups, all free groups, all nilpotent groups as well as all linear groups are residually finite.

In the following we investigate approximability properties of quotient groups G/G_n. The calculations are rather basic and presented in full detail. Let G be a residually finite group, generated by the finite and symmetric set S. Furthermore, let (G_n) be the sequence satisfying the conditions (R1) to (R4). For each $n \in \mathbb{N}$ we define H_n to be the quotient group

$$H_n := G/G_n = \{gG_n \mid g \in G\}.$$

Then H_n is generated by the set $S_n := \{sG_n \mid s \in S\}$. Furthermore, let $d := d_S$ be the word metric on G and $d_n := d_{S_n}$ be the word metric on H_n. We write

$$B_r = B_r^G = \{x \in G \mid d(x, \mathrm{id}) \leq r\}$$

and

$$B_r^{(n)} = B_r^{H_n} = \{x \in H_n \mid d_n(x, \mathrm{id}) \leq r\}$$

for the balls of radius $r \in \mathbb{N}_0$ centered at the unit elements.

Let us assume that for all $n \in \mathbb{N}$ one has $B_2 \cap G_n = \{\mathrm{id}\}$. This is possible by conditions (R3) and (R4) and as we are only interested in large n. Thus for $s, s' \in S$ with $sG_n = s'G_n$ we get $s^{-1}s' \in G_n$. This implies $s^{-1}s' \in B_2 \cap G_n = \{\mathrm{id}\}$ and hence $s = s'$.

Now, let $\vec{\Gamma} = \vec{\Gamma}(G, S)$ and $\vec{\Gamma}_n = \vec{\Gamma}_n(H_n, S_n) = (V_n, \vec{E}_n)$ be the associated directed Cayley graphs. We assume that $\vec{\Gamma}$ is canonically edge labeled. Let us define an labeling of the edges of $\vec{\Gamma}_n$. If $(gG_n, hG_n) \in \vec{E}_n$, then $gh^{-1}G_n \in S_n$. Thus there exists $s \in S$ with $gh^{-1}G_n = sG_n$. By the above considerations, this element s is uniquely defined and we set $\vartheta_n((gG_n, hG_n)) := s$. This construction gives a function $\vartheta_n : \vec{E}_n \to S$, which labels the edges of $\vec{\Gamma}_n$ by elements of S.

The following Lemma shows that for increasing n, the Cayley graphs of the quotient groups equal the Cayley graph of the group G on larger and larger scales.

Lemma 2.5. *Let G be a finitely generated, residually finite group. Then G is sofic. In particular, if $\vec{\Gamma}$ and $\vec{\Gamma}_n$, $n \in \mathbb{N}$, are given as above, we have that for all $r \in \mathbb{N}$ there is $n(r) \in \mathbb{N}$ such that for all $n \geq n(r)$:*

$$\vec{\Gamma}|_{B_r} \simeq_S \vec{\Gamma}_n|_{B_r^{(n)}}. \tag{2.5}$$

Proof. Let $r \in \mathbb{N}$ be given. Then by conditions (R3) and (R4) we can choose $n(r)$ such that $B_{2r+1} \cap G_n = \{\mathrm{id}\}$ for all $n \geq n(r)$. Then for $n \geq n(r)$, $g \in G$ and $h, h' \in gG_n \cap B_r$, one gets that $h^{-1}h' \in G_n \cap B_{2r} = \{\mathrm{id}\}$ which shows that $|gG_n \cap B_r| \leq 1$. If one additionally assumes that $g \in B_r$, then $|gG_n \cap B_r| = |\{g\}| = 1$. The r-ball in H_n around the identity element can be written as

$$B_r^{(n)} = \{gG_n \mid g \in B_r\}.$$

For $n \geq n(r)$ we set

$$\Psi_r^{(n)} : B_r^{(n)} \to B_r \quad \text{with} \quad \{\Psi_r^{(n)}(gG_n)\} = gG_n \cap B_r,$$

which is well-defined by the above considerations. In particular we obtain for $g \in B_r$ that $\Psi_r^{(n)}(gG_n) = g$ and hence $\Psi_r^{(n)}$ is bijective. Let (gG_n, hG_n) be an edge in $\vec{\Gamma}_n|_{B_r^{(n)}}$. Without loss of generality we assume $g, h \in B_r$. Then we have $gh^{-1}G_n \in S_n$. This and $S_n = \{sG_n \mid s \in S\}$ shows that there exists $s \in S$ with $s^{-1}gh^{-1} \in G_n$. Since $s^{-1}gh^{-1} \in B_{2r+1}$ and $n \geq n(r)$, we obtain $g = sh$. Thus, (g, h)

is an element of the edge set of $\vec{\Gamma}$. The canonical label of (g,h) is s, which coincides with the label of (gG_n, hG_n). Hence, we obtained (2.5). Moreover, as H_n is homogeneous, (2.5) holds also for balls with translated center, i.e. for all $x \in H_n$ we have

$$\vec{\Gamma}|_{B_r} \simeq_S \vec{\Gamma}_n|_{B_r^{H_n}(x)}.$$

This proves that G is sofic. ∎

2.1.3 Amenable groups

Let us start this subsection with the definition of amenability.

Definition 2.6. Let G be a finitely generated group. A sequence $(Q_n)_{n\in\mathbb{N}}$ of finite subsets of G is called *Følner sequence* if for any $K \in \mathcal{F}(G)$:

$$\lim_{n\to\infty} \frac{|KQ_n \triangle Q_n|}{|Q_n|} = 0.$$

The group G is called *amenable* if there exists a Følner sequence in G.

Here $KQ_n \triangle Q_n$ denotes the symmetric difference of the sets KQ_n and Q_n, i.e.

$$KQ_n \triangle Q_n = (KQ_n \setminus Q_n) \cup (Q_n \setminus KQ_n).$$

This term can be seen as a boundary of the set Q_n. With this interpretation one can say that a Følner sequence is a sequence of finite sets, where the ratio between the boundary and the volume of the sets tends to zero. Originally, amenability was defined as the existence of a left-invariant mean on the group. In his paper [Fø55], Følner was the first who gave a combinatorial characterization of amenability via the boundary of a set. For a discussion and reformulation of this characterization see [Ada93]. For a survey on amenability up to the year 1988 we refer to [Pat88]. In this work, we will use several formulations of Følner sequences. Most of them are provided by the next Lemma.

Lemma 2.7. *Let G be a finitely generated group and let (Q_n) be a sequence of finite subsets in G. Then the following are equivalent:*

(i) (Q_n) *is a Følner sequence,*

(ii) for all $K \in \mathcal{F}(G)$*:* $\lim_{n\to\infty} |\partial_K(Q_n)|/|Q_n| = 0$*,*

(iii) for all $r > 0$*:* $\lim_{n\to\infty} |\partial^r(Q_n)|/|Q_n| = 0$*,*

(iv) for all $r > 0$*:* $\lim_{n\to\infty} |\partial^r_{\text{int}}(Q_n)|/|Q_n| = 0$*,*

(v) for all $r > 0$*:* $\lim_{n\to\infty} |\partial^r_{\text{ext}}(Q_n)|/|Q_n| = 0$*.*

Proof. Let (Q_n) be a Følner sequence and let $K \in \mathcal{F}(G)$ be given. Then we set $\bar{K} := K \cup K^{-1} \cup \{\text{id}\}$ and claim that for any $F \in \mathcal{F}(G)$ we have

$$\partial_K(F) \subseteq \partial_{\bar{K}}(F) \subseteq \bar{K}(\bar{K}F \triangle F). \tag{2.6}$$

If this holds true, then $|\partial_K(F)| \leq |\bar{K}(\bar{K}F \triangle F)|$ which shows that (i) implies (ii). The first inclusion of (2.6) follows from Lemma 2.1. In order to show the second inclusion, let $g \in \partial_{\bar{K}}(F)$ be given. Then, using symmetry of $\bar{K}$, we have $g \in \bar{K}^{-1}F = \bar{K}F$. If $g \notin F$, then $\text{id} \in \bar{K}$ implies

$$g \in \bar{K}F \setminus F \subseteq \bar{K}F \triangle F \subseteq \bar{K}(\bar{K}F \triangle F).$$

Next, we consider the case $g \in F$. Then for all $k \in \bar{K}$ we have $kg \in \bar{K}F$. Since $\bar{K}g \cap (G \setminus F) \neq \emptyset$, there exists some $\bar{k} \in \bar{K}$ with $\bar{k}g \notin F$ which yields $\bar{k}g \in \bar{K}F \setminus F$ and hence

$$g \in \bar{k}^{-1}(\bar{K}F \setminus F) \subseteq \bar{K}(\bar{K}F \triangle F),$$

which proves (2.6).

In order to show that (ii) implies (iii), use that $\partial_{B^G_r}(F) = \partial^r(F)$. By definition, assertion (iii) implies (iv).

Let us prove that (iv) implies (v). Assume (iv) and let $F \in \mathcal{F}(G)$, $r > 0$ and $g \in \partial^r_{\text{ext}}(F)$ be given. Then $g \notin F$ and there exists $x \in F$ with $d_S(g, x) \leq r$. Therefore $x \in \partial^r_{\text{int}}(F)$ and $gx^{-1} \in B_r$, which implies $g \in B_r \partial^r_{\text{int}}(F)$. Hence we conclude that

$$|\partial^r_{\text{ext}}(F)| \leq |B_r||\partial^r_{\text{int}}(F)|.$$

This obviously implies (v).

It remains to show that (v) implies (i). Let $K \in \mathcal{F}(G)$ be arbitrary and set $r := \max\{d_S(k, \mathrm{id}) \mid k \in K\}$ and again $\bar{K} := K \cup K^{-1} \cup \{\mathrm{id}\}$. It suffices to prove

$$KF \triangle F \subseteq \bar{K} \partial^r_{\mathrm{ext}}(F) \tag{2.7}$$

for arbitrary $F \in \mathcal{F}(G)$. To this end let $g \in KF \triangle F$ be given. Then either $g \in KF \setminus F$ or $g \in F \setminus KF$. In the first case we have $g = kg'$ for some $k \in K$ and $g' \in F$. As $g \notin F$, this implies $d_S(g, F) \leq d_S(g, g') \leq r$ and hence $g \in \partial^r_{\mathrm{ext}}(F) \subseteq \bar{K}\partial^r_{\mathrm{ext}}(F)$. Now assume that $g \in F \setminus KF$. Then we have for all $k \in K$ that $g \notin kF$ or equivalently $\bar{g} := k^{-1}g \notin F$. As $g \in F$ we have $d_S(F, \bar{g}) \leq d_S(g, \bar{g}) \leq r$ and hence $\bar{g} \in \partial^r_{\mathrm{ext}}(F)$. This shows

$$g = k\bar{g} \in K\partial^r_{\mathrm{ext}}(F) \subseteq \bar{K}\partial^r_{\mathrm{ext}}(F). \qquad \blacksquare$$

The next Lemma gives another equivalent condition for being a Følner sequence. It has been proven in a similar form already in [Ada93]. It is quite useful for showing that a given sequence is a Følner sequence.

Lemma 2.8. *Let G be a group generated by the finite and symmetric set S and let (Q_n) be a sequence of finite subsets in G. Then (Q_n) is a Følner sequence if and only if*

$$\lim_{n \to \infty} \frac{|SQ_n \setminus Q_n|}{|Q_n|} = 0. \tag{2.8}$$

Proof. If (Q_n) is a Følner sequence then by the definition of the symmetric difference (2.8) obviously holds. In order to prove the converse implication, assume that (2.8) holds true and let $K \in \mathcal{F}(G)$ be arbitrary. For any $g \in K$ we have

$$|KQ_n \setminus Q_n| \geq |gQ_n \setminus Q_n| = |Q_n \setminus gQ_n| \geq |Q_n \setminus KQ_n|$$

which implies $|KQ_n \triangle Q_n| \leq 2|KQ_n \setminus Q_n|$. Hence, we only need to control the difference $KQ_n \setminus Q_n$. Next, choose m large enough such that K is contained in the ball B_m and estimate

$$KQ_n \setminus Q_n \subseteq B_m Q_n \setminus Q_n \subseteq \bigcup_{j=0}^{m-1} B_{j+1}Q_n \setminus B_j Q_n, \tag{2.9}$$

where $B_0 = \{\mathrm{id}\}$. Besides this, we have for $j \in \{0, \dots, m-1\}$:

$$\begin{aligned} B_{j+1}Q_n \setminus B_jQ_n &= \bigcup_{g \in B_j} gB_1Q_n \setminus B_jQ_n \\ &\subseteq \bigcup_{g \in B_j} gB_1Q_n \setminus gQ_n = \bigcup_{g \in B_j} g(SQ_n \setminus Q_n). \end{aligned}$$

Thence, with (2.9) we end up with

$$|KQ_n \setminus Q_n| \le \sum_{j=0}^{m-1} |B_{j+1}Q_n \setminus B_jQ_n| \le \sum_{j=0}^{m-1} |B_j||SQ_n \setminus Q_n|,$$

which proves the claim. ∎

Recall that for a given set $Q \in \mathcal{F}(G)$ and $r > 0$ we denote the set $Q \setminus \partial^r Q$ by $Q^{(r)}$.

Lemma 2.9. *Let G be a finitely generated group and let $r > 0$ be given. Then if (Q_n) is a Følner sequence in G, the sequence $(Q_n^{(r)})$ is a Følner sequence as well.*

Proof. Let (Q_n) be a Følner sequence. First we claim that for any $Q \in \mathcal{F}(G)$

$$\partial^r_{\mathrm{ext}}(Q^{(r)}) \subseteq \partial^r_{\mathrm{int}}(Q). \tag{2.10}$$

To see this, let $x \in \partial^r_{\mathrm{ext}}(Q^{(r)})$ be arbitrary. Then, $d_S(x, Q^{(r)}) \le r$, which means that there exists $y \in Q^{(r)}$ such that $d_S(x,y) \le r$. Suppose that $x \notin Q$, then we have $y \in \partial^r_{\mathrm{int}}(Q)$ since $y \in Q$. This would imply that $y \notin Q^{(r)}$, which is a contradiction. Therefore we have $x \in Q$. We use $x \notin Q^{(r)}$ to obtain (2.10). Observe that by Lemma 2.7 we get

$$\lim_{n \to \infty} \frac{|Q_n^{(r)}|}{|Q_n|} = 1.$$

Hence there exists a constant $n_0 \in \mathbb{N}$ such that $|Q_n|^{-1}|Q_n^{(r)}| \ge \frac{1}{2}$ for all $n \ge n_0$. Now, the above facts imply

$$0 \le \frac{|\partial^r_{\mathrm{ext}}(Q_n^{(r)})|}{|Q_n^{(r)}|} \le \frac{|\partial^r_{\mathrm{int}}(Q_n)|}{|Q_n^{(r)}|} \le 2\frac{|\partial^r_{\mathrm{int}}(Q_n)|}{|Q_n|}$$

for all $n \geq n_0$. Since (Q_n) is a Følner sequence, the result follows by Lemma 2.7. ■

It is easy to construct an example, showing that the converse of Lemma 2.9 is not true.

A Følner sequence (Q_n) is said to be *tempered* if for some $C > 0$ and all $n \in \mathbb{N}$

$$\left| \bigcup_{k<n} Q_k^{-1} Q_n \right| \leq C|Q_n|$$

holds true. It can be shown that each Følner sequence has a tempered subsequence, see e.g. [Lin01]. We call a sequence (Q_n) *strictly increasing* if $|Q_n| < |Q_{n+1}|$ for all $n \in \mathbb{N}$. Again one can show, that each Følner sequence has a strictly increasing subsequence. As each subsequence of a strictly increasing sequence is strictly increasing as well, this yields that there is a strictly increasing tempered Følner sequence in each amenable group. A Følner sequence (Q_n) is said to be *nested* if $\mathrm{id} \in Q_1$ and for all $n \in \mathbb{N}$ we have $Q_n \subseteq Q_{n+1}$. Obviously, a nested Følner sequence is strictly increasing. The next Lemma shows that each amenable group contains a nested Følner sequence.

Lemma 2.10. *Each finitely generated amenable group contains a nested Følner sequence.*

Proof. Let G be a finitely generated amenable group. Then there exists Følner sequence (Q_n) in G. Choose some $x \in Q_1$ and define $U_1 := Q_1 x^{-1}$. Now, we proceed inductively. If $U_1, \ldots, U_k$ are chosen, then by Lemma 2.7 there exists an $n \in \mathbb{N}$ such that $|\partial_{U_k}(Q_n)| < |Q_n|$. By

$$|Q_n \setminus \partial_{U_k}(Q_n)| \geq |Q_n| - |\partial_{U_k}(Q_n)| > 0$$

we obtain that $Q_n \setminus \partial_{U_k}(Q_n)$ is non-empty. We choose some $y \in Q_n \setminus \partial_{U_k}(Q_n)$. Then $U_k y \subseteq Q_n$ or equivalently $U_k \subseteq Q_n y^{-1}$. Thus, setting $U_{k-1} := Q_n y^{-1}$ gives a set which contains U_k. In this way we obtain a sequence $(U_n)_{n \in \mathbb{N}}$ which consists translates of a subsequence of (Q_n). Therefore, (U_n) is a Følner sequence and nested by construction. ■

The class of amenable groups is quite large. It contains all groups of polynomial growth as well as all groups of intermediate growth. Furthermore, there exist groups of exponential volume growth, which are amenable, e.g. the Lamplighter group. A famous example for a non-amenable group is the free group. The next Lemma shows that each amenable group is sofic.

Lemma 2.11. *Each finitely generated amenable group is sofic.*

Proof. Let G be finitely generated and amenable, $\varepsilon > 0$ and $r \in \mathbb{N}$ be given. Amenability implies that there exists a Følner sequence (Q_n) in G. By Lemma 2.7 there exist $n = n(r, \varepsilon) \in \mathbb{N}$ such that $|\partial^r_{\text{int}}(Q_n)| \leq \varepsilon |Q_n|$. Then, define $\Gamma_{r,\varepsilon} := (V_{r,\varepsilon}, E_{r,\varepsilon})$ as the restriction $\Gamma(G,S)|_{Q_n}$ of the Cayley graph Γ of G to Q_n. This means, that $V_{r,\varepsilon} = Q_n$ and two vertices are connected in $\Gamma_{r,\varepsilon}$ if and only if they are connected in Γ. Furthermore, set $V^{(0)}_{r,\varepsilon} := Q^{(r)}_n = Q_n \setminus \partial^r_{\text{int}}(Q_n)$. Then, for each $x \in Q^{(r)}_n = V^{(0)}_{r,\varepsilon}$ the ball $B_r(x)$ is contained in $Q_n = V_{r,\varepsilon}$. This proves (S2). Condition (S1) follows immediately from the choice of n:

$$|V^{(0)}_{r,\varepsilon}| = |Q^{(r)}_n| = |Q_n| - |\partial^r_{\text{int}}(Q_n)| \leq (1-\varepsilon)|V_{r,\varepsilon}|. \qquad \blacksquare$$

Studying spectral properties of discrete operators, the geometric setting of amenable groups can be seen as the natural generalization of $\mathbb{Z}^d$. This is due to the fact, that many proofs rely on the property, that boxes or balls in $\mathbb{Z}^d$ have a vanishing boundary (in comparison with its volume), if one increases the diameter. This property remains true for the above defined Følner sequences.

In this context it is also important that many tools, which have been established for the euclidean setting, can be generalized to amenable groups. The most prominent example of these tools is the pointwise ergodic theorem which is due to Lindenstrauss [Lin01, Theorem 1.2]. In Theorem 2.12 we cite a special (and for our purposes sufficient) case of it. Before we do so, let use give some more definitions.

We say that a group G *acts from the left* on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ by *measure preserving transformations* T_g, $g \in G$, if for any $g \in G$ the mapping $T_g : \Omega \to \Omega$ is a bijection satisfying:

(i) for all $\omega \in \Omega$ and $g, h \in G$ we have $T_{gh}(\omega) = T_g(T_h(\omega))$,

(ii) for all $\omega \in \Omega$ we have $T_{\text{id}}(\omega) = \omega$ and

(iii) for all $A \in \mathcal{A}$ and $g \in G$ we have $T_g(A) = \{T_g(\omega) \mid \omega \in A\} \in \mathcal{A}$ and $\mathbb{P}(A) = \mathbb{P}(T_g(A))$.

In this situation we also say that $\mathcal{T} = (T_g)_{g\in G}$ is a measure preserving left action of G on $(\Omega, \mathcal{A}, \mathbb{P})$. Furthermore, this action $\mathcal{T}$ of G is called *ergodic*, if for any $A \in \mathcal{A}$ with

$$T_g(A) = A$$

for all $g \in G$, one has $\mathbb{P}(A) \in \{0, 1\}$. Note that (i) and (ii) imply that for any $g \in G$ we have $T_g^{-1} = T_{g^{-1}}$.

Theorem 2.12 (Lindenstrauss)**.** *Let G be an amenable group and let $\mathcal{T} = (T_g)_{g\in G}$ be a measure preserving and ergodic left action of G on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Furthermore, let (Q_n) be a tempered Følner sequence in G. Then for any $f \in L^1(\mathbb{P})$*

$$\lim_{n\to\infty} \frac{1}{|Q_n|} \sum_{g\in Q_n} f(T_g(\omega)) = \int_\Omega f(\omega) d\mathbb{P}(\omega)$$

holds almost surely.

2.2 Operators on groups

In this section we give some well-known definitions and facts about linear operators on Hilbert spaces. In the first subsection, we concentrate on deterministic operators, whereas in the second subsection we consider the random setting.

2.2.1 Deterministic operators on groups

Let A be an linear operator mapping from its domain $D(A)$ to X, where $D(A) \subseteq X$ and X is a Hilbert space. Without indicating it at each specific situation, in this work we always assume that the operator under consideration is linear and the domain is a linear subspace of the associated Hilbert space. We say that an operator is *densely defined* when $D(A)$ is dense in X with respect to the norm which is induced by the scalar product on X. Furthermore A is called *symmetric*, if for all $x, y \in D(A)$ one has $\langle x, Ay\rangle = \langle Ax, y\rangle$. If D is a subspace of $D(A)$, then we denote by $A|_D$ the *restriction of A to D*, i.e. $A|_D : D \to X$, $A|_D x := Ax$ for all $x \in D$. The *graph* $G(A)$ of an operator A is a subset of $X^2 := X \times X$ and given by

$$G(A) := \{(x, Ax) \mid x \in D(A)\}.$$

We say that the operator A is *closed* if $G(A)$ is closed in X^2 with respect to the norm defined by $\|(x,y)\| := (\|x\|^2 + \|y\|^2)^{1/2}$. The operator A is called *closable* if the closure $\overline{G(A)}$ of $G(A)$ is the graph of an operator $\overline{A}$. This operator $\overline{A}$ is unique and will be called the *closure* of A. Note that each symmetric operator is closable.

Let A be a closed operator. A subspace D of the domain $D(A)$ is called *core* of A, if the closure of $A|_D$ equals A. If D is a core of A, then the closure of D with respect to the *graph norm* given by $\|x\|_A = (\|x\|^2 + \|Ax\|^2)^{1/2}$ is the domain $D(A)$.

The operator A is called *self-adjoint* if A is densely defined and $A = A^*$. Here A^* is the adjoint of a densely defined operator A given by

$$\begin{aligned} D(A^*) &:= \{x \in X \mid \exists\, y \in X : \langle x, Az\rangle = \langle y, z\rangle \text{ for all } z \in D(A)\}, \\ A^*x &:= y. \end{aligned}$$

Note that each symmetric and bounded operator is self-adjoint and each self-adjoint operator is closed. Furthermore, a symmetric operator is called *essentially self-adjoint*, if its closure is self-adjoint. In order to prove essential self-adjointness, it is good to know that a symmetric operator A on a Hilbert space X is essentially self-adjoint, if and only if for all $z \in \mathbb{C} \setminus \mathbb{R}$ one has that $(z - A)D(A)$ is dense in X, see for instance [RS80].

Given a complex number $z \in \mathbb{C}$, we denote by $\Re(z) \in \mathbb{R}$ the real part of z and by $\Im(z) \in \mathbb{R}$ the imaginary part, i.e. $z = \Re(z) + \mathrm{i}\Im(z)$. Let us specify the Hilbert space on which our operators will be defined. Let G be a finitely generated group. For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $p \in \{1, 2\}$ we will use the notation

$$\begin{aligned} \ell^p(G, \mathbb{K}) &:= \Big\{ f : G \to \mathbb{K} \mid \sum_{g \in G} |f(g)|^p < \infty \Big\}, \text{ and} \\ C_{\mathrm{c}}(G, \mathbb{K}) &:= \big\{ f : G \to \mathbb{K} \mid |\mathrm{spt}(f)| < \infty \big\}, \end{aligned}$$

where $\mathrm{spt}(f)$ denotes the *support* of f, i.e. $\mathrm{spt}(f) := \{g \in G \mid f(g) \neq 0\}$. The norm in $\ell^p(G, \mathbb{K})$ is defined by setting for $f \in \ell^p(G)$:

$$\|f\|_p := \Big(\sum_{g \in G} |f(g)|^2 \Big)^{1/p}.$$

Furthermore, we set $\ell^p(G) := \ell^p(G,\mathbb{C})$ and $C_{\mathrm{c}}(G) := C_{\mathrm{c}}(G,\mathbb{C})$. In many situations we consider operators on the Hilbert space $\ell^2(G)$. The scalar product in $\ell^2(G)$ is given as follows: for $f,g \in \ell^2(G)$ set

$$\langle f,g\rangle = \sum_{x\in G} \overline{f(x)}g(x).$$

Thus, the following relation holds: $\|f\|_2^2 = \langle f,f\rangle$. Moreover, the Cauchy-Schwarz inequality gives for $f,g \in \ell^2(G)$:

$$|\langle f,g\rangle| \le \|f\|_2 \, \|g\|_2.$$

Note that as G is finitely generated, we have that G is countable and hence $\ell^2(G)$ is separable. For $x \in G$ we define $\delta_x \in \ell^2(G)$ by setting $\delta_x(z) = 1$ if $x = z$ and $\delta_x(z) = 0$ if $x \neq z$. Furthermore, for $Q \in \mathcal{F}(G)$ we use the mapping $\pi_Q : \ell^2(G) \to C_{\mathrm{c}}(G)$,

$$(\pi_Q\phi)(x) = \begin{cases} \phi(x) & \text{if } x \in Q, \\ 0 & \text{otherwise.} \end{cases} \tag{2.11}$$

If the set $C_{\mathrm{c}}(G)$ is a subset of the domain of an operator A, then for $x,y \in G$ the expression $\langle \delta_x, A\delta_y\rangle$ is well defined. We will often refer to $\langle \delta_x, A\delta_y\rangle$ as the *matrix element* of A (with respect to the canonical basis of $\ell^2(G)$). Sometimes we also use the notation $a(x,y) := \langle \delta_x, A\delta_y\rangle$. We say that an operator A with $C_{\mathrm{c}}(G) \subseteq D(A)$ is of *finite hopping range* $r \in \mathbb{N}$, if for all $x,y \in G$ with $d_S(x,y) \ge r$ one has $a(x,y) = \langle \delta_x, A\delta_y\rangle = 0$. Moreover, an operator A with $C_{\mathrm{c}}(G) \subseteq D(A)$ is called *translation invariant* if for all $x,y,z \in G$ one has $a(x,y) = a(xz,yz)$.

Example 2.13. Here we verify many of the above defined properties at the example of the graph Laplacian of a Cayley graph. Let G be a finitely generated group and $S \subseteq G$ a finite and symmetric set of generators. The *Laplacian* or *Laplace operator* of the Cayley graph $\Gamma = \Gamma(G,S)$ is the operator $\Delta : \ell^2(G) \to \ell^2(G)$ which acts for given $f \in \ell^2(G)$ as follows

$$(\Delta f)(x) := \sum_{s\in S} \left(f(sx) - f(x)\right).$$

Let $f \in \ell^2(G)$ be given. With this definition we have for each $x \in G$:

$$|(\Delta f)(x)|^2 = \Big|\sum_{s\in S}(f(sx) - f(x))\Big|^2 \leq |S|\sum_{s\in S}|f(sx) - f(x)|^2.$$

We define $g \in \ell^2(G)$ by setting for each $x \in G$: $g(x) := f(sx)$. Then obviously we have $\|g\|_2 = \|f\|_2$. By the triangle inequality we obtain

$$\sum_{x\in G}|f(sx) - f(x)|^2 = \langle g - f, g - f\rangle = \|g - f\|_2^2 \leq 4\|f\|_2^2.$$

The combination of the previous calculations gives

$$\sum_{x\in G}|(\Delta f)(x)|^2 \leq |S|\sum_{s\in S}\sum_{x\in G}|f(sx) - f(x)|^2 \leq 4|S|^2\|f\|_2^2 < \infty,$$

which shows that for each $f \in \ell^2(G)$ we obtain that $\Delta f \in \ell^2(G)$. Thus, the Laplacian Δ is well-defined on its domain $D(\Delta) = \ell^2(G)$. Let us consider the matrix elements of the Laplacian. We calculate for $x, y \in G$ as follows:

$$\begin{aligned} \langle \delta_x, \Delta\delta_y\rangle &= \sum_{z\in G}\delta_x(z)\sum_{s\in S}(\delta_y(sz) - \delta_y(z)) \\ &= \sum_{s\in S}(\delta_y(sx) - \delta_y(x)) = \delta_{xy^{-1}}(s) - |S|\delta_y(x) \qquad (2.12) \end{aligned}$$

Thus, if $\mathrm{id} \notin S$, each diagonal element equals $-|S|$. Non-diagonal elements are either zero or one. They are one if and only if the corresponding vertices are connected by an edge in the Cayley graph. Moreover, equation (2.12) implies that for any $x, y, z \in G$ we have

$$\langle \delta_{xz}, \Delta\delta_{yz}\rangle = \delta_{xz(yz)^{-1}}(s) - |S|\delta_{yz}(xz) = \langle \delta_x, \Delta\delta_y\rangle\,.$$

This shows that the Laplacian of Γ is translation invariant. Besides this, the calculation of the matrix elements immediately gives that Δ is of finite hopping range 2. Let us check that Δ is symmetric. To this end let $f, g \in G$ be arbitrary. Then we have by symmetry of S:

$$\sum_{x\in G}\sum_{s\in S}\overline{f(sx)}g(x) = \sum_{x\in G}\sum_{s\in S}\overline{f(x)}g(sx).$$

This is used to obtain

$$\begin{aligned}\langle \Delta f, g\rangle &= \sum_{x\in G}\Big(\sum_{s\in S}\overline{f(sx)} - \overline{f(x)}\Big)g(x)\\ &= \sum_{x\in G}\sum_{s\in S}\overline{f(x)}g(sx) - \sum_{x\in G}\sum_{s\in S}\overline{f(x)}g(x)\\ &= \sum_{x\in G}\overline{f(x)}\Big(\sum_{s\in S} g(sx) - g(x)\Big) = \langle f, \Delta g\rangle\,.\end{aligned}$$

Thus, the Laplacian is symmetric. Since $D(\Delta) = \ell^2(G)$ it also is self-adjoint and closed.

The following well-known Lemma shows how to express matrix elements of powers of an operator via matrix elements of the operator itself.

Lemma 2.14. *Let G be a finitely generated group, $Q \subseteq G$ and let A be a bounded operator on $\ell^2(Q)$ with finite hopping range r. Then for each $m \in \mathbb{N}$ and $x, y \in Q$ one has*

$$\langle \delta_x, A^m\delta_y\rangle = \sum_{v_1\dots,v_{m-1}\in B_{r(m-1)}(x)\cap Q} \langle \delta_x, A\delta_{v_1}\rangle\langle \delta_{v_1}, A\delta_{v_2}\rangle\cdots\left\langle \delta_{v_{m-1}}, A\delta_y\right\rangle.$$

Here the elements δ_x, $x \in Q$ are the canonical basis of $\ell^2(Q)$.

Proof. We show the claim by induction. For $m = 1$ it is clear and for $m = 2$ it follows from the fact that for $x, y \in Q$ we have:

$$\left\langle \delta_x, A^2\delta_y\right\rangle = \sum_{v\in Q}\langle A^*\delta_x, \delta_v\rangle\langle \delta_v, A\delta_y\rangle = \sum_{v\in B_r(x)\cap Q}\langle \delta_x, A\delta_v\rangle\langle \delta_v, A\delta_y\rangle\,.$$

Assume that the claimed equality holds for $m - 1$. Then we have

$$\begin{aligned}&\langle \delta_x, A^m\delta_y\rangle = \left\langle (A^{m-1})^*\delta_x, A\delta_y\right\rangle\\ &= \sum_{v_{m-1}\in Q}\left\langle \delta_x, A^{m-1}\delta_{v_{m-1}}\right\rangle\left\langle \delta_{v_{m-1}}, A\delta_y\right\rangle\\ &= \sum_{v_1\dots,v_{m-1}\in B_{r(m-1)}(x)\cap Q}\langle \delta_x, A\delta_{v_1}\rangle\cdots\left\langle \delta_{v_{m-2}}, A\delta_{v_{m-1}}\right\rangle\left\langle \delta_{v_{m-1}}, A\delta_y\right\rangle,\end{aligned}$$

which proves the lemma. ■

In order to deal with resolvents of self-adjoint operators the following Lemma will be helpful.

Lemma 2.15. *Let $A : D(A) \subseteq \ell^2(G) \to \ell^2(G)$ be a self-adjoint operator, let $z \in \mathbb{C} \setminus \mathbb{R}$ and assume that $C_c(G)$ is a core of A. Then for each $\kappa > 0$ and $\xi \in \ell^2(G)$ there exists $\psi \in \ell^2(G)$ such that*

$$\|\xi - \psi\|_2 < \kappa \qquad \text{and} \qquad (z - A)^{-1}\psi \in C_c(G).$$

Proof. Let $\kappa > 0$ and $\xi \in \ell^2(G)$ be given. As $C_c(G)$ is a core of A, it is dense in $D(A)$ with respect to the norm $\|\cdot\|_A$. The map

$$z - A\colon (D(A), \|\cdot\|_A) \to (\ell^2(G), \|\cdot\|_2)$$

is continuous and surjective, and so

$$(z - A)(C_c(G)) = \{\psi \in \ell^2(G) \mid (z - A)^{-1}\psi \in C_c(G)\} \tag{2.13}$$

is dense in $\ell^2(G)$. This construction allows to find an element $\psi \in (z - A)(C_c(G))$ such that $\|\xi - \psi\|_2 < \kappa$. Furthermore, equation (2.13) shows that $(z - A)^{-1}\psi$ is compactly supported. ∎

2.2.2 Random operators on groups

In this section we give precise definitions of random and ergodic operators on groups. Here we stick to the notation of [PF92]. Let G be a finitely generated group and $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space. Recall that the space $\ell^2(G)$ is separable. Denote by $L(\ell^2(G))$ the space of linear operators on $\ell^2(G)$ and by $\mathcal{L}(\ell^2(G))$ the subspace of $L(\ell^2(G))$ which consists of all bounded linear operators, i.e.

$$L(\ell^2(G)) := \{A : D(A) \to \ell^2(G) \mid D(A) \subseteq \ell^2(G),\ A \text{ linear}\}, \text{ and}$$
$$\mathcal{L}(\ell^2(G)) := \{A \in L(\ell^2(G)) \mid D(A) = \ell^2(G),\ A \text{ bounded}\}.$$

We say that $\psi\colon \Omega \to \ell^2(G)$ is a *random vector* in $\ell^2(G)$ or that ψ is *weakly measurable*, if for any $\phi \in \ell^2(G)$ the function $\langle \phi, \psi\rangle : \Omega \to \mathbb{C}$, $\omega \mapsto \langle \phi, \psi(\omega)\rangle$ is measurable. Sometimes, a ψ which fulfills this condition is simply called *measurable.* If $\psi\colon \Omega \to \ell^2(G)$ satisfies $\psi^{-1}(U) \in \mathcal{A}$ for any open set $U \subseteq \ell^2(G)$, then ψ is called *norm measurable.* This notion refers to the fact that the open sets in $\ell^2(G)$

are defined using the norm $\| \cdot \|_2$. As $\ell^2(G)$ is separable, ψ is norm measurable if and only if it is weakly measurable, cf. [Con99, § 52].

Let D be a dense linear subspace of $\ell^2(G)$. Then a mapping

$$A\colon \Omega \to L(\ell^2(G)), \qquad A \mapsto A^{(\omega)},$$

or the family $(A^{(\omega)})_{\omega\in\Omega}$, is called a *random operator* on the domain D, if there is a set $\tilde{\Omega} \subseteq \Omega$ of full measure such that for all $\omega \in \tilde{\Omega}$ the set D is a subset of the domain of $A^{(\omega)}$ and if for any $\phi \in D$ the mapping $A\phi\colon \tilde{\Omega} \to \ell^2(G)$, $(A\phi)(\omega) := A^{(\omega)}\phi$ is a random vector on the probability space $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbb{P}})$. Here $\tilde{\mathcal{A}} := \{A \cap \tilde{\Omega} \mid A \in \mathcal{A}\}$ and $\tilde{\mathbb{P}}(A \cap \tilde{\Omega}) = \mathbb{P}(A)$ for all $A \in \mathcal{A}$. Again, a family of operators $(A^{(\omega)})_{\omega\in\Omega}$ which satisfies the conditions of a random operator on the domain D is sometimes simply called *measurable*. Note that the domain of a random operator is not uniquely determined.

Let us consider the special case, where A is mapping from Ω to the set of bounded linear operators on $\ell^2(G)$, i.e.

$$A : \Omega \to \mathcal{L}(\ell^2(G)), \qquad A \mapsto A^{(\omega)}.$$

Here one does not have to care about the domains, such that in order to show that A is measurable (or a random operator on the domain $\ell^2(G)$), it is sufficient to show that for all $\phi, \psi \in \ell^2(G)$ the mapping $\Omega \ni \omega \mapsto \langle \phi, A^{(\omega)}\psi \rangle$ is measurable.

This, and the fact that we are in particular interested in random operators where D equals $C_c(G)$ leads to the following definition.

Definition 2.16. Let A be a mapping from Ω to $L(\ell^2(G))$. We say that A is a *proper random operator* if for all $\omega \in \Omega$ one has $C_c(G) \subseteq D(A^{(\omega)})$ and for all $\phi \in C_c(G)$ the mapping $A\phi : \Omega \to \ell^2(G), \omega \mapsto A^{(\omega)}\phi$ is a random vector.

In particular, a proper random operator is a random operator on the domain $C_c(G)$. The random operators which we will consider in the following will usually have (at least) the domain $C_c(G)$. For these operators we define what it means to be ergodic. To this end let again $\tilde{\Omega}$ be a set of full measure such that $C_c(G) \subseteq D(A^{(\omega)})$ for all $\omega \in \tilde{\Omega}$. If the action $\mathcal{T} = (T_x)_{x\in G}$ of G on $(\Omega, \mathcal{A}, \mathbb{P})$ is a measure preserving and ergodic left action, we define

$$\Omega_{\mathcal{T}} := \bigcap_{x\in G} T_x(\tilde{\Omega}), \tag{2.14}$$

which is a set of full measure as G is countable.

Definition 2.17. Let $A = (A^{(\omega)})_{\omega\in\Omega}$ be a random operator on the domain $C_c(G)$ mapping each element of the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ to a linear operator on $\ell^2(G)$. Then A is called *ergodic* or *metrically transitive*, if $\mathcal{T} = (T_x)_{x\in G}$ is a measure preserving and ergodic left action of G on $(\Omega, \mathcal{A}, \mathbb{P})$ such that for all $\omega \in \Omega_{\mathcal{T}}$ and all $x, y, z \in G$ one has

$$a^{(T_z\omega)}(x, y) = a^{(\omega)}(xz, yz),$$

where for all $\omega \in \Omega_{\mathcal{T}}$ we use the notion $a^{(\omega)}(x, y) := \langle \delta_x, A^{(\omega)}\delta_y \rangle$.

Note that the choice of $\Omega_{\mathcal{T}}$ in (2.14) ensures that for each $\omega \in \Omega_{\mathcal{T}}$ and $y \in G$ we have $\delta_y \in D(A^{(\omega)})$. If A is an ergodic operator with $\mathcal{T}$ as in the definition, we define the family $\mathcal{U} := (U_x)_{x\in G}$ of unitary operators on $\ell^2(G)$ by setting for $\phi \in \ell^2(G)$ and $x, z \in G$

$$(U_z\phi)(x) := \phi(xz). \tag{2.15}$$

This yields $U_z^{-1} = U_{z^{-1}}$ for all $z \in G$ and

$$A^{(T_z\omega)} = U_z A^{(\omega)} U_z^{-1}$$

for all $\omega \in \Omega_{\mathcal{T}}$. Let A be a random operator with matrix elements $a^{(\omega)}(x, y)$, $x, y \in G$, $\omega \in \Omega$. We call A *translation invariant (in distribution)* if for any $z \in G$, $F \in \mathcal{F}(G)$ and $E \in \mathcal{B}(\mathbb{C}^{F\times F})$ one has

$$\mathbb{P}\Big(\big(a(x, y)\big)_{x,y\in F} \in E\Big) = \mathbb{P}\left(\big(a(xz, yz)\big)_{x,y\in F} \in E\right). \tag{2.16}$$

Note that ergodicity implies translation invariance in distribution. This property can also hold simultaneously for two operators. To define this, let B be another random operator with matrix elements $b^{(\omega)}(x, y)$, $x, y \in G$, $\omega \in \Omega$. Then we say that A and B are *jointly translation invariant (in distribution)* if for any $F \in \mathcal{F}(G)$, $E \in \mathcal{B}(\mathbb{C}^{F\times F} \times \mathbb{C}^{F\times F})$:

$$\begin{aligned}\mathbb{P}&\left(\big(a(x, y), b(x, y)\big)_{x,y\in F} \in E\right)\\ &= \mathbb{P}\left(\big(a(xz, yz), b(xz, yz)\big)_{x,y\in F} \in E\right).\end{aligned} \tag{2.17}$$

If A and B are ergodic operators on the same probability space and with the same family $\mathcal{T}$ of transformations, then they are jointly translation invariant in distribution.

Another important condition which our operators oftentimes have to fulfill is the following:

$$\mathbb{E}\left(\|A\delta_{\mathrm{id}}\|_1^2\right) = \mathbb{E}\left(\Big(\sum_{x\in G}|a(x,\mathrm{id})|\Big)^2\right) < \infty. \tag{2.18}$$

Note that in the case where the operator is translation invariant in distribution condition (2.18) implies that $\mathbb{E}\left(\|A\delta_x\|_1^2\right) = \mathbb{E}\left(\|A\delta_{\mathrm{id}}\|_1^2\right) < \infty$ holds for any $x \in G$. This gives immediately $\mathbb{E}\left(\|A\phi\|_1^2\right) < \infty$ for any $\phi \in C_{\mathrm{c}}(G)$.

The following lemma is adapted from [PF92, Proposition 4.1]. For a random operator B we denote by $\|B\|_\infty$ the $L^\infty(\mathbb{P})$-norm of the random variable $\omega \mapsto \|B^{(\omega)}\| \in \mathbb{R}$, i.e.

$$\|B\|_\infty = \operatorname*{ess\,sup}_{\omega\in\Omega}\|B^{(\omega)}\|,$$

where $\|\cdot\|$ is the operator norm.

Lemma 2.18. *Let A and B be random operators on the domain $C_{\mathrm{c}}(G)$, which are jointly translation invariant and let both satisfy (2.18). Furthermore let $\|B\|_\infty$ be finite. Then for all $x \in G$ and $r \in \mathbb{N}$,*

$$\mathbb{E}\big(\|A\pi_{B_r^G}B\delta_x\|_2^2\big) \le \|B\|_\infty^2\mathbb{E}\big(\|A\delta_{\mathrm{id}}\|_1^2\big),$$

holds true.

Proof. Let $\Omega_{\mathrm{c}} := \{\omega \in \Omega \mid C_{\mathrm{c}}(G) \subseteq D(A^{(\omega)}) \cap D(B^{(\omega)})\}$ and set for all $\omega \in \Omega_{\mathrm{c}}$ and $x, y \in G$ as before $a^{(\omega)}(x,y) := \langle\delta_x, A\delta_y\rangle$ and $b^{(\omega)}(x,y) := \langle\delta_x, B\delta_y\rangle$ the matrix elements of A and B. We have for $\psi \in \ell^2(G)$ the equality $\psi = \sum_{z\in G}\langle\delta_z,\psi\rangle\delta_z$. Using this, the triangle inequality and monotone convergence, we obtain

$$\begin{aligned}\mathbb{E}\big(\|A\pi_{B_r^G}B\delta_x\|_2^2\big) &= \mathbb{E}\big(\langle A\pi_{B_r^G}B\delta_x, A\pi_{B_r^G}B\delta_x\rangle\big)\\ &\le \sum\nolimits_{y,z\in G}\mathbb{E}\big(|\langle A\delta_y, A\delta_z\rangle b(y,x)b(x,z)|\big)\\ &= \sum\nolimits_{y,z}\mathbb{E}\big(|\langle A\delta_{yz^{-1}}, A\delta_{\mathrm{id}}\rangle b(yz^{-1},xz^{-1})b(xz^{-1},\mathrm{id})|\big),\end{aligned}$$

where we used joint translation invariance of A and B in the last step. Substitution, monotone convergence and Cauchy-Schwarz inequality leads to

$$\begin{aligned}\mathbb{E}\big(\|A\pi_{B_r^G}B\delta_x\|_2^2\big) &= \sum\nolimits_{y',z}\mathbb{E}\big(|\langle A\delta_{y'},A\delta_{\mathrm{id}}\rangle b(y',xz^{-1})b(xz^{-1},\mathrm{id})|\big)\\ &= \sum\nolimits_{y'}\mathbb{E}\big(|\langle A\delta_{y'},A\delta_{\mathrm{id}}\rangle|\sum\nolimits_{z'}|b(y',z')b(z',\mathrm{id})|\big)\\ &\le \sum\nolimits_{y'}\mathbb{E}\big(|\langle A\delta_{y'},A\delta_{\mathrm{id}}\rangle|\|B\delta_{y'}\|_2\|B\delta_{\mathrm{id}}\|_2\big).\end{aligned}$$

We estimate, using the norm $\|B\|_\infty$ and again translation invariance in distribution, to get

$$\begin{aligned}&\mathbb{E}\big(|\langle A\delta_{y'},A\delta_{\mathrm{id}}\rangle|\|B\delta_{y'}\|_2\|B\delta_{\mathrm{id}}\|_2\big)\\ &\quad\le \|B\|_\infty^2\mathbb{E}\big(|\langle A\delta_{y'},A\delta_{\mathrm{id}}\rangle|\big)\\ &\quad= \|B\|_\infty^2\mathbb{E}\Big(\Big|\sum\nolimits_z a(z,y')a(z,\mathrm{id})\Big|\Big)\\ &\quad= \|B\|_\infty^2\mathbb{E}\Big(\Big|\sum\nolimits_z a(\mathrm{id},y'z^{-1})a(\mathrm{id},z^{-1})\Big|\Big).\end{aligned}$$

Next, we apply the triangle inequality and reorder the sum to obtain

$$\begin{aligned}&\mathbb{E}\big(\|A\pi_{B_r^G}B\delta_x\|_2^2\big)\\ &\quad\le \|B\|_\infty^2\sum\nolimits_{y',z}\mathbb{E}\big(|a(\mathrm{id},y'z^{-1})a(\mathrm{id},z^{-1})|\big)\\ &\quad= \|B\|_\infty^2\mathbb{E}\Big(\sum\nolimits_{y''}|a(\mathrm{id},y'')|\sum\nolimits_{z'}|a(\mathrm{id},z')|\Big)\\ &\quad= \|B\|_\infty^2\mathbb{E}\Big(\Big(\sum\nolimits_y|a(\mathrm{id},y)|\Big)^2\Big) = \|B\|_\infty^2\mathbb{E}\big(\|A\delta_{\mathrm{id}}\|_1^2\big). \qquad\blacksquare\end{aligned}$$

In the following theorem we use the previous result to obtain essential self-adjointness.

Theorem 2.19. *Let $A=(A^{(\omega)})_{\omega\in\Omega}$ be a symmetric random operator on the domain $C_c(G)$ which is translation invariant in distribution and which satisfies (2.18). Then, for $\mathbb{P}$-almost all $\omega\in\Omega$ the operator $A^{(\omega)}$ is essentially self-adjoint.*

Proof. We generalize the proof of [PF92, Theorem 4.2] to our more general setting. As discussed before, in order to show almost sure essential self-adjointness, it is enough to prove that for $\mathbb{P}$-almost all ω and all $z \in \mathbb{C} \setminus \mathbb{R}$

$$(z - A^{(\omega)})D(A^{(\omega)})$$

is dense in $\ell^2(G)$. Note that by assumption we have $C_{\mathrm{c}}(G) \subseteq D(A^{(\omega)})$ for almost all $\omega \in \Omega$. Therefore it suffices to show that $(z - A^{(\omega)})C_{\mathrm{c}}(G)$ is almost surely dense in $\ell^2(G)$. To this end, choose some $g \in G$. Hence, it is enough to find for almost all ω a sequence $(\phi_k^{(\omega)})_{k\in\mathbb{N}}$ of finitely supported functions, such that $\lim_{k\to\infty} \|(z - A^{(\omega)})\phi_k^{(\omega)} - \delta_g\|_2 = 0$.

Define for $r \in \mathbb{N}$ and $\omega \in \Omega$ an approximating operator $A_r^{(\omega)} : C_{\mathrm{c}}(G) \to \ell^2(G)$ by setting for $\phi \in C_{\mathrm{c}}(G)$ and $x \in G$:

$$(A_r^{(\omega)}\phi)(x) := \sum_{y\in G} a_r^{(\omega)}(x,y)\phi(y)$$

where

$$a_r^{(\omega)}(x,y) := \begin{cases} a^{(\omega)}(x,y) & \text{if } |a^{(\omega)}(x,y)| \leq r \text{ and } d_S(x,y) \leq r, \\ 0 & \text{otherwise.} \end{cases}$$

Then there is a constant $b_r \geq 0$ such that for all ω we have $\|A_r^{(\omega)}\|_2 \leq b_r$. Hence for each ω the operator $A_r^{(\omega)}$ is self-adjoint. Now we introduce the element $\phi_{g,n,r}^{(\omega)} \in C_{\mathrm{c}}(G)$, which will, for an appropriate choice of r and n, help to find an approximant for δ_g. We set

$$\phi_{g,n,r}^{(\omega)} := \pi_{B_n^G}(z - A_r^{(\omega)})^{-1}\delta_g$$

and estimate

$$\begin{aligned} &\|(z - A^{(\omega)})\phi_{g,n,r}^{(\omega)} - \delta_g\|_2 \\ &= \big\|(z - A^{(\omega)})\pi_{B_n^G}(z - A_r^{(\omega)})^{-1}\delta_g - \\ &\qquad\qquad (z - A_r^{(\omega)})(\pi_{B_n^G} + \pi_{G\setminus B_n^G})(z - A_r^{(\omega)})^{-1}\delta_g\big\|_2 \\ &\leq \|(A_r^{(\omega)} - A^{(\omega)})\pi_{B_n^G}(z - A_r^{(\omega)})^{-1}\delta_g\|_2 + \\ &\qquad\qquad (|z| + b_r)\|\pi_{G\setminus B_n^G}(z - A_r^{(\omega)})^{-1}\delta_g\|_2, \end{aligned} \tag{2.19}$$

where we used $\|z - A_r^{(\omega)}\|_2 \leq |z| + b_r$ for all $\omega \in \Omega$. In order to estimate the expectation of the last summand in (2.19), note that the boundedness of $(z - A_r^{(\omega)})^{-1}$ implies $(z - A_r^{(\omega)})^{-1}\delta_g \in \ell^2(G)$. Hence we have

$$\lim_{n\to\infty} \|\pi_{G\setminus B_n^G}(z - A_r^{(\omega)})^{-1}\delta_g\|_2 = 0$$

and for all $r \in \mathbb{N}$. Moreover, for all $n \in \mathbb{N}$ we obtain

$$\|\pi_{G\setminus B_n^G}(z - A_r^{(\omega)})^{-1}\delta_g\|_2 \leq |\Im(z)|^{-1}.$$

Therefore, Lebesgues theorem yields

$$\lim_{n\to\infty} \mathbb{E}\left(\|\pi_{G\setminus B_n^G}(z - A_r)^{-1}\delta_g\|_2\right) = 0.$$

Thus we can find $\tilde{n} = \tilde{n}(g, r)$ such that

$$\mathbb{E}\left(\|\pi_{G\setminus B_{\tilde{n}}^G}(z - A_r)^{-1}\delta_g\|_2\right) \leq \frac{1}{r(|z| + b_r)}. \tag{2.20}$$

In order to control the expectation of the first summand in (2.19), we make use of Jensen's inequality and Lemma 2.18:

$$\begin{aligned}
&\left(\mathbb{E}\left(\|(A_r - A)\pi_{B_n^G}(z - A_r)^{-1}\delta_g\|_2\right)\right)^2 \\
&\qquad \leq \mathbb{E}\left(\|(A_r - A)\pi_{B_n^G}(z - A_r)^{-1}\delta_g\|_2^2\right) \\
&\qquad \leq \|(z - A_r)^{-1}\|_\infty^2 \mathbb{E}\left(\|(A_r - A)\delta_{\mathrm{id}}\|_2^2\right).
\end{aligned} \tag{2.21}$$

Furthermore, by definition we have

$$\sum_{g\in G} |a^{(\omega)}(g, \mathrm{id}) - a_r^{(\omega)}(g, \mathrm{id})| \leq \sum_{g\in G} |a^{(\omega)}(g, \mathrm{id})| = \|A^{(\omega)}\delta_{\mathrm{id}}\|_1.$$

Using (2.18), this gives integrable bounds for the following application of Lebesgue's theorem:

$$\begin{aligned}
\lim_{r\to\infty} \mathbb{E}\left(\|(A - A_r)\delta_{\mathrm{id}}\|_1^2\right) &= \lim_{r\to\infty} \mathbb{E}\left(\left(\sum_{g\in G} |a(g, \mathrm{id}) - a_r(g, \mathrm{id})|\right)^2\right) \\
&= \mathbb{E}\left(\left(\sum_{g\in G} \lim_{r\to\infty} |a(g, \mathrm{id}) - a_r(g, \mathrm{id})|\right)^2\right) = 0.
\end{aligned}$$

This, the fact that for each ω we have $\|(z - A_r^{(\omega)})^{-1}\| \leq |\Im(z)|^{-1}$ and (2.21) imply

$$\lim_{r\to\infty} \mathbb{E}\left(\|(A_r - A)\pi_{B_n^G}(z - A_r)^{-1}\delta_g\|_2\right) = 0.$$

The last equality, (2.20) and (2.19) yield

$$\lim_{r\to\infty} \mathbb{E}\left(\|(z-A)\phi_{g,\tilde{n}(g,r),r} - \delta_g\|_2\right) = 0,$$

which is L^1-convergence. This implies the existences of a sequence $(r_k)_{k\in\mathbb{N}}$ such that for $\mathbb{P}$-almost all ω we have

$$\lim_{k\to\infty} \|(z-A^{(\omega)})\phi^{(\omega)}_{g,\tilde{n}(g,r_k),r_k} - \delta_g\|_2 = 0,$$

which proves essential self-adjointness of $A^{(\omega)}$. ■

Remark 2.20. In the theory of random operators it is usual that certain properties can not be verified for all, but only for almost all realizations. This is for instance the case in Theorem 2.19, where we showed essential self-adjointness almost surely. Similarly, we often obtain that the operator in question is defined on $C_c(G)$ only for almost all ω. Let us briefly discuss two ways in order to deal with the realization where we are not able to verify the desired properties.

Let A be a random operator on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let (P) be a property which is only fulfilled on $\tilde{\Omega}$, a set of full measure. In this situation we can restrict our probability space to $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbb{P}})$, where $\tilde{\mathcal{A}} := \{D \cap \tilde{\Omega} \mid D \in \mathcal{A}\}$ and $\tilde{\mathbb{P}} \colon \tilde{\mathcal{A}} \to [0,1]$ is given by $\tilde{\mathbb{P}}(D \cap \tilde{\Omega}) := \mathbb{P}(D)$ for all $D \in \mathcal{A}$. Thus the operator $\tilde{A} : \tilde{\Omega} \to L(\ell^2(G))$, $\omega \mapsto A^{(\omega)}$ has property (P) for *all* $\omega \in \tilde{\Omega}$. Of course, proceeding this way, one has to keep in mind that all proven results on this probability space may only hold with probability one on the original one.

In this book, we rather pursue a second way to deal with realizations of an operator A where a certain property (P) is not satisfied. We redefine the operator A on the set of measure zero as the identity if the property (P) holds for the identity. This is done for instance in (4.6) and (6.1). In this way we obtain a "new" operator on the original probability space, where (P) is satisfied for all ω.

In particular, we will oftentimes pass from a random operator on the domain $C_c(G)$ which is almost surely self-adjoint, to a proper random operator which is self-adjoint for all realizations.

When dealing with resolvents of self-adjoint random operators it is worth knowing that they are measurable. This fact is provided by the next Lemma.

Lemma 2.21. *Let $A = (A^{(\omega)})_{\omega\in\Omega}$ be a proper random operator on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that for each $\omega \in \Omega$ the operator $A^{(\omega)}$ is self-adjoint. Then for all $z \in \mathbb{C} \setminus \mathbb{R}$ the mapping*

$$(z - A)^{-1} : \Omega \to \mathcal{L}(\ell^2(G)), \qquad \omega \mapsto (z - A^{(\omega)})^{-1}$$

is a proper random operator and in particular measurable.

Proof. For each $\omega \in \Omega$ we denote by $a^{(\omega)}(x, y) := \langle \delta_x, A^{(\omega)}\delta_y\rangle$ the matrix elements of $A^{(\omega)}$. We fix $z \in \mathbb{C} \setminus \mathbb{R}$. Define for $r > 0$ and $x, y \in G$

$$a_r^{(\omega)}(x, y) := \begin{cases} a^{(\omega)}(x, y) & \text{if } |a^{(\omega)}(x, y)| \le r \text{ and } x, y \in B_r^G, \\ 0 & \text{otherwise.} \end{cases}$$

Denote by $A_r^{(\omega)} \colon \ell^2(G) \to \ell^2(G)$ the operator with these matrix elements. Note that this operator is not translation invariant in distribution and has only finitely many non-zero matrix elements. Besides this, for each $\omega \in \Omega$ and $r > 0$ the operator $A_r^{(\omega)}$ is self-adjoint.

Since A is assumed to be a proper random operator, for all $x, y \in G$ the mappings $\Omega \ni \omega \mapsto a^{(\omega)}(x, y)$ are measurable.

Note that there are only finitely many matrix elements of $((z - A_r^{(\omega)})^{-1})$ which depend on ω. By Cramer's rule, each such element is a quotient of polynomials of measurable functions and hence measurable, too. Hence, for given $\phi, \psi \in \ell^2(G)$ the mapping

$$\omega \mapsto \left\langle \phi, (z - A_r^{(\omega)})^{-1}\psi \right\rangle$$

is measurable. This implies that $(z - A_r)^{-1}$ is a proper random operator and in particular measurable. Thus, for arbitrary $\phi \in \ell^2(G)$ the mapping $\Omega \ni \omega \mapsto (z - A_r^{(\omega)})^{-1}\phi \in \ell^2(G)$ in norm-measurable. Here we used that $\ell^2(G)$ is separable.

Our next aim is to prove that these mappings converge strongly to the resolvents of A, which will imply that they are measurable, too. Note that for all $\omega \in \Omega$ and $r > 0$ we have by self-adjointness

$$\|(z - A^{(\omega)})^{-1}\| \le |\Im(z)|^{-1} \quad \text{and} \quad \|(z - A_r^{(\omega)})^{-1}\| \le |\Im(z)|^{-1}.$$

We fix some $\omega \in \Omega$, $\xi \in \ell^2(G)$ and $\kappa > 0$. Using the above estimates, triangle inequality and the second resolvent identity we obtain for any $\psi \in \ell^2(G)$

$$\begin{aligned}
&\big\|\big((z - A^{(\omega)})^{-1} - (z - A_r^{(\omega)})^{-1}\big)\xi\big\|_2 \\
&\leq \big\|\big((z - A^{(\omega)})^{-1} - (z - A_r^{(\omega)})^{-1}\big)\psi\big\|_2 + 2\|\xi - \psi\|_2/|\Im(z)| \\
&\leq \|(z - A_r^{(\omega)})^{-1}(A^{(\omega)} - A_r^{(\omega)})(z - A^{(\omega)})^{-1}\psi\|_2 + 2\|\xi - \psi\|_2/|\Im(z)| \\
&\leq \big(\|(A^{(\omega)} - A_r^{(\omega)})(z - A^{(\omega)})^{-1}\psi\|_2 + 2\|\xi - \psi\|_2\big)/|\Im(z)|.
\end{aligned}$$

Since $A^{(\omega)}$ is self-adjoint and $C_{\mathrm{c}}(G)$ is a core, we can choose this ψ according to Lemma 2.15, i.e. we have

$$\|\xi - \psi\|_2 < \kappa \qquad \text{and} \qquad \phi := (z - A^{(\omega)})^{-1}\psi \in C_{\mathrm{c}}(G).$$

For each $r \in \mathbb{N}$ we set

$$\rho(r) := \max\{s \leq r \mid a^{(\omega)}(x, y) = a_r^{(\omega)}(x, y) \text{ for all } x, y \in B_s^G\}$$

and obtain $\rho(r) \to \infty$ if $r \to \infty$. Using this quantity we calculate

$$\begin{aligned}
\|(A^{(\omega)} - A_r^{(\omega)})\phi\|_2^2 &= \sum_{x \in G \setminus B_{\rho(r)}^G} \Big| \sum_{y \in \mathrm{spt}(\phi)} (a^{(\omega)}(x, y) - a_r^{(\omega)}(x, y))\phi(y) \Big|^2 \\
&\leq \|\phi\|_\infty^2 |\mathrm{spt}(\phi)| \sum_{x \in G \setminus B_{\rho(r)}^G} \sum_{y \in \mathrm{spt}(\phi)} |a^{(\omega)}(x, y) - a_r^{(\omega)}(x, y)|^2 \\
&\leq \|\phi\|_\infty^2 |\mathrm{spt}(\phi)| \sum_{y \in \mathrm{spt}(\phi)} \sum_{x \in G \setminus B_{\rho(r)}^G} |a^{(\omega)}(x, y)|^2.
\end{aligned}$$

As $C_{\mathrm{c}}(G) \subseteq D(A^{(\omega)})$ we have $A^{(\omega)}\delta_y \in \ell^2(G)$. This implies for arbitrary $y \in G$:

$$\lim_{r \to \infty} \sum_{x \in G \setminus B_{\rho(r)}^G} |a^{(\omega)}(x, y)|^2 = 0$$

and thus

$$\lim_{r \to \infty} \|(A^{(\omega)} - A_r^{(\omega)})\phi\|_2 = 0.$$

Therefore, we have

$$\limsup_{r\to\infty}\left\|\left((z-A^{(\omega)})^{-1}-(z-A_r^{(\omega)})^{-1}\right)\xi\right\|_2 \le \frac{2\kappa}{|\Im(z)|},$$

which shows the desired convergence of the resolvents, as $\kappa > 0$ was arbitrary. This proves that $(z-A)^{-1}$ is a proper random operator and in particular that $(z-A)^{-1}$ is measurable. ■

We apply the previous lemma to obtain another result pointing in this direction.

Theorem 2.22. *Let $A = (A^{(\omega)})$ be a proper random operator which is self-adjoint for all $\omega \in \Omega$, let $\kappa > 0$ and $z \in \mathbb{C}\setminus\mathbb{R}$. Then there exists $n \in \mathbb{N}$ and a random vector $\psi\colon \Omega \to \ell^2(G)$ such that*

$$\mathbb{E}\left(\|\delta_{\mathrm{id}}-\psi\|_2\right) \le \kappa \quad \textit{and} \quad \operatorname{spt}\left((z-A^{(\omega)})^{-1}\psi(\omega)\right) \subseteq B_n^G.$$

for all $\omega \in \Omega$.

Proof. For each $n \in \mathbb{N}$ we denote the set

$$\left\{\omega \in \Omega \;\middle|\; \exists f \in \ell^2(G)\colon \operatorname{spt}\left((z-A^{(\omega)})^{-1}f\right) \subseteq B_n^G, \|\delta_{\mathrm{id}}-f\|_2 \le \frac{\kappa}{2}\right\}$$

by $M_{n,\kappa}$. In order to verify the measurability of $M_{n,\kappa} \subseteq \Omega$ we claim that one can rewrite this set in the following way

$$M_{n,\kappa} = \bigcap_{m\in\mathbb{N}} \bigcup_{\substack{f\in D\\ \|f-\delta_{\mathrm{id}}\|_2\\ <\frac{\kappa}{2}+\frac{1}{m}}} \bigcap_{g\in G\setminus B_n^G} \left\{\omega \in \Omega \;\middle|\; |\langle \delta_g, (z-A^{(\omega)})^{-1}f\rangle| < m^{-1}\right\}, \tag{2.22}$$

where

$$D := \{\phi \in C_{\mathrm{c}}(G) \mid \Im(\phi(x)) \in \mathbb{Q},\ \Re(\phi(x)) \in \mathbb{Q} \text{ for all } x \in G\}.$$

Note that D is countable and dense in $\ell^2(G)$. By Lemma 2.21 we know that the mapping $\omega \mapsto \langle \delta_g, (z-A^{(\omega)})^{-1}f\rangle$ is measurable. As level sets of measurable functions are measurable and the expression in (2.22) contains only unions and intersections over countable index sets, equality (2.22) implies measurability of $M_{n,\kappa}$.

In order to prove (2.22), let us first verify the inclusion "$\subseteq$". To this end choose some $\omega \in M_{n,\kappa}$ and let $f \in \ell^2(G)$ be the corresponding element with the desired properties. Then one has $\|\delta_{\mathrm{id}} - f\|_2 \leq \kappa/2$ and for all $g \in G \setminus B_n^G$ that $\left\langle \delta_g, (z - A^{(\omega)})^{-1} f \right\rangle = 0$. Furthermore, since D is dense in $\ell^2(G)$, we can find for all $m \in \mathbb{N}$ an element $f_m \in D$ with

$$\|f - f_m\|_2 < \frac{1}{m} \min\{1, |\Im(z)|\},$$

such that we get

$$\|\delta_{\mathrm{id}} - f_m\|_2 \leq \|\delta_{\mathrm{id}} - f\|_2 + \|f - f_m\|_2 < \kappa/2 + m^{-1}.$$

Besides this, by Cauchy-Schwarz inequality we obtain for all $g \in G \setminus B_n^G$:

$$\left|\left\langle \delta_g, (z - A^{(\omega)})^{-1} f_m \right\rangle\right| \leq \left|\left\langle \delta_g, (z - A^{(\omega)})^{-1} (f_m - f) \right\rangle\right| < m^{-1}.$$

This proves the inclusion "$\subseteq$". Let us check the reverse inclusion "$\supseteq$". To this end let ω be an element of the set on the right hand side of (2.22). Hence, for all $m \in \mathbb{N}$ there exists $f_m \in D$ with $\|f_m - \delta_{\mathrm{id}}\|_2 < \kappa/2 + m^{-1}$ and $|((z - A^{(\omega)})^{-1} f_m)(g)| < m^{-1}$ for all $g \in G \setminus B_n^G$. For arbitrary $m \in \mathbb{N}$ we have

$$\|f_m\|_2 \leq \|f_m - \delta_{\mathrm{id}}\|_2 + 1 \leq \kappa/2 + 2. \tag{2.23}$$

Thus, for all $g \in G$ we have that $(f_m(g))_{m \in \mathbb{N}}$ is a bounded sequence and hence contains a convergent subsequence. Using a diagonal sequence we obtain a subsequence such that $(f_{m_k}(g))_{k \in \mathbb{N}}$ converges for all $g \in G$. We denote the pointwise limit of the sequence (f_{m_k}) by f. Fatou's Lemma yields

$$\sum_{g \in G} \lim_{k \to \infty} |f_{m_k}(g)|^2 \leq \liminf_{k \to \infty} \sum_{g \in G} |f_{m_k}(g)|^2 \leq (\kappa/2 + 2)^2.$$

This implies $f \in \ell^2(G)$. Furthermore, we have

$$\|f - \delta_{\mathrm{id}}\|_2 \leq \kappa/2 \qquad \text{and} \qquad \operatorname{spt}\left((z - A^{(\omega)})^{-1} f\right) \subseteq B_n^G,$$

which shows that $\omega \in M_{n,\kappa}$. Thus, we proved equality (2.22).

Note that for any $n \in \mathbb{N}$ we have $M_{n,\kappa} \subseteq M_{n+1,\kappa}$. For each $\omega \in \Omega$, the compactly supported functions $C_c(G)$ form a core for $A^{(\omega)}$. Lemma 2.15 shows that for each $\omega \in \Omega$ there exists $\psi \in \ell^2(G)$ with

$$\|\delta_{\mathrm{id}} - \psi\|_2 < \kappa/2 \qquad \text{and} \qquad (z - A^{(\omega)})^{-1}\psi \in C_c(G).$$

Hence, for arbitrary $\omega \in \Omega$ we can find $\bar{n} = \bar{n}(\kappa, \omega) \in \mathbb{N}$ such that $\omega \in M_{\bar{n},\kappa}$ or equivalently

$$\Omega = \bigcup_{n \in \mathbb{N}} M_{n,\kappa}.$$

This immediately yields $\lim_{n\to\infty} \mathbb{P}(M_{n,\kappa}) = 1$ and hence allows to find $n = n(\kappa)$ such that $\mathbb{P}(M_{n,\kappa}) > 1 - \kappa/2$. From now on we fix this n.

Finally, we define

$$\tilde{M}_{n,\kappa} := \left\{(\omega, f) \in M_{n,\kappa} \times B(\kappa) \mid \operatorname{spt}\big((z - A^{(\omega)})^{-1} f\big) \subseteq B_n^G\right\},$$

where

$$B(\kappa) := \left\{f \in \ell^2(G) \mid \|f - \delta_{\mathrm{id}}\|_2 \leq \kappa/2\right\}.$$

We defer the investigation of the measurability of the set $\tilde{M}_{n,\kappa}$ to the end of this proof. Assuming that $\tilde{M}_{n,\kappa}$ is measurable, the existence of a mapping $\tilde{\psi} : M_{n,k} \to \ell^2(G)$ which is norm-measurable and fulfills

$$\tilde{\psi}(\omega) \in B(\kappa) \quad \text{and} \quad \operatorname{spt}\big((z - A^{(\omega)})^{-1}\tilde{\psi}(\omega)\big) \subseteq B_n^G$$

for all $\omega \in M_{n,\kappa}$, follows directly from a selection theorem due to R. J. Aumann, see [AB06, Corollary 18.27]. Recall that norm-measurability means that $\tilde{\psi}$ is measurable with respect to the sigma-algebra $\mathcal{A}|_{M_{n,k}}$ on $M_{n,k}$ and the Borel sigma-algebra on $B(\kappa)$. Here we use the notion $\mathcal{A}|_{M_{n,k}} := \{C \cap M_{n,k} \mid C \in \mathcal{A}\}$. The desired vector $\psi : \Omega \to \ell^2(G)$ is given by

$$\psi(\omega) := \begin{cases} \tilde{\psi}(\omega) & \text{if } \omega \in M_{n,\kappa} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, ψ is norm-measurable and a random vector. Moreover, we have

$$\mathbb{E}\left(\|\delta_{\mathrm{id}} - \psi\|_2\right) \leq \int_{M_{n,\kappa}} \frac{\kappa}{2}\,\mathrm{d}\mathbb{P}(x) + \int_{\Omega\setminus M_{n,\kappa}} \|\delta_{\mathrm{id}}\|_2\,\mathrm{d}\mathbb{P}(x) \leq \kappa.$$

and for all $\omega \in \Omega$:

$$\mathrm{spt}\big((z - A^{(\omega)})^{-1}\psi(\omega)\big) \subseteq B_n^G.$$

Thus it remains to verify the measurability of $\tilde{M}_{n,\kappa}$. To this end, we rewrite this set, similar as before the set $M_{n,k}$, as

$$\tilde{M}_{n,\kappa} = \bigcap_{m\in\mathbb{N}} \bigcap_{g\in G\setminus B_n^G} \Big\{(\omega, f) \in \Omega \times B(\kappa) \mid \Big\langle(\bar{z} - A^{(\omega)})^{-1}\delta_g, f\Big\rangle \leq \frac{1}{m}\Big\}.$$

Thence, to see the measurability of $\tilde{M}_{n,\kappa}$, we need to show that the function $(\omega, f) \mapsto \big\langle(\bar{z} - A^{(\omega)})^{-1}\delta_g, f\big\rangle$ mapping elements from $\Omega \times \ell^2(G)$ to $\mathbb{C}$ is measurable. Here the space $\Omega \times \ell^2(G)$ is equipped with the product sigma-algebra $\mathcal{A} \otimes \mathcal{B}(\ell^2(G))$. But as the scalar product is continuous we only have to show that the mapping $V : \Omega \times \ell^2(G) \to \ell^2(G) \times \ell^2(G)$ where $V(\omega, f) := (V_1(\omega), V_2(f))$ and $V_1(\omega) := (\bar{z} - A^{(\omega)})^{-1}\delta_g$ and $V_2(f) = f$ is measurable, with respect to the associated product sigma-algebras. As the involved operator in V_1 is $(\bar{z} - A^{(\omega)})^{-1}$ the measurability of V follows from Lemma 2.21. This finishes the proof of the theorem. ■

2.3 Eigenvalue counting function

In this section we define the eigenvalue counting function and prove elementary properties. Later in this book, the results obtained here will be applied for operators on the Hilbert space $\ell^2(Q)$ where Q is some finite subset of a finitely generated group. Note that the results in this section are rather basic knowledge. For the sake of the reader we provide the proofs thereof.

The eigenvalue counting function of self-adjoint operators on finite dimensional Hilbert spaces is a distribution functions which encodes the distribution of the spectrum on the real axis. The precise definition reads as follows.

Definition 2.23. For a self-adjoint operator A on a finite dimensional Hilbert space $\mathcal{H}$ we define its *(cumulative) eigenvalue counting function* $\mathfrak{e}(A)\colon \mathbb{R} \to [0,\infty)$ by setting for $\lambda \in \mathbb{R}$:

$$\mathfrak{e}(A)(\lambda) := |\{\text{eigenvalues of } A \text{ not larger than } \lambda\}|,$$

where the eigenvalues of A are counted according to their multiplicity. The *normalized eigenvalue counting function* $\mathfrak{n}(A)\colon \mathbb{R} \to [0,1]$ is given by

$$\mathfrak{n}(A)(\lambda) := \frac{\mathfrak{e}(A)(\lambda)}{\dim(\mathcal{H})}$$

The next lemma controls the eigenvalue counting function for perturbed operators.

Lemma 2.24. *Let A and C be self-adjoint operators on a finite dimensional Hilbert space $\mathcal{H}$, then we have for all $\lambda \in \mathbb{R}$:*

$$|\mathfrak{e}(A)(\lambda) - \mathfrak{e}(A+C)(\lambda)| \leq \operatorname{rank}(C).$$

Before proving this lemma we mention some general facts concerning self-adjoint operators on finite dimensional Hilbert spaces. As in the setting of the Lemma let A and C be operators on the Hilbert space $\mathcal{H}$ and let $n := \dim(\mathcal{H})$ be finite. We fix a basis of $\mathcal{H}$. Thus, the operators A and C are hermitian matrices of dimension $n \times n$. Since $\mathfrak{e}(A)$ is defined as the eigenvalue counting function, we are interested in the relation between the size of the eigenvalues of A and $A + C$. This will be given with the help of the min-max principle of Courant and Fischer: let $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$ be the eigenvalues of A, then

$$\lambda_k(A) = \min_{\psi_1,\cdots,\psi_{n-k}\in\mathbb{C}^n} \max_{\substack{\phi\perp\psi_1,\cdots,\psi_{n-k}\\ \|\phi\|=1}} \langle \phi, A\phi\rangle \tag{2.24}$$

and

$$\lambda_k(A) = \max_{\psi_1,\cdots,\psi_{k-1}\in\mathbb{C}^n} \min_{\substack{\phi\perp\psi_1,\cdots,\psi_{k-1}\\ \|\phi\|=1}} \langle \phi, A\phi\rangle. \tag{2.25}$$

For the proof of this see for example [HJ90], where it is also stressed that the minimizing respectively maximizing vectors are exactly the eigenvectors. To be precise, denote for $k = 1, \dots, n$ by f_k the normalized eigenvector for the eigenvalue $\lambda_k(A)$. Then we have for $k \in \{1, \dots, n\}$:

$$\lambda_k(A) = \min_{\substack{\phi \perp f_1, \cdots, f_{k-1} \\ \|\phi\|=1}} \langle \phi, A\phi \rangle = \max_{\substack{\phi \perp f_{k+1}, \cdots, f_n \\ \|\phi\|=1}} \langle \phi, A\phi \rangle. \tag{2.26}$$

Proof of Lemma 2.24. For a given vector $\xi \in \mathbb{C}^n$ the matrix $B := \xi\xi^*$ is hermitian and of rank one. Given an arbitrary element $s \in \mathbb{C}$ the equality $\langle \phi, sB\phi \rangle = s\langle \phi, \xi \rangle \langle \xi, \phi \rangle = s|\langle \phi, \xi \rangle|^2$ holds for all $\phi \in \mathbb{C}^n$. Using (2.25) we get

$$\begin{aligned}\lambda_k(A + sB) &= \max_{\psi_1, \cdots, \psi_{k-1} \in \mathbb{C}^n} \min_{\substack{\phi \perp \psi_1, \cdots, \psi_{k-1} \\ \|\phi\|=1}} \langle \phi, A\phi \rangle + s\langle \phi, B\phi \rangle \\ &= \max_{\psi_1, \cdots, \psi_{k-1} \in \mathbb{C}^n} \min_{\substack{\phi \perp \psi_1, \cdots, \psi_{k-1} \\ \|\phi\|=1}} \langle \phi, A\phi \rangle + s|\langle \phi, \xi \rangle|^2.\end{aligned}$$

We estimate the maximum from below by setting $\psi_i = f_i$, $i = 1, \dots, k-2$ and $\psi_{k-1} = \xi$, where $f_i, i = 1, \dots, k-1$ are the normalized eigenvectors. This yields

$$\begin{aligned}\lambda_k(A + sB) &\geq \min_{\substack{\phi \perp f_1, \cdots, f_{k-2}, \xi \\ \|\phi\|=1}} \langle \phi, A\phi \rangle + s|\langle \phi, \xi \rangle|^2 \\ &\geq \min_{\substack{\phi \perp f_1, \cdots, f_{k-2} \\ \|\phi\|=1}} \langle \phi, A\phi \rangle = \lambda_{k-1}(A),\end{aligned}$$

where the last equality holds by (2.26). Similarly, by using the relation (2.24) we get an upper bound for the k-th eigenvalue of $A + sB$:

$$\begin{aligned}\lambda_k(A + sB) &= \min_{\psi_1, \cdots, \psi_{n-k} \in \mathbb{C}^n} \max_{\substack{\phi \perp \psi_1, \cdots, \psi_{n-k} \\ \|\phi\|=1}} \langle \phi, A\phi \rangle + s|\langle \phi, \xi \rangle|^2 \\ &\leq \max_{\substack{\phi \perp f_{k+2}, \cdots, f_n, \xi \\ \|\phi\|=1}} \langle \phi, A\phi \rangle \leq \lambda_{k+1}(A).\end{aligned}$$

This proves for an arbitrary matrix B of rank one and $s \in \mathbb{C}$ the inequalities

$$\lambda_{k-1}(A) \leq \lambda_k(A + sB) \leq \lambda_{k+1}(A). \tag{2.27}$$

In order to generalize this, let C be a hermitian matrix of rank m. With the eigendecomposition we get $C = \sum_{i=1}^n s_i \xi_i \xi_i^*$, where

s_i are the eigenvalues of C and ξ_i the normalized eigenvectors for $i = 1, \ldots, n$. Since the rank of C equals m, there are exactly m non-zero eigenvalues in the spectrum. Note that we count the eigenvalues according to their multiplicities. Thus, the above sum consists of m summands. Next we show by induction that

$$\lambda_{k-m}(A) \leq \lambda_k\Big(A + \sum_{i=1}^{m} s_i B_i\Big) \leq \lambda_{k+m}(A), \tag{2.28}$$

where for each $i = 1, \ldots, m$ the matrix B_i is given by $B_i = \xi_i \xi_i^*$. From (2.27) we know that this is true for $m = 1$. The inductive step follows from

$$\begin{aligned}\lambda_{k-m-1}(A) &\leq \lambda_{k-1}\Big(A + \sum_{i=1}^{m} s_i B_i\Big) \leq \lambda_k\Big(A + \sum_{i=1}^{m+1} s_i B_i\Big) \\ &\leq \lambda_{k+1}\Big(A + \sum_{i=1}^{m} s_i B_i\Big) \leq \lambda_{k+m+1}(A).\end{aligned}$$

Finally, we translate (2.28) into the language of the eigenvalue counting functions. Let $\lambda \in \mathbb{R}$ be given. Setting $k := \mathfrak{e}(A+C)(\lambda)$, leads to $\lambda_k(A+C) \leq \lambda < \lambda_{k+1}(A+C)$, which implies

$$\lambda_{k-m}(A) \leq \lambda < \lambda_{k+m+1}(A)$$

by using (2.28). This yields $k - m \leq \mathfrak{e}(A)(\lambda) \leq k + m$, and hence

$$\mathfrak{e}(A)(\lambda) - m \leq \mathfrak{e}(A+C)(\lambda) \leq \mathfrak{e}(A)(\lambda) + m$$

for all $\lambda \in \mathbb{R}$. The claim follows. ■

Applying this lemma to operators defined on a finite dimensional Hilbert space and their projections on a subspace leads to the following lemma, which has already been proven in [LS05].

Lemma 2.25. *Let $\mathcal{H}$ be a finite dimensional Hilbert space and $\mathcal{U}$ a subspace of $\mathcal{H}$. If $i : \mathcal{U} \to \mathcal{H}$ is the inclusion and $p : \mathcal{H} \to \mathcal{U}$ the orthogonal projection, we have*

$$|\mathfrak{e}(A)(\lambda) - \mathfrak{e}(pAi)(\lambda)| \leq 4 \cdot \operatorname{rank}(1 - ip)$$

for all self-adjoint operators A on $\mathcal{H}$ and all energies $\lambda \in \mathbb{R}$. Note that here $1 : \mathcal{H} \to \mathcal{H}$ is the identity.

Proof. We set $P := ip : \mathcal{H} \to \mathcal{H}$ and use the triangle inequality to obtain

$$\begin{aligned}|\mathfrak{e}(A)(\lambda) - \mathfrak{e}(pAi)(\lambda)| \\ \leq |\mathfrak{e}(A)(\lambda) - \mathfrak{e}(PAP)(\lambda)| + |\mathfrak{e}(PAP)(\lambda) - \mathfrak{e}(pAi)(\lambda)|. \quad (2.29)\end{aligned}$$

With the help of the equality

$$A - PAP = (1-P)AP + PA(1-P) + (1-P)A(1-P)$$

and Lemma 2.24 we get

$$\begin{aligned}&|\mathfrak{e}(A)(\lambda) - \mathfrak{e}(PAP)(\lambda)| \\ &\quad \leq \operatorname{rank}(PAP - A) \\ &\quad = \operatorname{rank}((1-P)AP + PA(1-P) + (1-P)A(1-P)) \\ &\quad \leq 3\operatorname{rank}(1-P). \quad (2.30)\end{aligned}$$

Let $\mathcal{U}^\perp$ denote the orthogonal complement of $\mathcal{U}$ and define $0_{\mathcal{U}^\perp} : \mathcal{U}^\perp \to \mathcal{U}^\perp$ with $f \mapsto 0$. It is obvious that

$$PAP = ipAip = (pAi) \oplus 0_{\mathcal{U}^\perp}$$

holds true. Therefore, we have

$$\begin{aligned}|\mathfrak{e}(PAP)(\lambda) - \mathfrak{e}(pAi)(\lambda)| &= |\mathfrak{e}((pAi) \oplus 0_{\mathcal{U}^\perp})(\lambda) - \mathfrak{e}(pAi)(\lambda)| \\ &= |\mathfrak{e}(0_{\mathcal{U}^\perp})(\lambda)| \leq \dim(\mathcal{U}^\perp).\end{aligned}$$

Note that the dimension of $\mathcal{U}^\perp$ equals the rank of $(1-P)$. Together with (2.30) and (2.29) we obtain the statement of the lemma. ∎

2.4 Convergence of measures

In this theses many results concern the convergence of certain probability measures on $\mathbb{R}$. In the following we define different types of convergence for probability measures or their distribution functions, respectively. Furthermore, we discuss necessary and sufficient conditions to verify them. By $\mathcal{B}(\mathbb{R})$ we denote the Borel sigma-algebra on $\mathbb{R}$.

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ be given. Then we set

$$\begin{aligned} C(\mathbb{R},\mathbb{K}) &:= \{f : \mathbb{R} \to \mathbb{K} \mid f \text{ is continuous}\}, \\ C_{\mathrm{b}}(\mathbb{R},\mathbb{K}) &:= \{f \in C(\mathbb{R},\mathbb{K}) \mid f \text{ is bounded}\}, \\ C_{\mathrm{c}}(\mathbb{R},\mathbb{K}) &:= \{f \in C(\mathbb{R},\mathbb{K}) \mid f \text{ is compactly supported}\}, \text{ and} \\ C_0(\mathbb{R},\mathbb{K}) &:= \{f \in C(\mathbb{R},\mathbb{K}) \mid \text{for all } \varepsilon > 0 \ \exists \text{ compact } B \subseteq \mathbb{K} \\ &\qquad\qquad \text{such that for all } x \in \mathbb{K} \setminus B \text{ one has } |f(x)| \le \varepsilon\}. \end{aligned}$$

If $\mathbb{K} = \mathbb{C}$ we write $C(\mathbb{R}) := C(\mathbb{R},\mathbb{C})$, $C_{\mathrm{b}}(\mathbb{R}) := C_{\mathrm{b}}(\mathbb{R},\mathbb{C})$, $C_{\mathrm{c}}(\mathbb{R}) := C_{\mathrm{c}}(\mathbb{R},\mathbb{C})$ and $C_0(\mathbb{R}) := C_0(\mathbb{R},\mathbb{C})$. For a function $f : \mathbb{R} \to \mathbb{R}$ we denote by $\mathrm{cont}(f)$ the subset of $\mathbb{R}$ where f is continuous and by $\mathrm{disc}(f)$ the set of points of discontinuity.

Beside these spaces we will need

$$B(\mathbb{R}) := \{f\colon \mathbb{R} \to \mathbb{R} \mid f \text{ right-continuous and bounded}\}, \tag{2.31}$$

which we will equip with supremum norm. Therefore, $B(\mathbb{R})$ is a Banach space containing the distribution functions of probability measures on $\mathbb{R}$. Let (ϕ_n) be a sequence of functions with $\phi_n \in B(\mathbb{R})$, $n \in \mathbb{N}$. If (ϕ_n) converges in $B(\mathbb{R})$ to some $\phi \in B(\mathbb{R})$, then these functions converge uniformly i.e. with respect to the supremum norm. An easy fact related to this gives the following lemma.

Lemma 2.26. *Let (ϕ_n) be a sequence of distribution functions of probability measures, which converge uniformly to some $\phi : \mathbb{R} \to \mathbb{R}$. Then ϕ is a distribution function of a probability measure as well.*

Proof. The monotonicity of ϕ is clear since we have for any $\lambda' \le \lambda$

$$\phi(\lambda) - \phi(\lambda') = \lim_{n\to\infty} (\phi_n(\lambda) - \phi_n(\lambda')) \ge 0,$$

as the functions ϕ_n are monotone. By the uniform convergence, the right-continuity of the functions ϕ_n carries over to the limit ϕ. In fact we use the uniform convergence to interchange the limits in the computation:

$$\lim_{\lambda'\searrow\lambda} \phi(\lambda') = \lim_{\lambda'\searrow\lambda} \lim_{n\to\infty} \phi_n(\lambda') = \lim_{n\to\infty} \lim_{\lambda'\searrow\lambda} \phi_n(\lambda') = \phi(\lambda).$$

Another application of uniform convergence yields

$$\lim_{\lambda\to\infty} \phi(\lambda) = \lim_{\lambda\to\infty} \lim_{n\to\infty} \phi_n(\lambda) = \lim_{n\to\infty} \lim_{\lambda\to\infty} \phi_n(\lambda) = 1.$$

Similarly, one obtains $\lim_{\lambda\to-\infty}\phi(\lambda)=0$. Hence, ϕ is the distribution function of a probability measure. ■

Definition 2.27. Let $\mu,\mu_1,\mu_2,\dots$ be probability measures on the measurable space $(\mathbb{R},\mathcal{B}(\mathbb{R}))$. We say that $(\mu_n)_{n\in\mathbb{N}}$ converges weakly to μ (and write $\mu=\text{w-lim}_{n\to\infty}\mu_n$) if for any $f\in C_\mathrm{b}(\mathbb{R},\mathbb{R})$ one has

$$\int_\mathbb{R} f(x)\mathrm{d}\mu_n(x)\to\int_\mathbb{R} f(x)\mathrm{d}\mu(x),\qquad n\to\infty.$$

The portmanteau theorem gives equivalent formulations of weak convergence. It can be found for instance in [Kle08, Bil99].

Theorem 2.28 (portmanteau theorem)**.** *Let $\mu,\mu_1,\mu_2,\dots$ be probability measures on the measurable space $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ and let $\phi,\phi_1,\phi_2,\dots$ be the associated distribution functions. Then the following are equivalent:*

(i) $\mu=\text{w-lim}_{n\to\infty}\mu_n$*;*

(ii) $\lim_{n\to\infty}\int_\mathbb{R} f(x)\mathrm{d}\mu_n(x)=\int_\mathbb{R} f(x)\mathrm{d}\mu(x)$ *for all bounded and Lipschitz continuous f;*

(iii) $\lim_{n\to\infty}\int_\mathbb{R} f(x)\mathrm{d}\mu_n(x)=\int_\mathbb{R} f(x)\mathrm{d}\mu(x)$ *for all bounded and measurable f with $\mu(disc(f))=0$;*

(iv) $\lim_{n\to\infty}\int_\mathbb{R} f(x)\mathrm{d}\mu_n(x)=\int_\mathbb{R} f(x)\mathrm{d}\mu(x)$ *for all $f\in C_\mathrm{c}(\mathbb{R},\mathbb{R})$;*

(v) one has $\limsup_{n\to\infty}\mu_n(F)\le\mu(F)$ *for all closed sets $F\subseteq E$;*

(vi) one has $\liminf_{n\to\infty}\mu_n(H)\ge\mu(H)$ *for all open sets $H\subseteq E$;*

(vii) $\lim_{n\to\infty}\phi_n(x)=\phi(x)$ *for all $x\in\mathrm{cont}(\phi)$.*

Note that item (iv) of the theorem actually defines vague convergence of measures, which is in the case of probability measures on $\mathbb{R}$ equivalent to weak convergence. The property (vii) of Theorem 2.28 is often referred to as *weak convergence of distribution functions.* If (vii) holds, we write

$$\phi=\operatorname*{w-lim}_{n\to\infty}\phi_n.$$

Note that if ϕ is the distribution function of a measure μ and f is integrable with respect to this measure, we have

$$\int_{\mathbb{R}} f(x)\mathrm{d}\mu(x) = \int_{\mathbb{R}} f(x)\mathrm{d}\phi(x)$$

in the sense of Lebesgue-Stieltjes integrals. Another important fact is that if a sequence of distribution functions (ϕ_n) of probability measures converges weakly to some continuous $\phi \in B(\mathbb{R})$, then this convergence is even uniform. In order to verify weak convergence it turns out that it is useful to investigate the limit

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)\mathrm{d}\mu_n(x) = \int_{\mathbb{R}} f(x)\mathrm{d}\mu(x) \tag{2.32}$$

for certain test functions f as done in (ii), (iii) and (iv) of the portmanteau theorem. The class of the test functions can be even more specified. In particular, in [CFKS09, Section 3] it is proposed to use $x \to (z-x)^{-1}$ for $z \in \mathbb{C}\setminus\mathbb{R}$ as test functions. This leads to the Stieltjes transform of a probability measure. Let μ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, then the *Stieltjes transform* $\mathfrak{r}(\mu) : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}$ of μ is given by setting for $z \in \mathbb{C}\setminus\mathbb{R}$:

$$\mathfrak{r}(\mu)(z) := \int_{\mathbb{R}} (z-x)^{-1}\mathrm{d}\mu(x). \tag{2.33}$$

Moreover, for $m \in \mathbb{N}$ the *m-th moment* of the probability measure μ is given by

$$M_m(\mu) := \int_{\mathbb{R}} x^m \mathrm{d}\mu(x). \tag{2.34}$$

If ϕ is the distribution function of μ, then we sometimes use the notions $\mathfrak{r}(\phi) := \mathfrak{r}(\mu)$ and $M_m(\phi) := M_m(\mu)$. In this situation we say that $\mathfrak{r}(\phi)$ is the Stieltjes transform of ϕ and $M_m(\phi)$ is the m-th moment of ϕ. In the next lemma we use these quantities to formulate other conditions which are equivalent to weak convergence.

Lemma 2.29. *Let $\mu, \mu_1, \mu_2, \dots$ be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then the following are equivalent* *(i)* $\mu = \text{w-lim}_{n\to\infty} \mu_n$*;*

(ii) for all $z \in \mathbb{C} \setminus \mathbb{R}$ *one has* $\lim_{n\to\infty} \mathfrak{r}(\mu_n)(z) = \mathfrak{r}(\mu)(z)$*.*

If, furthermore, there is a $k > 0$ *such that each of the measures* $\mu, \mu_1, \mu_2, \ldots$ *is supported on a subset of* $[-k, k] \subseteq \mathbb{R}$*, then also the following criterion is equivalent to (i) and (ii)*

(iii) for all $m \in \mathbb{N}$ *one has* $\lim_{n\to\infty} M_m(\mu_n) = M_m(\mu)$*.*

In the proof of this Lemma we will make use of the Stone-Weierstrass theorem in a complex-valued version to be found for instance in [dB59].

Theorem 2.30 (Stone-Weierstraß)**.** *Let* A *be a* $\mathbb{C}$*-subalgebra of* $C_0(\mathbb{R}, \mathbb{C})$ *such that*

(i) for any two points $x, y \in \mathbb{R}$ *with* $x \neq y$ *there is some* $f \in A$ *satisfying* $f(x) \neq f(y)$*;*

(ii) for any $x \in \mathbb{R}$ *there exists* $f \in A$ *with* $f(x) \neq 0$*;*

(iii) for any $f \in A$ *one has* $\bar{f} \in A$*.*

Then A *is dense in* $C_0(\mathbb{R}, \mathbb{C})$ *with respect to supremum norm.*

Proof of Lemma 2.29. First assume that (μ_n) converges weakly to μ. Then by definition of the Lebesgue-Stieltjes integral we have for $z \in \mathbb{C} \setminus \mathbb{R}$

$$\int_{\mathbb{R}} (z-x)^{-1} \mathrm{d}\mu_n(x) \\ = \int_{\mathbb{R}} \Re\left((z-x)^{-1}\right) \mathrm{d}\mu_n(x) + \mathrm{i} \int_{\mathbb{R}} \Im\left((z-x)^{-1}\right) \mathrm{d}\mu_n(x).$$

This expression tends by definition of weak convergence for $n \to \infty$ to

$$\int_{\mathbb{R}} (z-x)^{-1} \mathrm{d}\mu(x) \\ = \int_{\mathbb{R}} \Re\left((z-x)^{-1}\right) \mathrm{d}\mu(x) + \mathrm{i} \int_{\mathbb{R}} \Im\left((z-x)^{-1}\right) \mathrm{d}\mu(x),$$

which proves (ii). Assuming additionally that there is some $k > 0$ such that the support of each of the measures $\mu, \mu_1, \mu_2, \dots$ is contained in $[-k, k]$ we get for $m \in \mathbb{N}$

$$\begin{aligned}&\int_{\mathbb{R}} x^m \mathrm{d}\mu_n(x) \\ &= \int_{\mathbb{R}} x^m \mathbf{1}_{[-k,k]}(x)\mathrm{d}\mu_n(x) \overset{n\to\infty}{\to} \int_{\mathbb{R}} x^m \mathbf{1}_{[-k,k]}(x)\mathrm{d}\mu(x) = \int_{\mathbb{R}} x^m \mathrm{d}\mu(x),\end{aligned}$$

where we used (iii) of Theorem 2.28. This shows that (i) implies (iii).

Now we assume (iii). By linearity of the Lebesgue-Stieltjes integral we obtain that 2.32 holds for all polynomials f. The application of the approximation theorem of Weierstraß gives that (2.32) holds as well for each continuous f with support in $[-k, k]$. Again by the portmanteau theorem this proves weak convergence of the measures.

It remains to prove that (i) is implied by (ii). To this end, assume that (ii) holds and define the set

$$\mathcal{R} := \{\phi : \mathbb{R} \to \mathbb{C} \mid \exists z \in \mathbb{C} \setminus \mathbb{R} \text{ with } \phi(x) = (z - x)^{-1}\}.$$

We write $\mathrm{alg}(\mathcal{R})$ for the algebra which is generated by $\mathcal{R}$. Furthermore, denote the set of all functions $f : \mathbb{R} \to \mathbb{C}$ such that (2.32) holds by $\mathcal{K}$. Let us show that the space $\mathcal{K}$ is closed under limits with respect to supremum norm. To this end, choose a sequence (f_j) in $\mathcal{K}$ and some $f : \mathbb{R} \to \mathbb{C}$ satisfying $\lim_{j\to\infty} \|f_j - f\|_\infty = 0$ and calculate

$$\begin{aligned}\Big|\int_{\mathbb{R}} f(x)\mathrm{d}\mu_n(x) - \int_{\mathbb{R}} f(x)\mathrm{d}\mu(x)\Big| &\leq \Big|\int_{\mathbb{R}} f(x) - f_j(x)\mathrm{d}\mu_n(x)\Big| \\ &+ \Big|\int_{\mathbb{R}} f_j(x)\mathrm{d}\mu_n(x) - \int_{\mathbb{R}} f_j(x)\mathrm{d}\mu(x)\Big| + \Big|\int_{\mathbb{R}} f_j(x) - f(x)\mathrm{d}\mu(x)\Big| \\ &\leq 2\|f - f_j\|_\infty + \Big|\int_{\mathbb{R}} f_j(x)\mathrm{d}\mu_n(x) - \int_{\mathbb{R}} f_j(x)\mathrm{d}\mu(x)\Big|.\end{aligned}$$

This gives that the limit f is an element of $\mathcal{K}$ as well.

We claim that

$$\mathrm{alg}(\mathcal{R}) \subseteq \mathcal{K} \quad \text{and fulfills (i), (ii) and (iii) of Theorem 2.30.} \tag{2.35}$$

Then the Stone-Weierstrass theorem gives that the closure of $\mathrm{alg}(\mathcal{R})$ with respect to supremum norm equals $C_0(\mathbb{R}, \mathbb{C})$ and is a subset of $\mathcal{K}$.

As $C_c(\mathbb{R},\mathbb{C}) \subseteq C_0(\mathbb{R},\mathbb{C})$ we obtain using the portmanteau theorem $\text{w-lim}_{n\to\infty} \mu_n = \mu$. Therefore, it remains to prove (2.35).

Point (i) and (ii) of Theorem 2.30 are obvious. Furthermore $\mathcal{R}$ is by definition closed under conjugation, which carries over to $\text{alg}(\mathcal{R})$ and thus proves assumption (iii) of the Stone-Weierstrass theorem. Now we show $\text{alg}(\mathcal{R}) \subseteq \mathcal{K}$.

Let $z, z' \in \mathbb{C}\setminus\mathbb{R}$ be distinct. Then we have by partial fraction decomposition

$$\frac{1}{z-x}\frac{1}{z'-x} = \frac{1}{z'-z}\left(\frac{1}{z-x} - \frac{1}{z'-x}\right).$$

Therefore, products of distinct functions in $\mathcal{R}$ are in $\mathcal{K}$. If one considers integer powers of an elements of $\mathcal{R}$ the situation is more difficult. Let $z \in \mathbb{C}\setminus\mathbb{R}$ and $m \in \mathbb{N}$ be given. Then by Cauchy's integral formula we have

$$\frac{1}{(z-x)^m} = \frac{1}{2\pi\mathrm{i}} \oint_{\partial B_{\Im(z)/2}(z)} \frac{1}{(t-x)(z-t)^m}\mathrm{d}t,$$

where the integration over $\partial B_{|\Im(z)|/2}(z)$ means the integration along the circle of radius $|\Im(z)|/2$ around z. We set

$$\gamma_z : [0,1] \to \mathbb{C}, \qquad \gamma_z(s) := z + \frac{|\Im(z)|}{2}\mathrm{e}^{2\pi\mathrm{i}s}.$$

By definition of the curve integral we have

$$\oint_{\partial B_{\Im(z)/2}(z)} \frac{1}{(t-x)(z-t)^m}\mathrm{d}t = \int_0^1 \frac{|\gamma_z'(s)|}{(\gamma_z(s)-x)(z-\gamma_z(s))^m}\mathrm{d}s.$$

We write the integral as a limit of Riemann sums

$$\int_0^1 \frac{|\gamma_z'(s)|}{(\gamma_z(s)-x)(z-\gamma_z(s))^m}\mathrm{d}s$$
$$= \lim_{k\to\infty} \frac{1}{k}\sum_{j=0}^{k-1} \frac{|\gamma_z'(j/k)|}{(\gamma_z(j/k)-x)(z-\gamma_z(j/k))^m},$$

define for $x \in \mathbb{R}$ and $t \in [0,1]$

$$d(x,t) := \frac{|\gamma_z'(t)|}{(\gamma_z(t) - x)(z - \gamma_z(t))^m}$$

and consider the difference

$$\begin{aligned} D_k(x) &:= \left| \int\limits_0^1 d(x,s)\mathrm{d}s - \frac{1}{k}\sum_{j=0}^{k-1} d(x, j/k) \right| \\ &= \left| \sum_{j=0}^{k-1} \int\limits_{j/k}^{(j+1)/k} d(x,s)\mathrm{d}s - \frac{d(x,j/k)}{k} \right| \\ &\leq \sum_{j=0}^{k-1} \int_{j/k}^{(j+1)/k} \big| d(x,s) - d(x,j/k) \big| \mathrm{d}s. \end{aligned}$$

Now, choose $\varepsilon > 0$ arbitrary and consider two cases using the compact set

$$K := \left\{ x \in \mathbb{R} \;\middle|\; |z - x| \leq \frac{|\Im(z)|}{2}\left(1 + \frac{\pi 2^{m+2}}{\varepsilon |\Im(z)|^m}\right) \right\}.$$

First let $x \notin K$ then we have

$$\begin{aligned} |d(x,s)| &= \frac{\pi |\Im(z)|}{|z - x + 2^{-1}\Im(z)\mathrm{e}^{2\pi i t}| |2^{-1}\Im(z)|^m} \\ &\leq \frac{\pi 2^m |\Im(z)|^{1-m}}{|z-x| - |2^{-1}\Im(z)|} \leq \frac{\varepsilon}{2}, \end{aligned}$$

which in turn gives $D_k(x) \leq \varepsilon$ for all $k \in \mathbb{N}$. Now let $x \in K$ be given. Then as $K \times [0,1]$ is compact and $d : \mathbb{R} \times [0,1] \to \mathbb{C}$ is continuous we obtain that d restricted to $K \times [0,1]$ is even uniformly continuous. Therefore we can choose k_0 large enough such that for all $s, t \in [0,1]$ with $|s - t| \leq 1/k_0$ we have $|d(x,s) - d(x,t)| \leq \varepsilon$. This proves for $x \in K$ and all $k \geq k_0$ that $D_k(x) \leq \varepsilon$. We conclude that powers of elements in $\mathcal{R}$ are uniform limits of linear combinations of elements in $\mathcal{R}$ and are therefore elements of $\mathcal{K}$, which finally shows (2.35). This finishes the proof. ∎

3 Deterministic operators on sofic groups

In this chapter we assume that G is a sofic group, S is a finite and symmetric set of generators and we consider deterministic operators on the Cayley graph of G. The aim is to define classes of operators on the Cayley graph $\Gamma = \Gamma(G, S)$ which can be transferred to operators on the approximating graph Γ_r given in (2.2). These finite dimensional operators are supposed to approximate spectral properties of the original one. More precisely, we show that the normalized eigenvalue counting functions of the approximating operators converge weakly. Moreover, we obtain a Pastur-Shubin trance formula. We start with an investigation of non-random operators in Chapter 3 and prove in Chapter 4 similar results for the random setting. The results of both chapters are already published in [SS12], a joint work with Christoph Schumacher.

3.1 Weak convergence

We verify weak convergence of the eigenvalue counting functions for deterministic, translation invariant, self-adjoint operators on a sofic group G. An important part in the proof of this result is the appropriate choice of the approximating operators. To define these operators, we make use of the property that the Cayley graph of a sofic group can be approximated on arbitrary good scales by a finite graph, cf. Definition 2.2. Having one of these finite graphs at hand, we define the approximating operator by transferring certain matrix elements of the original operator to this approximation, see (3.2). After defining these approximations, we study the Stieltjes transforms of the associated eigenvalue counting functions. Their convergence implies by Lemma 2.29 weak convergence of the distribution functions or the measures, respectively. Let us start with the definition of the operator under consideration.

Let $A\colon D(A) \subseteq \ell^2(G) \to \ell^2(G)$ with $C_{\mathrm{c}}(G) \subseteq D(A)$ be a self-adjoint operator and set $a(x, y) := \langle \delta_x, A\delta_y \rangle$. We assume that A

is translation invariant and that $C_{\mathrm{c}}(G)$ is a core of A. Recall that translation invariance means that for all $x, y, z \in G$ we have $a(x, y) = a(xz, yz)$. These assumptions imply for all $x \in G$:

$$\|A\delta_x\|_2^2 = \sum_{y \in G} |a(x, y)|^2 = \sum_{y \in G} |a(\mathrm{id}, y)|^2 = \|A\delta_{\mathrm{id}}\|_2^2 < \infty. \tag{3.1}$$

Remark 3.1. Let $A\colon D(A) \subseteq \ell^2(G) \to \ell^2(G)$ be a self-adjoint and translation invariant operator with $C_{\mathrm{c}}(G) \subseteq D(A)$. Then the condition that $C_{\mathrm{c}}(G)$ is a core is in particular satisfied if A is bounded. However, the above formulated conditions do not imply boundedness of the operator. In Subsection 3.3 we present an example of an unbounded, self-adjoint, translation invariant operator on $\ell^2(\mathbb{Z})$ with core $C_{\mathrm{c}}(\mathbb{Z})$.

As in Section 2.1.1, we choose a function $\varepsilon : \mathbb{N} \to (0, \infty)$ with $\lim_{r\to\infty} \varepsilon(r) = 0$ and define for $r \in \mathbb{N}$ the objects $\Gamma_r = (V_r, E_r)$ and $V_r^{(0)}$ as in (2.2). We define the projection $A_r\colon \ell^2(V_r) \to \ell^2(V_r)$ of A to the graph Γ_r by

$$(A_r f)(x) := \sum_{y \in V_r} a_r(x, y) f(y),$$

where

$$a_r(x, y) := \begin{cases} a(\Psi_{v,r}(x), \Psi_{v,r}(y)) & \text{if } \exists v \in V_r^{(0)} \colon x, y \in B_{r/6}^{V_r}(v), \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

This operator is well-defined by Lemma 2.3. Note that A_r is a symmetric and hence self-adjoint operator on $\ell^2(V_r)$.

Remark 3.2. For the operators to be well-defined, it would have been sufficient to choose $r/2$ instead of $r/6$ in the definition of $a_r(x, y)$ in (3.2). The reason for the choice $r/6$ is a calculation in the proofs of Theorem 3.3 and Theorem 4.5. There we want that $y \in B_{r/2}^{V_r}(x_0)$ whenever $x_0 \in V_r^{(0)}$, $x \in B_{r/6}^{V_r}(x_0)$ and $a_r(x, y) \neq 0$. By triangle inequality this follows exactly from the above choice of $r/6$ in (3.2).

Define for each $r \in \mathbb{N}$ the normalized eigenvalue counting function $\mathfrak{n}_r$ of A_r by

$$\mathfrak{n}_r \colon \mathbb{R} \to [0,1], \quad \mathfrak{n}_r := \mathfrak{n}(A_r), \tag{3.3}$$

where $\mathfrak{n}(A_r)$ is given by Definition 2.23. If the (pointwise) limit of these functions for increasing r exists, it is called the *integrated density of states* of A. Given the operator A, we denote by E_λ the spectral projection on the interval $(-\infty, \lambda]$. Using this we set $\mathfrak{N} \colon \mathbb{R} \to [0,1]$ as

$$\mathfrak{N}(\lambda) := \langle \delta_{\mathrm{id}}, E_\lambda \delta_{\mathrm{id}} \rangle. \tag{3.4}$$

This is a distribution function of a probability measure, which we call the *spectral distribution function* (SDF). The next theorem shows that the integrated density of states exists and that it equals the spectral distribution function. In other words we show the *Pastur-Shubin trace formula.*

Theorem 3.3. *Let G be a finitely generated sofic group and $A : D(A) \to \ell^2(G)$ a self-adjoint, translation invariant operator with core $C_{\mathrm{c}}(G)$. Furthermore let $\mathfrak{N}$ and $\mathfrak{n}_r$ be given as above. Then*

$$\mathfrak{N} = \operatorname*{w-lim}_{r\to\infty} \mathfrak{n}_r .$$

Proof. In order to prove this theorem, we make use of the equivalence of (ii) in Lemma 2.29 to weak convergence of measures. Therefore we fix some arbitrary $z \in \mathbb{C} \setminus \mathbb{R}$ and only have to show

$$\lim_{r\to\infty} \mathfrak{r}(\mathfrak{n}_r)(z) = \mathfrak{r}(\mathfrak{N})(z).$$

Here $\mathfrak{r}(\mathfrak{n}_r)$ and $\mathfrak{r}(\mathfrak{N})$ are the Stieltjes transform of the distribution functions $\mathfrak{n}_r$ and $\mathfrak{N}$, respectively, cf (2.33). To this end we set

$$\begin{aligned} D_r &:= |\mathfrak{r}(\mathfrak{n}_r)(z) - \mathfrak{r}(\mathfrak{N})(z)| \\ &= \Big| \int_{\mathbb{R}} (z-\lambda)^{-1} \mathrm{d}\mathfrak{n}_r(\lambda) - \int_{\mathbb{R}} (z-\lambda)^{-1} \mathrm{d}\mathfrak{N}(\lambda) \Big| \end{aligned}$$

and use

$$\int_{\mathbb{R}} (z-\lambda)^{-1} \mathrm{d}\mathfrak{n}_r(\lambda) = \frac{1}{|V_r|} \sum_{\lambda \in \sigma(A_r)} m_\lambda (z-\lambda)^{-1}$$
$$= \frac{1}{|V_r|} \operatorname{Tr}((z-A_r)^{-1}) = \frac{1}{|V_r|} \sum_{x \in V_r} \langle \delta_x, (z-A_r)^{-1} \delta_x \rangle, \quad (3.5)$$

where for an eigenvalue λ we denote its multiplicity by m_λ. With the spectral theorem we obtain

$$D_r = \left| \frac{1}{|V_r|} \sum_{x \in V_r} \langle \delta_x, (z-A_r)^{-1} \delta_x \rangle - \left\langle \delta_{\mathrm{id}}, (z-A)^{-1} \delta_{\mathrm{id}} \right\rangle \right|$$
$$\leq \frac{1}{|V_r|} \sum_{x \in V_r^{(0)}} \left| \langle \delta_x, (z-A_r)^{-1} \delta_x \rangle - \left\langle \delta_{\mathrm{id}}, (z-A)^{-1} \delta_{\mathrm{id}} \right\rangle \right|$$
$$+ \frac{1}{|V_r|} \sum_{x \in V_r \setminus V_r^{(0)}} \left| \langle \delta_x, (z-A_r)^{-1} \delta_x \rangle - \left\langle \delta_{\mathrm{id}}, (z-A)^{-1} \delta_{\mathrm{id}} \right\rangle \right|.$$

Now the Cauchy-Schwarz inequality, and the fact that the norm of the resolvents is bounded from above by the absolute value of the inverse of the imaginary part of z imply

$$D_r \leq \frac{1}{|V_r|} \sum_{x \in V_r^{(0)}} \left| \langle \delta_x, (z-A_r)^{-1} \delta_x \rangle - \left\langle \delta_{\mathrm{id}}, (z-A)^{-1} \delta_{\mathrm{id}} \right\rangle \right| + 2 \frac{\varepsilon(r)}{|\Im(z)|}. \quad (3.6)$$

Here we also made use of property (S2) in the definition of sofic groups.

The aim of the next steps is to rewrite $\langle \delta_x, (z-A_r)^{-1} \delta_x \rangle$ as a resolvent of an operator evaluated at id $\in G$. We choose for all $x \in V_r^{(0)}$ an injective extension

$$\Psi'_{r,x} \colon V_r \to G$$

of the graph isomorphism $\Psi_{r,x} \colon B_r^{\Gamma_r}(x) \to B_r^G$ form (2.3). Note that Ψ' does not need to be a graph isomorphism itself. This map induces a bijection

$$\tilde{\Phi}_{r,x} : \ell^2(\Psi'_{r,x}(V_r)) \to \ell^2(V_r), \quad \tilde{\Phi}_{r,x} f := f \circ \Psi'_{r,x}.$$

Note that $\tilde{\Phi}_{r,x}$ is in fact a unitary operator. Since $\Psi'_{r,x}(x) = \mathrm{id}$ we obtain $\tilde{\Phi}_{r,x}\delta_x = \delta_{\mathrm{id}}$. We use this to define

$$\tilde{A}_{r,x} := \tilde{\Phi}^*_{r,x} A_r \tilde{\Phi}_{r,x} : \ell^2(\Psi'_{r,x}(V_r)) \to \ell^2(V_r).$$

Then we have

$$\begin{aligned}
\langle \delta_x, (z - A_r)^{-1}\delta_x\rangle &= \langle \delta_x, (z - \tilde{\Phi}_{r,x}\tilde{A}_{r,x}\tilde{\Phi}^*_{r,x})^{-1}\delta_x\rangle \\
&= \langle \delta_x, \tilde{\Phi}_{r,x}(z - \tilde{A}_{r,x})^{-1}\tilde{\Phi}^*_{r,x}\delta_x\rangle \\
&= \langle \tilde{\Phi}^*_{r,x}\delta_x, (z - \tilde{A}_{r,x})^{-1}\tilde{\Phi}^*_{r,x}\delta_x\rangle \\
&= \langle \delta_{\mathrm{id}}, (z - \tilde{A}_{r,x})^{-1}\delta_{\mathrm{id}}\rangle.
\end{aligned}$$

Next, we extend the operator $\tilde{A}_{r,x}$ to an operator acting on $\ell^2(G)$. To this end we set

$$\Phi_{r,x} \colon \ell^2(G) \to \ell^2(V_r), \qquad \Phi_{r,x} f := f \circ \Psi'_{r,x}$$

and

$$\hat{A}_{r,x} := \Phi^*_{r,x} A_r \Phi_{r,x} \colon \ell^2(G) \to \ell^2(G).$$

Comparing the operators in the sense of their matrix elements, we obtain for $a, b \in G$:

$$\langle \delta_a, \hat{A}_{r,x}\delta_b\rangle = \begin{cases} \langle \delta_a, \tilde{A}_{r,x}\delta_b\rangle & \text{if } a, b \in \Psi'_{r,x}(V_r), \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the operator $(z - \hat{A}_{r,x})$ is block diagonal. Therefore, a matrix element of the inverse of this operator can be obtained by inverting the corresponding block. In particular we have

$$\langle \delta_{\mathrm{id}}, (z - \tilde{A}_{r,x})^{-1}\delta_{\mathrm{id}}\rangle = \langle \delta_{\mathrm{id}}, (z - \hat{A}_{r,x})^{-1}\delta_{\mathrm{id}}\rangle.$$

This together with (3.6) gives

$$D_r \le \sup_{x \in V_r^{(0)}} |\langle \delta_{\mathrm{id}}, \big((z - \hat{A}_{r,x})^{-1} - (z - A)^{-1}\big)\delta_{\mathrm{id}}\rangle| + 2\frac{\varepsilon(r)}{|\Im(z)|}.$$

In the next step we introduce an element $\psi \in \ell^2(G)$ use the triangle inequality, the Cauchy-Schwarz inequality and again the estimation of the resolvents using the imaginary part of z. We get

$$D_r \leq \sup_{x \in V_r^{(0)}} |\langle \delta_{\mathrm{id}}, \big((z - \hat{A}_{r,x})^{-1} - (z - A)^{-1}\big)\psi\rangle| + 2\frac{\varepsilon(r) + \|\delta_{\mathrm{id}} - \psi\|_2}{|\Im(z)|}.$$

Now use the second resolvent identity and Cauchy-Schwarz inequality to estimate:

$$\begin{aligned} D_r &\leq \sup_{x \in V_r^{(0)}} |\langle \delta_{\mathrm{id}}, (z - \hat{A}_{r,x})^{-1}(A - \hat{A}_{r,x})(z - A)^{-1}\psi\rangle| \\ &\qquad + 2\frac{\varepsilon(r) + \|\delta_{\mathrm{id}} - \psi\|_2}{|\Im(z)|} \\ &\leq \frac{1}{|\Im(z)|} \sup_{x \in V_r^{(0)}} \|(A - \hat{A}_{r,x})(z - A)^{-1}\psi\|_2 + 2\frac{\varepsilon(r) + \|\delta_{\mathrm{id}} - \psi\|_2}{|\Im(z)|}. \end{aligned} \tag{3.7}$$

The next aim is to find an appropriate ψ such that $(z - A)^{-1}\psi$ is finitely supported and at the same time the norm $\|\delta_{\mathrm{id}} - \psi\|_2$ is small. To this end, fix some $\kappa > 0$ and make use of Lemma 2.15, which is applicable as A is self-adjoint with core $C_{\mathrm{c}}(G)$. We obtain $\psi \in \ell^2(G)$ with

$$\|\delta_{\mathrm{id}} - \psi\|_2 < \kappa \qquad \text{and} \qquad \phi := (z - A)^{-1}\psi \in C_{\mathrm{c}}(G).$$

Using this special choice of ϕ and ψ and choosing $r \geq 6 \operatorname{diam}(\operatorname{spt} \phi)$, we continue our estimation by considering $\|(A - \hat{A}_{r,x})\phi\|_2$. Applying the Cauchy-Schwarz inequality and the fact that the operators coincide on a ball $B^G_{r/6}$ yields

$$\begin{aligned} \|(A - \hat{A}_{r,x})\phi\|_2^2 &= \sum_{g \in G \setminus B^G_{r/6}} \Big| \sum_{h \in \operatorname{spt} \phi} \langle (A - \hat{A}_{r,x})\delta_g, \delta_h\rangle \phi(h)\Big|^2 \\ &\leq \|\phi\|_2^2 \sum_{g \in G \setminus B^G_{r/6}} \sum_{h \in \operatorname{spt} \phi} |a(g,h) - \hat{a}_{r,x}(g,h)|^2. \end{aligned}$$

The triangle inequality for norms leads to

$$\frac{\|(A-\hat{A}_{r,x})\phi\|_2}{\|\phi\|_2} \leq \Bigg(\sum_{h\in \operatorname{spt}\phi}\sum_{g\in G\setminus B^G_{r/6}} |a(g,h)|^2\Bigg)^{\frac{1}{2}} + \Bigg(\sum_{h\in \operatorname{spt}\phi}\sum_{g\in G\setminus B^G_{r/6}} |\hat{a}_{r,x}(g,h)|^2\Bigg)^{\frac{1}{2}}.$$

By Remark 3.2, $\hat{a}_{r,x}(g,h)\neq 0$ with $h\in \operatorname{spt}\phi\subseteq B_{r/6}$ implies $g\in B^G_{r/2}$. Therefore we get

$$\begin{aligned}\sum_{h\in \operatorname{spt}\phi}\sum_{g\in G\setminus B^G_{r/6}} |\hat{a}_{r,x}(g,h)|^2 &= \sum_{w\in\Psi^{-1}_{r,x}(\operatorname{spt}\phi)}\ \sum_{v\in B^{V_r}_{r/2}(x)\setminus B^{V_r}_{r/6}(x)} |a_r(v,w)|^2\\ &\leq \sum_{w\in\Psi^{-1}_{r,x}(\operatorname{spt}\phi)}\ \sum_{v\in B^{V_r}_{r/2}(x)\setminus B^{V_r}_{r/6}(x)} |a(\Psi_{r,x}(v),\Psi_{r,x}(w))|^2\\ &\leq \sum_{h\in \operatorname{spt}\phi}\sum_{g\in G\setminus B^G_{r/6}} |a(g,h)|^2. \end{aligned} \tag{3.8}$$

These considerations show that for arbitrary $x\in V_r^{(0)}$ we have

$$\|(A-\hat{A}_{r,x})\phi\|_2 \leq 2\|\phi\|_2\Bigg(\sum_{h\in \operatorname{spt}\phi}\sum_{g\in G\setminus B^G_{r/6}} |a(g,h)|^2\Bigg)^{1/2}.$$

This and (3.7) yield

$$D_r \leq \frac{2}{|\Im(z)|}\left(\|\phi\|_2\Bigg(\sum_{h\in \operatorname{spt}\phi}\sum_{g\in G\setminus B^G_{r/6}} |a(g,h)|^2\Bigg)^{\frac{1}{2}} + \varepsilon(r)+\kappa\right) \to \frac{2\kappa}{|\Im(z)|}$$

for $r\to\infty$. Here we used $\sum_{g\in G}|a(g,h)|^2<\infty$ for all $x\in G$, see (3.1). This finishes the proof, since $\kappa>0$ was arbitrary. ∎

3.2 Special case: the free group

Here we study the free group as an example of sofic groups. We present a specific example for a sequence of approximating finite graphs.

Let $k \in \mathbb{N}$ and F_k be the free group with set of generators

$$S_k := \{s_1, \ldots, s_k, s_1^{-1}, \ldots, s_k^{-1}\}.$$

For each element $g \in F_k$ there exists exactly one possibility to express g as a (reduced) word of elements in S_k. Here a word is called reduced, if no element is followed by its inverse. Note that the Cayley graph $\Gamma = \Gamma(F_k, S_k)$ is a $2k$-regular tree. The group F_k, $k \geq 2$ is is an example of a non-amenable residually finite group. Hence F_k is sofic.

In this subsection we give an explicit construction of a sequence of approximating graphs. The existence of this sequence of graphs in particular shows that F_k is residually finite and sofic. In fact we construct a sequence of normal subgroups and quotient groups fulfilling the conditions (R1) to (R4) in Definition 2.4. As shown in Lemma 2.5, the Cayley graphs of the quotient groups serve as approximating graphs fulfilling conditions (S1) and (S2) in Definition 2.2. The idea of the construction presented here goes back to [Big88].

As $\Gamma = \Gamma(F_k, S_k)$ is a $2k$-regular tree, the idea is to approximate Γ with finite $2k$-regular graphs Γ_n, $n \in \mathbb{N}$, which contain only large cycles. More precisely, if γ_n denotes the length of the shortest circle in Γ_n, we want γ_n tend to infinity if $n \to \infty$. The quantity γ_n is called the *girth* of the graph Γ_n. Here, the graphs will not just be regular, they will even be Cayley graphs of a group. The motivation of this idea is that, if Γ_n is the Cayley graph of group with girth γ_n, then the graphs Γ and Γ_n coincide on balls of radius $\gamma_n/2$, cf. property (2.5).

As before let $B_n := B_n^{F_k}(\mathrm{id})$ be the ball of radius $n \in \mathbb{N}$ in F_k centered at the identity. It is easy to see that

$$|B_n| = \frac{k(2k-1)^n - 1}{k-1}.$$

We will now consider permutations on B_n. Therefore denote by $\mathcal{S}_n$ the symmetric group on B_n. For each $s \in S_k$ we define $p_s^{(n)} \in \mathcal{S}_n$ by

$$p_s^{(n)} \colon B_n \to B_n, \qquad p_s^{(n)}(w) := \begin{cases} sw & \text{if } sw \in B_n \\ w_1^{-1} w_2^{-1} \cdots w_m^{-1} & \text{otherwise.} \end{cases} \tag{3.9}$$

Here, $w = w_1 \cdots w_m$ is expressed as reduced word with letters in $w_i \in S_k$, $i = 1, \ldots, k$. Thus, a permutation $p_s^{(n)}$ shifts an element $w \in B_n$ in direction of s, if this shifted element is still contained in B_n. If $sw \notin B_n$, the permutation $p_s^{(n)}$ maps w to the element which one obtains by point reflecting w at the center of the ball B_n. Note that this gives $(p_s^{(n)})^{-1} = p_{s^{-1}}^{(n)}$. The finite group H_n is now defined as the subgroup of $\mathcal{S}_n$ which is generated by $p_s^{(n)}$, $s \in S_k$, i.e

$$H_n := \langle S_k^{(n)} \rangle, \qquad \text{where} \quad S_k^{(n)} := \left\{ p_s^{(n)} \mid s \in S_k \right\}.$$

Now let us give a bound for the girth of this group. Therefore note that each circle in the Cayley graph $\Gamma_n = \Gamma_n(H_n, S_k^{(n)})$ corresponds to a reduced word with letters $p_i \in S_k^{(n)}$, $i = 1, \ldots, t$ with

$$p_t \cdots p_1 = \mathrm{id}_n \in H_n. \tag{3.10}$$

Hence, each such word maps the identity $\mathrm{id} \in B_n$ to itself, i.e.

$$(p_t \circ \cdots \circ p_1)(\mathrm{id}) = \mathrm{id} \in F_k.$$

However, by definition of the permutations in (3.9), p_1 maps id in the sphere $B_1 \setminus B_0$. Then inductively we get for $1 < i < n$ that each p_i maps an element from the sphere $B_{i-1} \setminus B_{i-2}$ in the sphere $B_{i-1} \setminus B_{i-2}$. This continues until one reaches some $g = s_1 \cdots s_n$ in the sphere $B_n \setminus B_{n-1}$. Here the element p_{n+1} maps g to $s_1^{-1} \cdots s_n^{-1}$. From there on, one needs again at least n further elements of $S_k^{(n)}$ to map this element again to the center of the ball. This shows that t in equation (3.10) has to be at least $2n + 1$. Therefore we have for the girth γ_n of H_n that $\gamma_n \geq 2n + 1$.

In this situation, we can define a group homomorphism $\rho_n \colon F_k \to H_n$ by setting for each $g = s_1 \cdots s_m \in F_k$

$$\rho_n(g) = p_{s_1}^{(n)} \cdots p_{s_m}^{(n)}.$$

We furthermore define the group $G_n := \ker \rho_n := \{g \in F_k \mid \rho_n(g) = \mathrm{id}_n\}$, where id_n is the identity in $\mathcal{S}_n$. Then we have

$$F_k / G_n = H_n,$$

which shows that G_n is normal in F_k and of finite index. Therefore conditions (R1) and (R2) hold. Furthermore the condition (R4) follows from the fact that $\lim_{n\to\infty}\gamma_n = \infty$, as for each $n \in \mathbb{N}$ the homomorphism γ_n maps none of the elements in $B_{\gamma_n} \setminus \{\mathrm{id}\}$ to the identity id_n. Referring to the short discussion after Definition 2.4, this shows that F_k is residually finite. Furthermore, for given $R \in \mathbb{N}$ there exists $n \in \mathbb{N}$ with $\gamma_n \geq R$. Then Γ_n fulfills (S1) and (S2) in Definition 2.2 with R and arbitrary ε.

The Cayley graph of (F_k, S_k) is a regular tree. For such graphs it is due to [McK81] that the spectral distribution function of the adjacency operator can explicitly be stated using the following density function

$$x \mapsto \frac{k\sqrt{4(2k-1)-x^2}}{\pi(4k^2-x^2)}\chi_{[0,2\sqrt{2k-1}]}(|x|).$$

This shows in particular that the spectral distribution function is continuous. Thus, the limit in Theorem 3.3 does not only exist for all $\lambda \in \mathbb{R}$, but it is even uniform in λ.

3.3 Special case: an unbounded operator

In this section we provide an example of an operator which is unbounded and fits in the setting of Section 3.1. In fact, we define an operator on $A\colon D(A) \to \ell^2(\mathbb{Z})$ which satisfies:

(a) A is self-adjoint,

(b) A is translation invariant,

(c) A is unbounded, and

(d) $C_c(\mathbb{Z})$ is a core of A.

This operator will be given via a certain multiplication operator in the Fourier space of $\ell^2(\mathbb{Z})$. Let

$$\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$$

be the unit circle in $\mathbb{C}$ and denote by $L^2(\mathbb{T})$ the space of the square integrable functions on $\mathbb{T}$, with respect to the normalized Haar measure. The scalar product and the norm in $L^2(\mathbb{T})$ are given by setting for $f, g \in L^2(\mathbb{T})$:

$$\langle f, g\rangle_{L^2} := \frac{1}{2\pi}\int_0^{2\pi} \overline{f(\mathrm{e}^{\mathrm{i}t})}g(\mathrm{e}^{\mathrm{i}t})\mathrm{d}t \quad \text{and} \quad \|f\|_{L^2} := \sqrt{\langle f, f\rangle_{L^2}}.$$

Let us define the *Fourier transform* $\mathfrak{F} : \ell^2(\mathbb{Z}) \to L^2(\mathbb{T})$. For $f \in \ell^2(\mathbb{Z})$ and $t \in [0, 2\pi)$ we set

$$(\mathfrak{F}f)(\mathrm{e}^{\mathrm{i}t}) := \sum_{x\in\mathbb{Z}} f(x)\mathrm{e}^{-\mathrm{i}tx}. \tag{3.11}$$

If $f \in \ell^1(\mathbb{Z})$, the sum converges absolutely and thus it is well-defined for each such f. If $f \in \ell^2(\mathbb{Z}) \setminus \ell^1(\mathbb{Z})$, the sum is to be considered as an $\ell^2(\mathbb{Z})$-limit. This limit exists since for $f \in C_\mathrm{c}(\mathbb{Z})$ we have Parseval's identity $\|\mathfrak{F}f\|_{L^2}^2 = \|f\|_2$. Let us prove this: for $f \in C_\mathrm{c}(\mathbb{Z})$ we have

$$\begin{aligned}
\|\mathfrak{F}f\|_{L^2}^2 &= \frac{1}{2\pi}\int_0^{2\pi} \overline{(\mathfrak{F}f)(\mathrm{e}^{\mathrm{i}t})}(\mathfrak{F}f)(\mathrm{e}^{\mathrm{i}t})\mathrm{d}t \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{x\in\mathbb{Z}} \overline{f(x)}\mathrm{e}^{\mathrm{i}tx} \sum_{y\in\mathbb{Z}} f(y)\mathrm{e}^{-\mathrm{i}ty}\mathrm{d}t \\
&= \frac{1}{2\pi} \sum_{x,y\in\mathbb{Z}} \overline{f(x)}f(y) \int_0^{2\pi} \mathrm{e}^{\mathrm{i}t(x-y)}\mathrm{d}t = \|f\|_2^2.
\end{aligned}$$

Here the last equality follows from

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}t(x-y)}\mathrm{d}t = \delta_x(y).$$

This shows that the Fourier transform defined on $C_\mathrm{c}(\mathbb{Z})$ is a bounded linear function. Thus, the B.L.T. theorem implies that $\mathfrak{F}$ is well-defined on the whole domain $\ell^2(\mathbb{Z})$ and we have $\|\mathfrak{F}\| = 1$, cf. [RS80, Theorem I.7]. Moreover, one can show that $\mathfrak{F}$ is bijective and the inverse $\mathfrak{F}^{-1}$ is given by

$$\mathfrak{F}^{-1} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z}), \qquad (\mathfrak{F}^{-1}\psi)(x) := \frac{1}{2\pi}\int_0^{2\pi} \psi(\mathrm{e}^{\mathrm{i}t})\mathrm{e}^{\mathrm{i}tx}\mathrm{d}t.$$

Note that $\mathfrak{F}^{-1}$ is an isometry as well. By Parseval's identity and polarization we have for $f, g \in C_\mathrm{c}(\mathbb{Z})$ that

$$\langle \mathfrak{F}^*\mathfrak{F}f, g\rangle_{L^2} = \langle \mathfrak{F}f, \mathfrak{F}g\rangle = \langle f, g\rangle_{L^2}.$$

Furthermore, we obtain

$$\begin{aligned}
\langle \mathfrak{F}\mathfrak{F}^*f, g\rangle_{L^2} = \left\langle \mathfrak{F}^{-1}\mathfrak{F}\mathfrak{F}^*f, \mathfrak{F}^{-1}g\right\rangle = \left\langle \mathfrak{F}^*f, \mathfrak{F}^{-1}g\right\rangle &= \left\langle f, \mathfrak{F}\mathfrak{F}^{-1}g\right\rangle_{L^2} \\
&= \langle f, g\rangle_{L^2}.
\end{aligned}$$

This yields that $\mathfrak{F}^* = \mathfrak{F}^{-1}$ and that $\mathfrak{F}$ is a unitary operator. Knowing that the Fourier transform is unitary, the following well-known lemma is very useful.

Lemma 3.4. *Let $\mathcal{H}$ and $\mathcal{V}$ be Hilbert spaces, $A : D(A) \subseteq \mathcal{H} \to \mathcal{H}$ a self-adjoint operator and $U : \mathcal{V} \to \mathcal{H}$ a unitary operator. Then the operator*

$$B := U^{-1}AU : D(B) \to \mathcal{V} \quad \textit{with} \quad D(B) = U^{-1}(D(A))$$

is self-adjoint and $\sigma(A) = \sigma(B)$. Moreover, if K is a core of A, then $U^{-1}(K)$ is a core of B.

Proof. The the self-adjointness of B and $\sigma(A) = \sigma(B)$ rely on rather basic calculations, see for instance [Wei00a, Satz 2.62]. Let us verify the assertion with the core. Let K be a core of A and set $L := U^{-1}(K)$. We need to show that $\overline{B|_L} = B$. As B is closed and $L \subseteq D(B)$, we obtain $\overline{B|_L} \subseteq B$. In order to show the reverse inclusion, let $h \in D(B)$ be given. Then $g := Uh \in D(A)$ and, as $\overline{A|_K} \subseteq A$, we find a sequence (g_n) in K such that

$$\lim_{n\to\infty} \|g - g_n\|_{\mathcal{H}} = 0 \quad \text{and} \quad \lim_{n\to\infty} \|Ag - Ag_n\|_{\mathcal{H}} = 0.$$

For each $n \in \mathbb{N}$ we set $h_n := U^{-1}g_n \in L$. Since unitary operators preserve norms, this yields

$$\lim_{n\to\infty} \|h - h_n\|_{\mathcal{V}} = 0 \quad \text{and} \quad \lim_{n\to\infty} \|Bh - Bh_n\|_{\mathcal{V}} = 0.$$

This implies $\overline{B|_L} \supseteq B$. ■

In the following, we define an appropriate unbounded and self-adjoint operator on $L^2(\mathbb{T})$. Using the Fourier transform, we will obtain an operator in $\ell^2(\mathbb{Z})$ with the desired properties (a) – (d).

Let $\phi : \mathbb{T} \to \mathbb{R}$ be a measurable function. We define the operator $M_\phi : D(M_\phi) \to L^2(\mathbb{T})$ by setting for $g \in D(M_\phi)$ and $x \in \mathbb{T}$:

$$(M_\phi g)(x) = \phi(x)g(x),$$

where

$$D(M_\phi) = \{g \in L^2(\mathbb{T}) \mid \phi g \in L^2(\mathbb{T})\}. \tag{3.12}$$

Then, one can show that M_ϕ is self-adjoint and that the spectrum of M_ϕ equals the essential range of ϕ, cf. [RS80, § VIII.3]. Thus, if $\phi \notin L^\infty(\mathbb{T})$, i.e. $\operatorname{ess\,sup}_{x\in\mathbb{T}}|\phi(x)| = \infty$, then the operator M_ϕ is unbounded. In the following we show that if additionally $\phi \in L^2(\mathbb{T})$, then the domain of M_ϕ contains the set $\Theta := \mathfrak{F}(C_{\mathrm{c}}(\mathbb{Z}))$. The elements in Θ are finite sums of the type (3.11) and are called *trigonometric polynomials*. Let $\phi \in L^2(\mathbb{T})$ and choose some $f \in C_{\mathrm{c}}(\mathbb{Z})$. Then we have by the triangle inequality

$$\begin{aligned}\|M_\phi \mathfrak{F} f\|_{L^2}^2 &= \frac{1}{2\pi}\int_0^{2\pi} \Big|\phi(\mathrm{e}^{\mathrm{i}t})\sum_{z\in\mathbb{Z}} f(z)\mathrm{e}^{-\mathrm{i}tz}\Big|^2 \mathrm{d}t \\ &\le \frac{1}{2\pi}\int_0^{2\pi} |\phi(\mathrm{e}^{\mathrm{i}t})|^2 \Big|\sum_{z\in\mathbb{Z}}|f(z)|\Big|^2 \mathrm{d}t = \|f\|_1^2\|\phi\|_{L^2}^2 < \infty.\end{aligned}$$

Using Lemma 3.4, this shows that for a given function $\phi : \mathbb{T} \to \mathbb{R}$, with $\phi \in L^2(\mathbb{T}) \setminus L^\infty(\mathbb{T})$ the operator

$$A := \mathfrak{F}^{-1} M_\phi \mathfrak{F} : D(A) \to \ell^2(\mathbb{Z}) \quad \text{with} \quad D(A) = \mathfrak{F}^{-1}(D(M_\phi)) \tag{3.13}$$

is unbounded and self-adjoint and the domain $D(A)$ contains the set $C_{\mathrm{c}}(\mathbb{Z})$. Let us check that this operator is translation invariant. To this end assume that $f \in C_{\mathrm{c}}(\mathbb{Z})$ and $x \in \mathbb{Z}$ are given and calculate similar as before

$$\begin{aligned}(Af)(x) = (\mathfrak{F}^{-1} M_\phi \mathfrak{F} f)(x) &= \frac{1}{2\pi}\int_0^{2\pi} \phi(\mathrm{e}^{\mathrm{i}t})\sum_{z\in\mathbb{Z}} f(z)\mathrm{e}^{\mathrm{i}t(x-z)}\mathrm{d}t \\ &= \sum_{m\in\mathbb{Z}} f(x-m)\frac{1}{2\pi}\int_0^{2\pi} \phi(\mathrm{e}^{\mathrm{i}t})\mathrm{e}^{\mathrm{i}tm}\mathrm{d}t \\ &= \sum_{m\in\mathbb{Z}} f(x-m)(\mathfrak{F}^{-1}\phi)(m).\end{aligned}$$

This proves that A acts as a convolution. Besides this we obtain for

$a, b \in \mathbb{Z}$:

$$\begin{aligned}\langle \delta_{a+z}, A\delta_{b+z}\rangle &= \sum_{x\in\mathbb{Z}} \delta_{a+z}(x) \sum_{m\in\mathbb{Z}} \delta_{b+z}(x-m)(\mathfrak{F}^{-1}\phi)(m)\\ &= \sum_{x\in\mathbb{Z}} \delta_{a}(x-z) \sum_{m\in\mathbb{Z}} \delta_{b}(x-z-m)(\mathfrak{F}^{-1}\phi)(m)\\ &= \langle \delta_a, A\delta_b\rangle \,.\end{aligned}$$

Thus, under the assumption $\psi \in L^2(\mathbb{T})\setminus L^\infty(\mathbb{T})$, the operator A given in (3.13) fulfills properties (a), (b) and (c). In order to verify (d) we need an additional assumption on ϕ.

Let $\psi : \mathbb{T} \to \mathbb{R}$ be a continuous function such that $\phi := 1/\psi$ is in $L^2(\mathbb{T}) \setminus L^\infty(\mathbb{T})$. For instance one can choose $\psi(\mathrm{e}^{\mathrm{i}t}) := |\sin(t)|^{1/4}$ for $t \in [0, 2\pi)$. We need to show that $C_{\mathrm{c}}(\mathbb{Z})$ is a core of A, i.e. that $\overline{A|_{C_{\mathrm{c}}(\mathbb{Z})}} = A$. By Lemma 3.4 it is sufficient to prove this in the Fourier space. Thus, our aim is to verify the following equality:

$$\overline{M_\phi|_\Theta} = M_\phi. \tag{3.14}$$

Denote by $C(\mathbb{T})$ the set of continuous functions mapping from $\mathbb{T}$ to $\mathbb{C}$. The first step to verify (3.14) is to show

$$\overline{M_\phi|_{C(\mathbb{T})}} = M_\phi. \tag{3.15}$$

As before we only need to show the inclusion $\overline{M_\phi|_{C(\mathbb{T})}} \supseteq M_\phi$. To this end let $g \in D(M_\phi)$ be given. By (3.12) we have $g \in L^2(\mathbb{T})$ and $M_\phi g = \phi g \in L^2(\mathbb{T})$. As $C(\mathbb{T})$ is dense in $L^2(\mathbb{T})$, we find a sequence (h_n) of elements in $C(\mathbb{T})$ with $\lim_{n\to\infty}\|M_\phi g - h_n\|_{L^2} = 0$. We set $g_n := \psi h_n$. As $\psi \in C(\mathbb{T})$, we have $g_n \in C(\mathbb{T})$. Besides this, the choice of (h_n) gives:

$$\lim_{n\to\infty}\|M_\phi g - M_\phi g_n\|_{L_2} = \lim_{n\to\infty}\|M_\phi g - h_n\|_{L_2} = 0.$$

Moreover, we obtain

$$\begin{aligned}\lim_{n\to\infty}\|g - g_n\|_{L_2} &= \lim_{n\to\infty}\|\psi(M_\phi g - h_n)\|_{L_2}\\ &\le \|\psi\|_\infty \lim_{n\to\infty}\|M_\phi g - h_n\|_{L_2} = 0.\end{aligned}$$

Here, $\|\psi\|_\infty = \sup_{x\in\mathbb{T}}|\psi(x)|$ is finite since ψ is continuous. Thus, we verified $\overline{M_\phi|_{C(\mathbb{T})}} \supseteq M_\phi$ and (3.15).

In order to show (3.14) it is now sufficient to prove

$$\overline{M_\phi|_{C(\mathbb{T})}} = \overline{M_\phi|_\Theta}. \tag{3.16}$$

As trigonometric polynomials are continuous we have $\overline{M_\phi|_{C(\mathbb{T})}} \supseteq \overline{M_\phi|_\Theta}$. In order to prove the reverse inclusion, it is sufficient to show $M_\phi|_{C(\mathbb{T})} \subseteq \overline{M_\phi|_\Theta}$. To this end, let $g \in C(\mathbb{T})$ be given. Note that by Weierstraß' theorem the trigonometric polynomials are dense in $C(\mathbb{T})$, with respect to supremum norm, cf. [Rud87, Theorem 4.25]. Thus, we find a sequence (g_n) of elements in Θ with $\lim_{n\to\infty}\|g - g_n\|_\infty = 0$. This clearly gives

$$\lim_{n\to\infty}\|g - g_n\|_{L^2} = 0.$$

Moreover, we obtain

$$\begin{aligned}\lim_{n\to\infty}\|M_\phi g - M_\phi g_n\|_{L^2} &= \lim_{n\to\infty}\|\phi(g - g_n)\|_{L^2} \\ &\leq \lim_{n\to\infty}\|\phi\|_{L^2}\|g - g_n\|_\infty = 0.\end{aligned}$$

This shows $M_\phi|_{C(\mathbb{T})} \subseteq \overline{M_\phi|_\Theta}$ and (3.16).

Therefore, we verified the claim (3.14), which implies using Lemma 3.4 that if $\psi \in C(\mathbb{T})$ with $\phi = 1/\psi \in L^2(\mathbb{T})\setminus L^\infty(\mathbb{T})$, then the operator $A : D(A) \to \ell^2(\mathbb{Z})$ given by (3.13) fulfills properties (a) to (d).

4 Random operators on sofic groups

This chapter is devoted to investigate random operators on sofic group. Similar as in the previous chapter we are interested in the approximation of the spectral distribution function via finite volume analogues. As before, we prove weak convergence of the normalized eigenvalue counting functions and verify a Pastur-Shubin trace formula.

The random operators under consideration are given via their matrix elements. More precisely, we define for each pair of vertices a real-valued random variable. Each such random variable gives rise to a non-diagonal matrix element. The diagonal matrix elements are given as a composition of a new random variable and the non-diagonal elements in the same row, cf. (4.3). In particular, this enables us to treat the graph Laplacian of a long-range percolation graph, as well as the Anderson model.

In Subsection 2.2.2 we already introduced random operators and proved certain crucial properties. Therefore, the first aim in this chapter is to show that the operators we consider here fit in setting of Subsection 2.2.2. Note that here we need to implement certain conditions on the underlying random variables, see (4.1) and (4.2). In particular, this implies that the operators are almost surely essentially self-adjoint and translation invariant in distribution. Afterwards and proceeding in two steps, we first show convergence results in mean, see Section 4.1, and improve this to almost sure convergence in Section 4.2. The reason why we need the intermediate step in Section 4.1 is that we do not have an ergodic theorem at hand. The random operators are not translation invariant for each realizations, but only in distribution. Thus, taking the expectation results in translation invariance. We use this to write the expectation of the normalized trace of an operator as the expectation of the matrix element at the group element id. This and the suitable choice of the approximating operators are basic ingredients for the proof of the convergence in mean.

In order to improve this convergence to a convergence which holds for almost all realizations we make use a concentration inequality by McDiarmid, cf. Theorem 4.6. To apply this for our purposes, we need to make sure that each approximating operator contains not too many random matrix elements. In particular, we need that an approximating operator on a graph with n vertices contains not more than $n^{3/2}$ random matrix elements. This is the reason for the rather involved definition of the finite dimensional operators at the beginning of Section 4.1

We start with the definition of the random operators. Let

$$E_{\mathrm{co}} := \{e \subseteq G \mid |e| \in \{1, 2\}\}$$

be the set of all edges of the complete graph with vertex set G. Furthermore, let $X_e \colon \Omega \to \mathbb{R}$, $e \in E_{\mathrm{co}}$, be independent random variables such that for each $g \in G$ the random variables in

$$\{X_{\{x,y\}} \mid x, y \in G, xy^{-1} = g\} \tag{4.1}$$

are identically distributed. We require further

$$\mathbb{E}\Big(\Big(\sum_{x\in G} |X_{\{\mathrm{id},x\}}|\Big)^2\Big) < \infty. \tag{4.2}$$

Let us emphasize that in this notation we have $X_{\{x,x\}} = X_{\{x\}}$. For some fixed $\alpha \in \mathbb{R}$ and using these random variables, we will define a random operator $\tilde{A} = \tilde{A}_\alpha = (\tilde{A}^{(\omega)})_{\omega\in\Omega} = (\tilde{A}_\alpha^{(\omega)})_{\omega\in\Omega}$. To this end, we set for $x, y \in G$:

$$a^{(\omega)}(x, y) := a_\alpha^{(\omega)}(x, y) := X_{\{x,y\}}(\omega) - \alpha\delta_x(y) \sum_{z\in G\setminus\{x\}} X_{\{x,z\}}(\omega). \tag{4.3}$$

The action of $\tilde{A}$ is given by setting for $\omega \in \Omega$, $f \in C_{\mathrm{c}}(G)$ and $x \in G$:

$$(\tilde{A}^{(\omega)} f)(x) := \sum_{y\in G} a^{(\omega)}(x, y) f(y). \tag{4.4}$$

The next aim is to show that this operator is well-defined as a mapping in the space $\ell^2(G)$ and fits in the setting of Section 2.2.2.

Lemma 4.1. *For each $\omega \in \Omega$ and $x, y \in G$ let $\tilde{a}^{(\omega)}(x, y)$ be given as in (4.3) and $\tilde{A} = (\tilde{A}^{(\omega)})_{\omega \in \Omega}$ as in (4.4). Then $\tilde{A}$ is a symmetric, translation invariant (in distribution) random operator on the domain $C_c(G)$ and the following three expectations are finite:*

$$\mathbb{E}\biggl(\Bigl(\sum_{x \in G} |a(x, \mathrm{id})|\Bigr)^2\biggr), \quad \mathbb{E}\biggl(\sum_{x \in G} |a(x, \mathrm{id})|\biggr) \quad \text{and} \quad \mathbb{E}\biggl(\sum_{x \in G} |a(x, \mathrm{id})|^2\biggr). \tag{4.5}$$

Furthermore, $\tilde{A}$ is almost surely essentially self-adjoint.

Proof. Let us first verify the finiteness of the three expectations. Using (4.2) we obtain

$$\begin{aligned} \mathbb{E}\biggl(\sum_{x \in G} |a(\mathrm{id}, x)|^2\biggr) &\le \mathbb{E}\biggl(\Bigl(\sum_{x \in G} |a(x, \mathrm{id})|\Bigr)^2\biggr) \\ &\le \mathbb{E}\biggl(\Bigl(\sum_{x \in G} |X_{\{\mathrm{id}, x\}}| + \alpha \sum_{z \in G} |X_{\{\mathrm{id}, z\}}|\Bigr)^2\biggr) \\ &= (1+\alpha)^2 \mathbb{E}\biggl(\Bigl(\sum_{x \in G} |X_{\{\mathrm{id}, x\}}|\Bigr)^2\biggr) < \infty. \end{aligned}$$

Furthermore, by Jensen's inequality we have

$$\biggl(\mathbb{E}\Bigl(\sum_{x \in G} |a(\mathrm{id}, x)|\Bigr)\biggr)^2 \le \mathbb{E}\biggl(\Bigl(\sum_{x \in G} |a(x, \mathrm{id})|\Bigr)^2\biggr) < \infty.$$

Next, we show that for almost all ω each $\phi \in C_c(G)$ is mapped by $A^{(\omega)}$ into $\ell^2(G)$. Note that $C_c(G) = \mathrm{lin}\{\delta_g \mid g \in G\}$. For fixed $g \in G$ we have

$$\mathbb{E}\biggl(\sum_{x \in G} |(\tilde{A}\delta_g)(x)|^2\biggr) = \mathbb{E}\biggl(\sum_{x \in G} |a(x, g)|\biggr) = \mathbb{E}\biggl(\sum_{x \in G} |a(x, \mathrm{id})|\biggr) < \infty.$$

Hence, for each $g \in G$ there exists a set $\Omega_g \subseteq \Omega$ of full measure such that $\|\tilde{A}^{(\omega)}\delta_g\|_2$ is finite. Setting $\tilde{\Omega} := \bigcap_{g \in G} \Omega_g$ and using linearity gives a set of full measure such that for all $\omega \in \tilde{\Omega}$ and all $\phi \in C_c(G)$ one has $\tilde{A}^{(\omega)}\phi \in \ell^2(G)$.

To prove that $\tilde{A}$ is a random operator on the domain $C_c(G)$ it remains to show that for all $\phi \in C_c(G)$ and $\psi \in \ell^2(G)$ the mapping

$\omega \mapsto \langle \psi, \tilde{A}^{(\omega)}\phi \rangle$ from $\tilde{\Omega}$ to $\mathbb{C}$ is measurable. Of course the corresponding sigma-algebras are $\tilde{\mathcal{A}} := \{D \cap \tilde{\Omega} \mid D \in \mathcal{A}\}$ on $\tilde{\Omega}$ and the Borel sigma-algebra on $\mathbb{C}$. For all $x, y \in G$ the mapping $\omega \mapsto a^{(\omega)}(x,y)$ on $(\tilde{\Omega}, \tilde{\mathcal{A}})$ is measurable by construction. This carries over to

$$\omega \mapsto \langle \psi, \tilde{A}^{(\omega)}\phi \rangle = \lim_{r\to\infty} \sum_{x \in B_r^G(\mathrm{id})} \psi(x) \sum_{y \in \operatorname{spt} \phi} a^{(\omega)}(x,y)\phi(y)$$

as limits and sums of measurable functions are measurable. The symmetry of $\tilde{A}$ follows from the definition of the matrix elements $a^{(\omega)}(x,y)$ in (4.3). The translation invariance in distribution is implied by the condition (4.1). Now, the fact that $\tilde{A}$ is almost surely essentially self-adjoint follows directly from Theorem 2.19. ■

Lemma 4.1 shows that there exists a set $\tilde{\Omega}$ of measure one such that for each $\omega \in \tilde{\Omega}$ the closure $\bar{A}^{(\omega)}$ of $\tilde{A}^{(\omega)}$ is self-adjoint and $C_c(G) \subseteq D(A^{(\omega)})$. In order to define A on the whole probability space we set for $\omega \in \Omega$:

$$A^{(\omega)} := \begin{cases} \bar{A}^{(\omega)} & \text{if } \omega \in \tilde{\Omega}, \\ \mathrm{Id} & \text{otherwise}, \end{cases} \tag{4.6}$$

where Id is the identity on $\ell^2(G)$. Thus $A = (A^{(\omega)})_{\omega\in\Omega}$ is a proper random operator. We will refer to the operator A as *random Hamiltonian*. The reason for that is described in the next remark.

Remark 4.2 (Random Hamiltonian). Let us briefly discuss the operator A for the different choices of α. In the case $\alpha = 0$ the operator is an adjacency matrix on graphs with vertex set G and random weights on the edges. For $\alpha = 1$ and $X_{\{x\}} = 0$ a.s. A can be interpreted a randomly weighted Laplace operator on such graphs. More generally, if the the diagonal terms $X_{\{x\}}$ do not equal zero, they can be understood as random potential. This setting is well studied under the term *Anderson model*.

Note that Lemma 4.1 implies

$$\mathbb{E}\left(\|A\delta_{\mathrm{id}}\|_1^2\right) < \infty, \quad \mathbb{E}\left(\|A\delta_{\mathrm{id}}\|_1\right) < \infty \quad \text{and} \quad \mathbb{E}\left(\|A\delta_{\mathrm{id}}\|_2^2\right) < \infty. \tag{4.7}$$

Remark 4.3 (Ergodicity). One can show, if one considers the canonical probability space with the canonical action of translations, that the operator A as given in (4.6) is ergodic. This is of special interest as one knows that ergodic operators exhibit a non-random spectrum. Therefore it is natural to expect that the limit of the eigenvalue counting functions, the integrated density of states, is non-random, too.

However, in the following we do not consider A on its canonical probability space. The reason for not using the canonical space is that we need to introduce more random variables for the approximating finite dimensional operators. Of course, this will not change spectral properties of the operator and one can still expect non-randomness the IDS.

The next well-known lemma gives conditions for boundedness and unboundedness of the operator in question.

Lemma 4.4. *Let A be a random Hamiltonian as given in* (4.6) *with random variables $X_e, E \in E_{\mathrm{co}}$ and $D := \sup_{x \in G} \|X_{\{\mathrm{id},x\}}\|_\infty \in [0, \infty]$.*

(i) If $D = \infty$, then $\|A^{(\omega)}\| = \infty$ for almost all $\omega \in \Omega$.

(ii) If $D < \infty$ and A is of finite hopping range R, i.e. for almost all $\omega \in \Omega$ we have $a^{(\omega)}(x, y) = 0$ whenever $d(x, y) \geq R$, then there exists $c > 0$ such that for almost all $\omega \in \Omega$ we have $\|A^{(\omega)}\| \leq c$.

Proof. Assume $D = \infty$. Condition (4.2) implies that

$$k := \mathbb{E}\Big(\sum_{z \neq \mathrm{id}} |X_{\{\mathrm{id},z\}}| \Big) < \infty.$$

Fix some $m \geq 2k|\alpha|$. As D is assumed to be infinite there exists $z \in G$ such that

$$\|X_{\{\mathrm{id},z\}}\|_\infty \geq 2m. \tag{4.8}$$

We distinguish two cases. If the z satisfying (4.8) is not $\mathrm{id} \in G$, the probability $\mathbb{P}(|a(\mathrm{id}, z)| \geq m)$ is strictly positive. If $z = \mathrm{id}$, i.e. $\|X_{\mathrm{id}}\|_\infty \geq 2m$, the same holds true, however we need a short calculation to see this. By definition of $a(\mathrm{id}, \mathrm{id})$ and using triangle inequality we have

$$\begin{aligned}\mathbb{P}(|a(\mathrm{id},\mathrm{id})| \geq m) &\geq \mathbb{P}\Big(|X_{\{\mathrm{id}\}}| - \Big|\alpha \sum_{z\in G\setminus\{\mathrm{id}\}} X_{\{\mathrm{id},z\}}\Big| \geq m\Big)\\ &\geq \mathbb{P}\Big(|X_{\{\mathrm{id}\}}| \geq 2m,\ \Big|\alpha \sum_{z\in G\setminus\{\mathrm{id}\}} X_{\{\mathrm{id},z\}}\Big| \leq m\Big)\\ &= \mathbb{P}\big(|X_{\{\mathrm{id}\}}| \geq 2m\big)\mathbb{P}\Big(\Big|\alpha \sum_{z\in G\setminus\{\mathrm{id}\}} X_{\{\mathrm{id},z\}}\Big| \leq m\Big).\end{aligned}$$

As $\|X_{\{\mathrm{id}\}}\|_\infty \geq 2m$, we get $\mathbb{P}(|X_{\{\mathrm{id}\}}| \geq 2m) > 0$. We use the Markov inequality to obtain

$$\mathbb{P}\Big(\Big|\alpha \sum_{z\in G\setminus\{\mathrm{id}\}} X_{\{\mathrm{id},z\}}\Big| \leq m\Big) \geq 1 - \frac{|\alpha|}{m}\mathbb{E}\Big(\Big|\sum_{z\in G\setminus\{\mathrm{id}\}} X_{\{\mathrm{id},z\}}\Big|\Big) \geq \frac{1}{2}.$$

This gives $\mathbb{P}(|a(\mathrm{id},\mathrm{id})| \geq m) > 0$. Thence, whenever $D = \infty$, there exists $z \in G$ such that $\mathbb{P}(|a(\mathrm{id},z)| \geq m)$ is positive. Furthermore, by construction we have that the random variables $a(x, zx)$, $x \in G$ are independent and identically distributed, so we get

$$\sum_{x\in G} \mathbb{P}(|a(x,zx)| \geq m) = \infty.$$

Hence, Borel-Cantelli gives that for almost all $\omega \in \Omega$ there are infinitely many $x \in G$ such that $|a^{(\omega)}(x, zx)| \geq m$. For each such ω, we choose one of these x and obtain with $(A^{(\omega)}\delta_{zx})(x) = a^{(\omega)}(x, zx)$ that

$$\|A^{(\omega)}\| \geq \|A^{(\omega)}\delta_{zx}\|_2 \geq m.$$

Since $m \geq 2k|\alpha|$ was arbitrary, $\|A^{(\omega)}\| = \infty$ for almost all ω.

Let $D < \infty$ and A be of finite hopping range R. We set $m := (1 + |\alpha||B_R^G|)D$. Then we have

$$\begin{aligned}\mathbb{P}(\exists x, y \in G \text{ with } |a(x, y)| \geq m) &= \mathbb{P}\Big(\bigcup_{x,y\in G}\big\{\omega \in \Omega \mid |a^{(\omega)}(x, y)| \geq m\big\}\Big)\\ &\leq \sum_{x,y\in G} \mathbb{P}\big(\big\{\omega \in \Omega \mid |a^{(\omega)}(x, y)| \geq m\big\}\big) = 0.\end{aligned}$$

Using this we get for $f \in \ell^2(G)$ and almost all realizations $\omega \in \Omega$

$$\begin{aligned}\|A^{(\omega)} f\|_2^2 &= \sum_{v\in G}\Big|\sum_{w\in B_R^G(v)} a^{(\omega)}(v,w)f(w)\Big|^2 \le \sum_{v\in G} m^2\Big(\sum_{w\in B_R^G(v)} |f(w)|\Big)^2 \\ &\le \sum_{v\in G} m^2 |B_R^G| \sum_{w\in B_R^G(v)} |f(w)|^2 \le m^2|B_R^G|^2\|f\|_2^2.\end{aligned}$$

This shows that for almost all ω the operator $A^{(\omega)}$ is bounded with constant $c := m^2|B_R^G|^2$. ■

4.1 Weak convergence in mean

In this section we investigate the operators $A = (A^{(\omega)})_{\omega\in\Omega}$ defined in (4.6), which we call random Hamiltonians. We define finite dimensional operators on the approximating graphs and study their eigenvalue counting functions. We are not yet able to show convergence of the eigenvalue counting functions itself, but we first concentrate on the convergence of the mean of these functions. This is easier since taking expectations induces translation invariance, which is crucial as we do not have an ergodic theorem for sofic groups.

We start with the definition of finite dimensional approximations to A. We consider the approximating graphs Γ_r, $r \in \mathbb{N}$, and use the simplified notation (2.2). As before in (2.3) the map $\Psi_{r,x} \colon B_r^{V_r}(x) \to B_r^G$ is a labeled graph isomorphism. As in the deterministic setting, we will use this function to transport the values of A to the approximation.

We define an increasing function $\rho : \mathbb{N} \to \mathbb{R}$ by setting for $r \in \mathbb{N}$:

$$\rho(r) := \max\left\{0, \frac{\ln r}{4\ln|S|} - 1\right\}.$$

This ρ will substitute the $r/6$ from the deterministic setting, which we discussed in Remark 3.2. Note that for all $r \in \mathbb{N}$ we have $\rho(r) \le r/6$ and $\rho(r) \to \infty$ if $r \to \infty$. In the deterministic setting it was enough to set $\rho(r) = r/6$. This would be still sufficient for the proof the main result of this section, namely Theorem 4.5. However, in the proof of Theorem 4.7 we need this slow growth of the function ρ, since there we require that our random matrices do not contain too many random elements.

For an element $e \in E_{\mathrm{co}}^{(r)} := \{e \subseteq V_r \mid |e| \in \{1,2\}\}$ of the edge set of the complete graph over V_r we define

$$C_r(e) := \{v \in V_r^{(0)} \mid e \subseteq B_{\rho(r)}^{V_r}(v)\}.$$

Thus, $C_r(e)$ consists of those elements in $V_r^{(0)}$, such that the $\rho(r)$-ball around these elements contains the vertices of the edge e. In order to introduce the approximating operator on $\ell^2(V_r)$, we need to choose a suitable random variable for each $e \in E_{\mathrm{co}}^{(r)}$. These random variable are selected in the following way. Let $e = \{x,y\} \in E_{\mathrm{co}}^{(r)}$ and $r \in \mathbb{N}$ be given. If $C_r(e) = \emptyset$, we set $X_e^r = 0$. Otherwise, we choose some $v \in C_r(e)$ and define X_e^r to be the a random variable which has the same distribution as $X_{\{\Psi_{r,v}(x),\Psi_{r,y}(w)\}}$. Moreover, we require that all random variables in

$$\big\{X_e \bigm| e \in E_{\mathrm{co}}\big\} \cup \big\{X_e^r \bigm| r \in \mathbb{N}, e \in E_{\mathrm{co}}^{(r)}\big\}$$

are independent.

Note that the distribution of X_e^r does not depend on the choice of $v \in C_r(e)$. To see this, let $r \in \mathbb{N}$ and $x, y \in V_r$ with $|C_r(\{x,y\})| \geq 2$ be given. Choose $v, w \in C_r(\{x,y\})$ with $v \neq w$. Then, by definition of $C_r(\{x,y\})$ we have that $x, y \in B_{\rho(r)}^{V_r}(v) \cap B_{\rho(r)}^{V_r}(w)$. Thus, by Lemma 2.3 the equality

$$\Psi_{r,v}(x)(\Psi_{r,v}(y))^{-1} = \Psi_{r,w}(x)(\Psi_{r,w}(y))^{-1},$$

holds. This gives that the random variables

$$X_{\{\Psi_{r,v}(x),\Psi_{r,v}(y)\}} \quad \text{and} \quad X_{\{\Psi_{r,w}(x),\Psi_{r,w}(y)\}}$$

are identically distributed, cf. (4.1). Hence, the distribution of $X_{\{x,y\}}^r$ does not depend on the $v \in C_r(\{x,y\})$.

We are now in the position to define a random approximating operator $A_r^{(\omega)} : \ell^2(V_r) \to \ell^2(V_r)$, $\omega \in \Omega$ depending on the parameter $\alpha \in \mathbb{R}$ from (4.3). We set for each $x, y \in V_r$ and $\omega \in \Omega$:

$$a_r^{(\omega)}(x,y) := X_{\{x,y\}}^r(\omega) - \alpha\delta_x(y) \sum_{z \in V_r \setminus \{x\}} X_{\{x,z\}}^r(\omega),$$

and for $x \in V_r$ and $\phi \in \ell^2(V_r)$:

$$(A_r^{(\omega)}\phi)(x) := \sum_{y \in V_r} a_r^{(\omega)}(x,y)\phi(y).$$

Note, that $A_r^{(\omega)}$ has hopping range $2\rho(r)$, i.e. $a_r^{(\omega)}(x,y) = 0$, as soon as $d_r(x,y) > 2\rho(r)$. The operator $A_r^{(\omega)}$ is symmetric and hence self-adjoint. As before, we define eigenvalue counting functions. For each $\omega \in \Omega$, $r \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ we set

$$\mathfrak{n}_r^{(\omega)} \colon \mathbb{R} \to [0,1], \quad \mathfrak{n}_r^{(\omega)}(\lambda) := \mathfrak{n}(A_r^{(\omega)})(\lambda),$$

where again $\mathfrak{n}(A_r^{(\omega)})$ is the normalized eigenvalue counting function as given in Definition 2.23. Besides this, we set for $\omega \in \Omega$ and $\lambda \in \mathbb{R}$

$$\mathfrak{N}^{(\omega)} \colon \mathbb{R} \to [0,1] \qquad \mathfrak{N}^{(\omega)}(\lambda) := \langle \delta_{\mathrm{id}}, E_\lambda^{(\omega)} \delta_{\mathrm{id}} \rangle, \tag{4.9}$$

where again $E_\lambda^{(\omega)}$ is the spectral projection of $A^{(\omega)}$ on the interval $(-\infty, \lambda]$. Furthermore, we define the functions $\bar{\mathfrak{n}}_r, \bar{\mathfrak{N}} \colon \mathbb{R} \to [0,1]$ by setting for $\lambda \in \mathbb{R}$ and $r \in \mathbb{N}$:

$$\bar{\mathfrak{N}}(\lambda) = \mathbb{E}(\mathfrak{N}(\lambda)) \quad \text{and} \quad \bar{\mathfrak{n}}_r(\lambda) = \mathbb{E}(\mathfrak{n}_r(\lambda)). \tag{4.10}$$

The function $\bar{\mathfrak{N}}$ is called *spectral distribution function* of the random operator A. If the limit $\mathfrak{I} := \lim_{r\to\infty} \mathfrak{n}_r^{(\omega)}$ exists, then $\mathfrak{I}$ is called *integrated density of states*.

Theorem 4.5. *Let G be a finitely generated sofic group and let A be given as in* (4.6)*. Furthermore let $\bar{\mathfrak{n}}_r, \bar{\mathfrak{N}} \colon \mathbb{R} \to [0,1]$ be as in* (4.10)*. Then*

$$\bar{\mathfrak{N}} = \operatorname*{w-lim}_{r\to\infty} \bar{\mathfrak{n}}_r.$$

Proof. By Lemma 2.29 it is sufficient to prove that the associated Stieltjes transforms converge pointwise, i.e. for all $z \in \mathbb{C} \setminus \mathbb{R}$ we have to show

$$\lim_{r\to\infty} \mathfrak{r}(\bar{\mathfrak{n}}_r)(z) = \mathfrak{r}(\bar{\mathfrak{N}})(z). \tag{4.11}$$

Fix $z \in \mathbb{C} \setminus \mathbb{R}$. The integral with respect to the distribution function $\bar{\mathfrak{n}}_r$ is actually a finite sum. This and linearity of the expectation yield

$$\begin{aligned}\mathbb{E}\big(\mathfrak{r}(\mathfrak{n}_r)(z)\big) &= \mathbb{E}\left(\int_{\mathbb{R}} (z-\lambda)^{-1} \mathrm{d}\mathfrak{n}_r(\lambda)\right) \\ &= \int_{\mathbb{R}} (z-\lambda)^{-1} \mathrm{d}\bar{\mathfrak{n}}_r(\lambda) = \mathfrak{r}(\bar{\mathfrak{n}}_r)(z). \end{aligned} \tag{4.12}$$

The Riemann-Stieltjes-Integral with respect to $\mathfrak{N}^{(\omega)}$ is as usual defined by

$$\begin{aligned}\int_{\mathbb{R}} (z-\lambda)^{-1}\, \mathrm{d}\mathfrak{N}^{(\omega)}(\lambda) = &\int_{-\infty}^{0} (z-\lambda)^{-1}\, \mathrm{d}\mathfrak{N}^{(\omega)}(\lambda) \\ &+ \int_0^{\infty} (z-\lambda)^{-1}\, \mathrm{d}\mathfrak{N}^{(\omega)}(\lambda) \end{aligned} \tag{4.13}$$

where

$$\begin{aligned}&\int_0^{\infty} (z-\lambda)^{-1}\, \mathrm{d}\mathfrak{N}^{(\omega)}(\lambda) \\ &\qquad := \lim_{L\to\infty} \lim_{\Delta\lambda\to 0} \sum_{j=0}^{k-1} (z-\lambda_j)^{-1} \big(\mathfrak{N}^{(\omega)}(\lambda_{j+1}) - \mathfrak{N}^{(\omega)}(\lambda_j)\big)\end{aligned}$$

with partitions $0 = \lambda_0 < \lambda_1 < \dots < \lambda_k = L$ and their mesh size $\Delta\lambda := \max_{j=0}^{k-1} \lambda_{j+1} - \lambda_j$. Since $|(z-\lambda)^{-1}| \le |\Im(z)|^{-1}$, we have

$$\left|\sum_{j=0}^{k-1} (z-\lambda_j)^{-1} \big(\mathfrak{N}^{(\omega)}(\lambda_{j+1}) - \mathfrak{N}^{(\omega)}(\lambda_j)\big)\right| \le \frac{1}{|\Im(z)|},$$

which gives an integrable bound. Therefore we get by Lebesgue's theorem

$$\begin{aligned}&\mathbb{E}\left(\int_0^{\infty} (z-\lambda)^{-1} \mathrm{d}\mathfrak{N}(\lambda)\right) \\ &= \lim_{K\to\infty} \lim_{\Delta\lambda\to 0} \sum_{j=0}^{k-1} (z-\lambda_j)^{-1} \big(\bar{\mathfrak{N}}(\lambda_{j+1}) - \bar{\mathfrak{N}}(\lambda_j)\big) = \int_0^{\infty} (z-\lambda)^{-1} \mathrm{d}\bar{\mathfrak{N}}(\lambda).\end{aligned}$$

The same is true for the other summand in (4.13), which gives

$$\begin{aligned}\mathbb{E}\big(\mathfrak{r}(\mathfrak{N})(z)\big) &= \mathbb{E}\left(\int_{\mathbb{R}} (z-\lambda)^{-1} \mathrm{d}\mathfrak{N}(\lambda)\right) \\ &= \int_{\mathbb{R}} (z-\lambda)^{-1} \mathrm{d}\bar{\mathfrak{N}}(\lambda) = \mathfrak{r}(\bar{\mathfrak{N}})(z). \end{aligned} \tag{4.14}$$

We define the difference D_r and apply (4.12) and (4.14):

$$D_r := \big|\mathfrak{r}(\bar{\mathfrak{n}}_r)(z) - \mathfrak{r}(\bar{\mathfrak{N}})(z)\big| = \big|\mathbb{E}\big(\mathfrak{r}(\mathfrak{n}_r)(z)\big) - \mathbb{E}\big(\mathfrak{r}(\mathfrak{N})(z)\big)\big|.$$

In order to estimate D_r we use that for each ω we have

$$\begin{aligned}&\int_{\mathbb{R}} (z-\lambda)^{-1} \mathrm{d}\mathfrak{n}_r^{(\omega)}(\lambda) = \frac{1}{|V_r|} \sum_{\lambda \in \sigma(A_r^{(\omega)})} m_\lambda (z-\lambda)^{-1} \\ &= \frac{1}{|V_r|} \operatorname{Tr}((z-A_r^{(\omega)})^{-1}) = \frac{1}{|V_r|} \sum_{x \in V_r} \langle \delta_x, (z-A_r^{(\omega)})^{-1} \delta_x \rangle,\end{aligned}$$

where again m_λ denotes the multiplicity of an eigenvalue λ. This and the spectral theorem lead to

$$D_r = \left| \mathbb{E}\left(\frac{1}{|V_r|} \sum_{x\in V_r} \langle \delta_x, (z-A_r)^{-1}\delta_x \rangle \right) - \mathbb{E}\big(\langle \delta_{\mathrm{id}}, (z-A)^{-1}\delta_{\mathrm{id}}\rangle\big) \right|.$$

Now, we apply the triangle inequality and the properties of the sofic approximation to obtain

$$\begin{aligned} D_r &\le \frac{1}{|V_r|} \sum_{x \in V_r^{(0)}} \big|\mathbb{E}\big(\langle \delta_x, (z-A_r)^{-1}\delta_x\rangle\big) - \mathbb{E}\big(\langle \delta_{\mathrm{id}}, (z-A)^{-1}\delta_{\mathrm{id}}\rangle\big)\big| \\ &\quad + \frac{1}{|V_r|} \sum_{x \in V_r \setminus V_r^{(0)}} \big|\mathbb{E}\big(\langle \delta_x, (z-A_r)^{-1}\delta_x\rangle\big) - \mathbb{E}\big(\langle \delta_{\mathrm{id}}, (z-A)^{-1}\delta_{\mathrm{id}}\rangle\big)\big| \\ &\le \sup_{x \in V_r^{(0)}} \big|\mathbb{E}\big(\langle \delta_x, (z-A_r)^{-1}\delta_x\rangle\big) - \mathbb{E}\big(\langle \delta_{\mathrm{id}}, (z-A)^{-1}\delta_{\mathrm{id}}\rangle)\big)\big| + \frac{2\varepsilon(r)}{|\Im(z)|}. \end{aligned}$$

Here we used $\|(z-H)^{-1}\| \le |\Im(z)|^{-1}$ for self-adjoint H. Again, we apply the construction of the proof of Theorem 3.3 to obtain an operator $A^{(\omega)}_{r,x}$ which transports our approximation to the space $\ell^2(G)$. As before we extend the graph isomorphism $\Psi_{r,x}$ at $x \in V_r^{(0)}$ from (2.3) injectively to

$$\Psi'_{r,x} \colon V_r \to G.$$

which induces a projection

$$\Phi_{r,x} \colon \ell^2(G) \to \ell^2(V_r), \quad \Phi_{r,x}(f) := f \circ \Psi'_{r,x}.$$

The operator $A^{(\omega)}_{r,x}$ is given by

$$A^{(\omega)}_{r,x} := \Phi^*_{r,x} A^{(\omega)}_r \Phi_{r,x} \colon \ell^2(G) \to \ell^2(G)$$

and satisfies as before for all $x \in V_r^{(0)}$

$$\langle \delta_x, (z - A^{(\omega)}_r)^{-1} \delta_x \rangle = \langle \delta_{\mathrm{id}}, (z - A^{(\omega)}_{r,x})^{-1} \delta_{\mathrm{id}} \rangle. \tag{4.15}$$

However, we still need to change some matrix elements of $A^{(\omega)}_{r,x}$ in order to control the difference of $A^{(\omega)}$ and its approximation. For each $x \in V_r^{(0)}$ we define a new approximating operator $\hat{A}^{(\omega)}_{r,x} \colon \ell^2(G) \to \ell^2(G)$ by its matrix elements

$$\hat{a}^{(\omega)}_{r,x}(g,h) := \begin{cases} a^{(\omega)}(g,h) & \text{if } g,h \in B^G_{\rho(r)}, g \neq h, \\ X_{\{g\}}(\omega) - \alpha \sum\limits_{k \in G \setminus \{g\}} \hat{a}^{(\omega)}_{r,x}(g,k) & \text{if } g = h \in B^G_{\rho(r)}, \\ a^{(\omega)}_{r,x}(g,h) & \text{otherwise.} \end{cases}$$

Here $X_{\{g\}}(\omega)$ is the random variable which equals the matrix element $a^{(\omega)}(g,g)$ of $A^{(\omega)}$ in the case $\alpha = 0$, see (4.3). This gives that still for each $g,h \in G$, the distribution of $a^{(\omega)}_{r,x}(g,h)$ equals the distribution of $\hat{a}^{(\omega)}_{r,x}(g,h)$. Thus, we have the following equality in expectation

$$\mathbb{E}\langle \delta_{\mathrm{id}}, (z - A_{r,x})^{-1} \delta_{\mathrm{id}} \rangle = \mathbb{E}\langle \delta_{\mathrm{id}}, (z - \hat{A}_{r,x})^{-1} \delta_{\mathrm{id}} \rangle. \tag{4.16}$$

Now, we choose an arbitrary $\kappa > 0$ and obtain by Theorem 2.22 an integer $n \in \mathbb{N}$ and a random vector ψ with $\mathbb{E}(\|\delta_{\mathrm{id}} - \psi\|_2) \leq \kappa$ and

$\operatorname{spt}\big((z-A^{(\omega)})^{-1}\psi(\omega)\big) \subseteq B_n^G$ for all $\omega \in \Omega$. Note that here we applied the fact that A is a proper random operator. We use the equalities (4.15) and (4.16) in the last estimate for D_r and insert the random vector ψ to get

$$\begin{aligned} D_r &\le \sup_{x\in V_r^{(0)}} \big|\mathbb{E}\langle \delta_{\mathrm{id}}, (z-\hat{A}_{r,x})^{-1}\delta_{\mathrm{id}}\rangle - \mathbb{E}\langle \delta_{\mathrm{id}}, (z-A)^{-1}\delta_{\mathrm{id}}\rangle\big| + \frac{2\varepsilon(r)}{|\Im(z)|} \\ &\le \sup_{x\in V_r^{(0)}} \big|\mathbb{E}\big(\langle \delta_{\mathrm{id}}, \big((z-\hat{A}_{r,x})^{-1} - (z-A)^{-1}\big)\psi\rangle\big)\big| \\ &\quad + \sup_{x\in V_r^{(0)}} \big|\mathbb{E}\big(\langle \delta_{\mathrm{id}}, \big((z-\hat{A}_{r,x})^{-1} - (z-A)^{-1}\big)(\delta_{\mathrm{id}}-\psi)\rangle\big)\big| + \frac{2\varepsilon(r)}{|\Im(z)|}. \end{aligned}$$

With another application of the Cauchy-Schwarz inequality and with the boundedness of the resolvents we estimate the supremum in the last expression. We deduce

$$\begin{aligned} D_r \le &\sup_{x\in V_r^{(0)}} \big|\mathbb{E}\big(\langle \delta_{\mathrm{id}}, \big((z-\hat{A}_{r,x})^{-1} - (z-A)^{-1}\big)\psi\rangle\big)\big| \\ &+ 2\frac{\varepsilon(r) + \mathbb{E}(\|\delta_{\mathrm{id}}-\psi\|_2)}{|\Im(z)|}. \end{aligned}$$

The second resolvent identity and the special choice of ψ according to Theorem 2.22 imply

$$\begin{aligned} D_r &\le \sup_{x\in V_r^{(0)}} \big|\mathbb{E}\big(\langle \delta_{\mathrm{id}}, (z-\hat{A}_{r,x})^{-1}(A-\hat{A}_{r,x})(z-A)^{-1}\psi\rangle\big)\big| + 2\frac{\varepsilon(r)+\kappa}{|\Im(z)|} \\ &\le \frac{1}{|\Im(z)|} \sup_{x\in V_r^{(0)}} \mathbb{E}\big(\|(A-\hat{A}_{r,x})\phi\|_2\big) + 2\frac{\varepsilon(r)+\kappa}{|\Im(z)|}, \end{aligned} \tag{4.17}$$

where ϕ is a random vector given by $\phi(\omega) := (z-A^{(\omega)})^{-1}\psi(\omega)$. Now assume that r is so large that $\rho(r) \ge n$, where still $n \in \mathbb{N}$ is the integer given by Theorem 2.22. Note that for all $\omega \in \Omega$ the vector $\phi(\omega)$ is supported in $B_n^G \subseteq B_{\rho(r)}^G$ and

$$\|\phi(\omega)\|_\infty \le \|\phi(\omega)\|_2 = \|(z-A^{(\omega)})^{-1}\psi(\omega)\|_2 \le \frac{1+\kappa}{|\Im(z)|}. \tag{4.18}$$

Now we continue the estimation for $x \in V_r^{(0)}$ using subadditivity of the square root:

$$\mathbb{E}\big(\|(A - \hat{A}_{r,x})\phi\|_2\big) = \mathbb{E}\Bigg(\Bigg(\sum_{g\in G}\Big|\sum_{h\in\mathrm{spt}(\phi)}(a(g,h) - \hat{a}_{r,x}(g,h))\phi(h)\Big|^2\Bigg)^{\frac{1}{2}}\Bigg)$$

$$\le \mathbb{E}\Bigg(\Bigg(\sum_{g\in B^G_{\rho(r)}}\Big|\sum_{h\in\mathrm{spt}(\phi)}(a(g,h) - \hat{a}_{r,x}(g,h))\phi(h)\Big|^2\Bigg)^{\frac{1}{2}}$$

$$+\Bigg(\sum_{g\in G\setminus B^G_{\rho(r)}}\Big|\sum_{h\in\mathrm{spt}(\phi)}(a(g,h) - \hat{a}_{r,x}(g,h))\phi(h)\Big|^2\Bigg)^{\frac{1}{2}}\Bigg).$$

By construction of $\hat{A}^{(\omega)}_{r,x}$ we have $a^{(\omega)}(g,h) = \hat{a}^{(\omega)}_{r,x}(g,h)$ for distinct $g,h \in B^G_{\rho(r)}$. This yields

$$\mathbb{E}\big(\|(A - \hat{A}_{r,x})\phi\|_2\big) \le T_1(r) + T_2(r) \tag{4.19}$$

with

$$T_1(r) := \mathbb{E}\Bigg(\Bigg(\sum_{g\in\mathrm{spt}(\phi)}\Big|(a(g,g) - \hat{a}_{r,x}(g,g))\phi(g)\Big|^2\Bigg)^{1/2}\Bigg)$$

and

$$T_2(r) := \mathbb{E}\Bigg(\Bigg(\sum_{g\in G\setminus B^G_{\rho(r)}}\Big|\sum_{h\in\mathrm{spt}(\phi)}(a(g,h) - \hat{a}_{r,x}(g,h))\phi(h)\Big|^2\Bigg)^{1/2}\Bigg).$$

Let us estimate these terms separately. In order to deal with $T_1(r)$, recall the definition of the diagonal terms of A and $\hat{A}_{r,x}$ to obtain for each $\omega \in \Omega$:

$$a^{(\omega)}(g,g) - \hat{a}^{(\omega)}_{r,x}(g,g) = \alpha \sum_{h\in G\setminus B^G_{\rho(r)}}\left(a(g,h) - \hat{a}^{(\omega)}_{r,x}(g,h)\right).$$

This gives, using the estimate (4.18), Cauchy Schwarz inequality and

Jensen inequality,

$$
\begin{aligned}
T_1(r) &\le |\alpha| \|\phi\|_\infty \mathbb{E}\Big(\Big(\sum_{g\in\operatorname{spt}(\phi)} \Big| \sum_{h\in G\setminus B^G_{\rho(r)}} \Big(a(g,h) - \hat{a}^{(\omega)}_{r,x}(g,h)\Big)\Big|^2\Big)^{\frac{1}{2}}\Big) \\
&\le \frac{|\alpha|(\kappa+1)}{\Im(z)} \mathbb{E}\Big(\Big(\sum_{g\in\operatorname{spt}(\phi)} \Big(\sum_{h\in G\setminus B^G_{\rho(r)}} |a(g,h)| + \sum_{h\in G\setminus B^G_{\rho(r)}} |\hat{a}^{(\omega)}_{r,x}(g,h)|\Big)^2\Big)^{\frac{1}{2}}\Big) \\
&\le \frac{\sqrt{2}|\alpha|(\kappa+1)}{\Im(z)} \Big(\mathbb{E}\Big(\sum_{g\in\operatorname{spt}(\phi)} \Big(\sum_{h\in G\setminus B^G_{\rho(r)}} |a(g,h)|\Big)^2 \\
&\qquad + \Big(\sum_{h\in G\setminus B^G_{\rho(r)}} |\hat{a}^{(\omega)}_{r,x}(g,h)|\Big)^2\Big)\Big)^{\frac{1}{2}}.
\end{aligned}
$$

A calculation similar as in the deterministic setting, see (3.8) and Remark 3.2, using $\rho(r) \le r/6$, shows that the sum over the $\hat{a}^{(\omega)}_{r,x}(g,h)$ can be estimated by a sum over $a(g,h)$. To be precise, we first use independence to obtain for $g \in \operatorname{spt}(\phi)$:

$$
\begin{aligned}
\mathbb{E}\Big(\Big(\sum_{h\in G\setminus B^G_{\rho(r)}} |\hat{a}_{r,x}(g,h)|\Big)^2\Big) &= \mathbb{E}\Big(\sum_{h,h'\in B^G_r\setminus B^G_{\rho(r)}} |\hat{a}_{r,x}(g,h)||\hat{a}_{r,x}(g,h')|\Big) \\
= \sum_{h\neq h'\in B^G_r\setminus B^G_{\rho(r)}} \mathbb{E}\big(|\hat{a}_{r,x}(g,h)|\big)&\mathbb{E}\big(|\hat{a}_{r,x}(g,h')|\big) + \sum_{h\in B^G_r\setminus B^G_{\rho(r)}} \mathbb{E}\big(|\hat{a}_{r,x}(g,h)|^2\big).
\end{aligned}
$$

Now, use that for $g \in \operatorname{spt}(\phi)$ and $h \in B^G_r \setminus B^G_{\rho(r)}$ we have $\hat{a}_{r,x}(g,h) = a_{r,x}(g,h)$. As $\rho(r) \le r/6$, the element $a_{r,x}(g,h)$ can only be non-zero if $h \in B^G_{r/2}$, cf. Remark 3.2. Thus, Lemma 2.3 ensures that we can replace the matrix elements $a_{r,x}(g,h)$ by matrix elements of A, i.e. we obtain

$$
\begin{aligned}
\mathbb{E}\Big(\Big(\sum_{g\in G\setminus B^G_{\rho(r)}} |\hat{a}_{r,x}(h,g)|\Big)^2\Big) &\le \sum_{g\neq g'\in G\setminus B^G_{\rho(r)}} \mathbb{E}\big(|a(h,g)|\big)\mathbb{E}\big(|a(h,g')|\big) \\
&\quad + \sum_{g\in G\setminus B^G_{\rho(r)}} \mathbb{E}\big(|a(h,g)|^2\big)
\end{aligned}
$$

$$= \mathbb{E}\Big(\Big(\sum_{g\in G\setminus B^G_{\rho(r)}} |a(h,g)|\Big)^2\Big).$$

This gives

$$T_1(r) \le \frac{2\sqrt{2}|\alpha|(\kappa+1)}{\Im(z)}\Big(\sum_{g\in \mathrm{spt}(\phi)} \mathbb{E}\Big(\Big(\sum_{h\in G\setminus B^G_{\rho(r)}} |a(g,h)|\Big)^2\Big)\Big)^{1/2},$$

which tends to zero as r tends to infinity, cf. Lemma 4.1.

Now we deal with the non-diagonal terms and estimate $T_2(r)$. By an application of Cauchy Schwarz inequality and (4.18) we get

$$T_2(r) \le \frac{1+\kappa}{|\Im(z)|}\mathbb{E}\Big(\Big(\sum_{g\in G\setminus B^G_{\rho(r)}}\sum_{h\in \mathrm{spt}(\phi)} |a(g,h)-\hat{a}_{r,x}(g,h)|^2\Big)^{1/2}\Big).$$

We set $c := (1+\kappa)/|\Im(z)|$ and use again triangle inequality and Jensen inequality to achieve

$$\begin{aligned} T_2(r) \le c\Big(\sum_{g\in G\setminus B^G_{\rho(r)}}\sum_{h\in \mathrm{spt}(\phi)} \mathbb{E}\big(|a(g,h)|^2\big)\Big)^{1/2} \\ + c\Big(\sum_{g\in G\setminus B^G_{\rho(r)}}\sum_{h\in \mathrm{spt}(\phi)} \mathbb{E}\big(|\hat{a}_{r,x}(g,h)|^2\big)\Big)^{1/2}. \end{aligned}$$

By a calculation as in (3.8) we obtain

$$T_2(r) \le 2c\Big(\sum_{h\in \mathrm{spt}(\phi)}\sum_{g\in G\setminus B^G_{\rho(r)}} \mathbb{E}\big(|a(g,h)|^2\big)\Big)^{\frac{1}{2}},$$

which tends to zero as r tends to infinity, see Lemma (4.1). This gives using (4.19) that uniformly in $x \in V_r^{(0)}$

$$\lim_{r\to\infty} \mathbb{E}\big(\|(A-\hat{A}_{r,x})\phi\|_2\big) = 0.$$

Now conclude from (4.17)

$$\limsup_{r\to\infty} D_r \le \frac{2\kappa}{|\Im(z)|}.$$

Since $\kappa > 0$ was arbitrary, this gives $\lim_{r\to\infty} D_r = 0$, which finishes the proof. ■

4.2 Weak convergence, almost sure

In order to obtain almost sure convergence we make use of a concentration inequality for functions of independent random variables. It is taken from [McD98].

Theorem 4.6 ([McD98, Theorem 3.1])**.** *Let $X = (X_1, \dots, X_n)$ be a family of independent random variables with values in $\mathbb{R}$, and let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a function, such that whenever $x \in \mathbb{R}^n$ and $x' \in \mathbb{R}^n$ differ only in one coordinate we have*

$$|f(x) - f(x')| \leq c.$$

Then, for $\mu := \mathbb{E}[f(X)]$ and any $\varepsilon \geq 0$,

$$\mathbb{P}(|f(X) - \mu| \geq \varepsilon) \leq 2\exp\Big(-\frac{2\varepsilon^2}{nc^2}\Big).$$

We use Theorem 4.6 to upgrade the convergence in Theorem 4.5. This is where we need the specifically slow growth of ρ. We obtain almost sure convergence as well as a Pastur-Shubin-trace formula.

Theorem 4.7. *Let G be a finitely generated sofic group and let A be given as in (4.6). Furthermore, let $\mathfrak{n}_r$ and $\bar{\mathfrak{N}}$ be as in (4.9) and (4.10). Then there is a set $\tilde{\Omega} \in \mathcal{A}$ with full probability $\mathbb{P}(\tilde{\Omega}) = 1$ such that for all $\omega \in \tilde{\Omega}$ we have*

$$\bar{\mathfrak{N}} = \operatorname*{w-lim}_{r\to\infty} \mathfrak{n}_r^{(\omega)}.$$

Proof. By definition, we need to show $\lim_{r\to\infty} \mathfrak{n}_r^{(\omega)}(\lambda) = \bar{\mathfrak{N}}(\lambda)$ for all $\lambda \in \operatorname{cont}(\bar{\mathfrak{N}})$. Let $\lambda \in \operatorname{cont}(\bar{\mathfrak{N}})$ and $\varepsilon > 0$ be given. By Theorem 4.5 there exists $r_0 > 0$ such that $|\bar{\mathfrak{n}}_r(\lambda) - \bar{\mathfrak{N}}(\lambda)| \leq \varepsilon/2$ for all $r \geq r_0$. Therefore, also for $r \geq r_0$, we have

$$\begin{aligned}\mathbb{P}\big(|\mathfrak{n}_r(\lambda) - \bar{\mathfrak{N}}(\lambda)| \geq \varepsilon\big) &\leq \mathbb{P}\big(|\mathfrak{n}_r(\lambda) - \bar{\mathfrak{n}}_r(\lambda)| \geq \varepsilon - |\bar{\mathfrak{n}}_r(\lambda) - \bar{\mathfrak{N}}(\lambda)|\big) \\ &\leq \mathbb{P}\big(|\mathfrak{n}_r(\lambda) - \bar{\mathfrak{n}}_r(\lambda)| \geq \varepsilon/2\big). \qquad (4.20)\end{aligned}$$

In order to apply Theorem 4.6, we need to show that the functions $\mathfrak{n}_r$ fit in the setting therein. To see this, note that by construction $\mathfrak{n}_r$ depends on the matrix elements of A_r. The non-zero matrix elements of A_r are constructed by random variables, and each random variable has influence in at most two rows of the matrix. Denote the number of these random variables by n. A change in one of these n random variables causes at most a rank two perturbation, which implies by Lemma 2.24 that the value of $\mathfrak{n}_r$ changes at most by $c := 2/|V_r|$. Furthermore, the number of random variables which are used in our approximation is limited using the function ρ. In fact we have for all $r \geq |S|^4$

$$n \leq |V_r||S|^{2(\rho(r)+1)} = |V_r||S|^{\frac{\ln r}{2 \ln |S|}} = |V_r|\sqrt{r}.$$

This gives with Theorem 4.6

$$\mathbb{P}\big(|\mathfrak{n}_r(\lambda) - \bar{\mathfrak{n}}_r(\lambda)| \geq \varepsilon/2\big) \leq 2\exp\left(-\frac{\varepsilon^2}{2nc^2}\right) \leq 2\exp\left(-\frac{\varepsilon^2|V_r|}{8\sqrt{r}}\right).$$

Now use $|V_r| \geq r$, which holds as a the r-balls around the elements in $V_r^{(0)}$ are isomorphic to the r-ball in G, to obtain

$$\sum_{r\in\mathbb{N}} \mathbb{P}(|\mathfrak{n}_r(\lambda) - \bar{\mathfrak{N}}(\lambda)| \geq \varepsilon) \leq 2\sum_{r\in\mathbb{N}} \exp\left(-\frac{\varepsilon^2\sqrt{r}}{8}\right) < \infty.$$

This is by definition complete convergence of $\mathfrak{n}_r(\lambda)$ to $\bar{\mathfrak{N}}(\lambda)$ and implies almost sure convergence, i.e., the existence of $\Omega_\lambda \in \mathcal{A}$ with $\mathbb{P}(\Omega_\lambda) = 1$ such that for all $\omega \in \Omega_\lambda$

$$\lim_{r\to\infty} \mathfrak{n}_r^{(\omega)}(\lambda) = \bar{\mathfrak{N}}(\lambda). \tag{4.21}$$

As a $\bar{\mathfrak{N}}$ is a monotone and bounded function, the set $\operatorname{disc}(\bar{\mathfrak{N}})$ is countable. Therefore, we can choose a set $M \subseteq \operatorname{cont}(\bar{\mathfrak{N}})$ which is countable and dense in $\mathbb{R}$. The set $\tilde{\Omega} := \bigcap_{\lambda\in M} \Omega_\lambda$ has measure one since it is an intersection of countably many sets of measure one. We fix $\omega \in \tilde{\Omega}$. By monotonicity of $\mathfrak{n}_r^{(\omega)}$ and (4.21), we get for all $\lambda \in \mathbb{R}$

$$\begin{aligned}\limsup_{r\to\infty} \mathfrak{n}_r^{(\omega)}(\lambda) &\leq \inf_{\lambda'\in M\cap[\lambda,\infty)} \lim_{r\to\infty} \mathfrak{n}_r^{(\omega)}(\lambda') \\ &= \inf_{\lambda'\in M\cap[\lambda,\infty)} \bar{\mathfrak{N}}(\lambda') = \bar{\mathfrak{N}}(\lambda).\end{aligned}$$

Here the last equality holds since $\bar{\mathfrak{N}}$ is monotone and continuous from the right and since M is dense in $\mathbb{R}$. The same arguments work for the other direction if we restrict ourselves to $\lambda \in \operatorname{cont}(\bar{\mathfrak{N}})$

$$\begin{aligned}\liminf_{r\to\infty} \mathfrak{n}_r^{(\omega)}(\lambda) &\geq \sup_{\lambda' \in M\cap(-\infty,\lambda]} \lim_{r\to\infty} \mathfrak{n}_r^{(\omega)}(\lambda') \\ &= \sup_{\lambda' \in M\cap(-\infty,\lambda]} \bar{\mathfrak{N}}(\lambda') = \bar{\mathfrak{N}}(\lambda).\end{aligned}$$

These facts together give for all $\omega \in \tilde{\Omega}$ and all $\lambda \in \operatorname{cont}(\bar{\mathfrak{N}})$

$$\lim_{r\to\infty} \mathfrak{n}_r^{(\omega)}(\lambda) = \bar{\mathfrak{N}}(\lambda),$$

which proves the claim. ∎

4.3 Special case: percolation

As an application we show in this section that a percolation model is covered by our abstract theory of random operators on sofic groups. We study the existence of the IDS of the corresponding graph Laplacian. The models in consideration contain short-range as well as long-range percolation on sofic groups.

As before let G be a finitely generated sofic group, S a finite, symmetric set of generators, and let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let $\Gamma_{\mathrm{co}} = (V, E_{\mathrm{co}})$ be the complete graph over the vertex set $V := G$, i.e. the edge set is

$$E_{\mathrm{co}} := \{e \subseteq G \mid 1 \leq |e| \leq 2\}.$$

Furthermore, let $p \in \ell^1(G, \mathbb{R})$ be such that for all $x \in G$ one has

$$0 \leq p(x) = p(x^{-1}) \leq 1$$

and define for distinct $x, y \in G$ the random variables $X_{\{x,y\}}$ on $(\Omega, \mathcal{A}, \mathbb{P})$ by

$$X_{\{x,y\}} = \begin{cases} 1 & \text{with probability } p(xy^{-1}), \\ 0 & \text{otherwise.} \end{cases} \tag{4.22}$$

We assume that all these random variables are independent. Using these random variables we define for each $\omega \in \Omega$ a random subgraph $\Gamma_\omega = (V, E_\omega)$ of Γ_{co} via

$$E_\omega = \{e \in E_{\mathrm{co}} \mid X_e(\omega) = 1\}.$$

Such a graph Γ_ω may contain edges between two arbitrary vertices. In particular, if p is not finitely supported, there is with probability one no uniform upper bound for the length of the edges which appear in the graph, cf. Lemma 4.10. In this situation this model is referred to as *long-range percolation model*. The following Lemma shows that Γ_ω is almost surely locally finite, i.e. each vertex is incident to only finitely many edges in Γ_ω.

Lemma 4.8. *The graph Γ_ω is locally finite for almost all $\omega \in \Omega$.*

Proof. Fix an element $x \in G$ and consider the events $A_y := \{X_{\{x,y\}} = 1\}$, $y \in G$. Then clearly

$$\sum_{y \in G} \mathbb{P}(A_y) = \sum_{y \in G} p(xy^{-1}) < \infty,$$

as $p \in \ell^1(G, \mathbb{R})$. Hence, the Borel-Cantelli Lemma gives a set Ω_x of full measure such that each $\omega \in \Omega_x$ is contained in only finitely many A_y, $y \in G$. As G is countable, $\tilde{\Omega} := \bigcap_{x \in G} \Omega_x$ is a set of full measure, too. By construction Γ_ω is locally finite for all $\omega \in \tilde{\Omega}$. ■

Note that a special case of this model is short-range percolation of the Cayley graph $\Gamma = \Gamma(G, S)$. Here one sets all $p(x) = 0$ for all $x \notin S$. Then, obviously p is finitely supported and the random graph Γ_ω is a subgraph of Γ.

The matrix elements of the operator in consideration are given by

$$a^{(\omega)}(x,y) = \begin{cases} X_{\{x,y\}}(\omega) & \text{if } x \neq y, \\ X_x(\omega) - \sum_{z \neq x} X_{\{x,z\}}(\omega) & \text{otherwise.} \end{cases} \tag{4.23}$$

In the following we define the Laplacian of this graph. For given $f \in C_{\mathrm{c}}(G)$ and $x \in G$ we set

$$(\tilde{\Delta}^{(\omega)} f)(x) := \sum_{y \in G} a^{(\omega)}(x,y) f(y). \tag{4.24}$$

See also Example 2.13. The next lemma show that this defines a random operator on the domain $C_{\mathrm{c}}(G)$.

Lemma 4.9. *The operator* $\tilde{\Delta}$ *given in* (4.24) *is a symmetric, translation invariant (in distribution) random operator with domain* $C_{\mathrm{c}}(G)$*. Moreover, this* $\tilde{\Delta}$ *is almost surely essentially self-adjoint.*

Proof. In order to prove this it is by Lemma 4.1 enough to show

$$\mathbb{E}\Bigg(\Big(\sum_{x\in G}|X_{\{\mathrm{id},x\}}|\Big)^2\Bigg) < \infty.$$

To this end, we calculate using monotone convergence

$$\begin{aligned}\mathbb{E}\Bigg(\Big(\sum_{x\in G}|X_{\{\mathrm{id},x\}}|\Big)^2\Bigg) &= \sum_{x,y\in G}\mathbb{E}\big(X_{\{\mathrm{id},x\}}X_{\{\mathrm{id},y\}}\big)\\ &\le \sum_{x,y\in G}\mathbb{E}\big(X_{\{\mathrm{id},x\}}\big)\mathbb{E}\big(X_{\{\mathrm{id},y\}}\big) + \sum_{x\in G}\mathbb{E}\big(X_{\{\mathrm{id},x\}}\big)\end{aligned}$$

With $\mathbb{E}(X_{\{\mathrm{id},x\}}) = p(x)$ we obtain

$$\mathbb{E}\Bigg(\Big(\sum_{x\in G}|X_{\{\mathrm{id},x\}}|\Big)^2\Bigg) \le \|p\|_1^2 + \|p\|_1,$$

which is finite by assumption on p. ∎

Thus, there is a set $\tilde{\Omega} \subset \Omega$ of full measure, such that there exists for all $\omega \in \tilde{\Omega}$ a unique self-adjoint operator $\bar{\Delta}^{(\omega)} \colon D(\bar{\Delta}^{(\omega)}) \to \ell^2(G)$ with matrix elements given by (4.23). We want to define a random operator on all ω and set

$$\Delta^{(\omega)} := \begin{cases}\bar{\Delta}^{(\omega)} & \text{if } \omega \in \tilde{\Omega}\\ \mathrm{Id} & \text{otherwise.}\end{cases} \tag{4.25}$$

This operator is called the *Laplacian* of Γ_ω. Thence, the theory developed in Chapter 4 is valid for this operator. In particular the IDS exists for almost all realizations ω and does not depend on ω.

The following lemma investigates the situation where p is not finitely supported. To formulate this we define for each $x \in G$ and $\omega \in \Omega$ by

$$m_x(\omega) := |\{y \in G \mid \{x, y\} \in E_\omega\}|$$

the *vertex degree* of x in Γ_ω.

Lemma 4.10. *Let* $\Delta = (\Delta^{(\omega)})$ *be given as in* (4.25) *and let* $|\mathrm{spt}(p)| = \infty$*. Then there exists a set* $\tilde{\Omega}$ *of measure one, such that for all* $\omega \in \tilde{\Omega}$ *we have*

(i) $\sup\{d_S(x,y) \mid \{x,y\} \in E_\omega\} = \infty$,

(ii) $\sup\{m_x(\omega) \mid x \in V\} = \infty$*, and*

(iii) $\sup\{\|A^{(\omega)} f\|_2 \mid \|f\|_2 = 1\} = \|A^{(\omega)}\| = \infty$.

Proof. Let $k \in \mathbb{N}$ be arbitrary. As $|\mathrm{spt}(p)| = \infty$, there exists $x \in G$ with $d(0,x) > k$ and $p(x) > 0$. Then $\mathbb{P}(\{\omega \in \Omega \mid \{y, xy\} \in E_\omega\}) = p(x)$ for all $y \in G$. Using independence, we obtain a set Ω_k of measure one, such that for all $\omega \in \Omega_k$ there exists $y \in G$ with $\{y, xy\} \in E_\omega$. By construction we have $d_S(y, xy) \geq k$. As $k \in \mathbb{N}$ was arbitrary, we get for each $k \in \mathbb{N}$ as set Ω_k with these properties. Setting

$$\Omega^{(1)} := \bigcap_{k \in \mathbb{N}} \Omega_k$$

we obtain set of measure one, such that (i) holds for each $\omega \in \Omega^{(1)}$.

In order to prove (ii) and (iii) let $K \in \mathbb{N}$ be arbitrary. As $|\mathrm{spt}(p)| = \infty$, there exist elements $x_1, \ldots, x_K \in G$ with $p(x_i) > 0$ for all $i = 1, \ldots, K$. For each $y \in G$ we define

$$A_y := \bigcap_{i=1}^{K} \{\omega \in \Omega \mid \{y, x_i y\} \in E_\omega\}$$

and obtain using independence

$$\mathbb{P}(m_y \geq K) \geq \mathbb{P}(A_y) = \prod_{i=1}^{K} p(x_i) > 0.$$

Note that if the distance between y and y' is big enough, one obtains independence of A_y and $A_{y'}$. Choose a sequence (y_n) such that for distinct $n, m \in \mathbb{N}$ the sets A_{y_n} and A_{y_m} are independent. Similar as above, the set $\Omega_K := \bigcup_{i=1}^{\infty} A_{y_n}$ is of measure one. Thus for each

$\omega \in \Omega_K$ there exists $n \in \mathbb{N}$ with $m_{y_n}(\omega) \geq K$. Moreover, for this ω and y_n the following holds true:

$$\|\Delta^{(\omega)}\delta_{y_n}\|_2^2 = \sum_{x \in G} |\Delta^{(\omega)}\delta_{y_n}(x)|^2 \geq K.$$

We define

$$\Omega^{(2)} := \bigcap_{K \in \mathbb{N}} \Omega_K$$

and obtain a set of measure one such that (ii) and (iii) is satisfied for each $\omega \in \Omega^{(2)}$. We define the desired set $\tilde{\Omega}$ as the intersection $\Omega^{(1)} \cap \Omega^{(2)}$. ■

The obtained result in Lemma 4.10 is complementary to the one in Lemma 4.4. Here we show that a random operator of type (4.6) can be unbounded, even if $\sup_{x \in G} \|X_{\mathrm{id},x}\|_\infty$ is finite. This shows in particular that the converse of (i) in Lemma 4.4 does not hold.

Note that here we show weak convergence of distribution functions for almost all ω. In more restricted settings one can obtain even more, i.e. uniform convergence for almost all realizations. This will be done for amenable groups in Chapters 5 and 6. However the methods rely massively on the existence of sets with an arbitrary small boundary, which is per definition not the case for non-amenable groups.

5 Deterministic operators on amenable groups

In this chapter we study deterministic operators on amenable groups. The operators are assumed to be of finite hopping range and invariant with respect to a given coloring of the group. These assumptions do not coincide with the ones in Chapter 3, where we for instance did not assume the operators to be of finite hopping range. However, the invariance with respect to a coloring, weakens the condition of translation invariance, which we assumed in Chapter 3. Roughly speaking, the operators here only have to be translation invariant, for points where the coloring in a certain neighborhood coincides. Hence, the operators in this section are neither more general nor more restricted than the ones we investigated for sofic groups.

The goal of this chapter is the verify uniform existence of the integrated density of states. To this end, we define the approximating operators by restricting the operator in question to the elements of a Følner sequence. As before, let $B(\mathbb{R})$ be the Banach space of the right continuous, bounded functions on $\mathbb{R}$, equipped with supremum norm. The eigenvalue counting functions can be interpreted as mappings which associate to given finite subset of the group (here an element of the Følner sequence) an element of $B(\mathbb{R})$. For such functions we prove a Banach space-valued ergodic theorem. This shows that the normalized eigenvalue counting functions converge uniformly to some limit function in $B(\mathbb{R})$. The idea to use a theorem of this type to obtain uniform existence of the IDS has been established in [LS05] of operators on Delone sets. Later in [LMV08] the authors presented an adapted version in the euclidean setting. The results of this chapter extend the latter work to the general case of amenable groups. This content is already published in [LSV11] and [PS12].

In a first step, we prove this ergodic theorem under a certain tiling condition on amenable groups. In fact we will verify this result for the so-called ST-amenable groups, see Definition (5.4). This class of groups allows to find a Følner sequence such that each element of the sequence is a monotile of the group and the corresponding grid is symmetric. This property is intensively used in Theorem 5.8, our first version of the ergodic theorem.

However, it is not clear whether each amenable group is ST-amenable. Thus, the aim of the second part of this chapter is to overcome the condition of Definition (5.4) and prove the ergodic theorem for *all* amenable groups. To this end, we present results from the theory of ε-quasi tilings. The ideas go back to [OW87] and have been extended to the versions we present here in [PS12]. These results allow to obtain in Theorem 5.24 the validity of the Banach space-valued ergodic theorem for all finitely generated amenable groups.

Moreover, in the last part of this chapter we provide additional results for the integrated density of states. We give characterizations of its discontinuity points and show that under certain assumptions the topological support of the associated measure is the spectrum of the operator.

Let $\mathcal{Z}$ be an arbitrary finite set, which we interpret as the set of possible colors. A *coloring* is a map $\mathcal{C} : G \to \mathcal{Z}$ and a *pattern* is a map $P : D(P) \to \mathcal{Z}$, where $D(P) \in \mathcal{F}(G)$ is called the domain of P. The *set of all patterns* is denoted by $\mathcal{P}$ and for a fixed $Q \in \mathcal{F}(G)$ the subset of $\mathcal{P}$ which only contains the patterns with domain Q is denoted by $\mathcal{P}(Q)$. Given a set $Q \subseteq D(P)$ and an element $x \in G$ we define a *restriction of a pattern* by $P|_Q : Q \to \mathcal{Z}$, $g \mapsto P|_Q(g) = P(g)$ and a *translation of a pattern* $Px : D(P)x \to \mathcal{Z}, yx \mapsto P(y)$. Two patterns are called *equivalent*, if one is a translation of the other. The equivalence class of a pattern P is denoted by $\tilde{P}$. We write $\tilde{\mathcal{P}}$ for the induced set of equivalence classes in $\mathcal{P}$. For two patterns P and P' the number of occurrences of the pattern P in P' is denoted by

$$\sharp_P(P') := \left|\{x \in G \mid D(P)x \subseteq D(P'), P'|_{D(P)x} = Px\}\right|.$$

Counting occurrences of patterns along a Følner sequence $(U_j)_{j\in\mathbb{N}}$ leads to the definition of frequencies. If for a pattern P and a Følner sequence $(U_j)_{j\in\mathbb{N}}$ the limit

$$\nu_P := \lim_{j\to\infty} \frac{\sharp_P(\mathcal{C}|_{U_j})}{|U_j|}$$

exists, we call ν_P the *frequency of P in the coloring $\mathcal{C}$ along $(U_j)_{j\in\mathbb{N}}$*.

Lemma 5.1. *Let G be a finitely generated group and assume that $\mathcal{C}$ is a coloring. If ν_P is the frequency of a pattern P along the Følner sequence $(U_j)_{j\in\mathbb{N}}$, then for any $r>0$ the value ν_P is the frequency of P along $(U_j^{(r)})_{j\in\mathbb{N}}$ as well.*

Proof. Let $r>0$ be given and let (U_j) be a Følner sequence with frequency ν_P. Then we have

$$\begin{aligned}\liminf_{j\to\infty}\frac{\sharp_P\big(\mathcal{C}|_{U_j^{(r)}}\big)}{|U_j^{(r)}|} &\geq \liminf_{j\to\infty}\frac{\sharp_P\big(\mathcal{C}|_{U_j^{(r)}}\big)}{|U_j|}\\ &\geq \liminf_{j\to\infty}\frac{\sharp_P\big(\mathcal{C}|_{U_j}\big)-\partial^r(U_j)}{|U_j|}=\nu_P.\end{aligned}$$

Furthermore, for arbitrary $\kappa>0$ we can find by Lemma 2.7 a number $j_\kappa\in\mathbb{N}$ with $|U_j^{(r)}|\geq(1-\kappa)|U_j|$ for all $j\geq j_\kappa$. Therefore we obtain

$$\limsup_{j\to\infty}\frac{\sharp_P\big(\mathcal{C}|_{U_j^{(r)}}\big)}{|U_j^{(r)}|}\leq\lim_{j\to\infty}\frac{\sharp_P\big(\mathcal{C}|_{U_j}\big)}{(1-\kappa)|U_j|}=\frac{\nu_P}{1-\kappa}.$$

As κ was arbitrary, the claim follows. ∎

Now, we introduce the space on which the operators will be defined. Let $\mathcal{H}$ be a finite dimensional Hilbert space with scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and induced norm $\|\cdot\|_{\mathcal{H}}$. We define

$$\ell^2(G,\mathcal{H}):=\left\{u:G\to\mathcal{H}\;\middle|\;\sum_{x\in G}\|u(x)\|_{\mathcal{H}}^2<\infty\right\},$$

which is a Hilbert space as well. Here the scalar product of two elements $u,v\in\ell^2(G,\mathcal{H})$ is given by

$$\langle u,v\rangle=\sum_{x\in G}\langle u(x),v(x)\rangle_{\mathcal{H}}\,.$$

As before let $C_{\mathrm{c}}(G,\mathcal{H})$ be the subset of $\ell^2(G,\mathcal{H})$ consisting of the finitely supported functions. For an arbitrary element $x\in G$ let

$$p_x:\ell^2(G,\mathcal{H})\to\mathcal{H},\ u\mapsto p_x(u):=u(x) \tag{5.1}$$

be the *natural projection* and

$$i_x : \mathcal{H} \to \ell^2(G, \mathcal{H}),\ h \mapsto i_x(h) \quad \text{with } (i_x(h))(y) := \begin{cases} h & \text{if } x = y, \\ 0 & \text{otherwise} \end{cases} \tag{5.2}$$

the *natural inclusion*. Note that i_x is the adjoint of p_x. These maps can be generalized for subsets $Q \subseteq G$. The support of $u \in \ell^2(G, \mathcal{H})$ is the set of those $x \in G$, such that $u(x) \neq 0$. We identify

$$\ell^2(Q, \mathcal{H}) = \left\{ u : Q \to \mathcal{H} \,\middle|\, \sum_{x \in Q} \|u(x)\|^2 < \infty \right\}$$

with the subspace of $\ell^2(G, \mathcal{H})$ consisting of all elements supported in Q. The map $p_Q : \ell^2(G, \mathcal{H}) \to \ell^2(Q, \mathcal{H})$ is given by setting

$$u \mapsto p_Q(u), \quad \text{where} \quad p_Q(u)(x) = u(x) \tag{5.3}$$

for $x \in Q$. Similarly, the inclusion $i_Q : \ell^2(Q, \mathcal{H}) \to \ell^2(G, \mathcal{H})$ is given by

$$u \mapsto i_Q(u), \quad \text{where} \quad i_Q(u)(x) := \begin{cases} u(x) & \text{if } x \in Q, \\ 0 & \text{otherwise} \end{cases} \tag{5.4}$$

for $x \in G$. Given $A : D(A) \subseteq \ell^2(G, \mathcal{H}) \to \ell^2(G, \mathcal{H})$, where $D(A)$ contains $C_c(G, \mathcal{H})$, we define for each $Q \in \mathcal{F}(G)$ the restricted operator $A[Q]$ by setting

$$A[Q] := p_Q A i_Q \colon \ell^2(Q, \mathcal{H}) \to \ell^2(Q, \mathcal{H}).$$

Note that for $x, y \in G$ the expression $p_y A i_x$ is an operator mapping $p_y A i_x : \mathcal{H} \to \mathcal{H}$.

Definition 5.2. Let $\mathcal{Z}$ be a finite set, $\mathcal{C} : G \to \mathcal{Z}$ a coloring and A an operator on $\ell^2(G, \mathcal{H})$, such that $D(A)$ contains $C_c(G, \mathcal{H})$. Then we say that

(a) A is of *finite hopping range* $:\Leftrightarrow \exists M > 0$ such that $p_y A i_x = 0$ for all $x, y \in G$ with $d_S(x, y) \geq M$,

(b) A is *$\mathcal{C}$-invariant* $:\Leftrightarrow \exists N \in \mathbb{N}$ such that $p_y A i_x = p_{yt} A i_{xt}$ for all $x, y, t \in G$ obeying

$$\big(\mathcal{C}|_{B_N(x) \cup B_N(y)}\big)t = \mathcal{C}|_{B_N(xt) \cup B_N(yt)},$$

(c) $R(A) := \max\{M, N\}$ is the *overall range* of A, if A is of finite hopping range with parameter M and $\mathcal{C}$-invariant with parameter N.

We observe that if A satisfies the condition (a) (or (b)) for some M (or N), then it does so for any $\tilde{M} > M$ (or $\tilde{N} > N$) as well.

Lemma 5.3. *Let $\mathcal{Z}$ be a finite set and $\mathcal{C} : G \to \mathcal{Z}$ a coloring. Then, any $\mathcal{C}$-invariant, finite hopping range operator A on $\ell^2(G, \mathcal{H})$ with $C_c(G, \mathcal{H}) \subseteq D(A)$ is bounded.*

Proof. Let A be of finite hopping range with constant M and $\mathcal{C}$-invariant with constant N. Moreover, assume that $C_c(G, \mathcal{H})$ is a subset of the domain of A. Set $R := \max\{M, N\}$. We fix a basis of the Hilbert space $\mathcal{H}$. Since $\mathcal{H}$ is of finite dimension, for each pair $x, y \in G$ the mapping $p_y A i_x : \mathcal{H} \to \mathcal{H}$ is given by a matrix of dimension $\dim(\mathcal{H}) \times \dim(\mathcal{H})$. The $\mathcal{C}$-invariance of A implies that the matrix corresponding to $p_y A i_x$ is a function which depends only on the values of $\mathcal{C}$ on $B_N(x) \cup B_N(y)$. Since A is of finite hopping range, the matrix is in fact a function of $\mathcal{C}|_{B_{2R}(x)}$ only. The reason for this is that for x and y with distance larger than M, $p_y A i_x$ vanishes identically, while for $d_S(x, y) \leq M$ the set $B_N(x) \cup B_N(y)$ is contained in $B_{2R}(x)$. Since $|\mathcal{Z}| < \infty$ and $|B_{2R}(x)| < \infty$ there are only finitely many functions $P : B_{2R}(x) \to \mathcal{Z}$ and hence only finitely many values which the matrix valued function $p_y A i_x$ can take. From this we conclude that

$$c := \sup_{x,y \in G} \sup \{\|(p_y A i_x) h\|_{\mathcal{H}} \mid h \in \mathcal{H}, \|h\|_{\mathcal{H}} \leq 1\} \tag{5.5}$$

is finite. For a given $\phi \in \ell^2(G, \mathcal{H})$ the finite hopping range of A

implies $A\phi(x) = \sum_{y \in B_R(x)} (p_x A i_y)\phi(y)$. Hence

$$\begin{aligned}\|A\phi\|^2 &= \sum_{x\in G} \langle A\phi(x), A\phi(x)\rangle \\ &= \sum_{x\in G} \Big\langle \sum_{y\in B_R(x)} (p_x A i_y)\phi(y), \sum_{z\in B_R(x)} (p_x A i_z)\phi(z)\Big\rangle\end{aligned}$$

holds. The Cauchy-Schwarz inequality implies

$$\begin{aligned}\|A\phi\|^2 &\le \sum_{x\in G} \sum_{y,z\in B_R(x)} \|(p_x A i_y)\phi(y)\| \|(p_x A i_z)\phi(z)\| \\ &\le \sum_{x\in G} \sum_{y,z\in B_R(x)} c^2 \|\phi(y)\|\|\phi(z)\|\end{aligned}$$

with c as in (5.5). Young's inequality $2\|\phi(x)\|\|\phi(y)\| \le \|\phi(x)\|^2 + \|\phi(y)\|^2$ yields that the last expression is less or equal to

$$\frac{c^2}{2} \sum_{x\in G} \left(\sum_{y,z\in B_R(x)} \|\phi(y)\|^2 + \sum_{y,z\in B_R(x)} \|\phi(z)\|^2 \right) = c^2 |B_R| \sum_{x\in G} \sum_{y\in B_R(x)} \|\phi(y)\|^2.$$

This shows the boundedness of A:

$$\|A\phi\| \le c|B_R| \|\phi\|. \qquad \blacksquare$$

Of course, if A is bounded, then $D(A) = \ell^2(G, \mathcal{H})$. Therefore, instead of assuming for A that $C_c(G) \subseteq D(A)$ and that A is self-adjoint and of finite hopping range, we can equivalently assume that A is a bounded, self-adjoint and finite hopping range operator. For such an operator A we will study functions $\mathfrak{e}(A[Q]) : \mathbb{R} \to \mathbb{R}$ where $Q \in \mathcal{F}(G)$. Dividing this function by the number of possible eigenvalues $\dim(\mathcal{H})|Q|$ of $A[Q]$ gives rise to a distribution function of a probability measure. It encodes the distribution of the spectrum of $A[Q]$. In the following we substitute Q by the elements of a Følner sequence (Q_j) and study the convergence of

$$\mathfrak{n}(A[Q_j]) = \frac{\mathfrak{e}(A[Q_j])}{|Q_j| \dim(\mathcal{H})}.$$

In order to do so, we will formulate an abstract Banach space-valued ergodic theorem. This theorem applies to cumulative eigenvalue counting functions and will imply convergence of these functions as elements in $B(\mathbb{R})$, i.e. with respect to supremum norm.

For general amenable groups the proof of this theorem is rather complex, such that for the sake of the reader we will first concentrate on a special case. In this special case the proofs are shorter and more accessible, however they already provide the major ideas. We will assume that in addition to amenability the group satisfies a certain tiling condition. To formulate this, we need some more notation.

Given a set $Q \subseteq G$, a *partition* of Q is a family of pairwise disjoint subsets Q_i, $i \in I$ of Q such that $\bigcup_{i\in I} Q_i = Q$, where I is some index set. We say that $Q \subseteq G$ *tiles* the group G or Q is a *monotile* of G, if there exists a set $B \subseteq G$ such that Qg, $g \in B$ is a partition of G. In this case Qg, $g \in B$ is called a *tiling* of the group along the *grid* B. If additionally $B = B^{-1}$ holds, we say that Q *symmetrically tiles* G or Qg, $g \in B$ is a *symmetric tiling* of G. If (Q_n) is a sequence of finite subsets, we say that (Q_n) is *symmetrically tiling*, if for each $n \in \mathbb{N}$ the set Q_n symmetrically tiles G. The announced additional assumption on the amenable group G is stated in the following definition.

Definition 5.4. A finitely generated groups G is called *ST-amenable* if there exists a symmetrically tiling Følner sequence (Q_n) in G.

Remark 5.5. Let us briefly discuss this definition. We assume that G contains a Følner sequence (Q_n) such that each Q_n symmetrically tiles G. This condition is particularly satisfied, if there exists a sequence of subgroups $(G_n)_{n\in\mathbb{N}}$, such that one can choose the associated fundamental domains $(Q_n)_{n\in\mathbb{N}}$ to be a Følner sequence. Based on a result of Weiss [Wei01], Krieger proves in [Kri07] that this is fulfilled for any residually finite, amenable group. This gives that in particular any group of polynomial volume growth fits in our framework.

Beside this, there is up to now no example of an amenable group known where no symmetrically tiling Følner sequence exists. However a verification of this property for all amenable groups seemed rather complicated. Therefore in [OW87] the authors introduced the theory of ε-quasi tilings, which can be established for *all* amenable groups. We will present results based on these ideas in Sections 5.2.1 and 5.2.2.

Before concentrating on the proof of a Banach space-valued ergodic theorem for ST-amenable groups and later for general amenable groups, let us introduce the class of functions with which such theorems may deal.

Definition 5.6. A function $b : \mathcal{F}(G) \to [0, \infty)$ is called a *boundary term* if

(a) $b(Q) = b(Qx)$ for all $x \in G$ and all $Q \in \mathcal{F}(G)$,

(b) $\lim_{j\to\infty} \frac{b(U_j)}{|U_j|} = 0$ for any Følner sequence (U_j),

(c) there exists $D > 0$ with $b(Q) \leq D|Q|$ for all $Q \in \mathcal{F}(G)$,

(d) one has for all $Q, Q' \in \mathcal{F}(G)$

$$\max\big\{b(Q \cap Q'), b(Q \cup Q'), b(Q \setminus Q')\big\} \leq b(Q) + b(Q').$$

For a pattern P we define $b(P) := b(D(P))$. Due to property (a) the value $b(P)$ depends only on the equivalence class of a pattern. Thus, $b(\tilde{P}) := b(P)$ is well-defined.

Definition 5.7. Let $(X, \|\cdot\|)$ be a Banach space and $\tilde{F}$ a function $\tilde{F} : \tilde{\mathcal{P}} \to X$. We call $\tilde{F}$ *almost-additive*, if there exists a boundary term b such that for any $\tilde{P} \in \tilde{\mathcal{P}}$ and $P \in \tilde{P}$ and any disjoint decomposition $D(P) = \bigcup_{k=1}^m D_k$ we have

$$\left\| \tilde{F}(\tilde{P}) - \sum_{k=1}^m \tilde{F}(\tilde{P}_k) \right\| \leq \sum_{k=1}^m b(\tilde{P}_k),$$

where $\tilde{P}_k := \widetilde{P|_{D_i}} \in \tilde{\mathcal{P}}$.

Let $\tilde{F} : \tilde{\mathcal{P}} \to X$ be an almost-additive function and $P : D(P) \to \mathcal{Z}$ an arbitrary pattern. For each $x \in D(P)$ define the pattern $P_x := P|_{\{x\}} : \{x\} \to \mathcal{Z}$. Then we obtain

$$\begin{aligned} \|\tilde{F}(\tilde{P})\| &\leq \Big\| \tilde{F}(\tilde{P}) - \sum_{x\in D(P)} \tilde{F}(\tilde{P}_x) \Big\| + \Big\| \sum_{x\in D(P)} \tilde{F}(\tilde{P}_x) \Big\| \\ &\leq \sum_{x\in D(P)} b(\tilde{P}_x) + \sum_{x\in D(P)} \|\tilde{F}(\tilde{P}_x)\|. \end{aligned} \tag{5.6}$$

Since each $D(P_x)$ contains exactly one element, $c_1 := b(\tilde{P}_x)$ is independent of x. Furthermore $F(\tilde{P}_x)$ can take at most $|\mathcal{Z}|$ different values. Let c_2 be the maximal norm of these values. It follows

$$\|\tilde{F}(\tilde{P})\| \leq C|D(P)|, \quad \text{where } C := c_1 + c_2. \tag{5.7}$$

A given coloring $\mathcal{C}$ on G and an almost-additive function $\tilde{F} : \tilde{\mathcal{P}} \to X$ give rise to a function

$$F : \mathcal{F}(G) \to X, \qquad F(Q) := \tilde{F}(\widetilde{\mathcal{C}|_Q}) \qquad \text{for } Q \in \mathcal{F}(G).$$

The following properties of F obviously hold:

(i) *$\mathcal{C}$-invariant*: if $x \in G$ is such that the patterns $\mathcal{C}|_Q$ and $\mathcal{C}|_{Qx}$ are equivalent, then we have

$$F(Q) = F(Qx),$$

(ii) *almost-additive*: if $Q_k, k = 1, \ldots, n$ are disjoint subsets of G, then we have

$$\left\| F\Big(\bigcup_{k=1}^m Q_k \Big) - \sum_{k=1}^m F(Q_k) \right\| \leq \sum_{k=1}^m b(Q_k),$$

(iii) *bounded*: there exists a $C > 0$ such that

$$\|F(Q)\| \leq C|Q| \qquad \text{for all } Q \in \mathcal{F}(G).$$

A $\mathcal{C}$-invariant and almost-additive function $F : \mathcal{F}(G) \to X$ is automatically bounded. This follows from an estimate analogous to (5.6). Instead of defining F based on $\tilde{F}$ one could also proceed the other way around: if a function $F : \mathcal{F}(G) \to X$ with the properties (i) and (ii) is given, define $\tilde{F} : \tilde{\mathcal{P}} \to X$ by the following procedure. If for $\tilde{P} \in \tilde{\mathcal{P}}$ there exists an $Q \in \mathcal{F}(G)$ such that $\widetilde{\mathcal{C}|_Q} = \tilde{P}$ set $\tilde{F}(\tilde{P}) = F(Q)$. This definition is independent of the particular choice of Q by the $\mathcal{C}$-invariance of F. If such a Q does not exist, set $F(\tilde{P}) = 0$. Therefore showing (i) and (ii) for F is the same as showing almost-additivity for $\tilde{F}$. To simplify the notation we will write $\tilde{F}(P)$ instead of $\tilde{F}(\tilde{P})$ for a given pattern $P \in \mathcal{P}$.

In order to be able to refer to it later, we now give a list of assumptions which will be needed in this chapter. The reason for introducing these assumptions here at once and not successively during the sections, is that in many results will refer to more than one of these assumptions.

Assumption 1. The group G is amenable and generated by a finite and symmetric set S, $\mathcal{Z}$ is a finite set and $\mathcal{C} : G \to \mathcal{Z}$ is a map, which we will call a coloring. The sequences (Q_n) and (U_j) are Følner sequences. The frequencies $\nu_P = \lim_{j\to\infty} |U_j|^{-1} \sharp_P(\mathcal{C}|_{U_j})$ exist for all patterns $P \in \bigcup_{n\in\mathbb{N}} \mathcal{P}(Q_n)$ along the Følner sequence $(U_j)_{j\in\mathbb{N}}$. We denote by $d(n) := \operatorname{diam}(Q_n)$ the diameter of Q_n. Furthermore $(X, \|\cdot\|)$ is a Banach-space.

Assumption 2. The group G is ST-amenable and the Følner sequence $(Q_n)_{n\in\mathbb{N}}$ symmetrically tiles G.

Assumption 3. The space $\mathcal{H}$ is a finite dimensional Hilbert space and the operator $A : \ell^2(G, \mathcal{H}) \to \ell^2(G, \mathcal{H})$ is bounded, self-adjoint, $\mathcal{C}$-invariant and of finite hopping range. Let $R = R(A)$ denote the overall range of A.

Assumption 4. The frequencies ν_P are strictly positive for all patterns $P \in \mathcal{P}$ which occur in $\mathcal{C}$, i.e. for which there exists $g \in G$ with $\mathcal{C}|_{D(P)g} = Pg$

5.1 Deterministic operators on ST-amenable groups

As announced before we will first restrict ourselves to ST-amenable groups. The results we present here are published in a joint work with Daniel Lenz and Ivan Veselić, see [LSV11] and [LSV12].

5.1.1 An ergodic theorem for ST-amenable groups

Given the setting outlined above, we are in the position to formulate and prove the announced ergodic type theorem for certain Banach space-valued functions on ST-amenable groups.

Theorem 5.8. *Let Assumptions 1 and 2 be satisfied. For a given $\mathcal{C}$-invariant and almost-additive function $F : \mathcal{F}(G) \to X$ and associated $\tilde{F} : \tilde{\mathcal{P}} \to X$ the following limits*

$$\lim_{j\to\infty} \frac{F(U_j)}{|U_j|} = \lim_{n\to\infty} \sum_{P\in\mathcal{P}(Q_n)} \nu_P \frac{\tilde{F}(P)}{|Q_n|}$$

exist and are equal. Furthermore, for $j, n \in \mathbb{N}$ the difference

$$\Delta(j,n) := \left\| \frac{F(U_j)}{|U_j|} - \sum_{P\in\mathcal{P}(Q_n)} \nu_P \frac{\tilde{F}(P)}{|Q_n|} \right\| \tag{5.8}$$

satisfies the estimate

$$\Delta(j,n) \le \frac{b(Q_n)}{|Q_n|} + (C+D)\frac{|\partial^{d(n)} U_j|}{|U_j|} + C \sum_{P\in\mathcal{P}(Q_n)} \left| \frac{\sharp_P(\mathcal{C}|_{U_j})}{|U_j|} - \nu_P \right|. \tag{5.9}$$

Remark 5.9. In the special case where the group equals $\mathbb{Z}^d$, it is convenient to think of the sets U_j as balls of radius j and of Q_n as cubes of side length n. While both of them are Følner sequences, (Q_n) has the additional property that each Q_n symmetrically tiles $\mathbb{Z}^d$. Here we require the frequencies of the patterns to exist along the sequence of balls.

Proof of Theorem 5.8. First, we prove (5.9). By adding a zero we get

$$\begin{aligned}\Delta(j,n) \le &\left\| \frac{F(U_j)}{|U_j|} - \sum_{P\in\mathcal{P}(Q_n)} \frac{\sharp_P(\mathcal{C}|_{U_j})}{|U_j|} \frac{\tilde{F}(P)}{|Q_n|} \right\| \\ &+ \left\| \sum_{P\in\mathcal{P}(Q_n)} \Big(\frac{\sharp_P(\mathcal{C}|_{U_j})}{|U_j|} - \nu_P \Big) \frac{\tilde{F}(P)}{|Q_n|} \right\|.\end{aligned}$$

With another application of the triangle inequality this gives

$$\Delta(j,n) \le D_1(j,n) + D_2(j,n),$$

where

$$D_1(j,n) := \left\| \frac{F(U_j)}{|U_j|} - \sum_{P\in\mathcal{P}(Q_n)} \frac{\sharp_P(\mathcal{C}|_{U_j})}{|U_j|} \frac{\tilde{F}(P)}{|Q_n|} \right\|$$

$$D_2(j,n) := \sum_{P\in\mathcal{P}(Q_n)} \left| \frac{\sharp_P(\mathcal{C}|_{U_j})}{|U_j|} - \nu_P \right| \frac{\|\tilde{F}(P)\|}{|Q_n|}.$$

We use the boundedness of $\tilde{F}$, see (5.7)

$$\|\tilde{F}(P)\| \le C|Q_n|, \tag{5.10}$$

to obtain

$$D_2(j,n) \le C \sum_{P\in\mathcal{P}(Q_n)} \left| \frac{\sharp_P(\mathcal{C}|_{U_j})}{|U_j|} - \nu_P \right|. \tag{5.11}$$

As G is ST-amenable, for each fixed $n \in \mathbb{N}$ the set Q_n (symmetrically) tiles the group G. Thus there exists a symmetric set $G_n \subseteq G$ such that $G = \bigcup_{g\in G_n} Q_n g$, where $Q_n g \cap Q_n h = \emptyset$ for all $g,h \in G_n$ with $g \neq h$. This property remains valid after shifting the grid by an arbitrary $x^{-1} \in G$. In fact we have

$$G = Gx^{-1} = \bigcup_{g\in G_n} Q_n g x^{-1} = \bigcup_{g\in G_n x^{-1}} Q_n g,$$

where $G_n x^{-1} := \{gx^{-1} | g \in G_n\}$. Still $Q_n g \cap Q_n h = \emptyset$ holds for all distinct $g,h \in G_n x^{-1}$, since $g = \tilde{g}x^{-1}$ and $h = \tilde{h}x^{-1}$ for some distinct $\tilde{g}, \tilde{h} \in G_n$ and

$$Q_n g \cap Q_n h = \emptyset \quad \Leftrightarrow \quad Q_n \tilde{g} x^{-1} \cap Q_n \tilde{h} x^{-1} = \emptyset \quad \Leftrightarrow \quad Q_n \tilde{g} \cap Q_n \tilde{h} = \emptyset.$$

Given a set $K \in \mathcal{F}(G)$ and an element $x \in G$, we introduce the set of elements $g \in G_n x^{-1}$ which gives rise to a translate $Q_n g$, which is not disjoint from K:

$$S(K,x,n) := \{g \in G_n x^{-1} \mid Q_n g \cap K \neq \emptyset\}.$$

We distinguish two types of elements in $S(K,x,n)$

$$\begin{aligned} &I(K,x,n) := \{g \in G_n x^{-1} \mid Q_n g \subseteq K\} \text{ and} \\ &\partial(K,x,n) := S(K,x,n) \setminus I(K,x,n). \end{aligned}$$

Since we have $Q_n g \subseteq \partial^{d(n)} K$ for all $g \in \partial(K,x,n)$ and $Q_n g \subseteq K$ for all $g \in I(K,x,n)$, the disjointness of the translates implies that the following inequalities hold:

$$|\partial(K,x,n)| \cdot |Q_n| \leq |\partial^{d(n)} K| \qquad \text{and} \qquad |I(K,x,n)| \cdot |Q_n| \leq |K|. \tag{5.12}$$

Given an $n \in \mathbb{N}$, $K \in \mathcal{F}(G)$ and $x \in G$ we have $Q_n g = Q_n g \cap K$ for $g \in I(K,x,n)$ and thus

$$\begin{aligned} &T(K,x,n) \\ &:= \Big\| F(K) - \sum_{g \in I(K,x,n)} F(Q_n g) \Big\| = \Big\| F(K) - \sum_{g \in I(K,x,n)} F(Q_n g \cap K) \Big\| \\ &\leq \Big\| F(K) - \sum_{g \in S(K,x,n)} F(Q_n g \cap K) \Big\| + \Big\| \sum_{g \in \partial(K,x,n)} F(Q_n g \cap K) \Big\|, \end{aligned} \tag{5.13}$$

where the last inequality holds since $S(K,x,n)$ is the disjoint union of $\partial(K,x,n)$ and $I(K,x,n)$. Now we use almost-additivity and the boundedness of F and later on the properties of the boundary term b to obtain

$$\begin{aligned} &T(K,x,n) \\ &\leq \Big(\sum_{g \in I(K,x,n)} b(Q_n g) + \sum_{g \in \partial(K,x,n)} b(Q_n g \cap K) \Big) + \sum_{g \in \partial(K,x,n)} C|Q_n g| \\ &\leq \sum_{g \in I(K,x,n)} b(Q_n) + \sum_{g \in \partial(K,x,n)} D|Q_n| + \sum_{g \in \partial(K,x,n)} C|Q_n| \\ &\leq |I(K,x,n)| b(Q_n) + |\partial(K,x,n)|(C+D)|Q_n|. \end{aligned}$$

The inequalities (5.12) yield the estimate

$$T(K,x,n) \leq \frac{|K|}{|Q_n|} b(Q_n) + (C+D)|\partial^{d(n)} K|. \tag{5.14}$$

Furthermore, we have the equality

$$\{z \in G \mid Q_n z \subseteq K\} = \dot{\bigcup_{x \in Q_n}} \{z \in G_n x^{-1} \mid Q_n z \subseteq K\} \tag{5.15}$$

since for each $z \in G$ there is $x \in Q_n$ and $g \in G_n$ with $z^{-1} = xg$. Hence, $z = g^{-1}x^{-1} \in G_n x^{-1}$, as G_n is a symmetric subset of G. To see that the union in (5.15) is disjoint, observe that for given $x, y \in Q_n$ with $x \neq y$ and $z \in G_n x^{-1}$ we have $z^{-1} \in xG_n$. Here we again used the symmetry of G_n. By the tiling property of Q_n this gives $z^{-1} \notin yG_n$ and hence $z \notin G_n y^{-1}$.

The $\mathcal{C}$-invariance of F and the equation (5.15) imply

$$\sum_{P \in \mathcal{P}(Q_n)} \sharp_P(\mathcal{C}|_{U_j})\tilde{F}(P) = \sum_{z \in G : Q_n z \subseteq U_j} F(Q_n z) = \sum_{x \in Q_n} \sum_{g \in I(U_j, x, n)} F(Q_n g), \tag{5.16}$$

from which we deduce

$$\begin{aligned} |U_j| D_1(j,n) &= \left\| F(U_j) - \sum_{P \in \mathcal{P}(Q_n)} \sharp_P(\mathcal{C}|_{U_j}) \frac{\tilde{F}(P)}{|Q_n|} \right\| \\ &= \left\| F(U_j) - \sum_{x \in Q_n} \sum_{g \in I(U_j, x, n)} \frac{F(Q_n g)}{|Q_n|} \right\|. \end{aligned}$$

Using $\sum_{x \in Q_n} 1 = |Q_n|$ we get

$$\begin{aligned} &\left\| F(U_j) - \sum_{x \in Q_n} \sum_{g \in I(U_j, x, n)} \frac{F(Q_n g)}{|Q_n|} \right\| \\ &= \frac{1}{|Q_n|} \left\| \sum_{x \in Q_n} \Big(F(U_j) - \sum_{g \in I(U_j, x, n)} F(Q_n g) \Big) \right\| \leq \frac{1}{|Q_n|} \sum_{x \in Q_n} T(U_j, x, n), \end{aligned}$$

where $T(U_j, x, n)$ is given as in (5.13). Now we use the estimate (5.14) for $T(U_j, x, n)$ to obtain

$$\begin{aligned} D_1(j,n) &\leq \frac{1}{|Q_n|} \sum_{x \in Q_n} \left(\frac{b(Q_n)}{|Q_n|} + (C + D) \frac{|\partial^{d(n)} U_j|}{|U_j|} \right) \\ &= \frac{b(Q_n)}{|Q_n|} + (C + D) \frac{|\partial^{d(n)} U_j|}{|U_j|}. \end{aligned}$$

Together with the upper bound for $D_2(j,n)$ in (5.11) we have

$$\begin{aligned}\Delta(j,n) &\leq D_1(j,n) + D_2(j,n)\\ &\leq \frac{b(Q_n)}{|Q_n|} + (C+D)\frac{|\partial^{d(n)}U_j|}{|U_j|} + C\sum_{P\in\mathcal{P}(Q_n)}\left|\frac{\sharp_P(\mathcal{C}|_{U_j})}{|U_j|} - \nu_P\right|,\end{aligned}$$

for all $j,n\in\mathbb{N}$. This proves (5.9). Now the main part of the theorem follows readily. One immediately sees that $\Delta(j,n)$ tends to zero, if j and n tend (in the right order) to infinity, i.e.

$$\lim_{n\to\infty}\lim_{j\to\infty}\Delta(j,n) = 0. \tag{5.17}$$

The triangle inequality shows that

$$\begin{aligned}&\left\|\frac{F(U_j)}{|U_j|} - \frac{F(U_m)}{|U_m|}\right\|\\ &= \left\|\frac{F(U_j)}{|U_j|} - \sum_{P\in\mathcal{P}(Q_n)}\nu_P\frac{\tilde{F}(P)}{|Q_n|} + \sum_{P\in\mathcal{P}(Q_n)}\nu_P\frac{\tilde{F}(P)}{|Q_n|} - \frac{F(U_m)}{|U_m|}\right\|\\ &\leq \Delta(j,n) + \Delta(m,n)\end{aligned}$$

holds for all $j,m,n\in\mathbb{N}$. This implies that $(|U_j|^{-1}F(U_j))_{j\in\mathbb{N}}$ is a Cauchy sequence and hence it is convergent in the Banach space X. We use again (5.17) to obtain that

$$\sum_{P\in\mathcal{P}(Q_n)}\nu_P\frac{\tilde{F}(P)}{|Q_n|}$$

converges to the same limit when n tends to infinity. ■

With the help of the above theorem we are able to give an explicit bound for the distance between the approximants and the limit term.

Corollary 5.10. *Let Assumptions 1 and 2 be satisfied and let a $\mathcal{C}$-invariant and almost-additive function $F:\mathcal{F}(G)\to X$ and associated $\tilde{F}:\tilde{\mathcal{P}}\to X$ be given. Denote the limit by $\bar{F} := \lim_{j\to\infty}|U_j|^{-1}F(U_j)$. Then we have for all $j,n\in\mathbb{N}$ the estimates*

$$\left\|\bar{F} - \frac{F(U_j)}{|U_j|}\right\| \leq 2\frac{b(Q_n)}{|Q_n|} + (C+D)\frac{|\partial^{d(n)}U_j|}{|U_j|} + C\sum_{P\in\mathcal{P}(Q_n)}\left|\frac{\sharp_P(\mathcal{C}|_{U_j})}{|U_j|} - \nu_P\right|$$

and

$$\left\| \bar{F} - \sum_{P \in \mathcal{P}(Q_n)} \nu_P \frac{\tilde{F}(P)}{|Q_n|} \right\| \leq \frac{b(Q_n)}{|Q_n|}.$$

Proof. We fix $j, n \in \mathbb{N}$. By definition of $\bar{F}$ and the triangle inequality we have that

$$\begin{aligned} D_1(j) := \left\| \bar{F} - \frac{F(U_j)}{|U_j|} \right\| &= \lim_{k\to\infty} \left\| \frac{F(U_k)}{|U_k|} - \frac{F(U_j)}{|U_j|} \right\| \\ &\leq \lim_{k\to\infty} (\Delta(k,n) + \Delta(j,n)) \end{aligned}$$

holds, where Δ is given as in (5.8). Using the estimate (5.9) for $\Delta(j,n)$ we obtain

$$\begin{aligned} D_1(j) \leq \lim_{k\to\infty} \Bigg(& \frac{b(Q_n)}{|Q_n|} + (C+D)\frac{|\partial^{d(n)} U_k|}{|U_k|} + C \sum_{P \in \mathcal{P}(Q_n)} \left| \frac{\sharp_P(\mathcal{C}|_{U_k})}{|U_k|} - \nu_P \right| \\ & + \frac{b(Q_n)}{|Q_n|} + (C+D)\frac{|\partial^{d(n)} U_j|}{|U_j|} + C \sum_{P \in \mathcal{P}(Q_n)} \left| \frac{\sharp_P(\mathcal{C}|_{U_j})}{|U_j|} - \nu_P \right| \Bigg) \\ = 2 & \frac{b(Q_n)}{|Q_n|} + (C+D)\frac{|\partial^{d(n)} U_j|}{|U_j|} + C \sum_{P \in \mathcal{P}(Q_n)} \left| \frac{\sharp_P(\mathcal{C}|_{U_j})}{|U_j|} - \nu_P \right|. \end{aligned}$$

We use the same ideas to estimate

$$D_2(n) := \left\| \bar{F} - \sum_{P \in \mathcal{P}(Q_n)} \nu_P \frac{\tilde{F}(P)}{|Q_n|} \right\| = \lim_{k\to\infty} \Delta(k,n) = \frac{b(Q_n)}{|Q_n|},$$

which proves the second claim. ∎

The assumptions of Theorem 5.8 and Corollary 5.10 are particularly satisfied if there exists a symmetrically tiling Følner sequence $(Q_n)_{n\in\mathbb{N}}$ along which the frequencies ν_P exist for all patterns $P \in \bigcup_{n\in\mathbb{N}} \mathcal{P}(Q_n)$. This corresponds to the special case of Assumption 1, where it is possible to choose $(U_j) = (Q_j)$.

5.1.2 Uniform convergence for ST-amenable groups

In this subsection we apply the obtained ergodic theorem of Section 5.1.1 to prove uniform convergence of the eigenvalue counting functions. This is stated in Theorem 5.11 below.

Theorem 5.11. *Let Assumptions 1, 2 and 3 be satisfied. Then there exists a unique probability measure μ_A on $\mathbb{R}$ with distribution function $\mathfrak{I}_A$, such that the estimate*

$$\Big\|\mathfrak{n}\big(A[U_j^{(R)}]\big) - \mathfrak{I}_A\Big\|_\infty \leq 8\frac{|\partial^R Q_n|}{|Q_n|} + (1+4|B_R|)\frac{|\partial^{d(n)} U_j|}{|U_j|} + \sum_{P\in\mathcal{P}(Q_n)} \left|\frac{\sharp_P(\mathcal{C}|_{U_j})}{|U_j|} - \nu_P\right| + \frac{|\partial^R_{\mathrm{int}} U_j|}{|U_j|}$$

holds for all $j, n \in \mathbb{N}$. This implies in particular the convergence

$$\mathfrak{n}\big(A[U_j^{(R)}]\big) \to \mathfrak{I}_A$$

with respect to the supremum norm for $j\to\infty$. The function $\mathfrak{I}_A$ is called the integrated density of states *(IDS).*

For the proof we establish a couple of auxiliary results. Before this we define the functions

$$F_R^A : \mathcal{F}(G) \to B(\mathbb{R}) \qquad F_R^A(Q) := \mathfrak{e}(A[Q^{(R)}])$$

and

$$b : \mathcal{F}(G) \to [0,\infty), \qquad b(Q) := 4|\partial^R Q| \dim(\mathcal{H}).$$

Lemma 5.12. *Let Assumptions 1 and 3 be satisfied and let F_R^A and b be given as above. Then F_R^A is $\mathcal{C}$-invariant and almost-additive with the boundary term b.*

Proof. Since R is the overall range of A, the values of $\mathfrak{e}(A[Q^{(R)}])$ only depend on the coloring of Q, namely $\mathcal{C}|_Q$, and hence F_R^A is $\mathcal{C}$-invariant. To show almost-additivity we use a decoupling argument. Let Q be a disjoint union of Q_k for $k = 1, \ldots, m$. By definition, R is big enough such that

$$A\Big[\bigcup\nolimits_{k=1}^m Q_k^{(R)}\Big] = \bigoplus_{k=1}^m A[Q_k^{(R)}]$$

holds. Therefore we can count the eigenvalues of $A[Q_k^{(R)}]$ for $k = 1, \dots, m$ separately

$$\mathfrak{e}\Big(A\Big[\bigcup\nolimits_{k=1}^m Q_k^{(R)}\Big]\Big) = \sum_{k=1}^m \mathfrak{e}(A[Q_k^{(R)}]).$$

Now, we apply Lemma 2.25 with the spaces $V = \ell^2(Q^{(R)}, \mathcal{H})$ and $U = \ell^2(\bigcup\nolimits_{k=1}^m Q_k^{(R)}, \mathcal{H})$. Hence we get

$$\begin{aligned}\left\| \mathfrak{e}(A[Q^{(R)}]) - \mathfrak{e}\Big(A\Big[\bigcup_{k=1}^m Q_k^{(R)}\Big]\Big) \right\|_\infty &\le 4 \sum_{k=1}^m |\partial^R Q_k| \dim(\mathcal{H}) \\ &= \sum_{k=1}^m b(Q_k).\end{aligned}$$

It remains to show that b given as above is a boundary term in the sense of Definition 5.6. Therefore use Lemma 2.1 and Lemma 2.7. In order to obtain $b(Q) \le D|Q|$ set $D := 4|B_R| \dim(\mathcal{H})$. This proves the almost-additivity with boundary term b. ■

From a calculation analogous to (5.6) it is clear that F_R^A is bounded. Since the operator $A[Q^{(R)}]$ has exactly $\dim(\mathcal{H})|Q^{(R)}|$ eigenvalues (counted with multiplicities), the boundedness holds with the constant $C = \dim(\mathcal{H})$.

Proof of Theorem 5.11. Since F_R^A is $\mathcal{C}$-invariant and almost-additive, we can apply Theorem 5.8 and Corollary 5.10 which gives the existence of a function $\tilde{\mathfrak{I}}_A$ with

$$\begin{aligned}&\left\| \frac{F_R^A(U_j)}{|U_j|} - \tilde{\mathfrak{I}}_A \right\|_\infty \\ &\le 2\frac{b(Q_n)}{|Q_n|} + (C+D)\frac{|\partial^{d(n)} U_j|}{|U_j|} + C \sum_{P \in \mathcal{P}(Q_n)} \left| \frac{\sharp_P(\mathcal{C}|_{U_j})}{|U_j|} - \nu_P \right| \\ &\le h\Big(8\frac{|\partial^R Q_n|}{|Q_n|} + (1 + 4|B_R|)\frac{|\partial^{d(n)} U_j|}{|U_j|} + \sum_{P \in \mathcal{P}(Q_n)} \left| \frac{\sharp_P(\mathcal{C}|_{U_j})}{|U_j|} - \nu_P \right| \Big)\end{aligned}$$

for all $j, n \in \mathbb{N}$, where $h = \dim(\mathcal{H})$. What remains to be done is to change the normalization of $F_R^A(U) = \mathfrak{e}(A[U^{(R)}])$. We know that $|U_j^{(R)}| = |U_j| - |\partial_{\text{int}}^R(U_j)|$ and by expansion one can show that

$$\frac{1}{|U_j|} = \frac{1}{|U_j| - |\partial_{\text{int}}^R(U_j)|} - \frac{|\partial_{\text{int}}^R(U_j)|}{|U_j|(|U_j| - |\partial_{\text{int}}^R(U_j)|)}$$

holds true. This gives for all $j, n \in \mathbb{N}$

$$\begin{aligned}\left\| \frac{F_R^A(U_j)}{|U_j|} - \tilde{\mathfrak{I}}_A \right\|_\infty &= \left\| \frac{F_R^A(U_j)}{|U_j^{(R)}|} - \frac{F_R^A(U_j)|\partial_{\text{int}}^R(U_j)|}{|U_j^{(R)}||U_j|} - \tilde{\mathfrak{I}}_A \right\|_\infty \\ &\geq \left\| \frac{F_R^A(U_j)}{|U_j^{(R)}|} - \tilde{\mathfrak{I}}_A \right\|_\infty - \left\| \frac{F_R^A(U_j)|\partial_{\text{int}}^R(U_j)|}{|U_j^{(R)}||U_j|} \right\|_\infty.\end{aligned}$$

By definition of F_R^A we have $\|F_R^A(U_j)\|_\infty = \dim(\mathcal{H})|U_j^{(R)}|$, which implies

$$\left\| \frac{F_R^A(U_j)|\partial_{\text{int}}^R(U_j)|}{|U_j^{(R)}||U_j|} \right\|_\infty = \dim(\mathcal{H}) \frac{|\partial_{\text{int}}^R(U_j)|}{|U_j|}$$

for all $j \in \mathbb{N}$. Finally, dividing everything by $\dim(\mathcal{H})$ and using

$$F_R^A(U_j) = \dim(\mathcal{H})|U_j^{(R)}|\mathfrak{n}(A[U_j^{(R)}])$$

leads to

$$\begin{aligned}\left\| \mathfrak{n}\big(A[U_j^{(R)}]\big) - \mathfrak{I}_A \right\|_\infty &\leq 8 \frac{|\partial^R Q_n|}{|Q_n|} + (1 + 4|B_R|) \frac{|\partial^{d(n)} U_j|}{|U_j|} \\ &\quad + \sum_{P \in \mathcal{P}(Q_n)} \left| \frac{\sharp_P(\mathcal{C}|_{U_j})}{|U_j|} - \nu_P \right| + \frac{|\partial_{\text{int}}^R U_j|}{|U_j|},\end{aligned}$$

which holds for all $j, n \in \mathbb{N}$ with $\mathfrak{I}_A := \tilde{\mathfrak{I}}_A / \dim(\mathcal{H})$. Using Lemma 2.7, this shows the claimed convergence. As we obtained uniform convergence of distribution functions of probability measures we get by Lemma 2.26 that $\mathfrak{I}_A$ is a distribution function of a probability measure as well. ■

Since the eigenvalue counting function $\mathfrak{e}$ is $\mathcal{C}$-invariant, the function $\tilde{\mathfrak{e}}$ on the set the equivalence classes of all patterns given by

$$\tilde{\mathfrak{e}}\colon \tilde{\mathcal{P}} \to B(\mathbb{R}) \quad \tilde{\mathfrak{e}}(\tilde{P}) := \begin{cases} \mathfrak{e}(A[Q^{(R)}]) & \text{if } Q \in \mathcal{F}(G) \text{ s.t. } \tilde{P} = \widetilde{\mathcal{C}|_Q}, \\ 0 & \text{otherwise} \end{cases} \tag{5.18}$$

is well defined. As before we write $\tilde{\mathfrak{e}}(P)$ instead of $\tilde{\mathfrak{e}}(\tilde{P})$ for a given $P \in \mathcal{P}$. The following result is a direct consequence of the second estimate in Corollary 5.10 and the boundary term from Lemma 5.12.

Corollary 5.13. *Let Assumptions 1, 2 and 3 be satisfied and let $\mathfrak{I}_A$ be defined as in Theorem 5.11. Then the bound*

$$\left\| \mathfrak{I}_A - \sum_{P \in \mathcal{P}(Q_n)} \nu_P \frac{\tilde{\mathfrak{e}}(P)}{|Q_n| \dim(\mathcal{H})} \right\|_\infty \le 4 \frac{|\partial^R(Q_n)|}{|Q_n|}$$

holds for all $n \in \mathbb{N}$.

We give a simple example to show that in general the IDS depends on the choice of the Følner sequence (U_j).

Example 5.14. Consider the usual graph of $\mathbb{Z}$ with standard edges, the set $\mathcal{Z} = \{\text{black, white}\}$ and the coloring

$$\mathcal{C} : \mathbb{Z} \to \mathcal{Z}, \qquad \mathcal{C}(x) = \begin{cases} \text{white,} & \text{if } x \ge 0 \text{ or } x = 3k \text{ for } k \in \mathbb{Z}, \\ \text{black,} & \text{otherwise.} \end{cases}$$

Deleting all edges which are incident to a white vertex gives rise to a new graph and hence a new adjacency operator A. This operator is self-adjoint, of finite hopping range and $\mathcal{C}$-invariant. We choose two Følner sequences (U_j) and (V_j) as follows

$$U_j = \{1, \dots, 3j\} \qquad \text{and} \qquad V_j = \{-3j, \dots, -1\}. \tag{5.19}$$

Since for all $j \in \mathbb{N}$ all entries of the matrix $A[U_j]$ are equal to zero, the IDS $\mathfrak{I}_U$ with respect to the sequence (U_j) is

$$\mathfrak{I}_U(\lambda) = \begin{cases} 0 & \text{if } \lambda < 0 \\ 1 & \text{otherwise.} \end{cases}$$

Computing the IDS along the sequence (V_j) gives a completely different picture: The eigenvalues of the matrix $A[V_j]$ are $-1, 0$ and 1, each of them with multiplicity j. Therefore, the IDS $\mathfrak{I}_V$ with respect to the sequence (V_j) is the function

$$\mathfrak{I}_V(\lambda) = \begin{cases} 0 & \text{if } \lambda < -1, \\ 1/3 & \text{if } -1 \leq \lambda < 0, \\ 2/3 & \text{if } 0 \leq \lambda < 1, \\ 1 & \text{otherwise.} \end{cases}$$

5.2 Deterministic operators on general amenable groups

This section is devoted to show that the results from the previous section carry over to the setting of *all* finitely generated amenable groups. To this end, we need to overcome the assumption that there exists a Følner sequence such that each element of the group symmetrically tiles the group. This will be done using the theory of ε-quasi tilings. The results we present here are joint work with Felix Pogorzelski, see [PS12]. They will also be part of the PhD thesis of Felix Pogorzelski. For this reason we will be rather explicit with assigning originality.

5.2.1 Tiling theorems for general amenable groups

In this subsection we provide results concerning the existence of certain quasi-tilings which are valid for *all* amenable groups. We present two tiling theorems. In the first tiling theorem, namely Theorem 5.20 one finds contributions of both authors (for details see the appendix). Theorem 5.22 is a development of Felix Pogorzelski. The proofs of the announced theorems are to be found in the appendix and will appear in [Pog14] as well. Let us start with some definitions.

Definition 5.15. Let G be a finitely generated group, $\varepsilon > 0$ and I some index set. Then the sets $T_i \subseteq G$, $i \in I$ are called *ε-disjoint* if there are subsets $\mathring{T}_i \subseteq T_i$, $i \in I$ such that for any distinct $i, j \in I$ we have

(i) $\mathring{T}_i$ and $\mathring{T}_j$ are disjoint,

(ii) $|\mathring{T}_i| \geq (1 - \varepsilon)|T_i|$.

Definition 5.16. Let G be a finitely generated group. We say that $S \in \mathcal{F}(G)$ *α-covers* a set $T \in \mathcal{F}(G)$ for $0 < \alpha \leq 1$ if

$$|S \cap T| \geq \alpha |T|.$$

Putting these notions together we formulate the following definition.

Definition 5.17. Let G be a finitely generated group, $T \in \mathcal{F}(G)$ and $\varepsilon, \delta > 0$. A finite subset $K \in \mathcal{F}(G)$ with a set $C \in \mathcal{F}(G)$ is called a *small ε-quasi tiling of T with accuracy δ* if

(i) $KC \subseteq T$,

(ii) Kc, $c \in C$ are ε-disjoint,

(iii) $(\varepsilon - \delta)|T| \leq |KC| \leq (\varepsilon + \delta)|T|$.

Moreover, for given $B \in \mathcal{F}(G)$ and $\zeta > 0$ we call this small ε-quasi tiling *(B, ζ)-good* if

(iv) there are pairwise disjoint sets $K^{(c)} \subseteq K$, $c \in C$ with the equality $KC = \bigcup_{c \in C} K^{(c)} c$, such that for each $c \in C$ we have $|K^{(c)}| \geq (1 - \varepsilon)|K|$ and $K^{(c)}$ is (B, ζ)-invariant.

Note that (iii) in the last definition implies that T is at least $(\varepsilon - \delta)$-covered (and at most $(\varepsilon + \delta)$-covered) by KC. The notion *small* in Definition 5.17 refers to the fact that here we only cover a small portion of T. The set C in Definition 5.17 is called *center set* of the small ε-quasi tiling. Now we formulate a Definition where nearly everything of T can be covered.

Definition 5.18. Let G be a finitely generated group, $T \in \mathcal{F}(G)$ and $\beta, \varepsilon > 0$. The sets $K_i \in \mathcal{F}(G)$, $i = 1, \ldots, N$ with sets $C_i \in \mathcal{F}(G)$, $i = 1, \ldots, N$ are called *ε-quasi tiling of T with accuracy β and densities η_i, $i = 1, \ldots, N$* if

(i) for all $i = 1, \ldots, N$ we have $K_i C_i \subseteq T$,

(ii) for all $i = 1, \ldots, N$ we have that $K_i c$, $c \in C_i$ are ε-disjoint,

(iii) the sets $K_i C_i$, $i = 1, \ldots, N$ are pairwise disjoint,

(iv) $(\eta_i - \beta)|T| \leq |K_i C_i| \leq (\eta_i + \beta)|T|$.

Moreover, for given $B \in \mathcal{F}(G)$ and $\zeta > 0$ we call this ε-quasi tiling (B,ζ)-*good* if

(v) for each $i \in \{1,\dots,N\}$ there are pairwise disjoint sets $K_i^{(c)} \subseteq K_i$, $c \in C_i$ with the equality $K_iC_i = \bigcup_{c\in C_i} K_i^{(c)}c$ such that for each $c \in C_i$ we have $|K_i^{(c)}| \geq (1-\varepsilon)|K_i|$ and $K_i^{(c)}$ is (B,ζ)-invariant.

As before the sets C_i, $i = 1,\dots,N$ are called *center sets* of the ε-quasi tiling. For a given number $b \in \mathbb{R}$, we will use the notation $\lceil b \rceil$ for the smallest integer greater than or equal to b, i.e. $\lceil b \rceil := \inf\{m \in \mathbb{Z} \mid m \geq b\}$. Beside this, for given $0 < \varepsilon < 1$, the number $N(\varepsilon)$ is defined by

$$N(\varepsilon) := \left\lceil \frac{\log(\varepsilon)}{\log(1-\varepsilon)} \right\rceil. \tag{5.20}$$

and for $i \in \mathbb{N}_0$ we set

$$\eta_i(\varepsilon) := \varepsilon(1-\varepsilon)^{N(\varepsilon)-i}. \tag{5.21}$$

Remark 5.19. Let us discuss Definition 5.18.

(a) If K_i with center sets C_i, $i = 1,\dots,N$ is an ε-quasi tiling of T with accuracy β and densities η_i, $i = 1,\dots,N$, then the part of T, which is covered by translates of K_i, $i = 1,\dots,N$ is by (iv) at least $\sum_{i=1}^N \eta_i - N\beta$. This expression might be close to one if the parameters β and η_i, $i = 1,\dots,N$ are chosen appropriately.

(b) Item (iv) also explains why we call the values η_i "densities". This is emphasized by the fact, that with the special choice of $\eta_i(\varepsilon)$, $i \in \{1,\dots,N(\varepsilon)\}$ and $N(\varepsilon)$ in (5.21) and (5.20), the $\eta_i(\varepsilon)$ almost sum up to one (up to an ε). In fact we have for $\varepsilon \in (0,1)$

$$1-\varepsilon \leq \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \leq 1.$$

This is clear as $N(\varepsilon) = \lceil \log(\varepsilon)/\log(1-\varepsilon) \rceil$ and

$$\sum_{i=1}^{N(\varepsilon)} \varepsilon(1-\varepsilon)^{N(\varepsilon)-i} = \varepsilon \sum_{i=0}^{N(\varepsilon)-1} (1-\varepsilon)^i = 1-(1-\varepsilon)^{N(\varepsilon)} \leq 1. \tag{5.22}$$

Furthermore

$$1-(1-\varepsilon)^{N(\varepsilon)} \geq 1-(1-\varepsilon)^{\log(\varepsilon)/\log(1-\varepsilon)} = 1-\varepsilon$$

holds for all $\varepsilon \in (0,1)$.

(c) In the next theorem we will obtain an ε-quasi tiling with sets K_i and center sets C_i of a set T, where $N = N(\varepsilon)$ and $\eta_i = \eta_i(\varepsilon)$ as in (5.20) and (5.21). If in this situation $\beta \leq \varepsilon/N(\varepsilon)$ we get that T is $(1-2\varepsilon)$-covered by the corresponding translates of K_i. To see this, note that by the properties (i) and (iii) we have

$$\frac{\left|T \cap \bigcup_{i=1}^{N(\varepsilon)} T_i C_i^T\right|}{|T|} = \sum_{i=1}^{N(\varepsilon)} \frac{|T_i C_i^T|}{|T|} \geq \sum_{i=1}^{N(\varepsilon)} \left(\varepsilon(1-\varepsilon)^{N(\varepsilon)-i} - \frac{\varepsilon}{N(\varepsilon)}\right)$$

With the calculation of the previous item we obtain

$$\sum_{i=1}^{N(\varepsilon)} \left(\varepsilon(1-\varepsilon)^{N(\varepsilon)-i} - \frac{\varepsilon}{N(\varepsilon)}\right) \geq 1-\varepsilon-\varepsilon\sum_{i=1}^{N(\varepsilon)} \frac{1}{N(\varepsilon)} = 1-2\varepsilon,$$

which proves the claim.

Now let us state the first tiling theorem. The proof is to be found in the appendix.

Theorem 5.20. *Let G be a finitely generated amenable group and (Q_n) a nested Følner sequence. Then for any $0<\beta<\varepsilon\leq 1/10$ there are sets*

$$\{\mathrm{id}\} \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_{N(\varepsilon)}$$

with $K_i \in \{Q_n \mid n \geq i\}$ for all $i=1,\dots,N(\varepsilon)$, such that for any set T, which is $(K_{N(\varepsilon)}K_{N(\varepsilon)}^{-1}, \beta 6^{-N(\varepsilon)})$-invariant, there exist center sets C_i^T, $i=1,\dots,N(\varepsilon)$ with which the K_i, $i=1,\dots,N(\varepsilon)$ form an ε-quasi tiling of T with accuracy β and densities $\eta_i(\varepsilon)$, $i=1,\dots,N$. If additionally $\mathrm{id} \in B \in \mathcal{F}(G)$ *and $\zeta>0$ is given, then we can even ensure that the ε-quasi tiling we obtain is (B,ζ)-good. Here $N(\varepsilon)$ and $\eta_i(\varepsilon)$ are given as in* (5.20) *and* (5.21).

Definition 5.21. Let G be a finitely generated group, $T \in \mathcal{F}(G)$, $N \in \mathbb{N}$ and $\beta, \varepsilon, r > 0$. Furthermore let $\gamma = (\gamma_i)_{i=1}^N$ and $\eta = (\eta_i)_{i=1}^N$ be elements of $[0,1]^N$. The sets $K_i \in \mathcal{F}(G)$, $i = 1, \dots, N$ with index set Λ and sets $C_i^\lambda \in \mathcal{F}(G)$, $i = 1, \dots, N$, $\lambda \in \Lambda$ are called *uniform ε-quasi tiling of T with parameters* (β, r, γ, η), if for all $\lambda \in \Lambda$ the following holds

(i) for all $i = 1, \dots, N$ we have $K_i C_i^\lambda \subseteq T$,

(ii) for all $i = 1, \dots, N$ we have that $K_i c$, $c \in C_i^\lambda$ are ε-disjoint,

(iii) the sets $K_i C_i^\lambda$, $i = 1, \dots, N$ are pairwise disjoint,

(iv) $|\bigcup_{i=1}^N K_i C_i^\lambda| \geq (1 - 3\varepsilon)|T|$,

as well as the condition on the uniformity

(v) for all $i = 1, \dots, N$ and all $g \in T^{(r)}$

$$\left| \frac{1}{|\Lambda|} \sum_{\lambda \in \Lambda} \mathbf{1}_{C_i^\lambda}(g) - \frac{\eta_i(\varepsilon)}{|K_i|} \right| \leq \frac{3\beta}{|K_i|} + \varepsilon \gamma_i.$$

The sets $C_i^\lambda \in \mathcal{F}(G)$, $i = 1, \dots, N$, $\lambda \in \Lambda$ are called the *center sets* of the uniform ε-quasi tiling. Now we formulate the uniform tiling theorem, for which a proof is also to be found in the appendix.

Theorem 5.22. *Let G be a finitely generated amenable group and (U_j) and (Q_n) be Følner sequences, where we assume (Q_n) to be nested. Furthermore, let for $0 < \varepsilon \leq 1/10$ and $0 < \beta \leq \varepsilon / N(\varepsilon)$ the K_i, $i = 1, \dots, N(\varepsilon)$ be chosen according to Theorem 5.20. Then there exist $j_0, r \in \mathbb{N}$, such that for each $j \geq j_0$ we can find an index set Λ_j and center sets $C_i^\lambda(j)$ ($i = 1, \dots, N(\varepsilon)$, $\lambda \in \Lambda_j$) and $\gamma \in [0,1]^{N(\varepsilon)}$, such that we obtain together with the K_i, $i = 1, \dots, N(\varepsilon)$ a uniform ε-quasi tiling of U_j with parameters $(\beta, r, \gamma, \eta(\varepsilon))$. Besides this, the γ_i fulfill $\sum_{i=1}^{N(\varepsilon)} \gamma_i |K_i| \leq 2$. Here again $N(\varepsilon)$ and $\eta_i(\varepsilon)$ are given as in* (5.20) *and* (5.21) *and $\eta(\varepsilon) = (\eta_i(\varepsilon))_{i=1}^{N(\varepsilon)}$.*

5.2.2 An ergodic theorem for general amenable groups

This subsection is devoted to generalize the Banach space-valued ergodic theorem from Subsection 5.1.1. We will show that one can drop the assumption that the group needs to fulfill condition of Definition (5.4). In order to do so, we will make use of the tiling theorems given in previous subsection.

The next Lemma is a joint work with Felix Pogorzelski. It shows that almost-additive functions still fulfill a certain kind of almost-additivity if one inserts not disjoint, but only ε-disjoint sets.

Lemma 5.23. *Let G be a finitely generated group, $(X, \|\cdot\|)$ a Banach space and let $F\colon \mathcal{F}(G) \to X$ be almost-additive with boundary term b and let $\varepsilon \in (0, 1/2)$ and $C > 0$ be such that for all $Q \in \mathcal{F}(G)$ one has $F(Q) \leq C|Q|$. Then for any ε-disjoint sets Q_i, $i = 1, \dots, k$ we have*

$$\left\| F(Q) - \sum_{i=1}^{k} F(Q_i) \right\| \leq \varepsilon(3C + 9D)|Q| + 3\sum_{i=1}^{k} b(Q_i)$$

where $Q := \bigcup_{i=1}^{k} Q_i$ and D is the constant from property (c) of Definition 5.6

Proof. Let Q_i, $i = 1, \dots, k$ be ε-disjoint and set $Q := \bigcup_{i=1}^{k} Q_i$. Furthermore let $\mathring{Q}_i \subseteq Q_i$ be the sets from Definition 5.15. Moreover, we use the notation $\mathring{Q} := \bigcup_{i=1}^{k} \mathring{Q}_i$. By triangle inequality we obtain

$$\left\| F(Q) - \sum_{i=1}^{k} F(Q_i) \right\| \leq d_1 + d_2 + d_3$$

where

$$d_1 = \| F(Q) - F(\mathring{Q}) \|, \quad d_2 = \left\| F(\mathring{Q}) - \sum_{i=1}^{k} F(\mathring{Q}_i) \right\|$$

and

$$d_3 = \sum_{i=1}^{k} \| F(Q_i) - F(\mathring{Q}_i) \|.$$

By almost-additivity and the boundedness of F we get for arbitrary sets $V \subseteq U \in \mathcal{F}(G)$

$$\begin{aligned}\|F(U) - F(V)\| &\leq \|F(U) - F(V) - F(U \setminus V)\| + \|F(U \setminus V)\| \\ &\leq b(V) + b(U \setminus V) + C|U \setminus V| \\ &\leq b(V) + (C + D)|U \setminus V| \end{aligned} \tag{5.23}$$

and

$$b(V) \leq b(U) + b(U \setminus V) \leq b(U) + D|U \setminus V|. \tag{5.24}$$

We apply (5.23) as well as the inequalities $|Q \setminus \mathring{Q}| \leq \varepsilon|Q|$ and $|Q_i \setminus \mathring{Q}_i| \leq \varepsilon|Q_i|$ to obtain

$$d_1 \leq b(\mathring{Q}) + \varepsilon(C + D)|Q| \quad \text{and} \quad d_3 \leq \sum_{i=1}^{k} b(\mathring{Q}_i) + \varepsilon(C + D)|Q_i|.$$

Due to almost-additivity we also have $d_2 \leq \sum_{i=1}^{k} b(Q_i)$, such that we end up with

$$\left\| F(Q) - \sum_{i=1}^{k} F(Q_i) \right\| \leq 3 \sum_{i=1}^{k} b(\mathring{Q}_i) + \varepsilon(C + D) \left(|Q| + \sum_{i=1}^{k} |Q_i| \right).$$

By (5.24) we have $b(\mathring{Q}_i) \leq b(Q_i) + \varepsilon D|Q_i|$ for all $i = 1, \dots, n$ and obtain

$$\begin{aligned}&\left\| F(Q) - \sum_{i=1}^{k} F(Q_i) \right\| \\ &\leq 3 \sum_{i=1}^{k} b(Q_i) + \varepsilon(C + 4D) \sum_{i=1}^{k} |Q_i| + \varepsilon(C + D)|Q|. \end{aligned} \tag{5.25}$$

Finally we use again the ε-disjointness to estimate

$$\frac{1}{2} \sum_{i=1}^{k} |Q_i| \leq \sum_{i=1}^{k} |Q_i|(1 - \varepsilon) \leq \sum_{i=1}^{k} |\mathring{Q}_i| = |\mathring{Q}| \leq |Q|,$$

which we plug in at (5.25) to obtain the claimed bound. ■

Let us formulate an appropriate assumption, which will be needed in the Theorem.

Assumption 5. The sequence (Q_n) is nested. For $0 < \varepsilon < 1$ and $i \in \mathbb{N}_0$ we use the notion $N(\varepsilon)$ and $\eta_i(\varepsilon)$ as given in (5.20) and (5.21). Furthermore for given $0 < \varepsilon < 1/10$ we denote by K_i^ε, $i = 1, \dots, N(\varepsilon)$ the elements given by Theorem 5.20 where we set $\beta := \varepsilon / N(\varepsilon)$.

Theorem 5.24. *Let Assumptions 1 and 5 be satisfied. Let $F : \mathcal{F}(G) \to X$ be almost-additive and $\mathcal{C}$-invariant and the associated function $\tilde{F} : \tilde{\mathcal{P}} \to X$ be given as before. Then the following limits exist with respect to the Banach space norm and they are equal:*

$$\lim_{j \to \infty} \frac{F(U_j)}{|U_j|} = \lim_{\substack{\varepsilon \searrow 0 \\ \varepsilon < 1/10}} \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P \in \mathcal{P}(K_i^\varepsilon)} \nu_P \frac{\tilde{F}(P)}{|K_i^\varepsilon|}.$$

The main idea of the proof is to extend the ideas of Theorem 5.8 to the situation where one has only ε-quasi tilings at hand. To do so, we need an appropriate bound on an error term $\Delta(j, \varepsilon)$ defined below. This bound corresponds to Inequality (5.9) in the situation of ε-quasi tiles and will be given in Lemma 5.25. The extension to this more general setting is due to Felix Pogorzelski, cf. [Pog14]. Inspired by Theorem 5.8 we develop an adapted version of this result (namely Theorem 5.24) using the uniform tilings given by Theorem 5.22.

Lemma 5.25. *Let Assumptions 1 and 5 be satisfied. Let $F : \mathcal{F}(G) \to X$ be almost-additive and $\mathcal{C}$-invariant and let $\tilde{F} : \tilde{\mathcal{P}} \to X$ be the associated function given as above. Furthermore let some $0 < \varepsilon < 1/10$ be given. Then there exist some $j(\varepsilon) \in \mathbb{N}$ and $r(\varepsilon) \in \mathbb{N}$ such that for every $j \geq j(\varepsilon)$, the difference*

$$\Delta(j, \varepsilon) := \left\| \frac{F(U_j)}{|U_j|} - \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P \in \mathcal{P}(K_i^\varepsilon)} \nu_P \frac{\tilde{F}(P)}{|K_i^\varepsilon|} \right\|$$

satisfies the estimate

$$\Delta(j,\varepsilon) \leq (11C+32D)\varepsilon + C\sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P\in\mathcal{P}(K_i^\varepsilon)} \left| \frac{\sharp_P(\mathcal{C}|_{U_j})}{|U_j|} - \nu_P \right|$$
$$+ 4\sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon)\frac{b(K_i^\varepsilon)}{|K_i^\varepsilon|} + (C+4D)\frac{|\partial^{r(\varepsilon)}U_j|}{|U_j|}\sum_{i=1}^{N(\varepsilon)} |K_i^\varepsilon|. \quad (5.26)$$

Proof. By Assumption 5 we have $\beta = \varepsilon/N(\varepsilon)$ and K_i^ε, $i=1,\dots,N(\varepsilon)$ are chosen according to Theorem 5.22. Denote by $j_0 = j(\varepsilon)$ and $r = r(\varepsilon)$ the constants given by the Theorem. For the rest of the proof we fix some $j \geq j_0$. Let the sets Λ_j and $C_i^\lambda(j)$, $i=1,\dots,N(\varepsilon)$ and the numbers γ_i, $i=1,\dots,N$ be the associated objects given by Theorem 5.22. We start to estimate $\Delta(j,\varepsilon)$. Using triangle inequality we obtain

$$\Delta(j,\varepsilon) \leq \left\| \frac{F(U_j)}{|U_j|} - \frac{1}{|U_j||\Lambda_j|}\sum_{\lambda\in\Lambda_j}\sum_{i=1}^{N(\varepsilon)}\sum_{g\in C_i^\lambda(j)} F(K_i^\varepsilon g) \right\|$$
$$+ \left\| \sum_{\lambda\in\Lambda_j}\sum_{i=1}^{N(\varepsilon)}\sum_{g\in C_i^\lambda(j)} \frac{F(K_i^\varepsilon g)}{|U_j||\Lambda_j|} - \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P\in\mathcal{P}(K_i^\varepsilon)} \frac{\sharp_P(\mathcal{C}|_{U_j})}{|U_j|}\frac{\tilde{F}(P)}{|K_i^\varepsilon|} \right\|$$
$$+ \left\| \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P\in\mathcal{P}(K_i^\varepsilon)} \left(\frac{\sharp_P(\mathcal{C}|_{U_j})}{|U_j|} - \nu_P \right) \frac{\tilde{F}(P)}{|K_i^\varepsilon|} \right\|.$$

Some more applications of triangle inequality yield

$$\Delta(j,\varepsilon) \leq D_1(j,\varepsilon) + D_2(j,\varepsilon) + D_3(j,\varepsilon),$$

where

$$D_1(j,\varepsilon) := \frac{1}{|U_j||\Lambda_j|}\sum_{\lambda\in\Lambda_j} \left\| F(U_j) - \sum_{i=1}^{N(\varepsilon)}\sum_{g\in C_i^\lambda(j)} F(K_i^\varepsilon g) \right\|,$$

$$D_2(j,\varepsilon) := \left\| \sum_{\lambda\in\Lambda_j}\sum_{i=1}^{N(\varepsilon)}\sum_{g\in C_i^\lambda(j)} \frac{F(K_i^\varepsilon g)}{|U_j||\Lambda_j|} - \sum_{i=1}^{N(\varepsilon)} \frac{\eta_i(\varepsilon)}{|K_i^\varepsilon|} \sum_{P\in\mathcal{P}(K_i^\varepsilon)} \frac{\sharp_P(\mathcal{C}|_{U_j})}{|U_j|}\tilde{F}(P)\right\|,$$

$$D_3(j,\varepsilon) := \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P\in\mathcal{P}(K_i^\varepsilon)} \left| \frac{\sharp_P(\mathcal{C}|_{U_j})}{|U_j|} - \nu_P\right| \frac{\|\tilde{F}(P)\|}{|K_i^\varepsilon|}.$$

In (5.7) we showed that $\tilde{F}$ is bounded, i.e. that there exists $C>0$ with $\|\tilde{F}(P)\| \leq C|D(P)|$ for all $P\in\mathcal{P}$. As the patterns here all have domain K_i^ε, we get

$$D_3(j,\varepsilon) \leq C\sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P\in\mathcal{P}(K_i^\varepsilon)} \left| \frac{\sharp_P(\mathcal{C}|_{U_j})}{|U_j|} - \nu_P\right|. \tag{5.27}$$

In order to estimate the term $D_2(j,\varepsilon)$ we use $C_i^\lambda(j)\subseteq U_j$ which holds due to the fact that each K_i^ε contains id and property (i) in Definition 5.21. Furthermore, we reorder the sum over all patters in the same way as we did it in the first equality in (5.16). We obtain

$$\begin{aligned}
&D_2(j,\varepsilon)\\
&= \frac{1}{|U_j|}\left\| \sum_{i=1}^{N(\varepsilon)}\sum_{g\in U_j}\sum_{\lambda\in\Lambda_j} \frac{\mathbf{1}_{C_i^\lambda(j)}(g)}{|\Lambda_j|}F(K_i^\varepsilon g) - \sum_{i=1}^{N(\varepsilon)}\frac{\eta_i(\varepsilon)}{|K_i^\varepsilon|}\sum_{\substack{g\in U_j\\ K_i^\varepsilon g\subseteq U_j}} F(K_i^\varepsilon g)\right\|\\
&\leq \frac{1}{|U_j|}\sum_{i=1}^{N(\varepsilon)}\sum_{g\in U_j}\left|\frac{1}{|\Lambda_j|}\sum_{\lambda\in\Lambda_j}\mathbf{1}_{C_i^\lambda(j)}(g) - \frac{\eta_i(\varepsilon)}{|K_i^\varepsilon|}\right| \|F(K_i^\varepsilon g)\|.
\end{aligned}$$

Now we apply the boundedness of F and get

$$D_2(j,\varepsilon) \leq \frac{C}{|U_j|}\sum_{i=1}^{N(\varepsilon)}\sum_{g\in U_j}|K_i^\varepsilon|\left|\frac{1}{|\Lambda_j|}\sum_{\lambda\in\Lambda_j}\mathbf{1}_{C_i^\lambda(j)}(g) - \frac{\eta_i(\varepsilon)}{|K_i^\varepsilon|}\right|.$$

In the inner sum we need to distinguish between elements g which are in $\partial^{r(\varepsilon)}(U_j)$ and those g which are in $U_j^{(r(\varepsilon))}$. The reason for that is, that only for the last ones we can apply the important property

(v) in Definition 5.21 on the uniformity of the covering. With end up with

$$\begin{aligned} D_2(j,\varepsilon) &\leq \frac{C}{|U_j|}\sum_{i=1}^{N(\varepsilon)}\sum_{g\in U_j^{(r(\varepsilon))}}|K_i^\varepsilon|\left|\frac{1}{|\Lambda_j|}\sum_{\lambda\in\Lambda_j}\mathbf{1}_{C_i^\lambda(j)}(g)-\frac{\eta_i(\varepsilon)}{|K_i^\varepsilon|}\right| \\ &\quad+\frac{C}{|U_j|}\sum_{i=1}^{N(\varepsilon)}\sum_{g\in\partial^{r(\varepsilon)}U_j}|K_i^\varepsilon|\left|\frac{1}{|\Lambda_j|}\sum_{\lambda\in\Lambda_j}\mathbf{1}_{C_i^\lambda(j)}(g)-\frac{\eta_i(\varepsilon)}{|K_i^\varepsilon|}\right| \\ &\leq \frac{C}{|U_j|}\sum_{i=1}^{N(\varepsilon)}\sum_{g\in U_j}|K_i^\varepsilon|\left(\frac{3\beta}{|K_i^\varepsilon|}+\varepsilon\gamma_i\right)+C\frac{|\partial^{r(\varepsilon)}U_j|}{|U_j|}\sum_{i=1}^{N(\varepsilon)}|K_i^\varepsilon|, \end{aligned}$$

where in the second summand we estimated the difference in absolute values by one. Note that by the choice of β and Theorem 5.22 we have

$$\beta N(\varepsilon)=\varepsilon \qquad \text{and} \qquad \sum_{i=1}^{N(\varepsilon)}\gamma_i|K_i^\varepsilon|\leq 2, \tag{5.28}$$

which we apply to obtain

$$C\sum_{i=1}^{N(\varepsilon)}(3\beta+\varepsilon\gamma_i|K_i^\varepsilon|)=3CN(\varepsilon)\beta+\varepsilon C\sum_{i=1}^{N(\varepsilon)}\gamma_i|K_i^\varepsilon|\leq 5C\varepsilon.$$

This can be used to estimate the first summand in the last estimate of $D_2(j,\varepsilon)$, such that we end up with

$$D_2(j,\varepsilon)\leq 5C\varepsilon+C\frac{|\partial^{r(\varepsilon)}U_j|}{|U_j|}\sum_{i=1}^{N(\varepsilon)}|K_i^\varepsilon|. \tag{5.29}$$

It remains to estimate $D_1(j,\varepsilon)$. In order to do so, we first define the set of elements in U_j, which are covered by some translate for one specific $\lambda\in\Lambda_j$

$$A_\lambda=\bigcup_{i=1}^{N(\varepsilon)}\bigcup_{g\in C_i^\lambda(j)}K_i^\varepsilon g.$$

Using this and almost additivity, we estimate for fixed $\lambda \in \Lambda$

$$\begin{aligned}\|F(U_j) - F(A_\lambda)\| &\leq \|F(U_j \setminus A_\lambda)\| + b(U_j \setminus A_\lambda) + b(A_\lambda) \\ &\leq 3C\varepsilon|U_j| + 3D\varepsilon|U_j| + \sum_{i=1}^{N(\varepsilon)} \sum_{g \in C_i^\lambda(j)} b(K_i^\varepsilon g).\end{aligned} \tag{5.30}$$

In the last step we applied the boundedness of F, properties (c) and (d) of Definition 5.6 and property (iv) of Definition 5.21, which gives $|U_j \setminus A_\lambda| \leq 3\varepsilon|U_j|$. Moreover, we use Lemma 5.23 to estimate

$$\begin{aligned}&\left\| F(A_\lambda) - \sum_{i=1}^{N(\varepsilon)} \sum_{g \in C_i^\lambda(j)} F(K_i^\varepsilon g) \right\| \\ &\leq \varepsilon(3C + 9D)|U_j| + 3\sum_{i=1}^{N(\varepsilon)} \sum_{g \in C_i^\lambda(j)} b(K_i^\varepsilon g).\end{aligned} \tag{5.31}$$

Using triangle inequality and the Estimates (5.30) and (5.31), we obtain

$$\begin{aligned}D_1(j,\varepsilon) &\leq \frac{1}{|U_j||\Lambda_j|} \sum_{\lambda \in \Lambda_j} \left(\varepsilon(6C + 12D)|U_j| + 4\sum_{i=1}^{N(\varepsilon)} \sum_{g \in C_i^\lambda(j)} b(K_i^\varepsilon) \right) \\ &\leq \varepsilon(6C + 12D) + \frac{4}{|U_j||\Lambda_j|} \sum_{\lambda \in \Lambda_j} \sum_{i=1}^{N(\varepsilon)} |C_i^\lambda(j)| b(K_i^\varepsilon).\end{aligned}$$

Again with the application of property (v) in Definition 5.21, we get for each index $i \in \{1, \ldots, N(\varepsilon)\}$:

$$\begin{aligned}&\frac{1}{|\Lambda_j|} \sum_{\lambda \in \Lambda_j} |C_i^\lambda(j)| \\ &= \sum_{g \in U_j} \frac{1}{|\Lambda_j|} \sum_{\lambda \in \Lambda_j} \mathbf{1}_{C_i^\lambda(j)}(g) \\ &\leq \sum_{g \in U_j^{(r(\varepsilon))}} \frac{1}{|\Lambda_j|} \sum_{\lambda \in \Lambda_j} \mathbf{1}_{C_i^\lambda(j)}(g) + \sum_{g \in \partial^{r(\varepsilon)} U_j} \frac{1}{|\Lambda_j|} \sum_{\lambda \in \Lambda_j} \mathbf{1}_{C_i^\lambda(j)}(g)\end{aligned}$$

$$\leq |U_j|\left(\frac{3\beta}{|K_i^\varepsilon|} + \varepsilon\gamma_i + \frac{\eta_i(\varepsilon)}{|K_i^\varepsilon|}\right) + |\partial^{r(\varepsilon)}U_j|,$$

which gives

$$\begin{aligned}
&D_1(j,\varepsilon)\\
&\leq \varepsilon(6C+12D) + 4\sum_{i=1}^{N(\varepsilon)} b(K_i^\varepsilon)\left(\frac{3\beta}{|K_i^\varepsilon|} + \varepsilon\gamma_i + \frac{\eta_i(\varepsilon)}{|K_i^\varepsilon|} + \frac{|\partial^{r(\varepsilon)}U_j|}{|U_j|}\right)\\
&\leq \varepsilon(6C+24D) + 4\varepsilon D\sum_{i=1}^{N(\varepsilon)}\gamma_i|K_i^\varepsilon| + 4\sum_{i=1}^{N(\varepsilon)} b(K_i^\varepsilon)\left(\frac{\eta_i(\varepsilon)}{|K_i^\varepsilon|} + \frac{|\partial^{r(\varepsilon)}U_j|}{|U_j|}\right)\\
&\leq \varepsilon(6C+32D) + 4\sum_{i=1}^{N(\varepsilon)}\eta_i(\varepsilon)\frac{b(K_i^\varepsilon)}{|K_i^\varepsilon|} + 4D\frac{|\partial^{r(\varepsilon)}U_j|}{|U_j|}\sum_{i=1}^{N(\varepsilon)}|K_i^\varepsilon|, \quad (5.32)
\end{aligned}$$

where we used again (5.28). Putting the Estimates (5.32), (5.29) and (5.27) together finally gives the desired bound on $\Delta(j,\varepsilon)$. ■

Lemma 5.26. *Let a complex-valued null sequence $(\alpha_i)_{i\in\mathbb{N}}$ be given and let $N(\varepsilon)$ and $\eta_i(\varepsilon)$ be as in* (5.20) *and* (5.21). *Then,*

$$\lim_{\substack{\varepsilon\to 0\\ 0<\varepsilon<1}} \sum_{i=1}^{N(\varepsilon)} \alpha_i\eta_i(\varepsilon) = 0.$$

Proof. First note that for each $\varepsilon \in (0,1)$ we have the following estimate:

$$\sum_{i=1}^{N(\varepsilon)} \varepsilon(1-\varepsilon)^{N(\varepsilon)-i} = \sum_{i=0}^{N(\varepsilon)-1} \varepsilon(1-\varepsilon)^i \leq 1.$$

Define $k := \sup\{|\alpha_i| \mid i \in \mathbb{N}\} < \infty$ and let $\delta > 0$ be given. Choose a number $n(\delta) \in \mathbb{N}$ with $|\alpha_i| < \delta/2$ for all $i \geq n(\delta)$. Then for all $0 < \varepsilon < \delta/(2kn(\delta))$ we obtain using triangle inequality and the above estimate

$$
\begin{aligned}
&\left|\sum_{i=1}^{N(\varepsilon)} \varepsilon(1-\varepsilon)^{N(\varepsilon)-i}\alpha_i\right| \\
&= \left|\sum_{i=1}^{n(\delta)} \varepsilon(1-\varepsilon)^{N(\varepsilon)-i}\alpha_i + \sum_{i=n(\delta)+1}^{N(\varepsilon)} \varepsilon(1-\varepsilon)^{N(\varepsilon)-i}\alpha_i\right| \\
&\leq k\varepsilon n(\delta) + \frac{\delta}{2}\sum_{i=n(\delta)+1}^{N(\varepsilon)} \varepsilon(1-\varepsilon)^{N(\varepsilon)-i} \leq \delta.
\end{aligned}
$$

Since δ is arbitrary, the claim follows. ∎

Proof of Theorem 5.24. Since (Q_n) is a Følner sequence and b a boundary term we have

$$\frac{b(Q_n)}{|Q_n|} \to 0 \qquad \text{as} \qquad n \to \infty.$$

With no loss of generality we assume that this convergence is monotone. If it was not, then we would pass over to a subsequence of (Q_n) and apply the tiling theorems with this subsequence.

Now we make use of the fact $K_i^\varepsilon \in \{Q_k \mid k \geq i\}$. This gives together with Lemma 5.26

$$\lim_{\substack{\varepsilon\searrow 0\\ \varepsilon<1/10}} \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon)\frac{b(K_i^\varepsilon)}{|K_i^\varepsilon|} \leq \lim_{\substack{\varepsilon\searrow 0\\ \varepsilon<1/10}} \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon)\frac{b(Q_i)}{|Q_i|} = 0$$

This, the existence of the frequencies along (U_j) and the bound on $\Delta(j,\varepsilon)$ given by Lemma 5.25 imply

$$\lim_{\varepsilon\to 0}\lim_{j\to\infty} \Delta(j,\varepsilon) = 0. \tag{5.33}$$

In order to show the Cauchy property let $\kappa > 0$ be arbitrary. Using (5.33) we find an $\varepsilon_0 \in (0, 1/10)$ such that $\lim_{j\to\infty}\Delta(j,\varepsilon_0) \leq \kappa/4$. Hence, there exists $j_1 \geq j(\varepsilon_0)$ with $\Delta(j,\varepsilon_0) \leq \kappa/2$ for all $j \geq j_1$.

We consider for $j, m \geq j_1$ the following difference and use triangle inequality to obtain

$$\left\| \frac{F(U_j)}{|U_j|} - \frac{F(U_m)}{|U_m|} \right\| \leq \Delta(j,\varepsilon) + \Delta(m,\varepsilon) \leq \kappa.$$

This shows that$(|U_j|^{-1} F(U_j))_{j\in\mathbb{N}}$ is a Cauchy sequence and hence convergent in the Banach space X. We denote the limit element by $\bar{F} \in X$. It remains to prove the convergence of the second limit to $\bar{F}$. This follows from

$$\left\| \bar{F} - \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P\in\mathcal{P}(K_i^\varepsilon)} \nu_P \frac{\tilde{F}(P)}{|K_i^\varepsilon|} \right\| \leq \lim_{j\to\infty} \Delta(j,\varepsilon)$$

and (5.33). ∎

As in the setting of ST-amenable groups we can deduce a result concerning the speed of convergence.

Corollary 5.27. *Let Assumptions 1 and 5 be satisfied. Let $F : \mathcal{F}(G) \to X$ be almost-additive and $\mathcal{C}$-invariant and let the associated function $\tilde{F} : \tilde{\mathcal{P}} \to X$ be given as above. Denote the limit in Theorem 5.24 by $\bar{F}$. Then for given $\varepsilon \in (0, 1/10)$ and $j \geq j(\varepsilon)$, where $j(\varepsilon)$ is given by Lemma 5.25 we have*

$$\begin{aligned}\left\| \bar{F} - \frac{F(U_j)}{|U_j|} \right\| \leq (22C + 64D)\varepsilon + C \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P\in\mathcal{P}(K_i^\varepsilon)} \left| \frac{\sharp_P(\mathcal{C}|_{U_j})}{|U_j|} - \nu_P \right| \\ + 8 \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \frac{b(K_i^\varepsilon)}{|K_i^\varepsilon|} + (C + 4D) \frac{|\partial^{r(\varepsilon)} U_j|}{|U_j|} \sum_{i=1}^{N(\varepsilon)} |K_i^\varepsilon|\end{aligned}$$

and

$$\left\| \bar{F} - \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P\in\mathcal{P}(K_i^\varepsilon)} \nu_P \frac{\tilde{F}(P)}{|K_i^\varepsilon|} \right\| \leq (11C + 32D)\varepsilon + 4 \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \frac{b(K_i^\varepsilon)}{|K_i^\varepsilon|}.$$

Proof. Use Estimate (5.26) and proceed as in the proof of Corollary 5.10. ∎

Remark 5.28. Note that if one assumes that (Q_n) is a Følner sequence, such that the sequence $(b(Q_n)/|Q_n|)_{n\in\mathbb{N}}$ converges to 0 monotonically, then we have for all $\varepsilon \in (0, 1/10)$ and $i \in \{1, \dots, N(\varepsilon)\}$

$$\frac{b(K_i^\varepsilon)}{|K_i^\varepsilon|} \leq \frac{b(Q_i)}{|Q_i|}.$$

Therefore, in the estimates of Corollary 5.27 we can replace the fractions $b(K_i^\varepsilon)/|K_i^\varepsilon|$ by $b(Q_i)/|Q_i|$.

5.2.3 Uniform convergence for general amenable groups

In this subsection we show how one uses the Ergodic Theorem 5.24 to generalize Theorem 5.11 to all amenable groups.

Theorem 5.29. *Let Assumptions 1, 3 and 5 be satisfied. Assume additionally that $(|\partial^R Q_n|/|Q_n|)$, converges monotonically to zero. Then there exists a unique probability measure μ_A on $\mathbb{R}$ with distribution function $\mathfrak{I}_A$, such that for all $\varepsilon \in (0, 1/10)$ and $j \geq j(\varepsilon)$ we have*

$$\begin{aligned}
&\left\| \mathfrak{n}\big(A[U_j^{(R)}]\big) - \mathfrak{I}_A \right\|_\infty \\
&\leq (22 + 256|B_R|)\varepsilon + \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P \in \mathcal{P}(K_i^\varepsilon)} \left| \frac{\sharp_P(\mathcal{C}|_{U_j})}{|U_j|} - \nu_P \right| \\
&\quad + 32 \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \frac{|\partial^R K_i^\varepsilon|}{|K_i^\varepsilon|} + (1{+}16|B_R|) \frac{|\partial^{r(\varepsilon)} U_j|}{|U_j|} \sum_{i=1}^{N(\varepsilon)} |K_i^\varepsilon| + \frac{|\partial^R_{\mathrm{int}} U_j|}{|U_j|},
\end{aligned}$$

Here $j(\varepsilon)$ and $r(\varepsilon)$ are given by Lemma 5.25. This implies in particular the convergence

$$\mathfrak{n}\big(A[U_j^{(R)}]\big) \to \mathfrak{I}_A$$

with respect to the supremum norm for $j \to \infty$. As before, the function $\mathfrak{I}_A$ is called the integrated density of states.

Proof. The proof works completely analogously to the proof of Theorem 5.11. The only difference is that we use Corollary 5.27 instead of Corollary 5.10. Since F_R^A is $\mathcal{C}$-invariant and almost-additive, we can apply Corollary 5.27. We set $h := \dim(\mathcal{H})$ and have $D = 4h|B_R|$ and

$C = h$ and for any $Q \in \mathcal{F}(G)$: $b(Q) = 4h|\partial^R Q|$. Therefore, we find a function $\tilde{\mathfrak{I}}_A \in B(\mathbb{R})$ with

$$\begin{aligned}\left\| \frac{F_R^A(U_j)}{h|U_j|} - \frac{\tilde{\mathfrak{I}}_A}{h} \right\|_\infty &\\ \leq (22 + 256|B_R|)\varepsilon + \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P \in \mathcal{P}(K_i^\varepsilon)} &\left| \frac{\sharp_P(\mathcal{C}|_{U_j})}{|U_j|} - \nu_P \right| \\ + 32 \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \frac{|\partial^R K_i^\varepsilon|}{|K_i^\varepsilon|} + (1 + 16|B_R|) &\frac{|\partial^{r(\varepsilon)} U_j|}{|U_j|} \sum_{i=1}^{N(\varepsilon)} |K_i^\varepsilon|.\end{aligned}$$

Changing the normalization and setting $\mathfrak{I}_A := \tilde{\mathfrak{I}}_A / h$ we get

$$\begin{aligned}&\left\| \mathfrak{n}\big(A[U_j^{(R)}]\big) - \mathfrak{I}_A \right\|_\infty \\ &\leq (22 + 256|B_R|)\varepsilon + \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P \in \mathcal{P}(K_i^\varepsilon)} \left| \frac{\sharp_P(\mathcal{C}|_{U_j})}{|U_j|} - \nu_P \right| \\ &+ 32 \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \frac{|\partial^R K_i^\varepsilon|}{|K_i^\varepsilon|} + (1+16|B_R|) \frac{|\partial^{r(\varepsilon)} U_j|}{|U_j|} \sum_{i=1}^{N(\varepsilon)} |K_i^\varepsilon| + \frac{|\partial^R_{\mathrm{int}} U_j|}{|U_j|},\end{aligned}$$

which was to show. Now use the assumption on the monotonicity of $(|\partial^R Q_n|/|Q_n|)$ and proceed as in the proof of Theorem 5.24 to see that this implies uniform convergence. Again by Lemma 2.26 we get that $\mathfrak{I}_A$ is a distribution function of a probability measure. ■

5.2.4 Sufficient conditions for the existence of frequencies

In this section we use the Lindenstrauss pointwise ergodic theorem, i.e. Theorem 2.12 to prove the existence of frequencies in a randomly colored Cayley graph along a tempered Følner sequence. This is motivated by the Banach space-valued ergodic theorems in the previous sections, as the existence of the frequencies is a basic assumption for their validity.

We consider a finitely generated amenable group G and a finite set $\mathcal{Z}$, which we will as before interpret as the set of colors. The probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is given in the following way. The sample space is the set

$$\Omega = \mathcal{Z}^G = \{\omega = (\omega_g)_{g \in G} \mid \omega_g \in \mathcal{Z} \text{ for all } g \in G\}.$$

The sigma-algebra $\mathcal{A}$ is generated by the cylinder sets and $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{A})$. Setting for each $\omega \in \Omega$

$$\mathcal{C}_\omega : G \to \mathcal{Z}, \quad g \mapsto \omega_g,$$

shows that each ω can be interpreted as a coloring of G. Let $T : G \times \Omega \to \Omega$ be given by

$$(g, \omega) \mapsto T_g \omega = \omega g^{-1}, \tag{5.34}$$

where $\omega g^{-1} \in \Omega$ is the element satisfying for each $x \in G$:

$$(\omega g^{-1})_x = \omega_{xg}.$$

We will assume that the action T of G on Ω is measure preserving and ergodic. Using Theorem 2.12 we can prove the existence of the frequencies ν_P along any tempered Følner sequence (Q_j).

Theorem 5.30. *Let the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ be given and let the action T of G on Ω be measure preserving and ergodic. Furthermore let (Q_j) be a tempered Følner sequence. Then there exists a set $\tilde{\Omega}$ of full measure such that the limit*

$$\lim_{n \to \infty} \frac{\sharp_P \left(\mathcal{C}_\omega|_{Q_j}\right)}{|Q_j|}$$

exists for all $P \in \mathcal{P}$ and all $\omega \in \Omega$ and the limit is independent of the specific choice of ω.

Proof. Let $P : D(P) \to \mathcal{Z}$ be some pattern. As the number of occurrences of two equivalent patterns P_1 and P_2 in another pattern P_3 is the same, we can assume without loss of generality that $\mathrm{id} \in D(P)$. Set $A_P := \{\omega \in \Omega \mid \mathcal{C}_\omega|_{D(P)} = P\}$ and let $f_P : \Omega \to \{0, 1\}$ be the indicator function of A_P. Now we can estimate the number of occurrences of P in $\mathcal{C}_\omega|_{Q_j}$ by

$$\sum_{g \in Q_j \setminus (\partial_{D(P)} Q_j)} f_P(\omega g^{-1}) \leq \sharp_P \left(\mathcal{C}_\omega|_{Q_j}\right) \leq \sum_{g \in Q_j} f_P(\omega g^{-1}). \tag{5.35}$$

This proves on the one hand that

$$\limsup_{j\to\infty} \frac{\sharp_P(\mathcal{C}_\omega|_{Q_j})}{|Q_j|} \leq \limsup_{j\to\infty} \frac{1}{|Q_j|} \sum_{g\in Q_j} f_P(\omega g^{-1})$$

and on the other hand

$$\begin{aligned}
&\liminf_{j\to\infty} \frac{1}{|Q_j|} \sum_{g\in Q_j\setminus(\partial_{D(P)}Q_j)} f_P(\omega g^{-1}) \\
&\geq \liminf_{j\to\infty}\left(\frac{1}{|Q_j|}\sum_{g\in Q_j} f_P(\omega g^{-1}) - \frac{|\partial_{D(P)}Q_j|}{|Q_j|}\right) \\
&= \liminf_{j\to\infty} \frac{1}{|Q_j|} \sum_{g\in Q_j} f_P(\omega g^{-1}).
\end{aligned}$$

We apply Theorem 2.12, which is possible since $f_P \in L^1(\mathbb{P})$ and T is a measure preserving and ergodic action. This yields that there is a set Ω_P of full measure such that

$$\lim_{j\to\infty} \frac{1}{|Q_j|} \sum_{g\in Q_j} f_P(\omega g^{-1}) = \mathbb{E}(f_P)$$

holds for all $\omega \in \Omega_P$. Using this with (5.35) we obtain

$$\lim_{j\to\infty} \frac{\sharp_P\left(\mathcal{C}_\omega|_{Q_j}\right)}{|Q_j|} = \mathbb{E}(f_P)$$

for all $\omega \in \Omega_P$. Next, set $\tilde{\Omega} = \bigcup_{P\in\mathcal{P}} \Omega_P$ and use the fact that $\mathcal{P}$ is countable to get the desired set $\tilde{\Omega}$ of full measure such that the frequencies along (Q_j) exist for all patterns $P \in \mathcal{P}$ and all $\omega \in \tilde{\Omega}$. The independence of the specific choice of ω is clear as $\mathbb{E}(f_P)$ is independent of ω. ■

Remark 5.31. In the case where the measure $\mathbb{P}$ has a product structure $\mathbb{P} = \Pi_{g\in G}\mu$ and μ is some measure on $\mathcal{Z}$, it is easy to show that T, defined as in (5.34) is measure preserving and ergodic. This shows that Theorem 5.30 applies in particular to i.i.d. models. For a result in this direction see Lemma 6.30, where we prove ergodicity in the situation of independent random variables.

5.2.5 Additional results on the integrated density of states

Under certain assumptions we will be able to show that the spectrum of the operator in question is the topological support of the measure μ_A. Furthermore, we characterize the points of discontinuity of the IDS as the eigenvalues of A with an associated finitely supported eigenfunction. In order to do so, we will need the following two lemmas. The first one is a well-known dimension argument from linear algebra.

Lemma 5.32. *Let $\mathcal{H}$ be a finite dimensional Hilbert space, $\mathcal{U}$, $\mathcal{V}$ subspaces of $\mathcal{H}$ with* $\dim \mathcal{U} > \dim \mathcal{V}$*, then* $\dim \mathcal{V}^\perp \cap \mathcal{U} > 0$*.*

The next result can be found in [Sim87] and [LMV08]. It is a useful tool in the proof of the Theorem 5.34.

Lemma 5.33. *Let A be a self-adjoint operator on a finite-dimensional Hilbert space $\mathcal{V}$. Let $\lambda \in \mathbb{R}$ and $\varepsilon > 0$ be given and denote by $\mathcal{U}$ the subspace of $\mathcal{V}$ spanned by the eigenvectors of A belonging to the eigenvalues in the open interval $(\lambda-\varepsilon, \lambda+\varepsilon)$. If there exist k pairwise orthogonal and normalized vectors $u_1, \dots, u_k \in \mathcal{V}$ such that $(A-\lambda)u_j$, $j = 1, \dots, k$ are pairwise orthogonal and satisfy $\|(A-\lambda)u_j\| < \varepsilon$, then* $\dim(\mathcal{U}) \geq k$*.*

Proof. We assume $\dim(\mathcal{U}) < k$. Let S be the linear span of $u_1, \dots, u_k$. By Lemma 5.32 there exists an unit element $s \in S$, which is orthogonal to $\mathcal{U}$, e.g $s \in \mathcal{U}^\perp$. Hence, s is a combination of elements $\overline{u}_k$ with $\|(A-\lambda)\overline{u}_k\| \geq \varepsilon$. This gives $\|(A-\lambda)s\| \geq \varepsilon$. On the other hand we know that $s \in S$ is an unit element combined by elements u_j with $\|(A-\lambda)u_j\| < \varepsilon$, $j = 1, \dots, k$, which implies $\|(A-\lambda)s\| < \varepsilon$. This is a contradiction. ■

The proof of the following result is a generalization of a result in [LMV08] to the situation of amenable groups.

Theorem 5.34. *Let Assumptions 1, 3, 5 and 4 be satisfied. Then the spectrum of A is the topological support of μ_A.*

Proof. In this proof we apply Theorem 5.29, which gives uniform convergence of the normalized eigenvalue counting functions. Since the operator A is assumed to be of overall range R we have

$$\|(A-\lambda)u\| = \|(A[Q]-\lambda)p_Q u\|, \tag{5.36}$$

for all u with $\mathrm{spt}(u) \subseteq Q^{(R)}$. Let λ be an element of the spectrum $\sigma(A)$, then $A - \lambda$ is not invertible. Thus, for each $\varepsilon > 0$ we can find a subset $Q \in \mathcal{F}(G)$ and a normalized vector u with support in $Q^{(R)}$ such that $\|(A - \lambda)u\| < \varepsilon$ holds. From this we know that $(A - \lambda)u$ is supported in Q and $\|(A[Q] - \lambda)p_Q u\| < \varepsilon$ by (5.36). For each $j \in \mathbb{N}$ we denote the number of disjoint occurrences of translates of $\mathcal{C}|_Q$ in the set $U_j^{(R)}$ by $k(j)$. This ensures the existence of $k(j)$ pairwise orthogonal normalized vectors u_i, $i = 1, \dots, k(j)$, where also $(A - \lambda)u_i$ are pairwise orthogonal and of norm strictly less than ε. Applying Lemma 5.33 we get that there must be at least $k(j)$ eigenvalues in the interval $(\lambda - \varepsilon, \lambda + \varepsilon)$, i.e.

$$\mathfrak{e}(A[U_j^{(R)}])(\lambda + \varepsilon) - \mathfrak{e}(A[U_j^{(R)}])(\lambda - \varepsilon) \geq k(j).$$

For a pattern $P \in \mathcal{P}$ Lemma 5.1 yields that the frequency ν_P along (U_j) is the same as the frequency along $(U_j^{(R)})$. As these frequencies are assumed to be strictly positive for all patterns which occur in $\mathcal{C}$, the number of disjoint occurrences of $\mathcal{C}|_Q$ in $U_j^{(R)}$ grows linearly in the volume of $U_j^{(R)}$ for large j. Thus we can find a $c > 0$ such that $k(j) \geq c|U_j^{(R)}|$ holds for large j. Using the uniform convergence of $\mathfrak{n}(A[U_j^{(R)}])$ we see

$$\begin{aligned} \mu_A([\lambda - \varepsilon, \lambda + \varepsilon]) &= \lim_{j \to \infty} \mathfrak{n}(A[U_j^{(R)}])(\lambda + \varepsilon) - \mathfrak{n}(A[U_j^{(R)}])(\lambda - \varepsilon) \\ &\geq \lim_{j \to \infty} \frac{k(j)}{|U_j^{(R)}| \dim(\mathcal{H})} \geq \frac{c}{\dim(\mathcal{H})}. \end{aligned}$$

As c is strictly positive and $\varepsilon > 0$ was arbitrary, we conclude that λ is in the support of μ_A.

Now, we start with λ in the support of μ_A. Thus for each $\varepsilon > 0$ we have a $c > 0$ such that $\mu_A([\lambda - \varepsilon, \lambda + \varepsilon]) \geq c$. By uniform convergence this gives that

$$\mathfrak{n}(A[U_j^{(R)}])(\lambda + \varepsilon) - \mathfrak{n}(A[U_j^{(R)}])(\lambda - \varepsilon) \geq \frac{c}{2}$$

holds for large j. We use Lemma 2.25 to observe

$$\|\mathfrak{n}(A[U_j^{(2R)}]) - \mathfrak{n}(A[U_j^{(R)}])\|_\infty \leq 4 \frac{|\partial_{\mathrm{int}}^R U_j^{(R)}|}{|U_j^{(R)}|},$$

which leads together with triangle inequality to

$$\mathfrak{n}(A[U_j^{(2R)}])(\lambda+\varepsilon)-\mathfrak{n}(A[U_j^{(2R)}])(\lambda-\varepsilon)\geq\frac{c}{2}-8\frac{|\partial_{\mathrm{int}}^R U_j^{(R)}|}{|U_j^{(R)}|}$$

for large j. As the right hand side is positive for large j, there exists an eigenvalue $\overline{\lambda}\in[\lambda-\varepsilon,\lambda+\varepsilon]$ and a normalized eigenvector $\overline{u}\in\ell^2(U_j^{(2R)},\mathcal{H})$ such that $(A[U_j^{(2R)}]-\overline{\lambda})\overline{u}=0$ holds. From this we have

$$\begin{aligned}\|(A[U_j^{(R)}]-\lambda)p_{U_j^{(R)}}u\| &= \|(A[U_j^{(R)}]-\overline{\lambda})p_{U_j^{(R)}}u+(\overline{\lambda}-\lambda)p_{U_j^{(R)}}u\|\\ &\leq|\overline{\lambda}-\lambda|\leq\varepsilon\end{aligned}$$

with a normalized vector $u=i_{U_j^{(2R)}}\overline{u}\in\ell^2(G,\mathcal{H})$ which is supported in $U_j^{(2R)}$. By (5.36) we get $\|(A-\lambda)u\|\leq\varepsilon$ and $\sigma(A)\cap[\lambda-\varepsilon,\lambda+\varepsilon]\neq\emptyset$. Since $\varepsilon>0$ is arbitrary, we obtain that λ belongs to $\sigma(A)$. ■

The following example shows that the positivity of the frequencies is a necessary assumption.

Example 5.35. Consider the same situation as in Example 5.14 but now choose the coloring $\mathcal{C}:\mathbb{Z}\to\mathcal{Z}$

$$\mathcal{C}(x)=\begin{cases}\text{white} & \text{if } x\geq 0 \text{ or } x\leq -100 \text{ or } x=3k \text{ for } k\in\mathbb{Z}\\ \text{black} & \text{otherwise.}\end{cases}$$

Again we treat the case where edges only exist between black vertices with distance one. The restricted adjacency operator $A[V_j]$, with V_j as in (5.19), has for $1\leq j\leq 33$ the eigenvalues $-1,0$ and 1 each of them with multiplicity j. From this we get in particular that -1 and 1 are elements of the spectrum of A.
However the frequencies of the patterns that give rise to these eigenvalues is zero. For all $j\geq 34$ the multiplicities of the eigenvalues -1 and 1 remain 33 and the multiplicity of the eigenvalue 0 equals $33+3j$. Therefore, for increasing j the steps of the cumulative eigenvalue counting become relatively small. This implies that the IDS is the function

$$\mathfrak{I}(\lambda)=\begin{cases}0 & \text{if } \lambda<0\\ 1 & \text{otherwise.}\end{cases}$$

Thus the topological support of the induced measure μ_A equals $\{0\}$, though -1 and 1 are in the spectrum of A.

The next corollary characterizes the set of points at which the IDS is discontinuous. It has been obtained previously in [LV09] by different methods. For earlier results characterizing the set of jumps see e.g. [KLS03], [Ves05].

Corollary 5.36. *Let Assumptions 1, 3, 5 and 4 be satisfied and let $\lambda \in \mathbb{R}$. Then the following assertions are equivalent:*

(i) λ is a point of discontinuity of $\mathfrak{I}_A$,

(ii) there exists a compactly supported eigenfunction of A corresponding to λ.

Proof. Let λ be a point of discontinuity of $\mathfrak{I}_A$. As stated in the assumption $(U_j)_{j\in\mathbb{N}}$ is a Følner sequence along which the (strictly positive) frequencies exist. Theorem 5.29 shows that the distribution function $\mathfrak{n}(A[U_j^{(R)}])$ converges to $\mathfrak{I}_A$ with respect to the supremum norm. Since λ is a point of discontinuity, the jump at λ will not get small, i.e.

$$\begin{aligned}&\dim(\ker(A[U_j^{(R)}]-\lambda))\\&=\lim_{\varepsilon\to 0}(\mathfrak{e}(A[U_j^{(R)}])(\lambda+\varepsilon)-\mathfrak{e}(A[U_j^{(R)}])(\lambda-\varepsilon))\geq c|U_j^{(R)}|\end{aligned}$$

for a $c>0$ and all $j\in\mathbb{N}$. We also know

$$\dim(\ell^2(\partial^{2R}_{\text{int}}U_j^{(R)}))=|\partial^{2R}_{\text{int}}U_j^{(R)}|=\frac{|\partial^{2R}_{\text{int}}U_j^{(R)}|}{|U_j^{(R)}|}|U_j^{(R)}|$$

and since $U_j^{(R)}$ is a Følner sequence

$$\lim_{j\to\infty}\frac{|\partial^{2R}_{\text{int}}U_j^{(R)}|}{|U_j^{(R)}|}=0.$$

Thus we get that

$$\dim(\ker(A[U_j^{(R)}]-\lambda))>\dim(\ell^2(\partial^{2R}_{\text{int}}U_j^{(R)}))$$

holds for large j. Using Lemma 5.32, we find an eigenvector u of A with $\operatorname{spt} u \subseteq U_j^{(3R)}$ for some $j \in \mathbb{N}$.

Now we prove the converse implication. To this end, let u be an eigenfunction corresponding to λ with $r > 0$ such that $\operatorname{spt}(u) \subseteq B_r$ holds. Furthermore, let Q be some finite subset of G. Set $P := \mathcal{C}|_{B_r}$, then each copy of P in $\mathcal{C}|_Q$ adds a dimension to the eigenspace of $p_Q A i_Q$ belonging to λ. We denote the number of disjoint copies of P in Q by $\dot{\sharp}_P(\mathcal{C}|_Q)$. A simple combinatorial argument shows $|B_{3r}|\dot{\sharp}_P(\mathcal{C}|_Q) \geq \sharp_P(\mathcal{C}|_Q)$. With this we get

$$\begin{aligned}\frac{\mathfrak{e}(A[Q])(\lambda-\varepsilon)}{|Q|} &\leq \frac{\mathfrak{e}(A[Q])(\lambda+\varepsilon) - \dot{\sharp}_P(\mathcal{C}|_Q)}{|Q|} \\ &\leq \frac{\mathfrak{e}(A[Q])(\lambda+\varepsilon)}{|Q|} - \frac{\sharp_P(\mathcal{C}|_Q)}{|B_{3r}||Q|}.\end{aligned}$$

Now we substitute Q by the elements of the Følner sequence $(U_j^{(R)})_{j\in\mathbb{N}}$

$$\frac{\mathfrak{e}(A[U_j^{(R)}])(\lambda+\varepsilon)}{|U_j^{(R)}|} - \frac{\mathfrak{e}(A[U_j^{(R)}])(\lambda-\varepsilon)}{|U_j^{(R)}|} \geq \frac{\sharp_P(\mathcal{C}|_{U_j^{(R)}})}{|B_{3r}||U_j^{(R)}|}.$$

If $j \to \infty$ we get

$$\mathfrak{I}_A(\lambda+\varepsilon) - \mathfrak{I}_A(\lambda-\varepsilon) \geq \frac{\nu_P}{|B_{3r}|\dim(\mathcal{H})} > 0, \tag{5.37}$$

where we used Lemma 5.1. As $\varepsilon > 0$ is arbitrary, this yields that λ is a point of discontinuity of $\mathfrak{I}_A$. ∎

Remark 5.37. Let us discuss the quantitative estimates on the jump size in (5.37). In the situation where v is an eigenfunction corresponding to the value λ, which is supported in B_r, each pattern which is equivalent to $\mathcal{C}|_{B_r}$ gives rise to a translate of v, which is again an eigenfunction corresponding to λ. The frequency of a pattern describes how often this pattern occurs in $\mathcal{C}$. Thus, the frequency encodes the density of occurrences of translates of v which are again an eigenfunction for λ. Therefore, it is natural that the size of the jump depends linearly on the frequency of the pattern. In order to explain the denominator in (5.37) recall that the IDS measures the number of eigenstates per unit volume. Hence one could expect the term $|B_r|\dim(\mathcal{H})$ as normalization. The discrepancy to the actual denominator is due to the fact that in the proof of the above corollary we are interested in *disjoint* translates, whereas in the definition of frequencies we count all (and not just disjoint) occurrences.

Furthermore if one knows that there are m linearly independent eigenfunctions, which are corresponding to λ and all of them are supported on B_r, then one obtains using the arguments in the proof of Corollary 5.36 the estimate

$$\mathfrak{I}_H(\lambda+\varepsilon) - \mathfrak{I}_H(\lambda-\varepsilon) \geq \frac{m\nu_P}{|B_{3r}|\dim(\mathcal{H})},$$

for any $\varepsilon > 0$.

5.3 Special cases and applications

5.3.1 Abelian groups

In this subsection the main results of Section 5.1.2 are applied to the case where the group G equals $\mathbb{Z}^d$, as an example for a finitely generated abelian group. Let S be the usual set of generators given by $S = \{\pm s_1, \dots, \pm s_d\}$, where s_i is the i-th unit vector in $\mathbb{Z}^d$. It is easy to check that the sequence (Q_n) of cubes $Q_n = \{0, \dots, n-1\}^d$ is a Følner sequence. Moreover for each $n \in \mathbb{N}$ the set Q_n symmetrically tiles $\mathbb{Z}^d$ with grid $(n\mathbb{Z})^d$. This shows that $\mathbb{Z}^d$ is in fact an ST-amenable group. One obtains the following corollary as a special case of Theorem 5.8 by using the equalities

$$|Q_n| = n^d \qquad \text{and} \qquad \operatorname{diam}(Q_n) = dn. \tag{5.38}$$

This result recovers the main result of [LMV08].

Corollary 5.38. *Let (Q_n) and S be as above and assume that $\mathcal{Z}$ is a finite set of colors, $\mathcal{C} : \mathbb{Z}^d \to \mathcal{Z}$ a map called coloring and (U_j) a Følner sequence along which the frequencies of all patterns $P \in \bigcup_{n\in\mathbb{N}} \mathcal{P}(Q_n)$ exist. For a given $\mathcal{C}$-invariant and almost-additive function $F : \mathcal{F}(\mathbb{Z}^d) \to X$ the following limits*

$$\lim_{j\to\infty} \frac{F(U_j)}{|U_j|} = \lim_{n\to\infty} \sum_{P\in\mathcal{P}(Q_n)} \nu_P \frac{\tilde{F}(P)}{n^d}$$

exist and are equal. Furthermore, for $j, n \in \mathbb{N}$ the difference

$$\Delta(j,n) := \Big\| \frac{F(U_j)}{|U_j|} - \sum_{P \in \mathcal{P}(Q_n)} \nu_P \frac{\tilde{F}(P)}{n^d} \Big\|$$

satisfies the estimate

$$\Delta(j,n) \leq \frac{b(Q_n)}{n^d} + (C+D)\frac{|\partial^{nd} U_j|}{|U_j|} + C \sum_{P \in \mathcal{P}(Q_n)} \Big| \frac{\sharp_P(\mathcal{C}|_{U_j})}{|U_j|} - \nu_P \Big|. \tag{5.39}$$

Now, consider the case where the assumptions of Corollary 5.38 are satisfied and assume additionally that that $\mathcal{H}$ is a Hilbert space of dimension $k < \infty$ and $A : \ell^2(\mathbb{Z}^d, \mathcal{H}) \to \ell^2(\mathbb{Z}^d, \mathcal{H})$ a self-adjoint, $\mathcal{C}$-invariant operator of finite hopping range with overall range R. Then, by Theorem 5.11 there exists a unique distribution function $\mathfrak{I}_A$, called *integrated density of states*, such that the estimate

$$\begin{aligned} \big\| \mathfrak{n}(A[U_{j,R}]) - \mathfrak{I}_A \big\|_\infty \leq\; & 8 \frac{|\partial^R Q_n|}{|Q_n|} + (1 + 4|B_R|)\frac{|\partial^{dn} U_j|}{|U_j|} \\ & + \sum_{P \in \mathcal{P}(Q_n)} \Big| \frac{\sharp_P(\mathcal{C}|_{U_j})}{|U_j|} - \nu_P \Big| + \frac{|\partial^R_{\mathrm{int}} U_j|}{|U_j|} \end{aligned}$$

holds for all $j, n \in \mathbb{N}$. Using the equalities (5.38) and the inequalities

$$|B_R| \leq (2R)^d \qquad \text{and} \qquad |\partial^R(Q_n)| \leq (n+4R)^d - n^d \tag{5.40}$$

leads to a slightly weaker corollary.

Corollary 5.39. *Assume the situation of Corollary 5.38 and additionally that $\mathcal{H}$ is a Hilbert space of dimension $k < \infty$ and $A : \ell^2(\mathbb{Z}^d, \mathcal{H}) \to \ell^2(\mathbb{Z}^d, \mathcal{H})$ a self-adjoint, $\mathcal{C}$-invariant operator of finite hopping range with overall range R. Then there exists a unique distribution function $\mathfrak{I}_A$, such that $\mathfrak{n}(A[U_{j,R}])$ converges to $\mathfrak{I}_A$ with respect to the supremum norm as $j \to \infty$. In fact, the estimate*

$$\begin{aligned} \big\| \mathfrak{n}(A[U_{j,R}]) - \mathfrak{I}_A \big\|_\infty \leq\; & 8\left(\left(1 + \frac{4R}{n}\right)^d - 1\right) + (1 + 4(2R)^d)\frac{|\partial^{dn} U_j|}{|U_j|} \\ & + \sum_{P \in \mathcal{P}(Q_n)} \Big| \frac{\sharp_P(\mathcal{C}|_{U_j})}{|U_j|} - \nu_P \Big| + \frac{|\partial^R_{\mathrm{int}} U_j|}{|U_j|} \end{aligned}$$

holds for all $j, n \in \mathbb{N}$.

In the situation where the frequencies ν_P of all patterns $P \in \bigcup_{n\in\mathbb{N}} \mathcal{P}(Q_n)$ exist along the sequence of cubes (Q_n) we set $U_j := Q_j$ for all $j \in \mathbb{N}$. Again by using (5.38) and (5.40), the estimate in Corollary 5.39 can be replaced by

$$\begin{aligned}
&\big\|\mathfrak{n}(A[Q_{j,R}]) - \mathfrak{I}_A\big\|_\infty \\
&\leq 8\bigg(\Big(1+\frac{4R}{n}\Big)^d - 1\bigg) + (1+4(2R)^d)\bigg(\Big(1+\frac{4dn}{j}\Big)^d - 1\bigg) \\
&\quad + \sum_{P\in\mathcal{P}(Q_n)} \Big|\frac{\sharp_P(\mathcal{C}|_{Q_j})}{|Q_j|} - \nu_P\Big| + \bigg(\Big(1+\frac{4R}{j}\Big)^d - 1\bigg).
\end{aligned}$$

If furthermore $\mathcal{Z}$ consists of only one element, all information given by a pattern $P \in \mathcal{P}$ is its domain $D(P)$. Therefore, in this situation the frequencies ν_P exist for all patterns $P \in \mathcal{P}$ along any Følner sequence (U_j). In fact

$$1 \geq \frac{\sharp_P(\mathcal{C}|_{Q_j})}{|Q_j|} \geq \frac{|Q_j^{(\operatorname{diam} D(P))}|}{|Q_j|} \to 1 \qquad \text{for } j\to\infty$$

holds and hence $\nu_P = 1$ for all $P \in \mathcal{P}$. Note that $\mathcal{P}(Q_n)$ contains just one element. In this situation we get

$$\begin{aligned}
\sum_{P\in\mathcal{P}(Q_n)} \Big|\frac{\sharp_P(\mathcal{C}|_{Q_j})}{|Q_j|} - \nu_P\Big| \leq 1 - \frac{|Q_{j,d(n)}|}{|Q_j|} &\leq \frac{|\partial^{d(n)} Q_j|}{|Q_j|} \\
&\leq \bigg(\Big(1+\frac{4dn}{j}\Big)^d - 1\bigg)
\end{aligned} \tag{5.41}$$

and hence that the estimate

$$\big\|\mathfrak{n}(A[U_{j,R}]) - \mathfrak{I}_A\big\|_\infty \leq c\bigg(\Big(1+\frac{c}{n}\Big)^d + \Big(1+\frac{cn}{j}\Big)^d - 2\bigg)$$

holds for all $j, n \in \mathbb{N}$, where $c = 6(2R)^d$.

5.3.2 Heisenberg group

The discrete Heisenberg group H_3 is a prominent example for a non-abelian, finitely generated group. A finite set of generators S gives rise to the Cayley graph and the adjacency operator. We are interested in the spectral distribution of this operator. Applying Theorem 5.11 leads to a uniform approximation of the IDS. The elements of the discrete Heisenberg group are given by the set

$$H_3 := \left\{ (a,b,c) := \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{pmatrix} \middle| a,b,c \in \mathbb{Z} \right\}.$$

The group multiplication is induced by the usual matrix multiplication. Thus, the product and the inverse for two elements $(a,b,c),(a',b',c') \in H_3$ are given by

$$(a,b,c)(a',b',c') = (a+a', b+b', c+c'+ba')$$

and

$$(a,b,c)^{-1} = (-a,-b,ab-c).$$

It is easy to verify, that the symmetric set $S := \{s_1^{\pm 1}, s_2^{\pm 1}\}$ with $s_1 = (1,0,0), s_2 = (0,1,0)$ generates H_3. Let the sequence of subgroups (G_n) be given by $G_n := \{(a,b,c) | a,b \in n\mathbb{Z}, c \in n^2\mathbb{Z}\}$. One can show that for each $n \in \mathbb{N}$ the set $Q_n = \{(a,b,c) | a,b \in \mathbb{Z}_0^{n-1}, c \in \mathbb{Z}_0^{n^2-1}\}$ is a fundamental domain for G_n in H_3, where we use for $u,v \in \mathbb{Z}$ with $u \le v$ the notation $\mathbb{Z}_u^v := \{u, u+1, \dots, v\}$. Next, we prove that $(Q_n)_{n\in\mathbb{N}}$ is a Følner sequence. By Lemma 2.8 it is enough to show that

$$\lim_{n\to\infty} \frac{|SQ_n \setminus Q_n|}{|Q_n|} = 0.$$

For this special choice of sets we have the equality

$$|SQ_n \setminus Q_n| = \sum_{s\in S} |sQ_n \setminus Q_n|.$$

Hence we study the size of the four disjoint parts of the boundary $|sQ_n \setminus Q_n|$, $s \in S$ separately. For the first part we get

$$|s_1 Q_n \setminus Q_n| = \left| \left\{ (n,b,c) \middle| b \in \mathbb{Z}_0^{n-1}, c \in \mathbb{Z}_0^{n^2-1} \right\} \right| = n^3.$$

The boundary we obtain by shifting with s_2 calculates as follows

$$\begin{aligned}
&|s_2 Q_n \setminus Q_n| \\
&= \left| \left\{ (a, b+1, c+a) \middle| a \in \mathbb{Z}_0^{n-1}, b = n-1, c \in \mathbb{Z}_0^{n^2-1} \right\} \right| \\
&\quad + \left| \left\{ (a, b+1, c+a) \middle| a \in \mathbb{Z}_0^{n-1}, b \in \mathbb{Z}_0^{n-2}, c \in \mathbb{Z}_0^{n^2-1}, a + c \geq n^2 \right\} \right| \\
&= n^3 + (n-1) \sum_{a=0}^{n-1} \sum_{c=0}^{n^2-1} \mathbf{1}_{\{a+c \geq n^2\}} = \frac{3}{2} n^3 - n^2 + \frac{1}{2} n,
\end{aligned}$$

where we used $\sum_{i=1}^{n} i = \frac{1}{2} n(n+1)$. Similarly one can show

$$|s_1^{-1} Q_n \setminus Q_n| = n^3 \qquad \text{and} \qquad |s_2^{-1} Q_n \setminus Q_n| = \frac{3}{2} n^3 - n^2 + \frac{1}{2} n$$

for the other generators. Hence, we have

$$|S Q_n \setminus Q_n| = 5n^3 - 2n^2 + n$$

for the boundary of a set Q_n. As the volume of the fundamental domain Q_n is equal to n^4, we get

$$\lim_{n \to \infty} \frac{|S Q_n \setminus Q_n|}{|Q_n|} = 0.$$

Thus, the sequence (Q_n) is a Følner sequence and H_3 is amenable. We consider the trivial coloring on H_3, i.e. $|\mathcal{Z}| = 1$. In this case $\nu_P = 1$ for all patterns $P \in \mathcal{P}$, cf. (5.3.1). The adjacency operator $A : \ell^2(H_3) \to \ell^2(H_3)$ is defined pointwise for $x, y \in H_3$ and $f \in \ell^2(H_3)$ by

$$Af(x) = \sum_{y \in H_3} a(x, y) f(y), \quad \text{where} \quad a(x, y) := \begin{cases} 1 & \text{if } d_S(x, y) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

This operator is obviously self-adjoint, of finite hopping range and $\mathcal{C}$-invariant with overall range $R(A) = 2$. Hence, the assumptions of Theorem 5.11 are fulfilled, which yields the uniform convergence of the eigenvalue counting function and the estimate

$$\begin{aligned}
&\left\|\mathfrak{n}(A[Q_j^{(2)}]) - \mathfrak{I}_A\right\|_\infty \\
&\leq 8\frac{|\partial^2(Q_n)|}{|Q_n|} + (1+4|B_2|)\frac{|\partial^{\mathrm{diam}(Q_n)}Q_j|}{|Q_j|} \\
&\quad + \sum_{P\in\mathcal{P}(Q_n)}\left|\frac{\sharp_P(\mathcal{C}|_{Q_j})}{|Q_j|} - \nu_P\right| + \frac{|\partial^2_{\mathrm{int}}(Q_j)|}{|Q_j|} \\
&\leq 8\frac{|\partial^2(Q_n)|}{|Q_n|} + 70\frac{|\partial^{\mathrm{diam}(Q_n)}Q_j|}{|Q_j|} + \sum_{P\in\mathcal{P}(Q_n)}\left|\frac{\sharp_P(\mathcal{C}|_{Q_j})}{|Q_j|} - \nu_P\right|.
\end{aligned}$$

Here we used that the ball of radius two contains exactly 17 elements. Since there exists only one pattern with domain Q_n the last sum is not larger than $|Q_j|^{-1}|\partial^{\mathrm{diam}(Q_n)}Q_j|$, cf. (5.41). Thus,

$$\left\|\mathfrak{n}(A[Q_j^{(2)}]) - \mathfrak{I}_A\right\|_\infty \leq 8\frac{|\partial^2(Q_n)|}{|Q_n|} + 71\frac{|\partial^{\mathrm{diam}(Q_n)}Q_j|}{|Q_j|}$$

holds for all $j, n \in \mathbb{N}$.

5.3.3 Periodic operators

Periodic operators are an important class to which our theory applies. In the following, we consider a graph $\Gamma = (V, E)$, which is related to the group G, in the sense that G acts via graph isomorphisms on Γ. We can use these isomorphisms to transport an operator on $\ell^2(V)$ to $\ell^2(G, \mathcal{H})$, with an appropriately chosen finite dimensional Hilbert space $\mathcal{H}$.

Let G be a finitely generated group, $\Gamma = (V, E)$ a locally finite graph with a countable set of vertices V. For each $g \in G$ let $T_g : \Gamma \to \Gamma$ be a graph isomorphism. We denote the family $(T_g)_{g\in G}$ by T. We furthermore assume that the action T of G on Γ is free and cocompact. Here *free* means that for any distinct $g, h \in G$ and all $\gamma \in \Gamma$ we have $T_g\gamma \neq T_h\gamma$. By the *cocompactness* assumption we have that the quotient space Γ/T is compact and in this case even finite. Note that the quotient space Γ/T is the set of all equivalence classes in Γ, where two elements γ and γ' are called equivalent, if there exists $g \in G$ with $\gamma = T_g\gamma'$. We observe that cocompactness implies that Γ

needs to have bounded vertex degree. A fundamental domain $\mathcal{D} \subseteq \Gamma$ contains by definition exactly one element of each equivalence class. This set $\mathcal{D}$ is therefore finite. We define $\mathcal{H} := \ell^2(\mathcal{D})$.

Now let us consider a bounded operator $A : \ell^2(V) \to \ell^2(V)$, which we assume to be self-adjoint and invariant under the action T, i.e. for all $x, y \in V$ and $g \in G$ we have

$$a(x, y) = a(T_g x, T_g y), \tag{5.42}$$

where as usual $a(x, y) := \langle \delta_x, A\delta_y \rangle$ with the scalar product in $\ell^2(V)$. Furthermore, we assume that A is of finite hopping range, which means that whenever the graph distance $d_\Gamma(x, y)$ between x and y is larger than a constant ρ, we have $a(x, y) = 0$.

Our next aim is to transport this operator, using the action T, to the space $\ell^2(G, \mathcal{H})$, with $\mathcal{H}$ chosen as above. To this end let $\delta_k \in \ell^2(\mathcal{D})$, $k \in \mathcal{D}$ be the usual basis of this space. Hence for each $\psi \in \ell^2(G, \mathcal{H})$ and $g \in G$ we find uniquely determined complex numbers $\psi_k(g) \in$, $k \in \mathcal{D}$ with

$$\psi(g) = \sum_{k \in \mathcal{D}} \psi_k(g)\delta_k.$$

We use these elements to the define the following operators $U : \ell^2(G, \mathcal{H}) \to \ell^2(\Gamma)$ by setting for $\psi \in \ell^2(G, \mathcal{H})$ and $\gamma \in \Gamma$

$$U\psi(\gamma) := \psi_k(g) \quad \text{if} \quad \gamma = T_g k,$$

which is well defined as the action T is assumed to be free. It is not hard to check that the operator $U^* : \ell^2(\Gamma) \to \ell^2(G, \mathcal{H})$ given by $U^*\phi(g) = \sum_{k \in \mathcal{D}} \phi(T_g k)\delta_k$ for $\phi \in \ell^2(\Gamma)$ and $g \in G$ is the inverse and the adjoint of U. Now we are in the position to define

$$H := U^* A U : \ell^2(G, \mathcal{H}) \to \ell^2(G, \mathcal{H}).$$

If we would like to apply our theory, we need to check that H fits to the setting of Chapter 5, cf. Assumption 3. To be precise, we need to show that H is of finite hopping range and $\mathcal{C}$-invariant, for some coloring $\mathcal{C}$. The last property is easy to verify as here we can consider $\mathcal{C}$ to be the trivial coloring. Then one can show, using property (5.42) that for each $g, h, t \in G$ we have $p_g H i_h = p_{gt} H i_{ht}$. Here the natural projection p_a and inclusion i_b are defined as in (5.1) and (5.2).

Now let us show that H is of finite hopping range. For each $b \in G$ we have that Ui_b maps in the following way:

$$Ui_b : \mathcal{H} \to T_b\mathcal{D} := \{\gamma \in \Gamma \mid \exists k \in \mathcal{D} \text{ such that } \gamma = T_b k\}.$$

Besides this, for given $\phi \in \ell^2(V)$ and $a \in G$ the value of $p_a U^* \phi$ only depends on the elements $\phi(\gamma)$, $\gamma \in T_a\mathcal{D}$. Thus, if the distance between $T_a\mathcal{D}$ and $T_b\mathcal{D}$

$$d_\Gamma(T_a\mathcal{D}, T_b\mathcal{D}) = \min\{d_\Gamma(v, w) \mid v \in T_a\mathcal{D} \text{ and } w \in T_b\mathcal{D}\}$$

is larger than ρ, the operator $p_a U^* A U i_b$ is equal to zero. As $\mathcal{D}$ is finite we can find a $R > 0$ such that $d_S(a, b) \geq R$ implies $d_\Gamma(T_a\mathcal{D}, T_b\mathcal{D}) > \rho$ and hence $p_a H i_b = 0$.

6 Random operators on amenable groups

In this chapter G is assumed to be a finitely generated amenable group and $S \subseteq G$ a finite and symmetric set of generators. As in Chapter 4, we will consider random operators on G. However, the models we treat here will slightly differ from the ones treated before. In particular, we are able to prove convergence results for ergodic operators, where the matrix elements are not necessarily given via independent random variables.

In Section 6.1 we study ergodic operators and obtain weak convergence of the eigenvalue counting functions and a Pastur-Shubin trace formula. Before, similar results have been proven on sofic groups for more restricted operators, cf. Chapter 4. Besides the generality of the operators, another important difference to the procedure in Chapter 4 is the choice of the approximating operators. While in the setting of sofic groups approximations are obtained by a rather involved strategy of copying certain matrix elements, the setting of amenable groups allows to define the approximating operators as restrictions of the original operator.

To be precise, we obtain a sequence of finite dimensional operators by restricting the operator under consideration to the elements of a Følner sequence, cf. (6.2). However, this setting defies the application of the procedure presented in Chapter 4 in order to show weak convergence of the eigenvalue counting functions. There are two reasons for this: first, the method we presented for sofic groups relies massively on the fact that the non-diagonal matrix elements of the operator are independent random variables. Second, the approximating operators defined as restrictions of the original operator contain to many random matrix elements. This makes it impossible to obtain a useful error bound using the concentration inequality by McDiarmid. Therefore, we present in Section 6.1 a proof for weak convergence of the eigenvalue counting functions, which is independent of the results in Chapter 4.

In the subsequent section we consider a long-range percolation model on ST-amenable groups. The operator under consideration is the graph Laplacian, which is, due to long-range interactions, almost surely unbounded and not of finite hopping range. Here we obtain uniform convergence by adapting the Banach space-valued ergodic theorems of Chapter 5 to this random setting. A key tool is a result from the theory of large deviations, namely a Bernstein inequality. Moreover, we give a precise characterization for the points of discontinuity of the integrated density of states.

In Section 6.3 we consider general amenable groups and random operators, which can almost surely be unbounded and of unbounded hopping range. In comparison with the previous section, we allow more general operators. For instance, the non-diagonal elements are no longer elements of $\{0, 1\}$ but are now taken (randomly) from a possibly uncountable and unbounded subset of $\mathbb{R}$. In the proofs we extend ideas of [LV09] to obtain uniform convergence. Roughly speaking, we use weak convergence and additionally obtain control over the convergence at the jumps of the IDS. As we consider random operators which are not necessarily of finite hopping range, we go beyond the results of [LV09], where finite hopping range of the operator is a central assumption. To deal with long-range interactions it is again necessary to apply large deviations theory. The results of Section 6.2 and Section 6.3 are already published in [Sch12] and citeAyadiSV-12, respectively.

6.1 Weak convergence

In this section we verify weak convergence for the normalized eigenvalue counting functions. As mentioned before, the methods of Chapter 4 can not directly be adapted. Therefore, we rather follow the ideas of [PF92] where the authors proved weak convergence for operators on $\mathbb{Z}^d$. The reason why the procedure on $\mathbb{Z}^d$ can be generalized to operators on amenable groups is that here we have Lindenstrauss' ergodic theorem at hand.

In order to apply this to an unbounded operator A, we need to introduce an intermediate step of an approximating operator, namely the operator $A^{(t)}$, cf. (6.3). The operator $A^{(t)}$ is by definition bounded and of finite hopping range, such that Lindenstrauss' theorem is applicable to obtain weak convergence of the eigenvalue counting functions to the spectral distribution function of $A^{(t)}$, see Theorem 6.2. Moreover, we show in Theorem 6.4 that for increasing t the SDF of $A^{(t)}$ converges to the SDF of A. Thus, it remains to control the difference between the eigenvalue counting function of the restrictions of A and of $A^{(t)}$, respectively. This is provided in Lemma 6.3. Combining these results, we prove in Theorem 6.5 (weak) existence if the IDS of A and the validity of a Pastur-Shubin trace formula.

We start with the definition of the operator. Let $\tilde{A} = (\tilde{A}^{(\omega)})_{\omega\in\Omega}$ be a symmetric, random ergodic operator on the domain $C_c(G)$ which satisfies $\mathbb{E}(\|\tilde{A}\delta_{\mathrm{id}}\|_1^2) < \infty$. By Lemma 2.19, there exists $\tilde{\Omega} \subseteq \Omega$ of full measure such that for each $\omega \in \tilde{\Omega}$ the operator $\tilde{A}^{(\omega)}$ is essentially self-adjoint and $C_c(G) \subseteq D(A^{(\omega)})$. For these $\omega \in \tilde{\Omega}$ we denote the self-adjoint extension of $A^{(\omega)}$ by $\bar{A}^{(\omega)}$ and set

$$A^{(\omega)} := \begin{cases} \bar{A}^{(\omega)} & \text{if } \omega \in \tilde{\Omega} \\ \mathrm{Id} & \text{otherwise.} \end{cases} \tag{6.1}$$

Therefore $A = (A^{(\omega)})$ is a proper random operator, cf. Definition 2.16. Moreover, A is ergodic and self-adjoint for *all* realizations. As before we denote the matrix elements by $a^{(\omega)}(x,y) := \langle \delta_x, A^{(\omega)}\delta_y \rangle$.

As G is amenable, there exists a Følner sequence, which we denote by (Q_j). For each $\omega \in \Omega$ and $j \in \mathbb{N}$ we define the approximating operator $A_j^{(\omega)} : \ell^2(Q_j) \to \ell^2(Q_j)$ by setting

$$A_j^{(\omega)} := A^{(\omega)}[Q_j] := p_{Q_j} A^{(\omega)} i_{Q_j}. \tag{6.2}$$

Here the inclusion $i_Q : \ell^2(Q) \to \ell^2(G)$ and the projection $p_Q : \ell^2(G) \to \ell^2(Q)$ are given as in (5.4) and (5.3) with $\mathcal{H} = \mathbb{C}$. We define an intermediate approximation of A on the whole group. For $t > 0$ we set for $x, y \in G$

$$a^{(t,\omega)}(x,y) := \begin{cases} a^{(\omega)}(x,y) & \text{if } d_S(x,y) \le t \text{ and } |a^{(\omega)}(x,y)| \le t \\ 0 & \text{otherwise} \end{cases}$$

and use this to define $A^{(t,\omega)} : \ell^2(G) \to \ell^2(G)$ by setting for any $\phi \in \ell^2(G)$ and $x \in G$

$$(A^{(t,\omega)}\phi)(x) := \sum_{y\in G} a^{(t,\omega)}(x,y)\phi(y). \tag{6.3}$$

The operator $A^{(t)} = (A^{(t,\omega)})_{\omega\in\Omega}$ is ergodic and self-adjoint for all realizations. We also define the finite dimensional operator given by

$$A_j^{(t,\omega)} := A^{(t,\omega)}[Q_j] := p_{Q_j} A^{(t,\omega)} i_{Q_j}. \tag{6.4}$$

For the operators introduced in (6.2) and (6.4), we define the associated eigenvalue counting functions as before by

$$\mathfrak{n}_j^{(\omega)} := \mathfrak{n}(A_j^{(\omega)}) \qquad \text{and} \qquad \mathfrak{n}_j^{(t,\omega)} := \mathfrak{n}(A_j^{(t,\omega)}). \tag{6.5}$$

Additionally, for each $\omega \in \Omega$ and $t > 0$, we define the functions $\mathfrak{N}^{(\omega)} \colon \mathbb{R} \to [0,1]$ and $\mathfrak{N}^{(t,\omega)} \colon \mathbb{R} \to [0,1]$ by setting for $\lambda \in \mathbb{R}$:

$$\mathfrak{N}^{(\omega)}(\lambda) := \langle \delta_{\mathrm{id}}, E_\lambda^{(\omega)} \delta_{\mathrm{id}} \rangle \quad \text{and} \quad \mathfrak{N}^{(t,\omega)}(\lambda) := \langle \delta_{\mathrm{id}}, E_\lambda^{(t,\omega)} \delta_{\mathrm{id}} \rangle, \tag{6.6}$$

where $E_\lambda^{(\omega)}$ and $E_\lambda^{(t,\omega)}$ are again the spectral projections on the interval $(-\infty, \lambda]$ of $A^{(\omega)}$ and $A^{(t,\omega)}$, respectively. Furthermore, we define the distribution functions $\bar{\mathfrak{N}}, \bar{\mathfrak{N}}^{(t)} \colon \mathbb{R} \to [0,1]$ by setting for $\lambda \in \mathbb{R}$:

$$\bar{\mathfrak{N}}(\lambda) = \mathbb{E}(\mathfrak{N}(\lambda)) \quad \text{and} \quad \bar{\mathfrak{N}}^{(t)}(\lambda) = \mathbb{E}(\mathfrak{N}^{(t,\omega)}(\lambda)) \qquad (\lambda \in \mathbb{R}, t > 0). \tag{6.7}$$

As before, the function $\bar{\mathfrak{N}}$ is called *spectral distribution function* of the random operator A. If the limit $\lim_{j\to\infty} \mathfrak{n}_j^{(\omega)}$ exists, it is called the *integrated density of states*. We use the shorthand notation for the Stieltjes transforms:

$$\mathfrak{r}_j^{(\omega)} := \mathfrak{r}(\mathfrak{n}_j^{(\omega)}), \quad \mathfrak{r}_j^{(t,\omega)} := \mathfrak{r}(\mathfrak{n}_j^{(t,\omega)}), \quad \bar{\mathfrak{r}}^{(t)} := \mathfrak{r}(\bar{\mathfrak{N}}^{(t)}) \text{ and } \bar{\mathfrak{r}} := \mathfrak{r}(\bar{\mathfrak{N}}). \tag{6.8}$$

Lemma 6.1. *Let G be an amenable finitely generated group and let B be a bounded operator on $\ell^2(G)$ with finite hopping range r. Then we have for each Følner sequence (Q_j) and $m \in \mathbb{N}$*

$$\lim_{j\to\infty} \frac{1}{|Q_j|} \left| \mathrm{Tr}\left((B[Q_j])^m\right) - \mathrm{Tr}\left(\chi_{Q_j} B^m\right) \right| = 0,$$

where $B[Q_j] := p_{Q_j} B i_{Q_j} : \ell^2(Q_j) \to \ell^2(Q_j)$.

Proof. Let (Q_j) be a given Følner sequence and fix $m \in \mathbb{N}$. Then we have by Lemma 2.14

$$\begin{aligned}
&\operatorname{Tr}\left((B[Q_j])^m\right) \\
&= \sum_{x \in Q_j} \sum_{v_1,\dots,v_{m-1} \in B_{r(m-1)}(x) \cap Q_j} \langle \delta_x, B[Q_j]\delta_{v_1}\rangle \cdots \left\langle \delta_{v_{m-1}}, B[Q_j]\delta_x\right\rangle \\
&= \sum_{x \in Q_j^{(rm)}} \sum_{v_1,\dots,v_{m-1} \in B_{r(m-1)}(x) \cap Q_j} \langle \delta_x, B[Q_j]\delta_{v_1}\rangle \cdots \left\langle \delta_{v_{m-1}}, B[Q_j]\delta_x\right\rangle \\
&\quad + \sum_{x \in \partial_{\mathrm{int}}^{rm}(Q_j)} \sum_{v_1,\dots,v_{m-1} \in B_{r(m-1)}(x) \cap Q_j} \langle \delta_x, B[Q_j]\delta_{v_1}\rangle \cdots \left\langle \delta_{v_{m-1}}, B[Q_j]\delta_x\right\rangle.
\end{aligned}$$

Similarly we obtain

$$\begin{aligned}
\operatorname{Tr}\left(\chi_{Q_j} B^m\right) = &\sum_{x \in Q_j^{(rm)}} \sum_{v_1,\dots,v_{m-1} \in B_{r(m-1)}(x)} \langle \delta_x, B\delta_{v_1}\rangle \cdots \left\langle \delta_{v_{m-1}}, B\delta_x\right\rangle \\
&+ \sum_{x \in \partial_{\mathrm{int}}^{rm}(Q_j)} \sum_{v_1,\dots,v_{m-1} \in B_{r(m-1)}(x)} \langle \delta_x, B\delta_{v_1}\rangle \cdots \left\langle \delta_{v_{m-1}}, B\delta_x\right\rangle .
\end{aligned}$$

Now we use that for each $v, w \in Q_j$ we have the equality $\langle \delta_v, B\delta_w\rangle = \langle \delta_v, B[Q_j]\delta_w\rangle$, which gives

$$\begin{aligned}
\left|\operatorname{Tr}\left((B[Q_j])^m\right) - \operatorname{Tr}\left(\chi_{Q_j} B^m\right)\right| &\leq 2 \sum_{x \in \partial_{\mathrm{int}}^{rm}(Q_j)} \sum_{v_1,\dots,v_{m-1} \in B_{r(m-1)}(x)} \|B\|^m \\
&= 2|\partial_{\mathrm{int}}^{rm}(Q_j)| \binom{|B_{r(m-1)}|}{m-1} \|B\|^m.
\end{aligned}$$

This proves the Lemma, as (Q_j) is assumed to be a Følner sequence. ■

Theorem 6.2. *Let G be a finitely generated group, let (Q_j) be a tempered Følner sequence and $t > 0$. Let the operators A and $A^{(t)}$ be given by (6.1) and (6.3). Then there exists a set $\tilde{\Omega} \subseteq \Omega$ of full measure, such that for all $\omega \in \tilde{\Omega}$*

$$\bar{\mathfrak{N}}^{(t)} = \operatorname*{w-lim}_{j \to \infty} \mathfrak{n}_j^{(t,\omega)},$$

where $\mathfrak{n}_j^{(t,\omega)}$ and $\bar{\mathfrak{N}}^{(t)}$ are given by (6.5) and (6.7).

Proof. First note that for all $j \in \mathbb{N}$, $t > 0$ and $\omega \in \Omega$ the measures associated to the distribution functions $\mathfrak{n}_j^{(t,\omega)}$, $\mathfrak{N}^{(t,\omega)}$ and $\bar{\mathfrak{N}}^{(t)}$ are supported on the interval $[-K, K]$, where $K = \sup_{\omega\in\Omega} \|A^{(t,\omega)}\| < \infty$. Therefore, by Lemma 2.29 and Theorem 2.28 it is sufficient to show that for almost all ω and all $m \in \mathbb{N}$:

$$\lim_{j\to\infty} M_m(\mathfrak{n}_j^{(t,\omega)}) = \lim_{j\to\infty} \int_{\mathbb{R}} \lambda^m \mathrm{d}\mathfrak{n}_j^{(t,\omega)}(\lambda)$$
$$= \int_{\mathbb{R}} \lambda^m \mathrm{d}\bar{\mathfrak{N}}^{(t)}(\lambda) = M_m(\bar{\mathfrak{N}}^{(t)}). \tag{6.9}$$

In order to do so, we study for $j, m \in \mathbb{N}$, $\omega \in \Omega$ and $t > 0$ the following integral

$$|Q_j| \int_{\mathbb{R}} \lambda^m \mathrm{d}\mathfrak{n}_j^{(t,\omega)}(\lambda) = \sum_{\lambda\in\sigma\left(A_j^{(t,\omega)}\right)} m_\lambda \lambda^m = \mathrm{Tr}\Big(\big(A_j^{(t,\omega)}\big)^m\Big),$$

where m_λ denotes the multiplicity of the eigenvalue λ. Now we make use of Lemma 6.1 and obtain

$$\lim_{j\to\infty} \int_{\mathbb{R}} \lambda^m \mathrm{d}\mathfrak{n}_j^{(t,\omega)}(\lambda) = \lim_{j\to\infty} \frac{1}{|Q_j|} \mathrm{Tr}\Big(\big(A_j^{(t,\omega)}\big)^m\Big)$$
$$= \lim_{j\to\infty} \frac{1}{|Q_j|} \mathrm{Tr}\big(\chi_{Q_j}\big(A^{(t,\omega)}\big)^m\big).$$

By ergodicity of $A^{(t)}$ we have for almost all ω

$$\Big\langle \delta_x, (A^{(t,\omega)})^m \delta_x \Big\rangle = \Big\langle \delta_{\mathrm{id}}, (A^{(t,T_x\omega)})^m \delta_{\mathrm{id}} \Big\rangle .$$

Hence we obtain by Lindenstrauss' ergodic Theorem 2.12

$$\lim_{j\to\infty} \int_{\mathbb{R}} \lambda^m \mathrm{d}\mathfrak{n}_j^{(t,\omega)}(\lambda) = \lim_{j\to\infty} \frac{1}{|Q_j|} \sum_{x\in Q_j} \Big\langle \delta_x, (A^{(t,\omega)})^m \delta_x \Big\rangle$$
$$= \lim_{j\to\infty} \frac{1}{|Q_j|} \sum_{x\in Q_j} \Big\langle \delta_{\mathrm{id}}, (A^{(t,T_x\omega)})^m \delta_{\mathrm{id}} \Big\rangle$$
$$= \mathbb{E} \Big\langle \delta_{\mathrm{id}}, (A^{(t)})^m \delta_{\mathrm{id}} \Big\rangle$$

for almost all $\omega \in \Omega$. Now we investigate the moments of the distribution function $\bar{\mathfrak{N}}^{(t)}$. Therefore, we first realize that for all ω, the Riemann-Stieltjes integral against $\mathfrak{N}^{(t,\omega)}$ is as usual defined by

$$\begin{aligned}&\int_{\mathbb{R}} \lambda^m \mathrm{d}\mathfrak{N}^{(t,\omega)}(\lambda)\\&= \int_{-K}^{K} \lambda^m \mathrm{d}\mathfrak{N}^{(t,\omega)}(\lambda) := \lim_{\Delta x \to 0} \sum_{i=0}^{k-1} x_i^m \big(\mathfrak{N}^{(t,\omega)}(x_{i+1}) - \mathfrak{N}^{(t,\omega)}(x_i)\big),\end{aligned}$$

with partitions $-K =: x_0 < x_1 < \cdots < x_k := K$ and their mesh size $\Delta x := \max_{i=0}^{k-1} x_{i+1} - x_i$. Since

$$\left|\sum_{i=0}^{k-1} x_i^m \big(\mathfrak{N}^{(t,\omega)}(x_{i+1}) - \mathfrak{N}^{(t,\omega)}(x_i)\big)\right| \le K^m,$$

by Lebesgue's dominated convergence theorem we obtain

$$\int_{\mathbb{R}} \lambda^m \mathrm{d}\bar{\mathfrak{N}}^{(t)}(\lambda) = \mathbb{E}\left(\int_{\mathbb{R}} \lambda^m \mathrm{d}\langle \delta_{\mathrm{id}}, E_\lambda^{(t)} \delta_{\mathrm{id}}\rangle\right) = \mathbb{E}\big(\big\langle \delta_{\mathrm{id}}, (A^{(t)})^m \delta_{\mathrm{id}}\big\rangle\big).$$

Here the second equality follows from the spectral theorem. This proves the claimed convergence in (6.9). ■

For a selfadjoint operator B, $z \in \mathbb{C}\setminus\mathbb{R}$ and $x, y \in G$ let us introduce the notation

$$R_B^z(x,y) := \big\langle \delta_x, (z-A)^{-1}\delta_y \big\rangle\,.$$

Lemma 6.3. *Let G be a finitely generated group, let (Q_j) be a tempered Følner sequence and $z \in \mathbb{C} \setminus \mathbb{R}$. Let the operators A and $A^{(t)}$, $t > 0$ be given by* (6.1) *and* (6.3). *Furthermore, let $\mathfrak{r}_j^{(\omega)}$ and $\mathfrak{r}_j^{(t,\omega)}$ be given as in* (6.8). *Then we have*

$$\lim_{t\to\infty} \mathbb{E}\big(\|(A - A^{(t)})\delta_{\mathrm{id}}\|_1\big) = 0 \tag{6.10}$$

and for almost all $\omega \in \Omega$

$$\lim_{t\to\infty} \lim_{j\to\infty} \big|\mathfrak{r}_j^{(\omega)}(z) - \mathfrak{r}_j^{(t,\omega)}(z)\big| = 0. \tag{6.11}$$

Proof. To show (6.10) note that for each $\omega \in \Omega$

$$\sum_{g\in G} |a^{(\omega)}(g, \mathrm{id}) - a^{(t,\omega)}(g, \mathrm{id})| \le \sum_{g\in G} |a^{(\omega)}(g, \mathrm{id})| = \|A^{(\omega)}\delta_{\mathrm{id}}\|_1,$$

which is integrable by assumption on A. Therefore, using dominated convergence we get

$$\begin{aligned}\lim_{t\to\infty} \mathbb{E}\big(\|(A - A^{(t)})\delta_{\mathrm{id}}\|_1\big) &= \lim_{t\to\infty} \mathbb{E}\Big(\Big(\sum_{g\in G} |a(g, \mathrm{id}) - a^{(t)}(g, \mathrm{id})|\Big)\Big) \\ &= \mathbb{E}\Big(\Big(\sum_{g\in G} \lim_{t\to\infty} |a(g, \mathrm{id}) - a^{(t)}(g, \mathrm{id})|\Big)\Big) = 0.\end{aligned}$$

For $t > 0$ and $j \in \mathbb{N}$ we use a similar calculation as in (3.5) and the second resolvent identity to obtain

$$\begin{aligned}&\mathfrak{r}_j^{(\omega)}(z) - \mathfrak{r}_j^{(t,\omega)}(z) \\ &= \frac{1}{|Q_j|} \sum_{x\in Q_j} \Big\langle \delta_x, \Big(\big(z - A_j^{(\omega)}\big)^{-1} - \big(z - A_j^{(t,\omega)}\big)^{-1}\Big)\, \delta_x \Big\rangle \\ &= \frac{1}{|Q_j|} \sum_{x\in Q_j} \Big\langle \big(\bar{z} - A_j^{(\omega)}\big)^{-1} \delta_x, \big(A_j^{(t,\omega)} - A_j^{(\omega)}\big)\big(z - A_j^{(t,\omega)}\big)^{-1} \delta_x \Big\rangle .\end{aligned}$$

We have for $\psi \in \ell^2(G)$ the equality $\psi = \sum_{a\in G} \langle \delta_a, \psi\rangle \delta_a$. We apply this twice, which leads to

$$\begin{aligned}&|Q_j|\big(\mathfrak{r}_j^{(\omega)}(z) - \mathfrak{r}_j^{(t,\omega)}(z)\big) \\ &= \sum_{x,a,b\in Q_j} R^z_{A_j^{(\omega)}}(a, x) R^z_{A_j^{(t,\omega)}}(b, x) \Big\langle \delta_a, \big(A_j^{(t,\omega)} - A_j^{(\omega)}\big)\delta_b \Big\rangle \\ &= \sum_{a,b\in Q_j} \Big\langle \big(\bar{z} - A_j^{(\omega)}\big)^{-1} \delta_a, \big(z - A_j^{(t,\omega)}\big)^{-1} \delta_b \Big\rangle \Big\langle \delta_a, \big(A_j^{(t,\omega)} - A_j^{(\omega)}\big)\delta_b \Big\rangle.\end{aligned}$$

Thence, using triangle inequality and ergodicity we obtain

$$\begin{aligned}&\big|\mathfrak{r}_j^{(\omega)}(z) - \mathfrak{r}_j^{(t,\omega)}(z)\big| \\ &\le \frac{1}{|\Im(z)|^2 |Q_j|} \sum_{a,b\in Q_j} \Big|\Big\langle \delta_a, \Big(A^{(t,\omega)} - A^{(\omega)}\Big)\, \delta_b \Big\rangle\Big|\end{aligned}$$

$$\leq \frac{1}{|\Im(z)|^2|Q_j|} \sum_{a,b\in Q_j} \left|\left\langle \delta_{ab^{-1}}, \left(A^{(t,T_b\omega)} - A^{(T_b\omega)}\right)\delta_{\mathrm{id}}\right\rangle\right|$$
$$\leq \frac{1}{|\Im(z)|^2|Q_j|} \sum_{b\in Q_j}\sum_{c\in G} \left|\left\langle \delta_c, \left(A^{(t,T_b\omega)} - A^{(T_b\omega)}\right)\delta_{\mathrm{id}}\right\rangle\right|.$$

Now the ergodic Theorem 2.12 yields the inequality

$$\lim_{j\to\infty}\left|\mathfrak{r}_j^{(\omega)}(z) - \mathfrak{r}_j^{(t,\omega)}(z)\right| \leq |\Im(z)|^{-2}\mathbb{E}\big(\|(A - A^{(t)})\delta_{\mathrm{id}}\|_1\big),$$

which clearly implies (6.11) using (6.10). ■

Theorem 6.4. *Let G be a finitely generated group and let the operators A and $A^{(t)}$, $t > 0$ be given by* (6.1) *and* (6.3)*. Furthermore, let $\bar{\mathfrak{N}}$ and $\bar{\mathfrak{N}}^{(t)}$ be given as in* (6.7)*. Then we have*

$$\bar{\mathfrak{N}} = \operatorname*{w-lim}_{t\to\infty} \bar{\mathfrak{N}}^{(t)}.$$

Proof. Here we use a similar procedure as in the proof of Lemma 6.3. We fix some $z \in \mathbb{C}\setminus\mathbb{R}$. The definitions of $\bar{\mathfrak{N}}$ and $\bar{\mathfrak{N}}^{(t)}$, the spectral theorem and the second resolvent identity imply

$$\begin{aligned}\bar{\mathfrak{r}}^{(t)}(z) - \bar{\mathfrak{r}}(z) &= \mathbb{E}\left(\left\langle \delta_{\mathrm{id}}, (z - A^{(t)})^{-1}(A - A^{(t)})(z - A)^{-1}\delta_{\mathrm{id}}\right\rangle\right)\\ &= \mathbb{E}\Big(\sum_{x,y\in G} R_A^z(y,\mathrm{id}) R_{A^{(t)}}^{\bar z}(x,\mathrm{id}) \left\langle \delta_x, (A - A^{(t)})\delta_y\right\rangle\Big).\end{aligned}$$

Again, we used the equality $\psi = \sum_{a\in G}\langle \delta_a, \psi\rangle\delta_a$. As the operators A and $A^{(t)}$ are ergodic with respect to the same group $\mathcal{T}$ of automorphisms, we can apply the joint translation invariance in distribution and get for $t > 0$

$$\begin{aligned}&\bar{\mathfrak{r}}^{(t)}(z) - \bar{\mathfrak{r}}(z)\\ &= \mathbb{E}\Big(\sum_{x,y\in G} R_A^z(\mathrm{id}, y^{-1}) R_{A^{(t)}}^{\bar z}(xy^{-1}, y^{-1}) \left\langle \delta_{xy^{-1}}, (A - A^{(t)})\delta_{\mathrm{id}}\right\rangle\Big)\\ &= \mathbb{E}\Big(\sum_{x,y\in G} R_A^z(\mathrm{id}, y^{-1}) R_{A^{(t)}}^{\bar z}(x, y^{-1}) \left\langle \delta_x, (A - A^{(t)})\delta_{\mathrm{id}}\right\rangle\Big)\\ &= \mathbb{E}\Big(\sum_{x\in G}\left\langle \delta_x, (A - A^{(t)})\delta_{\mathrm{id}}\right\rangle \sum_{y\in G} R_A^z(\mathrm{id}, y) R_{A^{(t)}}^{\bar z}(x, y)\Big).\end{aligned}$$

Using

$$\Big|\sum_{y\in G} R_A^z(\mathrm{id},y)R_{A^{(t)}}^{\bar z}(x,y)\Big| \le |\Im(z)|^{-2}$$

we obtain for $t>0$

$$\lim_{t\to\infty}\big|\bar{\mathfrak{r}}^{(t)}(z)-\bar{\mathfrak{r}}(z)\big| \le \lim_{t\to\infty}\frac{1}{|\Im(z)|^2}\mathbb{E}\big(\|(A-A^{(t)})\delta_{\mathrm{id}}\|_1\big)=0,$$

where we applied again Lemma 6.3. Now Lemma 2.29 and portmanteau theorem imply the weak convergence of the corresponding distribution functions. ∎

The next theorem proves the existence of the integrated density of states and the validity of a Pastur-Shubin trace formula.

Theorem 6.5. *Let G be a finitely generated amenable group, (Q_j) a tempered Følner sequence and let the operators $A^{(\omega)}$ and $A_j^{(\omega)}$ for $\omega\in\Omega$ and $j\in\mathbb{N}$ be given as in* (6.1) *and* (6.2)*. Furthermore, let $\mathfrak{n}_j^{(\omega)}$ be given as in* (6.5) *and let the spectral distribution functions $\bar{\mathfrak{N}}$ be given as in* (6.7)*. Then for almost all $\omega\in\Omega$ we have*

$$\bar{\mathfrak{N}} = \operatorname*{w-lim}_{j\to\infty}\mathfrak{n}_j^{(\omega)}.$$

Proof. By Lemma 2.29 and portmanteau theorem it is enough to show for arbitrary $z\in\mathbb{C}\setminus\mathbb{R}$ and almost all ω the equality

$$\lim_{j\to\infty}|\mathfrak{r}_j^{(\omega)}(z)-\bar{\mathfrak{r}}(z)|=0.$$

Introducing the operator $A^{(t)}$ as in (6.3) and the associated Stieltjes transforms $\mathfrak{r}_j^{(t,\omega)}$ and $\bar{\mathfrak{r}}^{(t)}$ given in (6.8) we obtain for arbitrary $t>0$

$$\begin{aligned}&\big|\mathfrak{r}_j^{(\omega)}(z)-\bar{\mathfrak{r}}(z)\big|\\&\le \big|\mathfrak{r}_j^{(\omega)}(z)-\mathfrak{r}_j^{(t,\omega)}(z)\big|+\big|\mathfrak{r}_j^{(t,\omega)}(z)-\bar{\mathfrak{r}}^{(t)}(z)\big|+\big|\bar{\mathfrak{r}}^{(t)}(z)-\bar{\mathfrak{r}}(z)\big|.\end{aligned}$$

By Lemma 6.3 we have a set $\Omega_1\subseteq\Omega$ of full measure such that for all $\omega\in\Omega_1$

$$\lim_{t\to\infty}\lim_{j\to\infty}\big|\mathfrak{r}_j^{(\omega)}(z)-\mathfrak{r}_j^{(t,\omega)}(z)\big|=0.$$

Theorem 6.2 together with Lemma 2.29 imply that for each $t > 0$ there exists a set $\Omega'_t \subseteq \Omega$ of full measure such that for all $\omega \in \Omega'_t$

$$\lim_{j\to\infty} \left| \mathfrak{r}_j^{(t,\omega)}(z) - \bar{\mathfrak{r}}^{(t)}(z) \right| = 0.$$

Furthermore, we infer from Theorem 6.4 that

$$\lim_{t\to\infty} \left| \bar{\mathfrak{r}}^{(t)}(z) - \bar{\mathfrak{r}}(z) \right| = 0.$$

These facts imply the assertion of the theorem for all $\omega \in \Omega_1 \cap \bigcap_{t\in\mathbb{Q}\cap(0,\infty)} \Omega'_t$. ■

6.2 Random operators on ST-amenable groups

This section is devoted to prove uniform convergence for eigenvalue counting functions for certain random operators on ST-amenable groups, which are allowed to be of unbounded hopping range. The procedure we follow is based on the ideas of [LMV08] and [LSV11], where the authors prove uniform existence of the IDS for deterministic finite hopping range operators, cf. Chapter 5. The results presented in this section have already been published in [Sch12].

We consider the long-range percolation model from Subsection 4.3. Here G is a finitely generated ST-amenable group and $\Gamma_{\mathrm{co}} = (V, E_{\mathrm{co}})$ is the complete undirected graph over $V = G$, i.e.

$$E_{\mathrm{co}} := |\{e \subseteq G \mid 1 \leq |e| \leq 2\}|.$$

Let $p = (p(x))_{x\in G} \in \ell^1(G, \mathbb{R})$ be an arbitrary element satisfying

$$0 \leq p(x) \leq 1 \qquad \text{and} \qquad p(x) = p(x^{-1}) \qquad \text{for all } x \in G. \tag{6.12}$$

In order to generate a random subset $E_\omega \subseteq E_{\mathrm{co}}$ by a percolation process we define for each $e \in E_{\mathrm{co}}$ the probability that the edge $e = \{x, y\}$ is an element of E_ω to be equal to $p(xy^{-1})$.

More precisely we consider the following probability space: the sample space is given by $\Omega = \{0,1\}^{E_{\mathrm{co}}}$ the set of all possible configurations. We take $\mathcal{A}$ to be the sigma-algebra of subsets of Ω generated by the cylinder sets. Finally we define the product measure $\mathbb{P} = \prod_{e\in E_{\mathrm{co}}} \mathbb{P}_e$

where for each $e = \{x, y\} \in E_{\mathrm{co}}$ the probability measure $\mathbb{P}_e$ on $\{0, 1\}$ is given by

$$\mathbb{P}_e(\omega(e) = 1) = p(xy^{-1}) \qquad \text{and} \qquad \mathbb{P}_e(\omega(e) = 0) = 1 - p(xy^{-1}).$$

Define for each $\omega \in \Omega$ the set E_ω by

$$E_\omega := \{e \in E_{\mathrm{co}} \mid \omega(e) = 1\}.$$

Thus each $\omega \in \Omega$ gives rise to a graph $\Gamma_\omega = (V, E_\omega)$. Now we discuss an alternative definition of the long-range percolation process.

Remark 6.6. We introduced the distribution of the probabilities via an arbitrary function $p \in \ell^1(G, \mathbb{R})$ satisfying (6.12). There is an equivalent way to do so, which is more common in the physics community.

For each pair of vertices $x, y \in G$ let $J_{x,y}$ be a real number such that

- $J_{xz,yz} = J_{x,y}$ for all $z \in G$,
- $J := J_x := \sum_{y \in G} J_{x,y}$ is finite and independent of $x \in G$.

We fix $\beta > 0$ and declare an edge $\{x, y\}$ to be open with probability $1 - \mathrm{e}^{-\beta J_{x,y}}$. To see the equivalence to the above definition it suffices to show that $\sum_{x \in G} p(x) < \infty$ holds if and only if $\sum_{y \in G} J_{x,y} < \infty$, where $p(xy^{-1}) = 1 - \mathrm{e}^{-\beta J_{x,y}}$. Using that $1 - \mathrm{e}^{-s} \leq s$ for all $s \in \mathbb{R}$ one obtains

$$\sum_{x \in G} p(x) = \sum_{y \in G} p(xy^{-1}) = \sum_{y \in G} 1 - \mathrm{e}^{-\beta J_{x,y}} \leq \beta \sum_{y \in G} J_{x,y}.$$

To prove the converse direction we apply Taylor's formula, which shows that there exists a constant $T > 0$ such that

$$1 - \mathrm{e}^{-\beta J_{x,y}} = \beta J_{x,y} - \sum_{k=2}^{\infty} \frac{(-\beta J_{x,y})^k}{k!} \geq \frac{1}{2} \beta J_{x,y}$$

holds for all $x, y \in G$ satisfying $d(x,y) \geq T$. Thus we get

$$\begin{aligned}\sum_{y\in G} J_{x,y} &= \sum_{\substack{y\in G\\ d(x,y)\leq T}} J_{x,y} + \sum_{\substack{y\in G\\ d(x,y)>T}} J_{x,y}\\ &\leq \sum_{\substack{y\in G\\ d(x,y)\leq T}} J_{x,y} + \frac{2}{\beta}\sum_{\substack{y\in G\\ d(x,y)>T}} \left(1-\mathrm{e}^{-\beta J_{x,y}}\right) \leq c\sum_{x\in G} p(x)\end{aligned}$$

for $c > 0$ large enough.

Note that by definition $\mathbb{P}(\{x,y\} \in E_\omega) = \mathbb{P}(\{xz, yz\} \in E_\omega) = p(xy^{-1})$. We define

$$\varepsilon(R) := \sum_{y\in G\setminus B_R} p(y), \tag{6.13}$$

which gives $\lim_{R\to\infty} \varepsilon(R) = 0$ since $p \in \ell^1(G,\mathbb{R})$. Moreover we have for all $x \in G$: $\varepsilon(R) := \sum_{y\in G\setminus B_R(x)} p(xy^{-1})$. We infer from Lemma 4.8 that there exists a set $\Omega_{\mathrm{lf}} \subseteq \Omega$ of full measure such that for each $\omega \in \Omega_{\mathrm{lf}}$ the graph Γ_ω is locally finite.

The operator we study is the Laplace operator $\Delta^{(\omega)}$ given as in (4.25). It is self-adjoint for all $\omega \in \Omega$. Similarly the Laplacian $\Delta_S : \ell^2(V_S) \to \ell^2(V_S)$ on a finite subgraph $S = (V_S, E_S)$ of the complete graph Γ_{co} is given by

$$\Delta_S f(x) = \sum_{y\in V_S:\{x,y\}\in E_S} \left(f(y) - f(x)\right).$$

We denote the set of all finite subgraphs of the complete undirected graph Γ_{co} by $\mathcal{S}$. The subset of $\mathcal{S}$ consisting of all subgraphs with vertex set $Q \in \mathcal{F}(G)$ is called $\mathcal{S}(Q)$. For a subgraph $S = (V_S, E_S)$ of Γ_{co} and $Q \subseteq V_S$ the induced subgraph of S on Q is denoted by $S[Q] := S|_Q$, i.e. $S[Q]$ is the graph on vertex set Q, where two vertices are adjacent in $S[Q]$ if and only if they are adjacent in S. Note that this definition coincides with the one at the beginning of Section 2.1. Given a subgraph $S = (V_S, E_S)$ of Γ_{co} and an element $x \in G$ the *translation of S by x* is the graph Sx whose vertex set is $V_{Sx} = V_S x = \{yx \in G \mid y \in V_S\}$ and the edges are $E_{Sx} = \{\{y, y'\} \in E \mid \{yx^{-1}, y'x^{-1}\} \in E_S\}$.

In order to define the restriction of the Laplacian to a subset $Q \subseteq G$, we introduce again the mappings p_Q and i_Q called projection and inclusion. The map $p_Q : \ell^2(G) \to \ell^2(Q)$ is given by $u \mapsto p_Q(u)$, where $p_Q(u)(x) = u(x)$ for $x \in Q$. Similarly $i_Q : \ell^2(Q) \to \ell^2(G)$ is given by

$$i_Q(u)(x) := \begin{cases} u(x) & \text{if } x \in Q, \\ 0 & \text{otherwise.} \end{cases}$$

Note that these definitions coincide with the ones in (5.4) and (5.3) in the special case $\mathcal{H} = \mathbb{C}$. For given $\omega \in \Omega$ and $S = (V_S, E_S) \in \mathcal{S}$ we will be particularly interested in the restricted operators $p_Q \Delta^{(\omega)} i_Q : \ell^2(Q) \to \ell^2(Q)$ and $p_U i_{V_S} \Delta_S p_{V_S} i_U : \ell^2(U) \to \ell^2(U)$, where $Q \subseteq G$ and $U \subseteq V_S$ are finite. For this we will use the notation

$$\Delta^{(\omega)}[Q] := p_Q \Delta^{(\omega)} i_Q \quad \text{and} \quad \Delta_S[U] := p_U i_{V_S} \Delta_S p_{V_S} i_U.$$

Note that these operators are symmetric with real-valued matrix elements, hence their eigenvalues are a subset of the real axis. Given $\omega \in \Omega$, $R \in \mathbb{N}_0$ and $Q \in \mathcal{F}(G)$, we will be interested in the difference

$$D_\omega^R(Q) := \Delta_{\Gamma_\omega[Q]}[Q^{(R)}] - \Delta^{(\omega)}[Q^{(R)}], \tag{6.14}$$

where as before $Q^{(R)} = Q \setminus \partial_{\mathrm{int}}^R(Q)$. Let us emphasize that the boundary $\partial_{\mathrm{int}}^R(Q)$ is deterministic and does not depend on the specific choice of $\omega \in \Omega$.

For given $Q \in \mathcal{F}(G)$, $R \in \mathbb{N}_0$, $\omega \in \Omega_{\mathrm{lf}}$ and $S = (V_S, E_S) \in \mathcal{S}$ we define $F_\omega^R, F_\omega : \mathcal{F}(G) \to B(\mathbb{R})$ by

$$F_\omega^R(Q) := \mathfrak{e}(\Delta^{(\omega)}[Q^{(R)}]) \quad \text{and} \quad F_\omega(Q) := F_\omega^0(Q) = \mathfrak{e}(\Delta^{(\omega)}[Q]), \tag{6.15}$$

as well as $\tilde{F}^R, \tilde{F} : \mathcal{S} \to B(\mathbb{R})$ by

$$\tilde{F}^R(S) := \mathfrak{e}(\Delta_S[(V_S)^{(R)}]) \quad \text{and} \quad \tilde{F}(S) := \tilde{F}^0(S) = \mathfrak{e}(\Delta_S). \tag{6.16}$$

In the following we study the question whether for a given Følner sequence (Q_j) the limit

$$\lim_{j\to\infty} \frac{F_\omega(Q_j)}{|Q_j|}$$

exists. In order to do so, we need to control the long-range interactions. Therefore, our next aim is to define random variables counting edges exceeding a certain length R. Given an edge $e \in E_{\mathrm{co}}$, we define X_e as the random variable which is equal to one if e is an element of E_ω and zero otherwise. If an edge is given by a pair of vertices $\{x, y\}$ it is obvious that $X_{\{x,y\}} = X_{\{y,x\}}$ and its distribution depends only on the value xy^{-1}.

For fixed $R \in \mathbb{N}$ and a finite subset $Q = \{x_1, \dots, x_{|Q|}\} \subseteq G$ we define random variables $Y_i, i = 1, \dots, |Q|$ by

$$Y_i(\omega) = \sum_{y \in M_i^R} X_{\{x_i,y\}}(\omega), \tag{6.17}$$

where

$$M_i^R := \{x \in G \mid d_S(x, x_i) > R,\ x \neq x_j\ \forall j < i\}.$$

Thus, Y_i is the random variable counting the edges of length larger than R, being incident to x_i and not counted by any $Y_j, j = 1, \dots, i-1$. Note that the variables Y_i are independent and Lemma 4.8 yields $\mathbb{P}(Y_i = \infty) = 0,\ i = 1, \dots, |Q|$. Furthermore, the distribution functions of these random variables fulfill $F_{Y_1}(z) \leq F_{Y_i}(z)$ for all $i \in \{1, \dots, |Q|\}$ and all $z \in \mathbb{R}$. By equation (6.13) the expectation value $\mathbb{E}(Y_1)$ equals $\varepsilon(R)$. We denote the centered random variable $Y_i - \mathbb{E}(Y_i)$ by $\bar{Y}_i$ for all $i = 1, \dots, |Q|$ and set $Y := Y_1$, $\bar{Y} := \bar{Y}_1$. The aim of Lemma 6.7 is to estimate the tails of the distribution of the variables Y_i.

Lemma 6.7. *Let $R \in \mathbb{N}$, $Q = \{x_1, x_2, \dots, x_{|Q|}\} \in \mathcal{F}(G)$ and Y_i, $i = 1, \dots, |Q|$ be given as above. Then the estimate*

$$\mathbb{P}(Y_i \geq t) \leq c\mathrm{e}^{-t}$$

holds for all $t \in \mathbb{N}$ and all $i = 1, \dots, |Q|$, where $c \in \mathbb{R}$ is given by

$$c = \prod_{y \in G} \left(1 + p(y)(\mathrm{e} - 1)\right).$$

Proof. Let $y \in G$ be arbitrary and set $x := x_1$ as well as $Y = Y_1$, then

$$\mathbb{E}(\mathrm{e}^{X_{\{x,y\}}}) = p(xy^{-1})\mathrm{e} + (1 - p(xy^{-1}))\mathrm{e}^0 = 1 + p(xy^{-1})(\mathrm{e} - 1)$$

holds. The independence of X_e, $e \in E_{\mathrm{co}}$ implies that

$$\begin{aligned}\mathbb{E}(\mathrm{e}^Y) &= \prod_{y \in G \setminus B_R(x)} \mathbb{E}(\mathrm{e}^{X_{\{x,y\}}}) \\ &= \prod_{y \in G \setminus B_R(x)} \big(1 + p(xy^{-1})(\mathrm{e}-1)\big) \le \prod_{y \in G} (1 + p(y)(\mathrm{e}-1))\end{aligned}$$

since $Y = \sum_{y \in G \setminus B_R(x)} X_{\{x,y\}}$. The product converges to a finite number since

$$\begin{aligned}\prod_{y \in G} (1 + p(y)(\mathrm{e}-1)) &= \exp\left(\sum_{y \in G} \ln(1 + p(y)(\mathrm{e}-1))\right) \\ &\le \exp\left((\mathrm{e}-1)\sum_{y \in G} p(y)\right) < \infty\end{aligned}$$

holds by assumption on p. Now we use Markov's inequality to obtain for given $i \in \{1, \dots, |Q|\}$

$$\mathbb{P}(Y_i \ge t) \le \mathbb{P}(Y \ge t) \le \mathrm{e}^{-t}\mathbb{E}(\mathrm{e}^Y).$$

This implies the claimed inequality with constant c not depending on R. ■

Lemma 6.7 implies that for each $k \in \mathbb{N}$ and $i \in \{1, \dots, |Q|\}$ the moments $\mathbb{E}(Y_i^k)$ and $\mathbb{E}(\bar{Y}_i^k)$ exist. This is clear from

$$|\mathbb{E}(Y_i^k)| = \sum_{t=0}^{\infty} t^k \mathbb{P}(Y_i = t) \le \sum_{t=0}^{\infty} t^k \mathbb{P}(Y_i \ge t) \le c \sum_{t=0}^{\infty} t^k \mathrm{e}^{-t} < \infty$$

and

$$\begin{aligned}|\mathbb{E}(\bar{Y}_i^k)| &= \left| \sum_{t=0}^{\infty} (t - \mathbb{E}(Y_i))^k \mathbb{P}(Y_i = t) \right| \\ &\le \sum_{t=0}^{\infty} |t - \mathbb{E}(Y_i)|^k \, \mathbb{P}(Y_i \ge t) \\ &\le c \sum_{t=0}^{\infty} |t - \mathbb{E}(Y_i)|^k \mathrm{e}^{-t} < \infty.\end{aligned}$$

6.2.1 Bernstein inequality

In this section we verify a Bernstein inequality for independent random variables ξ_i. This is a result from the theory of large deviations. It estimates the probability that the sum of the random variables differs too much from its expectation value. The proof follows ideas from [AZ88], where similar estimates are shown.

Theorem 6.8 (Bernstein inequality)**.** *Let $\xi_1, \dots, \xi_n$ be independent random variables satisfying*

$$\mathbb{E}(\xi_i) = 0 \quad \textit{and} \quad |\mathbb{E}(\xi_i^k)| \leq \frac{1}{2}\tau^{k-2}k! \tag{6.18}$$

for all $i = 1, \dots, n$, all $k \in \mathbb{N} \setminus \{1\}$ and some constant $\tau > 0$. Then

$$\mathbb{P}(S \geq \alpha) \leq \begin{cases} \exp\left(-\frac{\alpha^2}{4n}\right) & , 0 \leq \alpha \leq n/\tau \\ \exp\left(-\frac{\alpha}{4\tau}\right) & , \alpha > n/\tau \end{cases} ,$$

where $S = \sum_{i=1}^n \xi_i$.

Proof. We first prove that if a random variable ξ satisfies (6.18) then we have for all $k \in \mathbb{N} \setminus \{1\}$

$$\mathbb{E}(|\xi|^k) \leq \sqrt{\frac{1}{3}}\tau^{k-2}k!. \tag{6.19}$$

If k is even, then obviously (6.19) holds by condition (6.18). Let $k \geq 3$ be odd. Then we can write $k = 2m + 1$ for some $m \in \mathbb{N}$ and Hölder inequality gives

$$\mathbb{E}(|\xi|^k) = \mathbb{E}(|\xi|^m |\xi|^{m+1}) \leq \left(\mathbb{E}(|\xi|^{2m})\mathbb{E}(|\xi|^{2m+2})\right)^{1/2}.$$

Using condition (6.18) leads to

$$\begin{aligned}
\mathbb{E}(|\xi|^k) &\leq \frac{1}{2}\tau^{k-2}((2m)!(2m+2)!)^{1/2} \\
&\leq \frac{1}{2}\tau^{k-2}((k-1)!(k+1)!)^{1/2} \\
&= \frac{1}{2}\tau^{k-2}k!(1+k^{-1})^{1/2}.
\end{aligned}$$

As $k \geq 3$ we have

$$\mathbb{E}(|\xi|^k) \leq \frac{1}{2}\sqrt{\frac{4}{3}}\tau^{k-2}k! = \sqrt{\frac{1}{3}}\tau^{k-2}k!.$$

Now, fix some $i \in \{1, \dots, n\}$ and $h \in (0, \frac{1}{2\tau}]$. Then we have using monotone convergence

$$\begin{aligned}\mathbb{E}(\exp(|h\xi_i|)) = \mathbb{E}\left(\sum_{k=0}^{\infty}\frac{|h\xi_i|^k}{k!}\right) &= \sum_{k=0}^{\infty}\frac{h^k\mathbb{E}(|\xi_i|^k)}{k!}\\ &= 1 + h\mathbb{E}(|\xi_i|) + \sum_{k=2}^{\infty}\frac{h^k\mathbb{E}(|\xi_i|^k)}{k!}\end{aligned}$$

and with (6.19) and $\mathbb{E}(|\xi_i|) \leq \mathbb{E}(\xi_i^2) + 1 \leq 2$ we obtain

$$\mathbb{E}(\exp(|h\xi_i|)) \leq 1 + 2h + h^2\sum_{k=2}^{\infty}\frac{(h\tau)^{k-2}}{\sqrt{3}} = 1 + 2h + \frac{2h^2}{\sqrt{3}} < \infty.$$

This allows to use Lebesgue's theorem in the following calculation

$$\mathbb{E}(\exp(h\xi_i)) = \sum_{k=0}^{\infty}\frac{\mathbb{E}((h\xi_i)^k)}{k!} \leq 1 + h^2\sum_{k=2}^{\infty}h^{k-2}\frac{|\mathbb{E}(\xi_i^k)|}{k!},$$

which gives together with condition (6.18)

$$\mathbb{E}(\exp(h\xi_i)) \leq 1 + \frac{h^2}{2}\sum_{k=2}^{\infty}(h\tau)^{k-2} \leq 1 + h^2 \leq \exp(h^2).$$

Furthermore, the independence of the random variables implies

$$\mathbb{E}(\exp(hS)) = \prod_{i=1}^{n}\mathbb{E}(\exp(h\xi_i)) \leq \prod_{i=1}^{n}\exp(h^2) = \exp(nh^2).$$

Using this and Markov inequality we obtain

$$\mathbb{P}(S \geq \alpha) \leq \exp(-\alpha h)\mathbb{E}(\exp(hS)) \leq \exp(nh^2 - \alpha h) \qquad (6.20)$$

for each $\alpha > 0$.

In the case $0 < \alpha \leq \frac{n}{\tau}$ set $h = \frac{\alpha}{2n} \leq \frac{1}{2\tau}$. Then (6.20) can be written as

$$\mathbb{P}(S \geq \alpha) \leq \exp\Big(-\frac{\alpha^2}{4n}\Big).$$

If $\alpha \geq \frac{n}{\tau}$ we set $h = \frac{1}{2\tau}$ and conclude

$$\mathbb{P}(S \geq \alpha) \leq \exp\Big(-\frac{\alpha}{4\tau}\Big),$$

which proves the claimed estimate. ■

The next Lemma shows that the variables Y_i, $i = 1, \ldots, |Q|$ fulfill the conditions (6.18) with some parameter $\tau > 0$, which is independent of R and Q. This allows to apply Theorem 6.8 in order to prove an adapted inequality in Corollary 6.11.

Lemma 6.9. *There exists an $R_0 \in \mathbb{N}$ such that for each $R \geq R_0$ the following holds: for any set $Q = \{x_1, \ldots, x_{|Q|}\} \in \mathcal{F}(G)$ and associated random variables $Y_i, i = 1, \ldots, |Q|$ given in (6.17) each $\bar{Y}_i = Y_i - \mathbb{E}(Y_i)$ satisfies the conditions* (6.18) *with $\tau = 6 \prod_{y \in G} (1 + p(y)(\mathrm{e} - 1))$.*

Remark 6.10. Notice that the existence of the moments $\mathbb{E}(\bar{Y}_i^k)$, $k \in \mathbb{N}$, $i \in \{1, \ldots, |Q|\}$ is already clear from Lemma 6.7. However it is not obvious that the conditions (6.18) hold with τ given as above. Furthermore, we see $\tau = 6c$, where c is the constant given by Lemma 6.7. If p is finitely supported then the second moment of $\bar{Y}_i$ is zero for large R. In this situation the conditions (6.18) are clearly fulfilled since then $\mathbb{E}(\bar{Y}_i^k) = 0$ for all $k \in \mathbb{N}$, $i \in \{1, \ldots, |Q|\}$.

Proof of Lemma 6.9. Assume that $Q = \{x_1, \ldots, x_{|Q|}\} \in \mathcal{F}(G)$ and $i \in \{1, \ldots, |Q|\}$ are given and set $x := x_1$. We first choose a certain constant $T \in \mathbb{N}$ and give a condition for R_0 in order to prove that $\mathbb{E}(\bar{Y}_i^2)$ does not exceed one for all $i = 1, \ldots, n$ and all $R \geq R_0$. Let $T \in \mathbb{N}$ be such that

$$\sum_{t=T+1}^{\infty} t^2 \mathrm{e}^{-t} \leq \frac{1}{3c}, \tag{6.21}$$

where $c > 0$ is the constant given by Lemma 6.7. Now choose $R_0 \in \mathbb{N}$ such that

$$\varepsilon(R) \leq -\frac{1}{2} \ln \left(1 - \left(3 \sum_{t=1}^{T} t^2 \right)^{-1} \right) \tag{6.22}$$

for all $R \geq R_0$. This choice implies

$$\mathbb{E}(Y_i) \leq \mathbb{E}(Y) = \varepsilon(R) \leq \frac{1}{3} \quad \text{and} \quad p(y) \leq \frac{1}{2} \tag{6.23}$$

for all $R \geq R_0$ and all $y \in G \setminus B_{R_0}$. Furthermore we get for $R \geq R_0$

$$\mathbb{P}(Y_i = 0) \geq \mathbb{P}\Bigg(\sum_{y \in G \setminus B_R(x)} X_{\{x,y\}} = 0 \Bigg) = \prod_{y \in G \setminus B_R(x)} (1 - p(xy^{-1})).$$

and substitution leads to

$$\prod_{y \in G \setminus B_R(x)} (1 - p(xy^{-1})) = \prod_{y \in G \setminus B_R} (1 - p(y)) = \exp\Bigg(\sum_{y \in G \setminus B_R} \ln(1 - p(y)) \Bigg).$$

Now, we use the inequality $1 - z \geq \mathrm{e}^{-2z}$, which holds for all $z \in [0, 0.5]$ and obtain

$$\mathbb{P}(Y_i = 0) \geq \exp\left(-2 \sum_{y \in G \setminus B_R} p(y) \right) = \exp\left(-2\varepsilon(R)\right).$$

Using (6.22), this shows that

$$\mathbb{P}(Y_i \geq 1) = 1 - \mathbb{P}(Y_i = 0) \leq 1 - \exp(-2\varepsilon(R)) \leq \left(3 \sum_{t=1}^{T} t^2 \right)^{-1}. \tag{6.24}$$

As $\mathbb{E}(\bar{Y}_i^2)$ can be written as

$$\mathbb{E}(\bar{Y}_i^2) = \left| \mathbb{E}\left((Y_i - \mathbb{E}(Y_i))^2 \right) \right| = \sum_{t=0}^{\infty} (t - \mathbb{E}(Y_i))^2 \mathbb{P}(Y_i = t),$$

the estimates in (6.21),(6.23),(6.24) and Lemma 6.7 imply

$$\begin{aligned}
&\mathbb{E}(\bar{Y}_i^2) \\
&\leq (\mathbb{E}(Y_i))^2 + \sum_{t=1}^{T} (t - \mathbb{E}(Y_i))^2 \mathbb{P}(Y_i = t) + \sum_{t=T+1}^{\infty} (t - \mathbb{E}(Y_i))^2 \mathbb{P}(Y_i = t) \\
&\leq (\varepsilon(R))^2 + \mathbb{P}(Y_i \geq 1) \sum_{t=1}^{T} t^2 + \sum_{t=T+1}^{\infty} t^2 \mathbb{P}(Y_i \geq t) \leq \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1.
\end{aligned}$$

Fix some $k \geq 3$. The k-th moment of $\bar{Y}_i$ is by definition the k-th central moment of Y_i, which yields

$$\left|\mathbb{E}(\bar{Y}_i^k)\right| = \left|\mathbb{E}((Y_i - \mathbb{E}(Y_i))^k)\right| = \left|\sum_{t=0}^{\infty}(t - \mathbb{E}(Y_i))^k \mathbb{P}(Y_i = t)\right|.$$

Since $0 \leq \mathbb{E}(Y_i) \leq \frac{1}{3}$, by (6.23), we have that

$$\begin{aligned}\left|\sum_{t=0}^{\infty}(t - \mathbb{E}(Y_i))^k \mathbb{P}(Y_i = t)\right| &\leq (\mathbb{E}(Y_i))^k \mathbb{P}(Y_i = 0) + \sum_{t=1}^{\infty} t^k \mathbb{P}(Y_i = t) \\ &\leq (\mathbb{E}(Y_i))^k + \sum_{t=1}^{\infty} t^k \mathbb{P}(Y_i \geq t)\end{aligned}$$

holds. Using $\mathbb{P}(Y_i \geq t) \leq \mathbb{P}(Y \geq t)$ and $\mathbb{E}(Y_i) \leq \mathbb{E}(Y)$, this implies

$$\left|\mathbb{E}(\bar{Y}_i^k)\right| \leq (\mathbb{E}(Y))^k + c\sum_{t=1}^{\infty} t^k \mathrm{e}^{-t},$$

where the last inequality holds with constant $c > 0$ from the Lemma 6.7. The function $f : [0, \infty] \to \mathbb{R}$, $x \mapsto x^k \mathrm{e}^{-x}$ takes its maximal value at the argument $x = k$. Therefore we get

$$\begin{aligned}\sum_{t=1}^{\infty} t^k \mathrm{e}^{-t} &= \sum_{t=1}^{k-1} t^k \mathrm{e}^{-t} + k^k \mathrm{e}^{-k} + \sum_{t=k+1}^{\infty} t^k \mathrm{e}^{-t} \\ &\leq \int_0^k x^k \mathrm{e}^{-x} \mathrm{d}x + k^k \mathrm{e}^{-k} + \int_k^{\infty} x^k \mathrm{e}^{-x} \mathrm{d}x \\ &= \int_0^{\infty} x^k \mathrm{e}^{-x} \mathrm{d}x + k^k \mathrm{e}^{-k}.\end{aligned}$$

Using partial integration proves that

$$\int_0^{\infty} x^k \mathrm{e}^{-x} \mathrm{d}x = \int_0^{\infty} k x^{k-1} \mathrm{e}^{-x} \mathrm{d}x = \cdots = k! \int_0^{\infty} \mathrm{e}^{-x} dx = k!$$

holds true. Now, it is enough to show that

$$2(\mathbb{E}(Y))^k + 2ck! + 2c\left(\frac{k}{\mathrm{e}}\right)^k \leq \tau^{k-2} k! \tag{6.25}$$

holds for $\tau = 6c$. To this end we consider the three summands separately. The first one gives by (6.21) and as $\tau > 1$

$$\frac{2(\mathbb{E}(Y))^k}{\tau^{k-2}k!} \leq \frac{1}{3}.$$

The second summand gives

$$\frac{2ck!}{\tau^{k-2}k!} = \frac{2c}{(6c)^{k-2}} \leq \frac{1}{3}$$

and for the third summand we use the Stirling formula $k! \geq k^k \mathrm{e}^{-k}$ to obtain

$$\frac{2ck^k}{\mathrm{e}^k \tau^{k-2}k!} \leq \frac{2c}{(6c)^{k-2}} \leq \frac{1}{3}.$$

This shows that (6.25) holds, which finishes the proof. ■

Given a finite set $Q = \{x_1, \dots, x_{|Q|}\} \subseteq G$, we will use this result to show that the probability that "too many long edges" are incident to a vertex in Q is very small. To be precise, let $R \in \mathbb{N}$ and $\delta > 0$ be constants and set $\varepsilon = \varepsilon(R) = \mathbb{E}(Y)$ as in (6.13). We decompose the probability space $\Omega = \Omega_1(\delta, R, Q) \cup \Omega_2(\delta, R, Q)$ by setting

$$\Omega_1(\delta, R, Q) := \left\{ \omega \in \Omega \;\middle|\; \sum_{i=1}^{|Q|} Y_i(\omega) \geq |Q|(\varepsilon + \delta) \right\} \tag{6.26}$$

and

$$\Omega_2(\delta, R, Q) := \Omega \setminus \Omega_1(\delta, R, Q). \tag{6.27}$$

where Y_i, $i = 1, \dots, |Q|$ are given by (6.17). Thus the set $\Omega_1(\delta, R, Q)$ consists of all configurations where the number of edges of length longer than R that are incident to at least one vertex in Q is at least $|Q|(\varepsilon(R) + \delta)$.

Corollary 6.11. *Let R_0 and τ be as in Lemma 6.9, let $\delta > 0$, $R \geq R_0$ and $Q \in \mathcal{F}(G)$ be given and define $\Omega_1(\delta, R, Q)$ as in* (6.26)*. Then the following inequality holds*

$$\mathbb{P}(\Omega_1(\delta, R, Q)) \leq \begin{cases} \exp\left(-\frac{\delta^2|Q|}{4}\right) & , 0 \leq \delta \leq \frac{1}{\tau}, \\ \exp\left(-\frac{\delta|Q|}{4\tau}\right) & , \delta > \frac{1}{\tau}. \end{cases} \tag{6.28}$$

Proof. By definition of Y_i, $\bar{Y}_i$ and $\varepsilon = \varepsilon(R)$ we have

$$\mathbb{P}(\Omega_1(\delta, R, Q)) = \mathbb{P}\left(\sum_{i=1}^{|Q|} Y_i \geq |Q|(\mathbb{E}(Y) + \delta)\right) \leq \mathbb{P}\left(\sum_{i=1}^{|Q|} \bar{Y}_i \geq |Q|\delta\right).$$

As the variables $\bar{Y}_i, i = 1, \ldots, |Q|$ are independent and fulfill conditions (6.18) this term can be estimated using Theorem 6.8. Setting $\alpha = \delta|Q|$ we get

$$\mathbb{P}(\Omega_1(\delta, R, Q)) \leq \begin{cases} \exp\left(-\frac{\delta^2|Q|^2}{4|Q|}\right) & , 0 \leq \delta|Q| \leq \frac{|Q|}{\tau}, \\ \exp\left(-\frac{\delta|Q|}{4\tau}\right) & , \delta|Q| > \frac{|Q|}{\tau}, \end{cases}$$

which gives the desired estimate. ∎

6.2.2 Almost additivity

This section provides crucial properties of the functions F_ω^R and $\tilde{F}^R$. As in the deterministic setting, the most important condition which needs to be proven is a version of almost additivity. This will be done in Lemma 6.13.

Lemma 6.12. *Let $R \in \mathbb{N}_0$, $\omega \in \Omega$ and the functions $F_\omega^R : \mathcal{F}(G) \to B(\mathbb{R})$ and $\tilde{F}^R : \mathcal{S} \to B(\mathbb{R})$ be given as in* (6.15) *and* (6.16). *Then the following holds true:*

(i) the functions F_ω^R and $\tilde{F}^R$ are linearly bounded, in fact

$$\|F_\omega^R(Q)\| \leq |Q| \qquad \text{and} \qquad \|\tilde{F}^R(S)\| \leq |V_S|,$$

(ii) the function $\tilde{F}^R$ is invariant under translation, i.e. for any $S \in \mathcal{S}$ and $x \in G$ we have

$$\tilde{F}^R(S) = \tilde{F}^R(Sx).$$

Proof. This follows easily from the definition. ∎

The next results are devoted to prove further properties of these functions for $R \geq R_0$ with R_0 from Lemma 6.9. We will not be able to verify these properties for all $\omega \in \Omega$ but only for all $\omega \in \tilde{\Omega}$ where

$$\tilde{\Omega} := \tilde{\Omega}(\delta, R, Q) := \Omega_2(\delta, R, Q) \cap \Omega_{\mathrm{lf}} \quad \text{and} \quad \Omega_2(\delta, R, Q) \text{ as in (6.27)} \tag{6.29}$$

By Corollary 6.11 we have $\mathbb{P}(\tilde{\Omega}) \geq 1 - \exp(-\delta^2 |Q|/4)$ for $\delta \leq \tau^{-1}$. The function $F_\omega^R : \mathcal{F}(G) \to B(\mathbb{R})$, $Q \mapsto F_\omega^R(Q)$ satisfies a weak form of additivity, described in the next lemma.

Lemma 6.13. *Let $Q \in \mathcal{F}(G)$, $R \geq R_0$ and $\delta > 0$ be given and set $\tilde{\Omega} = \tilde{\Omega}(\delta, R, Q)$ as in (6.29) and $\varepsilon = \varepsilon(R) = \sum_{y \in G \setminus B_R} p(y)$ as in (6.13). Then for any disjoint sets $Q_i, i = 1, \ldots k$ with $Q = \bigcup_i Q_i$ the inequality*

$$\left\| F_\omega^R(Q) - \sum_{i=1}^k F_\omega^R(Q_i) \right\| \leq 4|Q|(\varepsilon + \delta) + 4 \sum_{i=1}^k |\partial^R(Q_i)|$$

holds for all $\omega \in \tilde{\Omega}$. Here R_0 is the constant given in Lemma 6.9.

Proof. Let $\omega \in \tilde{\Omega}$ and disjoint sets $Q_i, i = 1, \ldots k$ with $Q = \bigcup_i Q_i$ be given. During the proof we will call the edges of length longer than R the *long edges*. For given $U \in \mathcal{F}(G)$ we define an operator $L_\omega[U] : \ell^2(U) \to \ell^2(U)$ which does only respect the long edges by

$$(L_\omega[U]f)(x) = - \sum_{\substack{y \in U : \{x,y\} \in E_\omega \\ d(x,y) > R}} f(y)$$

and use the notation

$$\Delta_L^{(\omega)}[U] := \Delta^{(\omega)}[U] - L_\omega[U].$$

As ω is an element of $\Omega_2(\delta, R, Q)$, the number of long edges in Γ_ω which are incident to a vertex in Q is smaller than $|Q|(\varepsilon + \delta)$. Hence, the matrices $L_\omega[Q]$ and $L_\omega[Q^{(R)}]$ (with respect to the canonical basis) contain not more than $2|Q|(\varepsilon + \delta)$ non-zero elements and we get

$$\operatorname{rank}(L_\omega[Q]) \leq 2|Q|(\varepsilon + \delta) \quad \text{and} \quad \operatorname{rank}(L_\omega[Q^{(R)}]) \leq 2|Q|(\varepsilon + \delta).$$

This combined with Lemma 2.24 gives

$$\|\mathfrak{e}(\Delta^{(\omega)}[Q^{(R)}]) - \mathfrak{e}(\Delta_L^{(\omega)}[Q^{(R)}])\| \leq \operatorname{rank}(L_\omega[Q^{(R)}]) \leq 2|Q|(\varepsilon+\delta), \tag{6.30}$$

which immediately implies

$$\begin{aligned}&\left\|\mathfrak{e}(\Delta^{(\omega)}[Q^{(R)}]) - \sum_{i=1}^{k}\mathfrak{e}(\Delta^{(\omega)}[Q_i^{(R)}])\right\| \\ &\quad\leq \left\|\mathfrak{e}(\Delta_L^{(\omega)}[Q^{(R)}]) - \sum_{i=1}^{k}\mathfrak{e}(\Delta^{(\omega)}[Q_i^{(R)}])\right\| + 2|Q|(\varepsilon+\delta).\end{aligned}$$

Here the first term can be estimated by

$$\begin{aligned}&\left\|\mathfrak{e}(\Delta_L^{(\omega)}[Q^{(R)}]) - \sum_{i=1}^{k}\mathfrak{e}(\Delta^{(\omega)}[Q_i^{(R)}])\right\| \\ &\qquad\leq \left\|\mathfrak{e}(\Delta_L^{(\omega)}[Q^{(R)}]) - \sum_{i=1}^{k}\mathfrak{e}(\Delta_L^{(\omega)}[Q_i^{(R)}])\right\| \\ &\qquad\qquad + \left\|\sum_{i=1}^{k}\Big(\mathfrak{e}(\Delta_L^{(\omega)}[Q_i^{(R)}]) - \mathfrak{e}(\Delta^{(\omega)}[Q_i^{(R)}])\Big)\right\|.\end{aligned}$$

For the next step recall that $\sum_i \operatorname{rank}(L_\omega[Q_i^{(R)}])$ is also bounded by the number of non-zero matrix elements in $L_\omega[Q]$. This and another application of Lemma 2.24 yield

$$\begin{aligned}&\left\|\mathfrak{e}(\Delta^{(\omega)}[Q^{(R)}]) - \sum_{i=1}^{k}\mathfrak{e}(\Delta^{(\omega)}[Q_i^{(R)}])\right\| \\ &\quad\leq \left\|\mathfrak{e}(\Delta_L^{(\omega)}[Q^{(R)}]) - \sum_{i=1}^{k}\mathfrak{e}(\Delta_L^{(\omega)}[Q_i^{(R)}])\right\| + 4|Q|(\varepsilon+\delta).\end{aligned} \tag{6.31}$$

Now we use a decoupling argument very similar to the one in Lemma 5.12. By definition of $\Delta_L^{(\omega)}[\cdot]$ and $L_\omega[\cdot]$ we get

$$\Delta_L^{(\omega)}\left[\bigcup\nolimits_{i=1}^{k} Q_i^{(R)}\right] = \bigoplus_{i=1}^{k}\left(\Delta_L^{(\omega)}[Q_i^{(R)}]\right).$$

Therefore we can count the eigenvalues of $\Delta_L^{(\omega)}[Q_i^{(R)}]$ for $i = 1, \ldots, k$ separately

$$\mathfrak{e}\left(\Delta_L^{(\omega)}\left[\bigcup\nolimits_{i=1}^k Q_i^{(R)}\right]\right) = \sum_{i=1}^k \mathfrak{e}\left(\Delta_L^{(\omega)}[Q_i^{(R)}]\right).$$

Next, we apply Proposition 2.25 with $V = \ell^2(Q^{(R)})$ and $U = \ell^2(\bigcup_{i=1}^k Q_i^{(R)})$ and obtain

$$\begin{aligned}
&\left\|\mathfrak{e}(\Delta_L^{(\omega)}[Q^{(R)}]) - \sum_{i=1}^k \mathfrak{e}(\Delta_L^{(\omega)}[Q_i^{(R)}])\right\| \\
&= \left\|\mathfrak{e}(\Delta_L^{(\omega)}[Q^{(R)}]) - \mathfrak{e}\left(\Delta_L^{(\omega)}\left[\bigcup\nolimits_{i=1}^k Q_i^{(R)}\right]\right)\right\| \le 4\sum_{i=1}^k |\partial^R Q_i|.
\end{aligned}$$

This together with (6.31) finishes the proof. ∎

The next lemma shows that the functions F_ω^R and $\tilde{F}^R$ behave similarly with high probability.

Lemma 6.14. *Let $Q \in \mathcal{F}(G)$, $R \ge R_0$ and $\delta > 0$ be given and set $\tilde{\Omega} = \tilde{\Omega}(\delta, R, Q)$ as in (6.29) and $\varepsilon = \varepsilon(R) = \sum_{y \in G \setminus B_R} p(y)$ as in (6.13). Then*

$$\left\|F_\omega^R(Q) - \tilde{F}^R(\Gamma_\omega[Q])\right\| \le |Q|(\varepsilon + \delta)$$

holds for all $\omega \in \tilde{\Omega}$. Here R_0 is the constant given in Lemma 6.9.

Proof. Let $\omega \in \tilde{\Omega}$ be given. By definition of $\tilde{F}^R$, F_ω^R and $D_\omega^R(\cdot)$, see (6.14)

$$\begin{aligned}
\left\|F_\omega^R(Q) - \tilde{F}^R(\Gamma_\omega[Q])\right\| &= \left\|\mathfrak{e}(\Delta^{(\omega)}[Q^{(R)}]) - \mathfrak{e}(\Delta_{\Gamma_\omega[Q]}[Q^{(R)}])\right\| \\
&= \left\|\mathfrak{e}(\Delta^{(\omega)}[Q^{(R)}]) - \mathfrak{e}(\Delta^{(\omega)}[Q^{(R)}] + D_\omega^R(Q))\right\|
\end{aligned}$$

holds. Lemma 2.24 yields that

$$\begin{aligned}
\left\|\mathfrak{e}(\Delta^{(\omega)}[Q^{(R)}]) - \mathfrak{e}(\Delta^{(\omega)}[Q^{(R)}] + D_\omega^R(Q))\right\| &\le \operatorname{rank}(D_\omega^R(Q)) \\
&\le \sum_{x \in Q^{(R)}} |D_\omega^R(Q)(x,x)|.
\end{aligned} \tag{6.32}$$

Note that $(-D^R_\omega(Q)) : \ell^2(Q^{(R)}) \to \ell^2(Q^{(R)})$ is a diagonal matrix (with respect to the canonical basis) where the entry at (x,x) denotes the number of edges in Γ_ω from $x \in Q^{(R)}$ to $G \setminus Q$. The sum of these entries is bounded from above by the number of all edges of length longer than R which are incident to some $x \in Q^{(R)}$. Therefore, the last term in inequality (6.32) is not larger then $|Q|(\varepsilon + \delta)$ as ω is an element of Ω_2. ∎

6.2.3 Uniform convergence

At the beginning of this subsection we introduce some notation concerning frequencies of finite subgraphs in infinite graphs. Note that bond percolation can be interpreted as a random coloring of the edges in two colors. Therefore, the following notation is very similar to the notation in the deterministic setting on amenable groups, where we considered colorings of vertices, cf. the first pages of Chapter 5.

For two graphs $S, S' \in \mathcal{S}$ the number of occurrences of translations of the graph S in S' is denoted by

$$\sharp_S(S') := |\{x \in G \mid V_S x \subseteq V_{S'},\ S'[V_S x] = Sx\}|.$$

Counting occurrences of graphs along a Følner sequence $(U_j)_{j\in\mathbb{N}}$ leads to the definition of frequencies. Let $S \in \mathcal{S}$, $(U_j)_{j\in\mathbb{N}}$ be a Følner sequence and let $\Gamma' = (V, E')$ be a subgraph of Γ_{co} on the full vertex set V. If the limit

$$\nu_S(\Gamma') := \lim_{j\to\infty} \frac{\sharp_S(\Gamma'[U_j])}{|U_j|}$$

exists, we call $\nu_S(\Gamma')$ the *frequency of S in the graph Γ' along $(U_j)_{j\in\mathbb{N}}$*. Similarly frequencies can be defined for subgraphs which are not (or sparsely) connected to the rest of the graph. Given $R \in \mathbb{N}$ and a graph $\Gamma' = (V, E')$ on the full vertex set V, we say that a graph $S = (V_S, E_S)$ is *R-isolated in Γ'* if $\Gamma[V_S] = S$ and $\{g,h\} \notin E'$ for all $g \in V_S$, $h \in G \setminus V_S$ satisfying $d(g,h) \geq R$. Thence, a 1-isolated graph S has no edge connecting it with the rest of the graph. For a given graph $S = (V_S, E_S) \in \mathcal{S}$, a set $Q \in \mathcal{F}(G)$, $R \in \mathbb{N}$ and Γ' as above we write

$$\sharp_{S,R}(\Gamma', Q) := |\{x \in G \mid V_S x \subseteq Q \text{ and } Sx \text{ is } R\text{-isolated in } \Gamma'\}|$$

for the number of occurrences of R-isolated copies of S in Q. The *frequency of an R-isolated graph S* along a Følner sequence (U_j) in Γ' is defined by

$$\nu_{S,R}(\Gamma') := \lim_{j\to\infty} \frac{\sharp_{S,R}(\Gamma', U_j)}{|U_j|},$$

if the limit exists. In the following the graph Γ' will always be given by a percolation graph Γ_ω, $\omega \in \Omega$. However, Lemma 6.15 shows that the frequencies $\nu_S(\Gamma_\omega)$ will coincide for almost all $\omega \in \Omega$. The same holds true for the frequencies $\nu_{S,R}(\Gamma_\omega)$.

We define the action T of G on $(\Omega, \mathcal{A}, \mathbb{P})$ by

$$T : G \times \Omega \to \Omega, \qquad (g, \omega) \mapsto T_g(\omega) := \omega g^{-1} \tag{6.33}$$

where $\omega g^{-1} \in \Omega$ is given pointwise by

$$\omega g^{-1}(\{x, y\}) = \omega(\{xg, yg\}) \quad \text{for all } x, y \in G.$$

Note that T is an ergodic and measure preserving left-action on $(\Omega, \mathcal{A}, \mathbb{P})$.

Lemma 6.15. *Given $R \in \mathbb{N}$ and a tempered Følner sequence (Q_n), there exists a set $\Omega_{\mathrm{fr}} \subseteq \Omega$ of full measure such that the frequencies $\nu_S(\Gamma_\omega)$ and $\nu_{S,R}(\Gamma_\omega)$ along (Q_n) exist for all $S = (V_S, E_S) \in \mathcal{S}$ and all $\omega \in \Omega_{\mathrm{fr}}$, in particular*

$$\nu_S(\Gamma_\omega) = \prod_{\{x,y\}\in E_S} p(xy^{-1}) \cdot \prod_{\substack{\{x,y\}\notin E_S \\ x,y\in V_S}} (1 - p(xy^{-1})) \text{ and}$$

$$\nu_{S,R}(\Gamma_\omega) = \nu_S(\Gamma_\omega) \cdot \prod_{\substack{\{x,y\}\in E, x\in V_S, \\ y\notin V_S, d(x,y)\geq R}} (1 - p(xy^{-1}))$$

holds true. The values of $\nu_S := \nu_S(\Gamma_\omega)$ and $\nu_{S,R} := \nu_{S,R}(\Gamma_\omega)$ do not depend on the specific choice of the sequence (Q_n).

Proof. Let $S = (V_S, E_S) \in \mathcal{S}$ be a finite graph such that $\mathrm{id} \in V_S$. We define $A_S = \{\omega \in \Omega \mid \Gamma_\omega[V_S] = S\}$ to be the subset of Ω consisting of all configurations where Γ_ω coincides with S on V_S and we denote the indicator function of A_S by f_S. Then, analogously to the proof of Theorem 5.30, we use Lindenstrauss' ergodic theorem, namely Theorem 2.12, to obtain a set $\Omega_S \subseteq \Omega$ of full measure, such that we have for all $\omega \in \Omega_S$ that $\nu_S(\Gamma_\omega) = \mathbb{E}(f_S)$. In the same way we get for each $R \in \mathbb{N}$ and $S \in \mathcal{S}$ a set $\Omega_S^R \subseteq \Omega$ of full measure, such that for each $\omega \in \Omega_S^R$ we have $\nu_{S,R}(\Gamma_\omega) = \mathbb{E}(f_{S,R})$. Here $f_{S,R}$ is the indicator function of the set

$$A_{S,R} = \{\omega \in \Omega \mid \Gamma_\omega[V_S] = S \text{ and } S \text{ is } R\text{-isolated in } \Gamma_\omega\}.$$

We define

$$\Omega_{\mathrm{fr}} := \left(\bigcap_{S \in \mathcal{S}} \Omega_S \right) \cap \left(\bigcap_{S \in \mathcal{S}} \bigcap_{R \in \mathbb{N}} \Omega_S^R \right),$$

which is of measure one, as the above index sets are countable. It remains to calculate the expectations $\mathbb{E}(f_S)$ and $\mathbb{E}(f_{S,R})$. For the first one we get using the probabilities given by $p \in \ell^1(G, \mathbb{R})$ in (6.12):

$$\mathbb{E}(f_S) = \mathbb{P}(f_S(\omega) = 1) = \prod_{\{x,y\} \in E_S} p(xy^{-1}) \cdot \prod_{\substack{\{x,y\} \notin E_S \\ x,y \in V_S}} (1 - p(xy^{-1})).$$

In the same manner we obtain

$$\mathbb{E}(f_{S,R}) = \prod_{\{x,y\} \in E_S} p(xy^{-1}) \cdot \prod_{\substack{\{x,y\} \notin E_S \\ x,y \in V_S}} (1 - p(xy^{-1})) \cdot \prod_{\substack{\{x,y\} \in E, x \in V_S, \\ y \notin V_S, d(x,y) \geq R}} (1 - p(xy^{-1})).$$

Here the last product is finite since $p \in \ell^1(G, \mathbb{R})$. ■

Theorem 6.16. *Let G be a finitely generated, ST-amenable group and let (Q_n) and (U_j) be Følner sequences fulfilling*

(a) (U_j) is strictly increasing and tempered;

(b) (Q_n) is symmetrically tiling.

Let the functions $F_\omega : \mathcal{F}(G) \to B(\mathbb{R})$ and $\tilde{F} : \mathcal{S} \to B(\mathbb{R})$ be given as in (6.15) and (6.16). Then the following limits

$$\mathfrak{I} := \lim_{j \to \infty} \frac{F_\omega(U_j)}{|U_j|} = \lim_{n \to \infty} \sum_{S \in \mathcal{S}(Q_n)} \nu_S \frac{\tilde{F}(S)}{|Q_n|} \tag{6.34}$$

exist and are equal almost surely. Furthermore, they do not depend on the specific choice of (Q_n) and (U_j). The function $\mathfrak{I}$ is called the integrated density of states of $\Delta^{(\omega)}$.

Remark 6.17. Given a group G and Følner sequences with the above properties, Theorem 6.16 ensures the existence of a set Ω' of measure one, such that the limits (6.34) exist for all $\omega \in \Omega'$. However, it is not possible to find a set Ω'' of full measure, such that the limit exists for all Følner sequences satisfying (a) and (b). This is due to the fact that for almost all $\omega \in \Omega$ we can construct (by translation) Følner sequences such that the associated sequences in (6.34) do not converge.

A similar and well known phenomenon occurs in the theory of Lebesgue measurable functions. Here one identifies functions which agree up to a set of measure zero, however it is not possible to find a set of full measure such that all functions in the same equivalence class are equal on this set.

The proof of Theorem 6.16 is based on the following Lemma.

Lemma 6.18. *Let G be a finitely generated, ST-amenable group and let (U_j) be a strictly increasing, tempered Følner sequence and let (Q_n) be symmetrically tiling. Let $j \in \mathbb{N}$, $R \geq R_0$ and $0 < \delta \leq \tau^{-1}$ be given, where R_0 and τ are constants given by Lemma 6.9. Set $\varepsilon = \varepsilon(R) = \sum_{y \in G \setminus B_R} p(y)$ as in (6.13) and $\Omega_j = \tilde{\Omega}(\delta, R, U_j) \cap \Omega_{\mathrm{fr}}$, where $\tilde{\Omega}(\delta, R, U_j)$ is as in (6.29) and Ω_{fr} as in Lemma 6.15. The functions $F_\omega^R : \mathcal{F}(G) \to B(\mathbb{R})$ and $\tilde{F}^R : \mathcal{S} \to B(\mathbb{R})$ are defined as in (6.15) and (6.16). Then the difference*

$$D_\omega(j, n, R) := \left\| \frac{F_\omega^R(U_j)}{|U_j|} - \sum_{S \in \mathcal{S}(Q_n)} \nu_S \frac{\tilde{F}^R(S)}{|Q_n|} \right\| .$$

satisfies the estimate

$$\begin{aligned} D_\omega(j, n, R) \leq{} & 4\frac{|\partial^R Q_n|}{|Q_n|} + \left(4\frac{|\partial^R Q_n|}{|Q_n|} + 1 \right) \frac{|\partial^{d(n)} U_j|}{|U_j|} \\ & + 5(\varepsilon + \delta) + \sum_{S \in \mathcal{S}(Q_n)} \left| \frac{\sharp_S(\Gamma_\omega[U_j])}{|U_j|} - \nu_S \right| \end{aligned}$$

for all $\omega \in \Omega_j$ and all $n \in \mathbb{N}$, where $d(n) := \mathrm{diam}(Q_n)$.

Proof. Let $n \in \mathbb{N}$ and $\omega \in \Omega_j$ be given. By inserting zeros we estimate the difference $D_\omega(j,n,R)$ in the following way

$$\begin{aligned} D_\omega(j,n,R) \leq &\left\| \frac{F_\omega^R(U_j)}{|U_j|} - \sum_{\substack{g\in G\\ Q_n g\subseteq U_j}} \frac{F_\omega^R(Q_n g)}{|U_j|\cdot|Q_n|} \right\| \\ &+ \left\| \sum_{\substack{g\in G\\ Q_n g\subseteq U_j}} \frac{F_\omega^R(Q_n g)}{|U_j|\cdot|Q_n|} - \sum_{S\in\mathcal{S}(Q_n)} \frac{\sharp_S(\Gamma_\omega[U_j])}{|U_j|} \frac{\tilde{F}^R(S)}{|Q_n|} \right\| \\ &+ \left\| \sum_{S\in\mathcal{S}(Q_n)} \frac{\sharp_S(\Gamma_\omega[U_j])}{|U_j|} \frac{\tilde{F}^R(S)}{|Q_n|} - \sum_{S\in\mathcal{S}(Q_n)} \nu_S \frac{\tilde{F}^R(S)}{|Q_n|} \right\|. \end{aligned}$$

With another application of the triangle inequality this gives

$$D_\omega(j,n,R) \leq D_\omega^{(1)}(j,n,R) + D_\omega^{(2)}(j,n,R) + D_\omega^{(3)}(j,n,R),$$

where

$$D_\omega^{(1)}(j,n,R) := \frac{1}{|U_j||Q_n|} \left\| \sum_{x\in Q_n} F_\omega^R(U_j) - \sum_{\substack{g\in G\\ Q_n g\subseteq U_j}} F_\omega^R(Q_n g) \right\|,$$

$$D_\omega^{(2)}(j,n,R) := \frac{1}{|U_j||Q_n|} \left\| \sum_{\substack{g\in G\\ Q_n g\subseteq U_j}} F_\omega^R(Q_n g) - \sum_{S\in\mathcal{S}(Q_n)} \sharp_S(\Gamma_\omega[U_j]) \tilde{F}^R(S) \right\|$$

and

$$D_\omega^{(3)}(j,n,R) := \sum_{S\in\mathcal{S}(Q_n)} \left| \frac{\sharp_S(\Gamma_\omega[U_j])}{|U_j|} - \nu_S \right| \frac{\|\tilde{F}^R(S)\|}{|Q_n|}.$$

We use the boundedness of $\tilde{F}^R(S)$ (see Lemma 6.12) to obtain

$$D_\omega^{(3)}(j,n,R) \leq \sum_{S\in\mathcal{S}(Q_n)} \left| \frac{\sharp_S(\Gamma_\omega[U_j])}{|U_j|} - \nu_S \right|. \tag{6.35}$$

To estimate the other terms we make use of the assumption that each Q_n symmetrically tiles G. Here we proceed analogously to the proof of Theorem 5.11. For each $n \in \mathbb{N}$ there exists a set $G_n = G_n^{-1} \subseteq G$ such that G is the disjoint union of the sets $Q_n t$, $t \in G_n$. For fixed $x \in G$ we shift the grid $G_n x^{-1} = \{tx^{-1} \mid t \in G_n\}$ and get

$$G = Gx^{-1} = \bigcup_{t \in G_n} Q_n t x^{-1} = \bigcup_{t \in G_n x^{-1}} Q_n t$$

and $Q_n t \cap Q_n t' = \emptyset$ for distinct $t, t' \in G_n^{-1}$. This shows that $\{Q_n t \mid t \in G_n x^{-1}\}$ is a tiling of G as well. Given a set $U \in \mathcal{F}(G)$ and an element $x \in G$, we set

$$T(U, x, n) := \{g \in G_n x^{-1} \mid Q_n g \cap U \neq \emptyset\}$$

and distinguish two types of elements in $T(U, x, n)$

$$I(U, x, n) := \{g \in G_n x^{-1} \mid Q_n g \subseteq U\}$$

and

$$\partial(U, x, n) := T(U, x, n) \setminus I(U, x, n).$$

Therefore, translations of Q_n by elements of $I(U, x, n)$ are completely contained in U, whereas translations of Q_n by elements of $\partial(U, x, n)$ have non-empty intersections with both U and $G \setminus U$. By construction we have the following equality

$$\{g \in G \mid Q_n g \subseteq U_j\} = \dot{\bigcup}_{x \in Q_n} I(U_j, x, n). \tag{6.36}$$

Here, the inclusion "$\supseteq$" is obvious. To obtain the other inclusion one, take an element $g \in G$ and choose $x \in Q_n$ and $t \in G_n$ such that $g^{-1} = xt$. This gives $g = t^{-1}x^{-1} \in G_n x^{-1}$ as G_n is symmetric. In order to show that the union in (6.36) is disjoint, let $x, y \in Q_n$ with $x \neq y$ be given. Then $xG_n \cap yG_n = \emptyset$ and again by symmetry of G_n we have $G_n x^{-1} \cap G_n y^{-1} = \emptyset$, which proves (6.36). We use the invariance of $\tilde{F}^R$ under translation, see Lemma 6.12 and then (6.36) to obtain

$$\begin{aligned} D_\omega^{(2)}(j, n, R) &= \frac{1}{|U_j||Q_n|} \Bigg\| \sum_{\substack{g \in G \\ Q_n g \subseteq U_j}} \Big(F_\omega^R(Q_n g) - \tilde{F}^R(\Gamma_\omega[Q_n g]) \Big) \Bigg\| \\ &\leq \frac{1}{|U_j||Q_n|} \sum_{x \in Q_n} \sum_{g \in I(U_j, x, n)} \Big\| F_\omega^R(Q_n g) - \tilde{F}^R(\Gamma_\omega[Q_n g]) \Big\|. \end{aligned}$$

As $\omega \in \tilde{\Omega}(\delta, R, U_j)$ and as $Q_n g \cap Q_n h = \emptyset$ for distinct $g, h \in I(U_j, x, n)$, Lemma 6.14 leads to

$$D_\omega^{(2)}(j,n,R) \le \frac{1}{|U_j| \cdot |Q_n|} \sum_{x \in Q_n} \sum_{g \in I(U_j,x,n)} |Q_n|(\varepsilon+\delta) \le \varepsilon+\delta. \quad (6.37)$$

To estimate $D_\omega^{(1)}(j,n,R)$, firstly note that the disjointness of the translates and the fact that $Q_n g \subseteq \partial^{d(n)} U_j$ holds for all $g \in \partial(U_j, x, n)$ imply the following inequalities:

$$|\partial(U_j,x,n)| \cdot |Q_n| \le |\partial^{d(n)} U_j| \quad \text{and} \quad |I(U_j,x,n)| \cdot |Q_n| \le |U_j|. \quad (6.38)$$

We use again (6.36) to obtain

$$D_\omega^{(1)}(j,n,R) \le \frac{1}{|U_j| \cdot |Q_n|} \sum_{x \in Q_n} \left\| F_\omega^R(U_j) - \sum_{g \in I(U_j,x,n)} F_\omega^R(Q_n g) \right\| \quad (6.39)$$

and analyze one summand

$$\begin{aligned} Z_\omega^R(U_j,x,n) &:= \left\| F_\omega^R(U_j) - \sum_{g \in I(U_j,x,n)} F_\omega^R(Q_n g) \right\| \\ &= \left\| F_\omega^R(U_j) - \sum_{g \in I(U_j,x,n)} F_\omega^R((Q_n g) \cap U_j) \right\| \\ &\le \left\| F_\omega^R(U_j) - \sum_{g \in T(U_j,x,n)} F_\omega^R((Q_n g) \cap U_j) \right\| \\ &\quad + \left\| \sum_{g \in \partial(U_j,x,n)} F_\omega^R((Q_n g) \cap U_j) \right\|, \end{aligned}$$

where the last inequality holds since $T(U_j, x, n)$ is the disjoint union of $\partial(U_j, x, n)$ and $I(U_j, x, n)$. Next we use the weak form of additivity given by Lemma 6.13. This is applicable since $\omega \in \Omega_j \subseteq \tilde{\Omega}(\delta, R, U_j)$ and it gives, together with the boundedness of F_ω^R (see Lemma 6.12) the following

$$
\begin{aligned}
&Z_\omega^R(U_j,x,n)\\
&\leq 4\Big(\sum_{g\in I(U_j,x,n)}|\partial^R(Q_n g)| + \sum_{g\in\partial(U_j,x,n)}|\partial^R((Q_n g)\cap U_j)| + |U_j|(\varepsilon+\delta)\Big)\\
&\quad + \sum_{g\in\partial(U_j,x,n)}|Q_n g|.
\end{aligned}
$$

The invariance of $\partial^R(\cdot)$ and $|\cdot|$ under translation and the inequalities (6.38) yield

$$
\begin{aligned}
&Z_\omega^R(U_j,x,n)\\
&\leq 4|\partial^R Q_n||I(U_j,x,n)| + 4|\partial^R Q_n||\partial(U_j,x,n)| + |Q_n||\partial(U_j,x,n)|\\
&\quad + 4|U_j|(\varepsilon+\delta)\\
&\leq 4|\partial^R Q_n|\frac{|U_j|}{|Q_n|} + 4|\partial^R Q_n|\frac{|\partial^{d(n)}U_j|}{|Q_n|} + |\partial^{d(n)}U_j| + 4|U_j|(\varepsilon+\delta),
\end{aligned}
$$

which we plug in (6.39) and obtain

$$
\begin{aligned}
&D_\omega^{(1)}(j,n,R)\\
&\leq \frac{1}{|U_j|}\left(4|\partial^R Q_n|\frac{|U_j|}{|Q_n|} + \left(4\frac{|\partial^R Q_n|}{|Q_n|}+1\right)|\partial^{d(n)}U_j| + 4|U_j|(\varepsilon+\delta)\right)\\
&= 4\frac{|\partial^R Q_n|}{|Q_n|} + \left(4\frac{|\partial^R Q_n|}{|Q_n|}+1\right)\frac{|\partial^{d(n)}U_j|}{|U_j|} + 4(\varepsilon+\delta).
\end{aligned}
\tag{6.40}
$$

The combination of the estimates in (6.35), (6.37) and (6.40) gives

$$
\begin{aligned}
D_\omega(j,n,R) \leq 4\frac{|\partial^R Q_n|}{|Q_n|} &+ \left(4\frac{|\partial^R Q_n|}{|Q_n|}+1\right)\frac{|\partial^{d(n)}U_j|}{|U_j|}\\
&+ 5(\varepsilon+\delta) + \sum_{S\in\mathcal{S}(Q_n)}\left|\frac{\sharp_S(\Gamma_\omega[U_j])}{|U_j|} - \nu_S\right|,
\end{aligned}
$$

which proves the desired estimate on $D_\omega(j,n,R)$. ■

Proof of Theorem 6.16. For given $j, n \in \mathbb{N}$, $R \geq R_0$, $0 < \delta \leq \tau^{-1}$ and $\omega \in \Omega_j := \tilde{\Omega}(\delta, R, U_j) \cap \Omega_{\mathrm{fr}}$ we set

$$B_\omega(j,n,R,\delta) := 4\frac{|\partial^R Q_n|}{|Q_n|} + \left(4\frac{|\partial^R Q_n|}{|Q_n|} + 1\right)\frac{|\partial^{\mathrm{diam}(Q_n)} U_j|}{|U_j|} \\ + 5(\varepsilon + \delta) + \sum_{S \in \mathcal{S}(Q_n)} \left|\frac{\sharp_S(\Gamma_\omega[U_j])}{|U_j|} - \nu_S\right|,$$

i.e. the upper bound for $D_\omega(j,n,R)$ given in the previous lemma. In the following we explain how to choose the mutual dependencies of the parameters j, n, R, δ in order to obtain sufficient control on $B_\omega(j,n,R,\delta)$ and $\mathbb{P}(\Omega_j)$ and be able to conclude the statement of the theorem.

Since (Q_n) is a Følner sequence we have for all $R \in \mathbb{N}$

$$\lim_{n\to\infty} \frac{|\partial^R Q_n|}{|Q_n|} = 0.$$

The function $R(n)$ is defined inductively in the following way: for all $k \in \mathbb{N}$ we choose n_k to be the smallest natural number such that $|Q_n|^{-1}|\partial^k Q_n| \leq k^{-1}$ for all $n \geq n_k$. Now we set $R(n) = R_0$ for all $n < n_{R_0}$ and $R(n) = k$ for all $n_k \leq n < n_{k+1}$, $k \geq R_0$. This gives a function $n \mapsto R(n)$ satisfying $R(n) \geq R_0$ for all $n \in \mathbb{N}$ as well as

$$\lim_{n\to\infty} R(n) = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{|\partial^{R(n)} Q_n|}{|Q_n|} = 0.$$

Furthermore recall that $\varepsilon = \varepsilon(R) = \sum_{y \in G\setminus B_R} p(y)$, as in (6.13). Thus we have $\lim_{n\to\infty} \varepsilon(R(n)) = 0$. Setting $\delta(j) := (j^{1/4}\tau)^{-1}$ implies for fixed $n \in \mathbb{N}$

$$\lim_{j\to\infty} \delta(j) = 0 \quad \text{and} \quad \delta(j) \leq \frac{1}{\tau} \quad \text{as well as} \quad -\frac{\delta(j)^2|U_j|}{4} \leq -\frac{j^{1/2}}{4\tau^2} \tag{6.41}$$

for all $j \in \mathbb{N}$. Here we used $j \leq |U_j|$, which holds since (U_j) is strictly increasing. Now for $j, n \in \mathbb{N}$ Lemma 6.18 implies that

$$D_\omega(j,n) := D_\omega(j,n,R(n)) \leq B_\omega(j,n,R(n),\delta(j)) =: B_\omega(j,n)$$

holds for all $\omega \in \Omega_j := \tilde{\Omega}(\delta(j), R(n), U_j) \cap \Omega_{\mathrm{fr}}$. Note that $\mathbb{P}(\Omega_j) \geq 1 - \exp(-j^{1/2}/4\tau^2)$ by (6.41) and Corollary 6.11. Furthermore for each $\omega \in \Omega_j$ we have

$$\lim_{n\to\infty} \lim_{j\to\infty} B_\omega(j,n) = 0.$$

Given $j, n \in \mathbb{N}$ we set

$$E_j^{(n)} := \{\omega \in \Omega_{\mathrm{lf}} \cap \Omega_{\mathrm{fr}} \mid D_\omega(j,n) > B_\omega(j,n)\}.$$

Therefore $\mathbb{P}(E_j^{(n)}) \leq \exp(-j^{1/2}/4\tau^2)$ for all $j \in \mathbb{N}$ and hence we obtain $\sum_j \mathbb{P}(E_j^{(n)}) < \infty$. Applying Borel-Cantelli lemma leads to $\mathbb{P}(A^{(n)}) = 0$, where

$$A^{(n)} := \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j^{(n)} = \{E_j^{(n)} \text{ infinitely often }\}.$$

Thus, we get for all $n \in \mathbb{N}$:

$$\mathbb{P}\left(\left\{\omega \in \Omega_{\mathrm{lf}} \cap \Omega_{\mathrm{fr}} \;\middle|\; \lim_{j\to\infty} (D_\omega(j,n) - B_\omega(j,n)) \leq 0\right\}\right) = 1.$$

Hence, there exists a set $\tilde{\Omega} \subseteq \Omega_{\mathrm{lf}} \cap \Omega_{\mathrm{fr}}$ with $\mathbb{P}(\tilde{\Omega}) = 1$ such that for all $\omega \in \tilde{\Omega}$ we have

$$\lim_{n\to\infty} \lim_{j\to\infty} (D_\omega(j,n) - B_\omega(j,n)) \leq 0,$$

which implies by definition of $B_\omega(j,n)$:

$$\lim_{n\to\infty} \lim_{j\to\infty} D_\omega(j,n) = 0. \tag{6.42}$$

Let $\kappa > 0$ and $\omega \in \tilde{\Omega}$ arbitrary. There exists an integer $n_0 = n_0(\omega,\kappa)$ satisfying $\lim_{j\to\infty} D_\omega(j,n_0) \leq \kappa/8$, thus there exists $j_0 = j_0(\omega,\kappa) \in \mathbb{N}$ such that $D_\omega(j,n_0) \leq \kappa/4$ for all $j \geq j_0$. Using triangle inequality gives that for all $j, m \geq j_0$ we have

$$\left\| \frac{F_\omega^{R(n_0)}(U_j)}{|U_j|} - \frac{F_\omega^{R(n_0)}(U_m)}{|U_m|} \right\| \leq D_\omega(j,n_0) + D_\omega(m,n_0) \leq \frac{\kappa}{2}.$$

Furthermore, we use Lemma 2.25 to obtain that there exists a $j_1 = j_1(\kappa) \in \mathbb{N}$ such that

$$\left\| \frac{F_\omega(U_j)}{|U_j|} - \frac{F_\omega^{R(n_0)}(U_j)}{|U_j|} \right\| = \left\| \frac{\mathfrak{e}(\Delta^{(\omega)}[U_j]) - \mathfrak{e}(\Delta^{(\omega)}[U_j^{(R(n_0))}])}{|U_j|} \right\| \leq \frac{4|\partial^{R(n_0)} U_j|}{|U_j|} \leq \frac{\kappa}{4} \tag{6.43}$$

for all $j \geq j_1$. Now, the triangle inequality yields for all $j, m \geq \max\{j_0, j_1\}$:

$$\begin{aligned}
&\left\| \frac{F_\omega(U_j)}{|U_j|} - \frac{F_\omega(U_m)}{|U_m|} \right\| \\
&\leq \left\| \frac{F_\omega(U_j)}{|U_j|} - \frac{F_\omega^{R(n_0)}(U_j)}{|U_j|} \right\| + \left\| \frac{F_\omega^{R(n_0)}(U_j)}{|U_j|} - \frac{F_\omega^{R(n_0)}(U_m)}{|U_m|} \right\| \\
&\quad + \left\| \frac{F_\omega^{R(n_0)}(U_m)}{|U_m|} - \frac{F_\omega(U_m)}{|U_m|} \right\| \\
&\leq \frac{\kappa}{4} + \frac{\kappa}{2} + \frac{\kappa}{4} = \kappa,
\end{aligned}$$

which implies for all $\omega \in \tilde{\Omega}$ that $|U_j|^{-1} F_\omega(U_j)$ is a Cauchy sequence and hence convergent in the Banach space $B(\mathbb{R})$. We denote the limit function by $\mathfrak{I}$.

It remains to show that $\sum_{S \in \mathcal{S}(Q_n)} \nu_S \frac{\tilde{F}(S)}{|Q_n|}$ converges to the same limit. To this end, we fix $\omega \in \tilde{\Omega}$ and consider

$$\lim_{n\to\infty} \left\| \mathfrak{I} - \sum_{S \in \mathcal{S}(Q_n)} \nu_S \frac{\tilde{F}(S)}{|Q_n|} \right\| = \lim_{n\to\infty} \lim_{j\to\infty} \left\| \frac{F_\omega(U_j)}{|U_j|} - \sum_{S \in \mathcal{S}(Q_n)} \nu_S \frac{\tilde{F}(S)}{|Q_n|} \right\|.$$

Adding zeros leads to the inequality

$$\begin{aligned}
&\left\| \frac{F_\omega(U_j)}{|U_j|} - \sum_{S \in \mathcal{S}(Q_n)} \nu_S \frac{\tilde{F}(S)}{|Q_n|} \right\| \\
&\leq \left\| \frac{F_\omega(U_j)}{|U_j|} - \frac{F_\omega^{R(n)}(U_j)}{|U_j|} \right\| + \left\| \frac{F_\omega^{R(n)}(U_j)}{|U_j|} - \sum_{S \in \mathcal{S}(Q_n)} \nu_S \frac{\tilde{F}^{R(n)}(S)}{|Q_n|} \right\|
\end{aligned}$$

$$+\left\| \sum_{S\in\mathcal{S}(Q_n)} \nu_S \frac{\tilde{F}^{R(n)}(S)}{|Q_n|} - \sum_{S\in\mathcal{S}(Q_n)} \nu_S \frac{\tilde{F}(S)}{|Q_n|} \right\|,$$

which is valid for all $j, n \in \mathbb{N}$. Now we take $\lim_{n\to\infty}\lim_{j\to\infty}$ on both sides and obtain that the three summands on the right vanish. The first one is zero by an estimate as in (6.43). Applying (6.42) gives that the second summand vanishes. The third summand tends to zero since Lemma 2.25 yields

$$\begin{aligned}
&\left\| \sum_{S\in\mathcal{S}(Q_n)} \nu_S \frac{\tilde{F}^{R(n)}(S)}{|Q_n|} - \sum_{S\in\mathcal{S}(Q_n)} \nu_S \frac{\tilde{F}(S)}{|Q_n|} \right\| \\
&\leq \sum_{S\in\mathcal{S}(Q_n)} \nu_S \frac{\left\|\tilde{F}^{R(n)}(S) - \tilde{F}(S)\right\|}{|Q_n|} \leq \sum_{S\in\mathcal{S}(Q_n)} \nu_S \frac{4\left|\partial^{R(n)} Q_n\right|}{|Q_n|}
\end{aligned}$$

and for some fixed $y \in Q_n$

$$\begin{aligned}
\sum_{S\in\mathcal{S}(Q_n)} \nu_S &= \lim_{j\to\infty} \frac{1}{|U_j|} \sum_{S\in\mathcal{S}(Q_n)} |\{x \in G \mid V_S x \subseteq U_j,\ \Gamma_\omega[V_S x] = Sx\}| \\
&\leq \lim_{j\to\infty} \frac{1}{|U_j|} \sum_{S\in\mathcal{S}(Q_n)} |\{z \in U_j \mid x := y^{-1}z,\ \Gamma_\omega[V_S x] = Sx\}| \\
&= \lim_{j\to\infty} \frac{1}{|U_j|} \left| \dot{\bigcup_{S\in\mathcal{S}(Q_n)}} \{z \in U_j \mid x := y^{-1}z,\ \Gamma_\omega[V_S x] = Sx\} \right|.
\end{aligned}$$

This proves the claimed convergence for all $\omega \in \tilde{\Omega}$, since the last term can not exceed 1. Finally we need to show the independence of the specific choice of the sequences. Therefore let (U_j') and (Q_n') be two other Følner sequences satisfying (a) and (b), respectively. By Lemma 6.15 we know that the frequencies ν_S do not depend on the choice of the Følner sequence. Hence we can repeat the arguments of this proof once with the sequences (U_j') and (Q_n) and afterwords with (U_j) and (Q_m') to obtain

$$\lim_{j\to\infty} \frac{F_\omega(U_j')}{|U_j'|} = \lim_{n\to\infty} \sum_{S\in\mathcal{S}(Q_n)} \nu_S \frac{\tilde{F}(S)}{|Q_n|} = \mathfrak{I}$$

and

$$\mathfrak{I} = \lim_{j\to\infty} \frac{F_\omega(U_j)}{|U_j|} = \lim_{n\to\infty} \sum_{S\in\mathcal{S}(Q'_n)} \nu_S \frac{\tilde{F}(S)}{|Q'_n|}$$

almost surely. This finishes the proof. ■

6.2.4 Discontinuities

In this subsection we investigate the points of discontinuity of the integrated density of states. We firstly prove a criterion for the IDS to have a jump at $\lambda \in \mathbb{R}$. Afterwards we characterize the set of points of discontinuity as a large subset of the real axis. The results of this subsection are closely related to the ones in [Ves05], where the author studied discontinuities of the IDS for (short-range) percolation models on $\mathbb{Z}^d$.

Note that in this subsection we are always in the setting of Theorem 6.16. The next Theorem is well-known in similar situations, see Corollary 5.36. Here we complement ideas from Corollary 5.36 with the properties of our specific model to obtain an additional equivalence.

Theorem 6.19. *There exists a set $\tilde{\Omega} \subseteq \Omega$ of full measure such that for each $\omega \in \tilde{\Omega}$ and $\lambda \in \mathbb{R}$ the following assertions are equivalent:*

(a) λ is a point of discontinuity of $\mathfrak{I}$,

(b) there exists a finitely supported eigenfunction corresponding to λ,

(c) there exist infinitely many mutually orthogonal finitely supported eigenfunctions corresponding to λ.

Proof. Let (U_j) be a strictly increasing, tempered Følner sequence and $\tilde{\Omega} \subseteq \Omega_{\mathrm{fr}} \cap \Omega_{\mathrm{lf}}$ a set of full measure such that Theorem 6.16 holds for all $\omega \in \tilde{\Omega}$. This implies in particular that for an arbitrary graph $S \in \mathcal{S}$ and $\omega \in \tilde{\Omega}$ the frequency ν_S in Γ_ω along (U_j) exists. As p is assumed to be an element of $\ell^1(G,\mathbb{R})$ there exists $R \in \mathbb{N}$ such that $p(xy^{-1})$ is strictly smaller than 1 for all $x, y \in G$ satisfying $d(x,y) \geq R$. We fix this $R \in \mathbb{N}$ and some $\omega \in \tilde{\Omega}$.

Let λ be a point of discontinuity of $\mathfrak{I}$. Theorem 6.16 yields that $\mathfrak{e}(\Delta^{(\omega)}[U_j])/|U_j|$ approximates the IDS $\mathfrak{I}$ uniformly in the energy variable. Hence there exists a constant $c > 0$ such that for all $j \in \mathbb{N}$:

$$\begin{aligned}&\dim(\ker(\Delta^{(\omega)}[U_j] - \lambda))\\ &= \lim_{\varepsilon\to 0}\Big(\mathfrak{e}(\Delta^{(\omega)}[U_j])(\lambda+\varepsilon) - \mathfrak{e}(\Delta^{(\omega)}[U_j])(\lambda-\varepsilon)\Big) \geq c|U_j|.\end{aligned}$$

Since (U_j) is a Følner sequence, we have $\lim_{j\to\infty} |\partial^R_{\mathrm{int}} U_j|/|U_j| = 0$, which implies the existence of $k \in \mathbb{N}$ such that

$$\dim(\ker(\Delta^{(\omega)}[U_k] - \lambda)) \geq c|U_k| > |\partial^R_{\mathrm{int}} U_k| = \dim(\ell^2(\partial^R_{\mathrm{int}} U_k))$$

holds true. Lemma 5.32 yields that there exists an element $0 \neq u \in \ell^2(U_k)$ satisfying $(\Delta^{(\omega)}[U_k] - \lambda)u = 0$ and $u \equiv 0$ on $\partial^R_{\mathrm{int}} U_k$. Now we consider the subgraph

$$S := (V_S, E_S) := \Gamma_\omega[U_k]. \tag{6.44}$$

Lemma 6.15 proves that the frequency of R-isolated occurrences of S in Γ_ω along (U_j) is given by

$$\nu_{S,R} = \prod_{\{x,y\}\in E_S} p(xy^{-1})\cdot \prod_{\substack{\{x,y\}\notin E_S\\ x,y\in V_S}} (1-p(xy^{-1}))\cdot \prod_{\substack{\{x,y\}\in E, x\in V_S,\\ y\notin V_S, d(x,y)\geq R}} (1-p(xy^{-1})). \tag{6.45}$$

Here the first two products have to be non-zero as S is a restriction of Γ_ω. The positivity of the infinite product follows from the choice of R and the summability condition on p. This implies that there is an infinite set $M \subseteq G$ such that $\Gamma_\omega[U_k x]$ is an R-isolated copy of S for each $x \in M$. Furthermore there exists an infinite subset $M' \subseteq M$ such that $U_k x \cap U_k y = \emptyset$ for all $x, y \in M'$. For $x \in M'$ we define $u_x \in \ell^2(G)$ by setting

$$u_x(g) = \begin{cases} u(gx^{-1}) & g \in U_k x,\\ 0 & \text{otherwise.}\end{cases}$$

Then u_x, $x \in M'$ are mutually orthogonal, finitely supported eigenfunctions of $\Delta^{(\omega)}$ corresponding to λ. This proves that (a) implies (c).

Obviously (c) implies (b), thus it remains to show that given a finitely supported eigenfunction u corresponding to $\lambda \in \mathbb{R}$ the IDS is discontinuous at λ. To this end let $r > 0$ be large enough such that $\mathrm{spt}(u) \subseteq B_r$. As $\omega \in \Omega_{\mathrm{lf}}$ the graph Γ_ω is locally finite. Therefore we find $s > r$ such that there are no edges connecting the sets B_r and $G \setminus B_s$ in Γ_ω. Now we consider the graph $S = (V_S, E_S) := \Gamma_\omega[B_t]$, where $t := s + R$. As S is a restriction of Γ_ω the frequency $\nu_{S,R}$ of R-isolated occurrences of S in Γ_ω along (U_j) is strictly positive. Thus there exists a constant $c > 0$ such that $\sharp_{S,R}(\Gamma_\omega, U_j) \geq c|U_j|$ for j large enough.

For given $Q \in \mathcal{F}(G)$ each disjoint R-isolated copy of S in $\Gamma_\omega[Q]$ adds a dimension to the eigenspace of $p_Q \Delta^{(\omega)} i_Q$ corresponding to λ. Therefore we define $\dot{\sharp}_{S,R}(\Gamma_\omega, Q)$ to be the maximal number of disjoint and R-isolated occurrences of the subgraph S in $\Gamma_\omega[Q]$. It is easy to verify that in this situation the inequality $|B_{3t}|\dot{\sharp}_{S,R}(\Gamma_\omega, Q) \geq \sharp_{S,R}(\Gamma_\omega, Q)$ holds. For each $\varepsilon > 0$ we get

$$\begin{aligned}\frac{\mathfrak{e}(\Delta^{(\omega)}[Q])(\lambda - \varepsilon)}{|Q|} &\leq \frac{\mathfrak{e}(\Delta^{(\omega)}[Q])(\lambda + \varepsilon) - \dot{\sharp}_{S,R}(\Gamma_\omega, Q)}{|Q|} \\ &\leq \frac{\mathfrak{e}(\Delta^{(\omega)}[Q])(\lambda + \varepsilon)}{|Q|} - \frac{\sharp_{S,R}(\Gamma_\omega, Q)}{|B_{3t}||Q|}.\end{aligned}$$

Replacing Q by elements of the sequence (U_j) yields

$$\frac{\mathfrak{e}(\Delta^{(\omega)}[U_j])(\lambda + \varepsilon)}{|U_j|} - \frac{\mathfrak{e}(\Delta^{(\omega)}[U_j])(\lambda - \varepsilon)}{|U_j|} \geq \frac{\sharp_{S,R}(\Gamma_\omega, U_j)}{|B_{3t}||U_j|}.$$

We let j tend to infinity and obtain

$$\mathfrak{I}(\lambda + \varepsilon) - \mathfrak{I}(\lambda - \varepsilon) \geq \frac{\nu_{S,R}}{|B_{3t}|},$$

which proves that λ is a point of discontinuity of $\mathfrak{I}$. ∎

Next, we study the set of points of discontinuity, which obviously depends on the specific choice of the function $p \in \ell^1(G, \mathbb{R})$. Here we consider the case where the given function p satisfies not just (6.12) but even

$$0 < p(x) < 1 \qquad \text{and} \qquad p(x) = p(x^{-1}) \tag{6.46}$$

for all $x \in G$ and define the set

$$W = \{\lambda \in \mathbb{R} \mid \exists\, S \in \mathcal{S} \text{ with } \lambda \in \sigma(\Delta_S)\}.$$

Corollary 6.20. *Let $p \in \ell^1(G, \mathbb{R})$ satisfying (6.46) and the associated probability space $(\Omega, \mathcal{A}, \mathbb{P})$ be given. Then the set of points of discontinuity of the IDS $\mathfrak{I}$ is equal to W almost surely.*

Proof. Let $\tilde{\Omega} \subseteq \Omega_{\mathrm{fr}} \cap \Omega_{\mathrm{lf}}$ be a set of full measure such that Theorem 6.16 holds for all $\omega \in \tilde{\Omega}$ and choose some $\omega \in \tilde{\Omega}$.

Let λ be a point of discontinuity of $\mathfrak{I}$. By Theorem 6.19 there is a finitely supported eigenfunction u corresponding to λ. As in the proof of Theorem 6.19 we find $r > 0$ such that $\mathrm{spt}(u) \subseteq B_r$ and $s > r$ such that there are no edges in Γ_ω connecting B_r with $G \setminus B_s$. We set $S = (V_S, E_S) = \Gamma_\omega[B_s]$. Therefore, λ is an eigenvalue of Δ_S with eigenfunction $p_{V_S} u$.

Let λ be an element in W, i.e. there exists $S = (V_S, E_S) \in \mathcal{S}$ such that λ is an eigenvalue of the associated Laplacian Δ_S. Let u be an associated eigenfunction. By Lemma 6.15 the frequency $\nu_{S,1}$ is given by

$$\nu_{S,1} = \prod_{\{x,y\} \in E_S} p(xy^{-1}) \cdot \prod_{\substack{\{x,y\} \notin E_S \\ x,y \in V_S}} (1 - p(xy^{-1})) \cdot \prod_{\substack{\{x,y\} \in E, x \in V_S, \\ y \notin V_S, d(x,y) \geq 1}} (1 - p(xy^{-1})),$$

which is strictly positive by assumption on p. Thus there exists a $x \in G$ such that Sx is a 1-isolated copy of S in Γ_ω. Then $u' \in \ell^2(G)$ given by

$$u'(g) = \begin{cases} u(gx^{-1}) & \text{if } g \in V_S x, \\ 0 & \text{otherwise} \end{cases}$$

is a finitely supported eigenfunction of $\Delta^{(\omega)}$ corresponding to λ. By Theorem 6.19 this implies the discontinuity of $\mathfrak{I}$ at λ. ■

6.3 Random operators on general amenable groups

In this section we assume that G is an arbitrary finitely generated amenable group and consider random operators on G. In comparison with Section 6.2 the geometric setting is less restricted: we consider *all* amenable groups. Moreover, the operators under consideration are more general. Recall that in Section 6.2 we studied the spectral properties of the graph Laplacian. Here the operators are taken from a certain class of ergodic operators.

As before, the aim is to verify uniform existence of the integrated density of states. Here, we use that we already know that the approximants converge weakly to the spectral distribution function (Theorem 6.5) and upgrade this to obtain uniform convergence. Recall that weak convergence means pointwise convergence at all points of continuity of the limit function. Thus, to obtain uniform convergence one needs to control the approximations at the points of discontinuity of the IDS. This will be done in Theorem 6.25. For similar reasoning see [LV09] and also [MSY03].

In the special case, where $G = \mathbb{Z}^d$ and with slightly more restricted operators the results of this section are joint work with Slim Ayadi and Ivan Veselić, see [ASV13].

Let $\tilde{A} = (\tilde{A}^{(\omega)})_{\omega\in\Omega}$ be a symmetric and ergodic operator on the domain $C_\mathrm{c}(G)$ on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Here, as usual, we define $\tilde{a}^{(\omega)}(x, y) := \langle \delta_x, \tilde{A}^{(\omega)}\delta_y\rangle$. Furthermore, we assume

$$\sum_{x\in G} \mathbb{E}(|\tilde{a}(x, \mathrm{id})|^2) < \infty \quad \text{and} \quad \sum_{x\in G} \mathbb{P}(\tilde{a}(x, \mathrm{id}) \neq 0) < \infty. \tag{6.47}$$

By a calculation as in the proof of Lemma 4.8 we get that there exists a set Ω_lf, such that for all $\omega \in \Omega_\mathrm{lf}$ and all $y \in G$

$$|\{x \in G \mid \tilde{a}^{(\omega)}(x, y) \neq 0\}| < \infty.$$

Using this we show that the above operator is almost surely essentially self-adjoint.

Lemma 6.21. *There exists a set $\tilde{\Omega}$ of full measure, such that for all $\omega \in \tilde{\Omega}$ the operator $\tilde{A}^{(\omega)}$ is essentially self-adjoint.*

Proof. As an ergodic operator is always translation invariant in distribution we only need to show $\mathbb{E}(\|\tilde{A}\delta_\mathrm{id}\|_1^2) < \infty$, cf. Lemma 2.19. To this end, we define $N(\omega) := \{x \in G \mid \tilde{a}^{(\omega)}(x, \mathrm{id}) \neq 0\}$ and calculate for $\omega \in \Omega_\mathrm{lf}$:

$$\|\tilde{A}^{(\omega)}\delta_\mathrm{id}\|_1^2 = \Bigg(\sum_{x\in N(\omega)} |\tilde{a}^{(\omega)}(x, \mathrm{id})|\Bigg)^2 \leq |N(\omega)| \sum_{x\in N(\omega)} |\tilde{a}^{(\omega)}(x, \mathrm{id})|^2.$$

Now, set $N_x(\omega) := |\{y \in G \setminus \{x\} \mid a^{(\omega)}(y, \mathrm{id}) \neq 0\}|$ to obtain

$$\begin{aligned}\mathbb{E}(\|\tilde{A}\delta_{\mathrm{id}}\|_1^2) &\leq \sum_{x \in G} \mathbb{E}(|N||\tilde{a}(x, \mathrm{id})|^2) \\ &\leq \sum_{x \in G} \mathbb{E}(N_x + 1)\mathbb{E}(|\tilde{a}(x, \mathrm{id})|^2) \\ &\leq \mathbb{E}(N+1) \sum_{x \in G} \mathbb{E}(|\tilde{a}(x, \mathrm{id})|^2) < \infty\end{aligned}$$

where the finiteness follows from (6.47). ■

Let $\tilde{\Omega}$ be given by the previous Lemma and let for all $\omega \in \tilde{\Omega}$ the operator $\bar{A}^{(\omega)} : D(\bar{A}^{(\omega)}) \to \ell^2(G)$ be the unique self-adjoint extension of $\tilde{A}^{(\omega)}$. Then we set for $\omega \in \Omega$

$$A^{(\omega)} := \begin{cases} \bar{A}^{(\omega)} & \text{if } \omega \in \tilde{\Omega}, \\ \mathrm{Id} & \text{otherwise} \end{cases} \qquad \text{and} \qquad a^{(\omega)}(x, y) := \left\langle \delta_x, A^{(\omega)} \delta_y \right\rangle. \tag{6.48}$$

Then $A = (A^{(\omega)})_{\omega \in \Omega}$ is an ergodic proper random operator which is self-adjoint for *all* realizations ω. Let (Q_j) be a tempered Følner sequence. Set as in (6.2) for $j \in \mathbb{N}$ and $\omega \in \Omega$

$$A_j^{(\omega)} := A^{(\omega)}[Q_j] := p_{Q_j} A^{(\omega)} i_{Q_j} \tag{6.49}$$

as well as for $\lambda \in \mathbb{R}$

$$\mathfrak{n}_j^{(\omega)} := \mathfrak{n}(A_j^{(\omega)}), \quad \mathfrak{N}^{(\omega)}(\lambda) := \langle \delta_{\mathrm{id}}, E_\lambda^{(\omega)} \delta_{\mathrm{id}} \rangle \quad \text{and} \quad \bar{\mathfrak{N}}(\lambda) := \mathbb{E}(\mathfrak{N}(\lambda)). \tag{6.50}$$

As before, the function $\bar{\mathfrak{N}}$ is called *spectral distribution function* of the operator. We obtain by Theorem 6.5 that $\bar{\mathfrak{N}}$ is almost surely the weak limit of $\mathfrak{n}_j^{(\omega)}$. The aim of this section is to upgrade this result to uniform convergence. Therefore we need to obtain control over the convergence of the distribution functions also at the points of discontinuity. This will be done in the next subsection.

6.3.1 Control of the jumps and uniform convergence

The aim of this subsection is to control the convergence at the jumps of the limit function $\bar{\mathfrak{N}}$ given in (6.50). The first result pointing in this direction is the next lemma, which is valid for *all* $\lambda \in \mathbb{R}$.

Here we introduce the following notion for an operator $A = (A^{(\omega)})_{\omega\in\Omega}$ as given in (6.48): for an interval $I \subseteq \mathbb{R}$ (which might consist of only one point) and $\omega \in \Omega$ we denote by $E_I(A^{(\omega)})$ the spectral projection of the operator $A^{(\omega)}$ on the interval I. This gives in particular $E_{(-\infty,\lambda]}(A^{(\omega)}) = E_\lambda^{(\omega)}$, where $E_\lambda^{(\omega)}$ is the spectral projection as used for instance in (4.9) and (6.6). Given $\omega \in \Omega$, let $\mathrm{Eig}(A^{(\omega)},\lambda)$ denote the *eigenspace of* $A^{(\omega)}$ corresponding to the value λ, which could possibly be empty if λ is not an eigenvalue.

Lemma 6.22. *Let A be the random operator defined in* (6.48) *and let (Q_j) be a tempered Følner sequence. Then there exists a set $\tilde\Omega \subseteq \Omega$ of full measure such that for all $\omega \in \tilde\Omega$ and all $\lambda \in \mathbb{R}$ we have*

$$\lim_{j\to\infty} \frac{\mathrm{Tr}(\chi_{Q_j} E_{\{\lambda\}}(A^{(\omega)}))}{|Q_j|} = \mathbb{E}(\langle \delta_{\mathrm{id}}, E_{\{\lambda\}}(A)\delta_{\mathrm{id}}\rangle).$$

Proof. Let $\lambda \in \mathbb{R}$ be fixed. By definition of the trace we have for each $\omega \in \Omega$:

$$\begin{aligned}\mathrm{Tr}(\chi_{Q_j} E_{\{\lambda\}}(A^{(\omega)})) &= \sum_{x\in G} \left\langle \delta_x, \chi_{Q_j} E_{\{\lambda\}}(A^{(\omega)})\delta_x \right\rangle \\ &= \sum_{x\in Q_j} \left\langle \delta_x, E_{\{\lambda\}}(A^{(\omega)})\delta_x \right\rangle. \end{aligned} \tag{6.51}$$

Given $z \in G$, we have

$$\phi \in \mathrm{Eig}(A^{(\omega)},\lambda) \quad \text{if and only if} \quad U_z\phi \in \mathrm{Eig}(U_z A^{(\omega)} U_z^{-1}, \lambda). \tag{6.52}$$

Here U_z is given as in (2.15). As A is ergodic there exists a set Ω' of full measure such that for each $\omega \in \Omega'$ we have $U_z A^{(\omega)} U_z^{-1} = A^{(T_z\omega)}$. Now we show for all $\omega \in \Omega'$

$$\left\langle \delta_{\mathrm{id}}, E_{\{\lambda\}}(A^{(T_z\omega)})\delta_{\mathrm{id}} \right\rangle = \left\langle \delta_z, E_{\{\lambda\}}(A^{(\omega)})\delta_z \right\rangle. \tag{6.53}$$

To this end, let $\delta'_{\mathrm{id}} \in \mathrm{Eig}(A^{(T_z\omega)},\lambda)$ and $\delta''_{\mathrm{id}} \in \mathrm{Eig}(A^{(T_z\omega)},\lambda)^\perp$ such that $\delta_{\mathrm{id}} = \delta'_{\mathrm{id}} + \delta''_{\mathrm{id}}$. Then we obtain

$$\begin{aligned}\langle \delta_{\mathrm{id}}, \delta'_{\mathrm{id}}\rangle &= \left\langle \delta_{\mathrm{id}}, E_{\{\lambda\}}(A^{(T_z\omega)})\delta'_{\mathrm{id}} \right\rangle + \left\langle \delta_{\mathrm{id}}, E_{\{\lambda\}}(A^{(T_z\omega)})\delta''_{\mathrm{id}} \right\rangle \\ &= \left\langle \delta_{\mathrm{id}}, E_{\{\lambda\}}(A^{(T_z\omega)})\delta_{\mathrm{id}} \right\rangle\end{aligned}$$

and with the above equivalence (6.52) we get for $\omega \in \Omega'$

$$\begin{aligned}
&\langle \delta_{\mathrm{id}}, \delta'_{\mathrm{id}} \rangle = \left\langle U_z^{-1}(\delta_{\mathrm{id}}), U_z^{-1}(\delta'_{\mathrm{id}}) \right\rangle \\
&= \left\langle U_z^{-1}(\delta_{\mathrm{id}}), E_{\{\lambda\}}(A^{(\omega)}) U_z^{-1}(\delta'_{\mathrm{id}}) \right\rangle + \left\langle U_z^{-1}(\delta_{\mathrm{id}}), E_{\{\lambda\}}(A^{(\omega)}) U_z^{-1}(\delta''_{\mathrm{id}}) \right\rangle \\
&= \left\langle \delta_z, E_{\{\lambda\}}(A^{(\omega)}) \delta_z \right\rangle ,
\end{aligned}$$

which implies (6.53). Applying (6.51) and (6.53) leads for all $\omega \in \Omega'$ to

$$\frac{\operatorname{Tr}(\chi_{Q_j} E_{\{\lambda\}}(A^{(\omega)}))}{|Q_j|} = \frac{1}{|Q_j|} \sum_{x \in Q_j} \left\langle \delta_{\mathrm{id}}, E_{\{\lambda\}}(A^{(T_x(\omega))}) \delta_{\mathrm{id}} \right\rangle .$$

Finally, we use Theorem 2.12 to obtain the existence of a set $\tilde{\Omega} \subseteq \Omega$ of measure one such that for each $\omega \in \tilde{\Omega}$ we have

$$\lim_{j \to \infty} \frac{\operatorname{Tr}(\chi_{Q_j} E_{\{\lambda\}}(A^{(\omega)}))}{|Q_j|} = \int_\Omega \left\langle \delta_{\mathrm{id}}, E_{\{\lambda\}}(A^{(\omega)}) \delta_{\mathrm{id}} \right\rangle d\mathbb{P}(\omega). \qquad \blacksquare$$

The following easy fact is taken from [LV09]

Lemma 6.23. *Let $r > 0$, $Q \subseteq G$ and $U \subseteq \ell^2(Q)$ be given and denote by U_r the subspace of U consisting of all functions which vanish on $\partial^r(Q)$. Then*

$$0 \leq \dim(U) - \dim(U_r) \leq |\partial^r_{\mathrm{int}}(Q)|.$$

Proof. Let $P : U \to \ell^2(\partial^r_{\mathrm{int}}(Q))$ be the natural projection with $(P\phi)(x) = \phi(x)$ for all $x \in \partial^r_{\mathrm{int}}(Q)$. Then we have

$$0 \leq \dim(U) - \dim(\ker P) = \dim(\operatorname{Ran} P) \leq |\partial^r_{\mathrm{int}}(Q)|,$$

which proves the claim as $\ker P = U_r$. $\blacksquare$

For given $\omega \in \Omega$, $R \in \mathbb{N}$ and $Q \subseteq G$ finite, let $L^{(\omega)}(R, Q)$ be given by

$$L^{(\omega)}(R,Q) := \Big| \Big\{ \{x,y\} \in E_{\mathrm{co}} \mid a^{(\omega)}(x,y) \neq 0, d(x,y) \geq R \text{ and } \{x,y\} \cap Q \neq \emptyset \Big\} \Big|. \qquad (6.54)$$

Note that this quantity is well-defined as $a^{(\omega)}(x,y)=0$ if and only if $a^{(\omega)}(y,x)=0$. Therefore $L^{(\omega)}(R,Q)$ counts the interactions of the elements of Q of length not less than R.

In the next lines we fix dependencies between certain parameters which appear in this section. We adjust these dependencies in a such a way that the approximation error in Theorem 6.25 vanishes. Let (Q_j) be a Følner sequence. Using a diagonal sequence, we choose a function $R:\mathbb{N}\to\mathbb{N}$ such that

$$\lim_{j\to\infty} R(j)=\infty \quad \text{and} \quad \lim_{j\to\infty}\frac{|\partial^{R(j)}Q_j|}{|Q_j|}=0 \tag{6.55}$$

and set

$$L_j^{(\omega)} := L^{(\omega)}(R(j),Q_j). \tag{6.56}$$

Additionally, we set for $R\geq 0$

$$\varepsilon_R := \sum_{x\in G, d(\mathrm{id},x)\geq R} \mathbb{P}(a^{(\omega)}(\mathrm{id},x)\neq 0))$$

and for $j\in\mathbb{N}_0$

$$\varepsilon(j):=\varepsilon_{R(j)} \quad \text{as well as} \quad \delta(j):=(j)^{-1/4}. \tag{6.57}$$

Note that by condition (6.47) and by the definition of $R(j)$ we have that

$$\lim_{j\to\infty}\varepsilon(j)=\lim_{j\to\infty}\delta(j)=0.$$

The next result estimates (independently of R) the probability that $L^{(\omega)}(R,Q)$ takes "large" values. The first part follows directly from the calculations in Section 6.2.1 and Corollary 6.11.

Lemma 6.24. *Let G be a finitely generated amenable group, (Q_j) a strictly increasing Følner sequence and the operator A be given as in* (6.48). *Then the following holds:*

(a) There exist constants $R_0\in\mathbb{N}$ and $\bar\delta>0$, such that for all $0<\delta<\bar\delta$, all $R\geq R_0$, and all finite $Q\subseteq G$:

$$\mathbb{P}\left(L^{(\omega)}(R,Q)\geq |Q|(\varepsilon_R+\delta)\right)\leq \exp\left(-\frac{\delta^2|Q|}{4}\right).$$

(b) *Let $R : \mathbb{N} \to \mathbb{N}$ be as in* (6.55)*. Then there exists a set $\tilde{\Omega} \subseteq \Omega$ of full measure such that for each $\omega \in \tilde{\Omega}$ there exists $j_0(\omega)$ satisfying for $j \geq j_0(\omega)$:*

$$L_j^{(\omega)} \leq |Q_j|(\varepsilon(j) + \delta(j)).$$

Proof. The proof of statement (a) carries over from the proof of Corollary 6.11. Let us prove part (b). Therefore, consider the events

$$A_j := \Big\{ \omega \in \Omega \mid L_j^{(\omega)} > |Q_j|(\varepsilon(j) + \delta(j)) \Big\}.$$

Then part (a) shows that for j large enough we have

$$\mathbb{P}(A_j) \leq \exp\left(-\delta(j)^2 |Q_j|/4\right) \leq \exp\bigl(-\sqrt{j}/4\bigr),$$

where the second inequality uses that (Q_j) is strictly increasing. This clearly gives $\sum_{j \in \mathbb{N}} \mathbb{P}(A_j) < \infty$. By the Borel-Cantelli Lemma we have

$$\mathbb{P}\Big(\limsup_{j \to \infty} A_j\Big) = 0,$$

which implies the claim of part (b). ∎

We use Lemmas 6.22, 6.23 and 6.24 to obtain a result similar as Lemma 6.2 in [LV09]. However, technically this is the point where we go far beyond the calculations of [LV09]. The reason is that long-range interactions force us to implement complex arguments to estimate dimensions of certain ℓ^2-subspaces. This was not necessary in [LV09] as the authors thereof dealt with finite hopping range operators.

Theorem 6.25. *Let G be a finitely generated amenable group, let A be given as in* (6.48) *and let (Q_j) be a strictly increasing, tempered Følner sequence. Furthermore let $\rho_j^{(\omega)}$ be the probability measure associated to the distribution function $\mathfrak{n}_j^{(\omega)}$. Then there exists a set $\tilde{\Omega} \subseteq \Omega$ of full measure such that for all $\omega \in \tilde{\Omega}$ and all $\lambda \in \mathbb{R}$ we have*

$$\lim_{j \to \infty} \rho_j^{(\omega)}(\{\lambda\}) = \mathbb{E}\bigl(\langle \delta_{\mathrm{id}}, E_{\{\lambda\}}(A)\delta_{\mathrm{id}} \rangle\bigr).$$

Proof. During the proof, we will rather use the measure $\bar{\rho}_j := |Q_j| \cdot \rho_j$, which is the measure associated to the cumulative eigenvalue counting function. Let $\tilde{\Omega} \subseteq \Omega$ be a set of full measure such that the results of Lemma 6.22 and of Lemma 6.24 (b) hold for all $\omega \in \tilde{\Omega}$. We fix some $\omega \in \tilde{\Omega}$ and $\lambda \in \mathbb{R}$. With the function $R : \mathbb{N} \to \mathbb{N}$ given in (6.55) we set

$$\begin{aligned} V_j^{(\omega)} &:= \left\{ v \in \ell^2(G) \mid (A^{(\omega)} - \lambda)v = 0 \text{ and } \operatorname{spt} v \subseteq Q_j^{(R(j))} \right\}, \\ D_j^{(\omega)} &:= \dim V_j^{(\omega)}. \end{aligned}$$

Note that $V_j^{(\omega)}$ consists of the elements $i_{Q_j} v$, where $v \in \ell^2(Q_j)$ satisfying $v \equiv 0$ on $\partial_{\text{int}}^{R(j)} Q_j$,

$$(p_{Q_j} A^{(\omega)} i_{Q_j} - \lambda)v = 0 \quad \text{and} \quad \sum_{y \in Q_j^{(R(j))}} (a^{(\omega)}(x,y) - \lambda \delta_x(y))v(y) = 0 \tag{6.58}$$

for all $x \notin Q_j$ with $x \overset{\omega}{\sim} Q_j^{(R(j))}$. Note that here we write $x \overset{\omega}{\sim} Q_j^{(R(j))}$ if one can find $y \in Q_j^{(R(j))}$ with $a^{(\omega)}(x,y) \neq 0$.

We consider the following difference

$$\begin{aligned} |\bar{\rho}_j^{(\omega)}(\{\lambda\}) - \operatorname{Tr}(\chi_{Q_j} E_{\{\lambda\}}(A^{(\omega)}))| \leq{}& |\bar{\rho}_j^{(\omega)}(\{\lambda\}) - D_j^{(\omega)}| \\ &+ |D_j^{(\omega)} - \operatorname{Tr}(\chi_{Q_j} E_{\{\lambda\}}(A^{(\omega)}))| \end{aligned} \tag{6.59}$$

and estimate the two summands on the right hand side separately. Let us estimate the first one. Therefore, consider the sets

$$U_j^{(\omega)} := \left\{ u \in \ell^2(Q_j) \mid (p_{Q_j} A^{(\omega)} i_{Q_j} - \lambda)u = 0 \right\}$$

and

$$U_{j,R}^{(\omega)} = \left\{ u \in U_j \mid u \equiv 0 \text{ on } Q_j \setminus Q_j^{(R(j))} \right\}.$$

Then clearly, $\bar{\rho}_j^{(\omega)}(\{\lambda\}) = \dim(U_j^{(\omega)}) \geq \dim(V_j^{(\omega)})$ and

$$\begin{aligned} \dim(U_{j,R}^{(\omega)}) - \dim(V_j^{(\omega)}) &\leq |\{y \notin Q_j \mid y \overset{\omega}{\sim} Q_j^{(R(j))}\}| \\ &\leq L^{(\omega)}(R(j), Q_j) = L_j^{(\omega)}, \end{aligned} \tag{6.60}$$

where we used the definition (6.54). The application of Lemma 6.23 gives

$$
\begin{aligned}
0 \le \bar\rho_j^{(\omega)}(\{\lambda\}) - D_j^{(\omega)} &= \dim(U_j^{(\omega)}) - \dim(V_j^{(\omega)}) \\
&\le \dim(U_j^{(\omega)}) - \dim(U_{j,R}^{(\omega)}) + L_j^{(\omega)} \\
&\le |\partial_{\mathrm{int}}^{R(j)} Q_j| + L_j^{(\omega)}. \qquad (6.61)
\end{aligned}
$$

Now we estimate the second summand in (6.59). Therefore let v_i, $i = 1, \dots, D_j^{(\omega)}$ be an orthonormal basis (ONB) of $V_j^{(\omega)}$ and let $\tilde v_i$, $i \in I$ be an ONB of the orthogonal complement of $V_j^{(\omega)}$ in the space $\mathrm{Eig}(A^{(\omega)}, \lambda)$. Furthermore, let $\bar v_i$, $i \in J$ be an ONB of $\mathrm{Eig}(A^{(\omega)}, \lambda)^\perp$. Then we have

$$
\begin{aligned}
&\mathrm{Tr}(\chi_{Q_j} E_{\{\lambda\}}(A^{(\omega)})) \\
&= \sum_{i=1}^{D_j^{(\omega)}} \left\langle \chi_{Q_j} E_{\{\lambda\}}(A^{(\omega)}) v_i, v_i \right\rangle + \sum_{i\in I} \left\langle \chi_{Q_j} E_{\{\lambda\}}(A^{(\omega)}) \tilde v_i, \tilde v_i \right\rangle \\
&\quad + \sum_{i \in J} \left\langle \chi_{Q_j} E_{\{\lambda\}}(A^{(\omega)}) \bar v_i, \bar v_i \right\rangle \\
&= \sum_{i=1}^{D_j^{(\omega)}} \langle v_i, v_i \rangle + \sum_{i \in I} \left\langle \chi_{Q_j} \tilde v_i, \chi_{Q_j} \tilde v_i \right\rangle ,
\end{aligned}
$$

which gives $D_j^{(\omega)} \le \mathrm{Tr}(\chi_{Q_j} E_{\{\lambda\}}(A^{(\omega)}))$. Next, let u_i, $i \in I$ be an ONB of

$$\bar U_j^{(\omega)} := \mathrm{Ran}(\chi_{Q_j} E_{\{\lambda\}}(A^{(\omega)}))$$

and $\tilde u_k$, $k \in J$ be an ONB of $(\bar U_j^{(\omega)})^\perp$. Then, using Cauchy-Schwarz inequality, we obtain

$$\left\langle \chi_{Q_j} E_{\{\lambda\}}(A^{(\omega)}) u_i, u_i \right\rangle \le \|u_i\| = 1$$

and

$$\left\langle \chi_{Q_j} E_{\{\lambda\}}(A^{(\omega)}) \tilde u_j, \tilde u_j \right\rangle = 0$$

for all $i \in I$ and all $j \in J$. This gives

$$\begin{aligned} D_j^{(\omega)} &\leq \operatorname{Tr}(\chi_{Q_j} E_{\{\lambda\}}(A^{(\omega)})) \\ &= \sum_{i \in I} \left\langle \chi_{Q_j} E_{\{\lambda\}}(A^{(\omega)}) u_i, u_i \right\rangle + \sum_{j \in J} \left\langle \chi_{Q_j} E_{\{\lambda\}}(A^{(\omega)}) \tilde{u}_j, \tilde{u}_j \right\rangle \\ &\leq \dim(\bar{U}_j^{(\omega)}), \end{aligned} \tag{6.62}$$

where we used $\dim(\bar{U}_j) = |I|$. As before we denote by $\bar{U}_{j,R}^{(\omega)}$ the subset of $\bar{U}_j^{(\omega)}$ consisting of those elements in $\bar{U}_j^{(\omega)}$ which vanish outside of $Q_j^{(R)}$. Therefore, we have

$$\bar{U}_{j,R}^{(\omega)} = \left\{ \chi_{Q_j} v \mid v \in \ell^2(G), (A^{(\omega)} - \lambda) v = 0, v \equiv 0 \text{ on } \partial_{\text{int}}^{R(j)} Q_j \right\}. \tag{6.63}$$

In the next step we define a set $\bar{\bar{U}}_{j,R}^{(\omega)} \supseteq \bar{U}_{j,R}^{(\omega)}$ by dropping conditions in (6.63), in the following way

$$\begin{aligned} \bar{\bar{U}}_{j,R}^{(\omega)} := \Bigg\{ \chi_{Q_j} v \Bigg| & v \in \ell^2(G), v \equiv 0 \text{ on } \partial_{\text{int}}^{R(j)} Q_j, \\ & \sum_{y \in G} (a^{(\omega)}(x,y) - \lambda \delta_x(y)) v(y) = 0 \text{ for all } x \in Z_j^{(\omega)} \Bigg\} \\ = \Bigg\{ \chi_{Q_j} v \Bigg| & v \in \ell^2(G), v \equiv 0 \text{ on } \partial_{\text{int}}^{R(j)} Q_j, \\ & \sum_{y \in Q_j} (a^{(\omega)}(x,y) - \lambda \delta_x(y)) v(y) = 0 \text{ for all } x \in Z_j^{(\omega)} \Bigg\}, \end{aligned}$$

where

$$Z_j^{(\omega)} = Q_j^{(R(j))} \setminus \{x \in Q_j^{(R(j))} \mid x \overset{\omega}{\sim} (G \setminus Q_j)\}.$$

Here we used that for all $x \in Z_j^{(\omega)}$ and $y \in G \setminus Q_j$ we have the equality $a^{(\omega)}(x,y) = 0$.

Comparing this representation of $\bar{\bar{U}}_{j,R}^{(\omega)}$ with the representation of $V_j^{(\omega)}$ in (6.58), we realize that they differ in at most $2L_j^{(\omega)} + |\partial_{\text{int}}^{R(j)} Q_j|$

conditions. As each of these conditions may change the dimension at most by one, we get

$$\dim(\bar{U}_{j,R}^{(\omega)}) \leq \dim(\bar{\bar{U}}_{j,R}^{(\omega)}) \leq D_j^{(\omega)} + 2L_j^{(\omega)} + |\partial_{\mathrm{int}}^{R(j)} Q_j|. \tag{6.64}$$

Applying (6.62), Lemma 6.23 and (6.64) gives

$$\begin{aligned} 0 &\leq \mathrm{Tr}(\chi_{Q_j} E_{\{\lambda\}}(A^{(\omega)})) - D_j^{(\omega)} \\ &\leq \dim(\bar{U}_j^{(\omega)}) - D_j^{(\omega)} \\ &\leq \dim(\bar{U}_{j,R}^{(\omega)}) - D_j^{(\omega)} + |\partial_{\mathrm{int}}^{R(j)} Q_j| \leq 2|\partial_{\mathrm{int}}^{R(j)} Q_j| + 2L_j^{(\omega)}. \end{aligned} \tag{6.65}$$

In the last step we apply Lemma 6.22, then we combine the estimates for the two summands in (6.59) given in (6.61) and (6.65) and finally use part (b) of Lemma 6.24 to obtain

$$\begin{aligned} &\lim_{j\to\infty} \frac{\bar{\rho}_j^{(\omega)}(\{\lambda\})}{|Q_j|} - \mathbb{E}\Big(\Big\langle \delta_0, E_{\{\lambda\}}(A^{(\omega)})\delta_0 \Big\rangle\Big) \\ &= \lim_{j\to\infty} \frac{|\bar{\rho}_j^{(\omega)}(\{\lambda\}) - \mathrm{Tr}(\chi_{Q_j} E_{\{\lambda\}}(A^{(\omega)}))|}{|Q_j|} \\ &\leq \lim_{j\to\infty} \frac{3|\partial_{\mathrm{int}}^{R(j)} Q_j| + 3L_j^{(\omega)}}{|Q_j|} \\ &\leq 3 \lim_{j\to\infty} \left(\frac{|\partial_{\mathrm{int}}^{R(j)} Q_j|}{|Q_j|} + \varepsilon(j) + \delta(j) \right) = 0. \end{aligned}$$

Here we used the definitions of $R(j)$, $\varepsilon(j)$ and $\delta(j)$ in (6.55) and (6.57). ∎

Remark 6.26. (a) Let us stress the fact that proof of Theorem 6.25 does not contain any probabilistic argument. We show the claimed convergence for any fixed choice of $\lambda \in \mathbb{R}$ and $\omega \in \tilde{\Omega}$, where $\tilde{\Omega}$ is a set given rather explicitly by Lemmas 6.22 and 6.24.

(b) Furthermore the proof gives an explicit error term on finite scales. To be precise, we have for any $j \in \mathbb{N}$, $\lambda \in \mathbb{R}$ and $\omega \in \tilde{\Omega}$

$$|\bar{\rho}_j^{(\omega)}(\{\lambda\}) - \mathrm{Tr}(\chi_{Q_j} E_{\{\lambda\}}(A^{(\omega)}))| \leq 3|\partial_{\mathrm{int}}^{R(j)} Q_j| + 3L_j^{(\omega)}$$

where $L_j^{(\omega)} = L^{(\omega)}(R(j), Q_j)$ as in (6.56).

The following result is essentially standard and has been used in the present context already in [LV09]. It shows that weak convergence of measures plus convergence of the measures at each point implies uniform convergence.

Lemma 6.27. *Let ρ be a probability measure on $\mathbb{R}$ and let (ρ_j) be a sequence of bounded measures on $\mathbb{R}$ which converge weakly to ρ and fulfill*

$$\lim_{j\to\infty} \rho_j(\{\lambda\}) = \rho(\{\lambda\})$$

for all $\lambda \in \mathbb{R}$. Then the distribution functions $F_j : \mathbb{R} \to \mathbb{R}$, $F_j(\lambda) := \rho_j((-\infty, \lambda])$ converge to the distribution function $F : \mathbb{R} \to \mathbb{R}$, $F(\lambda) := \rho((-\infty, \lambda])$ with respect to the supremum norm.

The proof of the main theorem is now basically a combination of the previous results. It shows that the integrated density of states exists uniformly and the validity of a Pastur-Shubin trace formula.

Theorem 6.28. *Let G be a finitely generated amenable group, let A be given as in* (6.48) *and let (Q_j) be a strictly increasing, tempered Følner sequence. Furthermore let $\mathfrak{n}_j^{(\omega)}$ and $\bar{\mathfrak{N}}$ be given as in* (6.50)*. Then there exists a set $\tilde{\Omega} \subseteq \Omega$ of full measure such that for all $\omega \in \tilde{\Omega}$ we have*

$$\mathfrak{n}_j^{(\omega)} \to \bar{\mathfrak{N}} \quad \text{as} \quad j \to \infty$$

with respect to the supremum norm.

Proof. Let $\rho, \rho_j^{(\omega)} : \mathcal{B}(\mathbb{R}) \to [0,1]$ be the measures associated to the distribution functions $\bar{\mathfrak{N}}$ respectively $\mathfrak{n}_j^{(\omega)}$. Then obviously ρ is a probability measure and the measures $\rho_j^{(\omega)}$ are bounded. As shown in Theorem 6.5, there exists a set $\Omega_1 \subseteq \Omega$ with $\mathbb{P}(\Omega_1) = 1$ such that for all $\omega \in \Omega_1$ the measure ρ is the weak limit of $\rho_j^{(\omega)}$. Furthermore we have by Theorem 6.25 a set $\Omega_2 \subseteq \Omega$ with $\mathbb{P}(\Omega_2) = 1$ such that for all $\omega \in \Omega_2$ and all $\lambda \in \mathbb{R}$ one has $\lim_{j\to\infty} \rho_j^{(\omega)}(\{\lambda\}) = \rho(\{\lambda\})$. Therefore, Lemma 6.27 yields the uniform convergence of the distribution functions for all $\omega \in \Omega_1 \cap \Omega_2$. ∎

6.3.2 Special case: randomly weighted Laplacians

In this subsection we consider a special case of the setting in Section 6.3. In fact we show that the results therein apply to randomly weighted Laplacians on a long-range percolation graph. For simplification we restrict ourselves to the case where $G = \mathbb{Z}^d$. However, the results of this subsection are independent of that choice and randomly weighted Laplacians can be defined on general amenable groups, in a completely analogous way.

Let Γ be the $\mathbb{Z}^d$ lattice and denote by $d^\Gamma : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{N}_0$ the graph distance in the lattice or equivalently the ℓ^1-distance in $\mathbb{Z}^d$. In the language of finitely generated groups, Γ is the Cayley graph of $\mathbb{Z}^d$ with respect to the standard generators and $d^\Gamma = d_S$ is the word metric. With this metric the R-boundary of a set $\Lambda \subseteq \mathbb{Z}^d$ is as before given by

$$\partial^R_{\mathrm{int}}\Lambda = \{x \in \Lambda \mid d(x,y) \leq R \text{ for some } y \in \mathbb{Z}^d \setminus \Lambda\}.$$

Furthermore, we let $E_{\mathrm{co}} := \{\{x,y\} \subseteq \mathbb{Z}^d \mid x, y \in \mathbb{Z}^d\}$ be the set of all subsets of $\mathbb{Z}^d$ containing either one or two elements. As in previous sections we interpret the set E_{co} as the edge set of the complete undirected graph over $\mathbb{Z}^d$, containing loops at each vertex.

The probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is given in the following way. The sample space is $\Omega = \prod_{e \in E_{\mathrm{co}}}(\mathbb{R} \times \{0,1\})$ and we denote the elements in Ω by $\omega = (\omega'_e, \omega''_e)_{e \in E_{\mathrm{co}}}$. The appropriate sigma-algebra is $\mathcal{A} = \bigotimes_{e \in E_{\mathrm{co}}}(\mathcal{B}(\mathbb{R}) \otimes \mathcal{P}(\{0,1\}))$. In order to define a measure on this space, we fix some $p \in \ell^1(\mathbb{Z}^d, \mathbb{R})$ with

$$0 \leq p(x) = p(-x) \leq 1 \tag{6.66}$$

for all $x \in \mathbb{Z}^d$. Let for each $z \in \mathbb{Z}^d$, ν_z be a Bernoulli measure with parameter $p(z)$. Besides this, let $v \in \mathbb{R}$ be some constant and μ_z, $z \in \mathbb{Z}^d$ be probability measures on $\mathbb{R}$ such that for all $z \in \mathbb{Z}^d$

$$\int_{\mathbb{R}} x^2 d\mu_z(x) \leq v^2. \tag{6.67}$$

We set

$$\mathbb{P} := \bigotimes_{\{x,y\} \in E_{\mathrm{co}}} (\mu_{x-y} \otimes \nu_{x-y}).$$

Remark 6.29. The sigma-algebra $\mathcal{A}$ is generated by the cylinder sets $\mathcal{Y}$, which are given the following way

$$\mathcal{Y} = \big\{ Z(A_{e_1}, B_{e_1}, \dots, A_{e_k}, B_{e_k}) \mid k \in \mathbb{N}, e_i \in E_{\text{co}}, A_{e_i} \in \mathcal{B}(\mathbb{R}), \\ B_{e_i} \in \mathcal{P}(\{0,1\}) \text{ for } i = 1, \dots, k \big\},$$

where

$$Z(A_{e_1}, B_{e_1}, \dots, A_{e_k}, B_{e_k}) \\ = \big\{ \omega \in \Omega \mid \omega'_{e_i} \in A_{e_i}, \omega''_{e_i} \in B_{e_i} \text{ for } i = 1, \dots, k \big\}.$$

Now for each $\omega = (\omega'_e, \omega''_e)_{e \in E_{\text{co}}}$ and $e \in E_{\text{co}}$ we set $X_e(\omega) := \omega'_e$ and $Y_e(\omega) := \omega''_e$. This procedure gives independent random variables X_e, Y_e, $e \in E_{\text{co}}$ satisfying $\mathbb{P}(X_e \in B) = \mu_e(B)$ as well as $\mathbb{P}(Y_e = 1) = \nu_e(\{1\}) = p(x - y)$ for arbitrary $e = \{x, y\} \in E_{\text{co}}$ and $B \in \mathcal{B}(\mathbb{R})$. Furthermore, by (6.67) we have for each $e \in E_{\text{co}}$

$$\mathbb{E}(|X_e|) \leq v^2 + 1.$$

These random variables induce for each $\omega \in \Omega$ a graph $\Gamma_\omega = (\mathbb{Z}^d, E_\omega)$ with weighted edges. Here $\mathbb{Z}^d$ is the vertex set and E_ω is the subset of E_{co}, where an edge $e \in E_{\text{co}}$ is an element of E_ω if and only if $Y_e(\omega) = 1$. In this case, one can think of $X_e(\omega)$ as the weight of the edge e.

As in the proof of Lemma 4.8 the assumption $p \in \ell^1(\mathbb{Z}^d, \mathbb{R})$ implies that Γ_ω is for almost all ω locally finite, i.e. each vertex is incident to only finitely many edges in Γ_ω. We denote by Ω_{lf} the set of measure one, such that for each ω of this set the graph is locally finite.

Given $\gamma \in \mathbb{Z}^d$, let us define translations $T_\gamma : \Omega \to \Omega$ by

$$T_\gamma(\omega) = T_\gamma((\omega'_e, \omega''_e)_{e \in E_{\text{co}}}) = (\omega'_{e+\gamma}, \omega''_{e+\gamma})_{e \in E_{\text{co}}},$$

where for $e = \{g, h\} \in E_{\text{co}}$ we mean by $e + \gamma$ the element $\{g + \gamma, h + \gamma\} \in E_{\text{co}}$. For $\gamma \in \mathbb{Z}^d$ and $B \in \mathcal{A}$ we denote the image and the preimage of B under T_γ by

$$T_\gamma(B) = \{T_\gamma(\omega) \in \Omega \mid \omega \in B\}$$

and

$$T_\gamma^{-1}(B) = \{\omega \in \Omega \mid T_\gamma(\omega) \in B\}.$$

Note that for $B \in \mathcal{A}$ we have $T_\gamma^{-1}(B) = T_{-\gamma}(B)$. By definition, the mapping $\gamma \mapsto T_\gamma$ maps each element of $\mathbb{Z}^d$ into the space of automorphisms on $(\Omega, \mathcal{A}, \mathbb{P})$. We denote the family $(T_\gamma)_{\gamma\in\mathbb{Z}^d}$ by T.

The next result shows that the action T of $\mathbb{Z}^d$ on $(\Omega, \mathcal{A}, \mathbb{P})$ is measure preserving and ergodic. This is rather elementary, but we do not know an explicit reference in the literature, so we include a proof for completeness sake.

Lemma 6.30. *T is a measure preserving, ergodic left-action on $(\Omega, \mathcal{A}, \mathbb{P})$.*

Proof. For an edge $e = \{g, h\} \in E_{\mathrm{co}}$, vertices $x, y \in \mathbb{Z}^d$ and $\omega \in \Omega$ we have $T_0(\omega) = \omega$ and

$$T_{x+y}(\omega) = (\omega'_{e+x+y}, \omega''_{e+x+y})_{e\in E_{\mathrm{co}}} = T_x(T_y(\omega)),$$

which shows that T is a left action of $\mathbb{Z}^d$ on Ω.

By definition of $\mathbb{P}$ and the random variables X_e and Y_e we have $\mathbb{P}(X_e \in B) = \mathbb{P}(X_{e+\gamma} \in B)$ as well as $\mathbb{P}(Y_e = 1) = \mathbb{P}(Y_{e+\gamma} = 1)$ for any $e \in E_{\mathrm{co}}$, $\gamma \in \mathbb{Z}^d$ and $B \in \mathcal{B}(\mathbb{R})$. Furthermore, as T_γ is a translation, $\mathbb{P}(Z) = \mathbb{P}(T_\gamma(Z))$ holds obviously for any $\gamma \in \mathbb{Z}^d$ and any cylinder set $Z \in \mathcal{Y}$, which implies the same property for any set $B \in \mathcal{A}$, cf. Remark 6.29.

To prove ergodicity let $B \in \mathcal{A}$ with $B = T_\gamma(B)$ for all $\gamma \in \mathbb{Z}^d$ and $\mathbb{P}(B) > 0$ be given. We need to show that this implies $\mathbb{P}(B) = 1$. In the following we apply the approximation lemma for measures, which belongs to the entourage of Carathéodory's extension theorem, cf. e.g. Theorem 1.65 in [Kle08]. Let $\varepsilon > 0$. As $B \in \mathcal{A} = \sigma(\mathcal{Y})$ and $\mathcal{Y}$ is a semiring we can find cylinder sets $Z_1, \dots, Z_n \in \mathcal{Y}$ such that

$$\mathbb{P}(B \triangle Z) < \varepsilon \quad \text{where} \quad Z := \bigcup_{k=1}^{n} Z_k.$$

This gives

$$\mathbb{P}(B)^2 - 2\mathbb{P}(B)\varepsilon \le \mathbb{P}(Z)^2 \le \mathbb{P}(B)^2 + 2\mathbb{P}(B)\varepsilon + \varepsilon^2. \tag{6.68}$$

Furthermore, we have for any $\gamma \in \mathbb{Z}^d$

$$\mathbb{P}\left((Z \setminus B) \cap T_\gamma Z\right) \le \varepsilon \quad \text{and} \quad \mathbb{P}\left(B \cap (T_\gamma Z \setminus T_\gamma B)\right) \le \varepsilon.$$

Thus we obtain

$$\mathbb{P}\left(B \cap T_\gamma Z\right) \le \mathbb{P}\left(B \cap \left(T_\gamma B \cup \left(T_\gamma Z \setminus T_\gamma B\right)\right)\right) \le \mathbb{P}\left(B \cap T_\gamma B\right) + \varepsilon$$

and hence

$$\begin{aligned}\mathbb{P}(Z \cap T_\gamma Z) &\le \mathbb{P}\left(\left(B \cup (Z \setminus B)\right) \cap T_\gamma Z\right)\\ &\le \mathbb{P}\left(B \cap T_\gamma Z\right) + \mathbb{P}\left(\left(Z \setminus B\right) \cap T_\gamma Z\right)\\ &\le \mathbb{P}\left(B \cap T_\gamma B\right) + 2\varepsilon.\end{aligned}$$

By symmetry, we get for all $\gamma \in \mathbb{Z}^d$

$$\mathbb{P}(B \cap T_\gamma B) - 2\varepsilon \le \mathbb{P}(Z \cap T_\gamma Z) \le \mathbb{P}(B \cap T_\gamma B) + 2\varepsilon.$$

The T-invariance of B implies

$$\mathbb{P}(B) - 2\varepsilon \le \mathbb{P}(Z \cap T_\gamma Z) \le \mathbb{P}(B) + 2\varepsilon. \tag{6.69}$$

As Z is a finite union of cylinder sets, it does only depend on finitely many edges. Hence, there exists an element $h \in \mathbb{Z}^d$ such that Z and $T_h Z$ are independent, which gives

$$\mathbb{P}(Z \cap T_h Z) = \mathbb{P}(Z)\mathbb{P}(T_h Z) = \mathbb{P}(Z)^2,$$

since T is measure preserving. This implies together with (6.68) and (6.69)

$$\mathbb{P}(B) - 2\mathbb{P}(B)\varepsilon - \varepsilon^2 - 2\varepsilon \le \mathbb{P}(B)^2 \le \mathbb{P}(B)$$

and dividing by $\mathbb{P}(B) > 0$ leads to

$$1 - 2\varepsilon - \frac{\varepsilon^2 + 2\varepsilon}{\mathbb{P}(B)} \le \mathbb{P}(B) \le 1.$$

As this holds true for arbitrary $\varepsilon > 0$ we get $\mathbb{P}(B) = 1$. ∎

Let us define the operator which is in the center of the investigations of this subsection. In order to do so, we follow the procedure of the beginning of Chapter 4. This means, we firstly define an operator $\tilde{A}$ by its matrix elements and afterwards show that this operator is almost surely essentially self-adjoint. Finally, we define the desired operator (for all possible ω) as the self-adjoint extension of $\tilde{A}$ and as the identity elsewhere.

Let $\alpha \in \mathbb{R}$ be some fixed number. In the following, we use the random variables X_e, Y_e, $e \in E_{\text{co}}$ to define a random operator $\tilde{A}^{(\omega)} = \tilde{A}^{(\omega)}_\alpha = (\tilde{A}^{(\omega)})_{\omega\in\Omega} = (\tilde{A}^{(\omega)}_\alpha)_{\omega\in\Omega}$. To this end, set

$$\tilde{a}^{(\omega)}_\alpha(x,y) := \begin{cases} X_{\{x,y\}}(\omega)Y_{\{x,y\}}(\omega) & \text{if } x \neq y, \\ X_{\{x\}}(\omega)Y_{\{x\}}(\omega) - \alpha \sum\limits_{z\neq x} X_{\{x,z\}}(\omega)Y_{\{x,z\}}(\omega) & \text{if } x = y, \end{cases}$$

and $\tilde{a}^{(\omega)}(x,y) := \tilde{a}^{(\omega)}_\alpha(x,y)$. Moreover, define for $\phi \in C_{\text{c}}(\mathbb{Z}^d)$ and $x \in \mathbb{Z}^d$:

$$(\tilde{A}^{(\omega)}\phi)(x) := (\tilde{A}^{(\omega)}_\alpha\phi)(x) := \sum_{y\in\mathbb{Z}^d} \tilde{a}^{(\omega)}(x,y)\phi(y). \tag{6.70}$$

Let $\phi \in C_{\text{c}}(\mathbb{Z}^d)$, $\omega \in \Omega_{\text{lf}}$, then $\tilde{A}^{(\omega)}\phi \in \ell^1(\mathbb{Z}^d) \subseteq \ell^2(\mathbb{Z}^d)$. To see this, we set $M := \operatorname{spt}\phi$, $m := \max_{x\in A}|\phi(x)|$ and

$$N_y(\omega) := \{x \in \mathbb{Z}^d \setminus \{y\} \mid \{x,y\} \in E_\omega\}$$

to estimate

$$\begin{aligned}\sum_{x\in\mathbb{Z}^d}\left|\sum_{y\in\mathbb{Z}^d} \tilde{a}^{(\omega)}(x,y)\phi(y)\right| &\le \sum_{x\in\mathbb{Z}^d}\sum_{y\in M}\left|\tilde{a}^{(\omega)}(x,y)\right||\phi(y)| \\ &\le m\sum_{y\in M}\sum_{x\in\mathbb{Z}^d}\left|\tilde{a}^{(\omega)}(x,y)\right|\end{aligned}$$

and

$$\begin{aligned}\sum_{x\in\mathbb{Z}^d}\left|\tilde{a}^{(\omega)}(x,y)\right| &\le |\tilde{a}^{(\omega)}(y,y)| + \sum_{\substack{x\in N_y(\omega)\\ x\neq y}}\left|\tilde{a}^{(\omega)}(x,y)\right| \\ &\le |X_{\{y\}}(\omega)| + (1+|\alpha|)\sum_{\substack{x\in N_y(\omega)\\ x\neq y}}\left|\tilde{X}_{\{x,y\}}(\omega)\right| < \infty.\end{aligned}$$

Note that here we used that $N_y(\omega)$ is finite, as $\omega \in \Omega_{\text{lf}}$ and the underlying graph Γ_ω is locally finite. Therefore, the mapping $\tilde{A}$:

$\Omega \to L(\ell^2(\mathbb{Z}^d))$, $\omega \mapsto \tilde{A}^{(\omega)}$ is a random operator on the domain $C_c(\mathbb{Z}^d)$. Here the measurability of $\tilde{A}$ can be shown as in Lemma 4.1. Moreover, it is easy to see that

$$(\tilde{A}^{(\omega)}\phi)(x) = \sum_{\substack{y \neq x \\ \{x,y\} \in E_\omega}} (\phi(y) - \alpha\phi(x))\, X_{\{x,y\}}(\omega) + \phi(x) X_{\{x\}}(\omega). \tag{6.71}$$

Remark 6.31. The operator $\tilde{A}^{(\omega)}$ depends on the choice of $\alpha \in \mathbb{R}$ and and the involved random variables. Later we will define the self-adjoint extension $A^{(\omega)}$ of this operator. Depending on α and the value $p(0)$ (since $p(0)$ determines the distribution of random variables $Y_{\{x\}}$, $x \in \mathbb{Z}^d$), we have in particular the following special cases for $A^{(\omega)}$:

- if $\alpha = 1$ and $p(0) = 0$, then $A^{(\omega)}$ is the randomly weighted Laplacian on the graph Γ_ω,
- if $\alpha = 1$ and $p(0) > 0$, then $A^{(\omega)}$ is the randomly weighted Laplacian on the graph Γ_ω plus a random diagonal,
- if $\alpha = 0$ and $p(0) > 0$, then $A^{(\omega)}$ is the randomly weighted adjacency operator of Γ_ω plus a random diagonal,
- if $\alpha = 0$ and $p(0) = 0$, then $A^{(\omega)}$ is the randomly weighted adjacency operator of Γ_ω with zeros on the diagonal.

The diagonal elements which appear if $p(0) > 0$, can be interpreted either as random weights on the loops, or as a random potential. For values $\alpha \in (0, 1)$ the operator can be seen as an interpolation between the adjacency operator and the Laplacian or the Schrödinger operator of the graph Γ_ω, respectively.

As in the situation for general groups, we define $U_\gamma : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ by setting for $\phi = (\phi(x))_{x\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$:

$$U_\gamma((\phi(x))_{x\in\mathbb{Z}^d}) := (\phi(x+\gamma))_{x\in\mathbb{Z}^d}.$$

For $x, y, \gamma \in \mathbb{Z}^d$ with $x \neq y$ and $\omega = (\omega'_e, \omega''_e)_{e\in E_{\mathrm{co}}}$ we set $s := \{x, y\}$

and have

$$\begin{aligned}\tilde{a}^{(T_\gamma(\omega))}(x,y) &= X_s(T_\gamma(\omega))Y_s(T_\gamma(\omega)) \\ &= X_s((\omega'_{e+\gamma},\omega''_{e+\gamma})_{e\in E_{\mathrm{co}}})Y_s((\omega'_{e+\gamma},\omega''_{e+\gamma})_{e\in E_{\mathrm{co}}}) \\ &= \omega'_{s+\gamma}\cdot\omega''_{s+\gamma} \\ &= \omega'_{\{x+\gamma,y+\gamma\}}\cdot\omega''_{\{x+\gamma,y+\gamma\}} \\ &= X_{\{x+\gamma,y+\gamma\}}(\omega)\cdot Y_{\{x+\gamma,y+\gamma\}}(\omega) = \tilde{a}^{(\omega)}(x+\gamma,y+\gamma).\end{aligned}$$

Furthermore, we obtain for the diagonal elements

$$\begin{aligned}&\tilde{a}^{(T_\gamma(\omega))}(x,x) \\ &= X_{\{x\}}(T_\gamma(\omega))Y_{\{x\}}(T_\gamma(\omega)) - \alpha\sum_{z\neq x} X_{\{x,z\}}(T_\gamma(\omega))Y_{\{x,z\}}(T_\gamma(\omega)) \\ &= X_{\{x+\gamma\}}(\omega)Y_{\{x+\gamma\}}(\omega) - \alpha\sum_{z\neq x} X_{\{x+\gamma,z+\gamma\}}(\omega)Y_{\{x+\gamma,z+\gamma\}}(\omega) \\ &= \tilde{a}^{(\omega)}(x+\gamma,x+\gamma).\end{aligned}$$

This yields

$$\tilde{A}^{(T_\gamma(\omega))} = U_\gamma\tilde{A}^{(\omega)}U_\gamma^{-1}. \tag{6.72}$$

The next lemma establishes the assumptions on the operator made in Section 6.3. We obtain essential self-adjointness and ergodicity. Furthermore we show that we can define an operator A as in (6.48).

Lemma 6.32. *Let $(\Omega,\mathcal{A},\mathbb{P})$ and the random operator $\tilde{A}$ be given as above. Then $\tilde{A}$ is a symmetric and ergodic random operator on the domain $C_{\mathrm{c}}(\mathbb{Z}^d)$ and we have*

$$\sum_{x\in\mathbb{Z}^d}\mathbb{E}(|\tilde{a}(x,0)|^2)<\infty \quad \textit{and} \quad \sum_{x\in\mathbb{Z}^d}\mathbb{P}(\tilde{a}(x,0)\neq 0)<\infty. \tag{6.73}$$

Proof. We have already seen that $\tilde{A}$ is a random operator on the domain $C_{\mathrm{c}}(\mathbb{Z}^d)$. Beside this, $\tilde{A}$ is obviously symmetric. The ergodicity of $\tilde{A}$ follows directly from Lemma 6.30 and equation (6.72). Furthermore, we calculate

$$\sum_{x\in\mathbb{Z}^d\setminus\{0\}}\mathbb{P}(\tilde{a}(x,0)\neq 0)\leq\sum_{x\in\mathbb{Z}^d\setminus\{0\}}\mathbb{P}(Y_{\{x,0\}}=1)\leq\|p\|_1<\infty.$$

It remains to consider the first expression in (6.73). We first study the diagonal term for some $\omega \in \Omega_{\mathrm{lf}}$:

$$\begin{aligned}|\tilde{a}^{(\omega)}(0,0)|^2 &= \Big(X_{\{0\}}(\omega)Y_{\{0\}}(\omega) - \alpha \sum_{x\in\mathbb{Z}^d\setminus\{0\}} X_{\{x,0\}}(\omega)Y_{\{x,0\}}(\omega)\Big)^2 \\ &\le (|\alpha|+1)^2\Big(\sum_{x\in N(\omega)}|X_{\{x,0\}}(\omega)|\Big)^2 \\ &\le (|\alpha|+1)^2|N(\omega)|\sum_{x\in N(\omega)}|X_{\{x,0\}}(\omega)|^2.\end{aligned}$$

Here we used again the notation $N(\omega) = \{x \in \mathbb{Z}^d \mid Y_{\{x,0\}}(\omega) = 1\}$. Moreover, we define $N_x(\omega) := |\{y \in \mathbb{Z}^d \setminus \{x\} \mid Y_{\{y,0\}}(\omega) = 1\}|$. We obtain

$$|N(\omega)|\sum_{x\in N(\omega)}|X_{\{x,0\}}(\omega)|^2 \le \sum_{x\in\mathbb{Z}^d}|X_{\{x,0\}}(\omega)|^2 Y_{\{x,0\}}(\omega)(N_x(\omega)+1)$$

and taking the expectation leads to

$$\mathbb{E}(|\tilde{a}(0,0)|^2) \le (|\alpha|+1)^2 v^2\|p\|_1(1+\|p\|_1) < \infty,$$

where v is the constant from (6.67). Using this we finally get

$$\begin{aligned}\mathbb{E}\Big(\sum_{x\in\mathbb{Z}^d}|\tilde{a}(x,0)|^2\Big) &\le \mathbb{E}(|\tilde{a}(0,0)|^2) + \mathbb{E}\Big(\sum_{x\in\mathbb{Z}^d\setminus\{0\}}|\tilde{a}(x,0)|^2\Big) \\ &\le \mathbb{E}(|\tilde{a}(0,0)|^2) + \mathbb{E}\Big(\sum_{x\in\mathbb{Z}^d\setminus\{0\}}|X_{\{x,0\}}|^2 Y_{\{x,0\}}\Big) \\ &\le (|\alpha|+1)^2 v^2\|p\|_1(1+\|p\|_1) + v^2\|p\|_1 < \infty,\end{aligned}$$

which finishes the proof. ∎

The previous Lemma shows that the assumptions (6.47) are satisfied for the operator $\tilde{A}$. Thus, Lemma 6.21 yields that there exists a set $\tilde{\Omega}$ of full measure such that $\tilde{A}^{(\omega)}$ is essentially self-adjoint for all $\omega \in \tilde{\Omega}$. For each such ω we denote the self-adjoint extension of $\tilde{A}^{(\omega)}$ by $\bar{A}^{(\omega)}$.

As in (6.48) we define the random operator $A = (A^{(\omega)})$ by setting for $\omega \in \Omega$:

$$A^{(\omega)} := \begin{cases} \bar{A}^{(\omega)} & \text{if } \omega \in \tilde{\Omega}, \\ \mathrm{Id} & \text{otherwise.} \end{cases}$$

We can choose an appropriate Følner sequence by setting for $n \in \mathbb{N}$:

$$\Lambda_n := ([0, n) \cap \mathbb{Z})^d \tag{6.74}$$

Then, it is easy to check that (Λ_n) is tempered and strictly increasing. We define for each $n \in \mathbb{N}$ the restriction

$$A_n^{(\omega)} := p_{\Lambda_n} A^{(\omega)} i_{\Lambda_n} \tag{6.75}$$

and for $\lambda \in \mathbb{R}$

$$\mathfrak{n}_n^{(\omega)} := \mathfrak{n}(A_n^{(\omega)}), \quad \mathfrak{N}^{(\omega)}(\lambda) := \langle \delta_0, E_\lambda^{(\omega)} \delta_0 \rangle \quad \text{and} \quad \bar{\mathfrak{N}}(\lambda) := \mathbb{E}(\mathfrak{N}(\lambda)), \tag{6.76}$$

as we did it in (6.50). The function $\bar{\mathfrak{N}}$ is called *spectral distribution function* of the operator.

Corollary 6.33. *Let $A = (A^{(\omega)})$, $A_n = (A_n^{(\omega)})$, $\mathfrak{n}_n$ and $\bar{\mathfrak{N}}$ be given as above. Then for almost all $\omega \in \Omega$:*

$$\lim_{n \to \infty} \|\mathfrak{n}_n^{(\omega)} - \bar{\mathfrak{N}}\|_\infty = 0.$$

Proof. This result follows directly from Theorem 6.28. ■

Thus, we also obtained uniform existence of the integrated density of states as well as the validity of the Pastur-Shubin trace formula.

Appendix: Tiling results for amenable groups

For the sake of the reader we provide here proofs of the tiling theorems of [PS12], which we stated in the Section 5.2.1. They are crucial for the understanding of the underlying decomposition approach, leading to the ergodic theorem in Section 5.2.2. These tiling results constitute a topic of importance by themselves, but not being in the center of interest of this book, this topic is separated from the main part. A second reason for locating these elaborations in the appendix is, that these theorems will also appear in the PhD thesis [Pog14] of Felix Pogorzelski, who contributed many ideas, as we will explain in the following (at the appropriate position) in detail.

The ideas we use here are based on the seminal work [OW87] of Ornstein and Weiss. We start with the two lemmas, which are minor adaptions of results, which have already been proven in [OW87, Section I.3].

Lemma A.1. *Let G be a finitely generated group, $\delta > 0$ and* $\mathrm{id} \in K \in \mathcal{F}(G)$. *Besides this, let $T \in \mathcal{F}(G)$ be (K, δ)-invariant. Then for the set*

$$S := \{g \in G \mid Kg \subseteq T\},$$

the following holds true:

(i) $|S| \geq (1 - \delta)|T|$,

(ii) $\sum_{c \in S} \mathbf{1}_{Kc}(g) \leq |K|$ *for all* $g \in G$.

Proof. The proof of (i) follows from the fact $S = T \setminus \partial_K(T)$ and $\mathrm{id} \in K$. In order to verify (ii) let $g \in G$ be arbitrary. We use that $g \in Kc$ if and only if $c \in K^{-1}g$ to obtain

$$\sum_{c \in S} \mathbf{1}_{Kc}(g) = \sum_{c \in S} \mathbf{1}_{K^{-1}g}(c) = |K^{-1}g \cap S| \leq |K|. \qquad \blacksquare$$

Lemma A.2. *Let G be a finitely generated group and let $K, S, T \in \mathcal{F}(G)$ be non-empty sets, where* $\mathrm{id} \in K$ *and* $|S| \geq (1-\delta)|T|$ *for some* $0 < \delta < 1$*. Then for a given set $A \in \mathcal{F}(G)$ there is some $c \in S$ such that*

$$|Kc \cap A| \leq \frac{|A|\,|K|}{|T|(1-\delta)}. \tag{A.1}$$

Proof. Let $A \in \mathcal{F}(G)$ be given. We prove the Lemma by contradiction. To this end, assume that there is no $c \in S$ such that (A.1) is satisfied. Then we get

$$\sum_{c \in S} |Kc \cap A| > \frac{|A|\,|K|\,|S|}{|T|(1-\delta)} \geq |A|\,|K|. \tag{A.2}$$

However, we also obtain

$$\sum_{c \in S} |Kc \cap A| = \sum_{a \in A} \sum_{c \in S} \mathbf{1}_K(ac^{-1}) \leq \sum_{a \in A} \sum_{g \in G} \mathbf{1}_K(g) = |A|\,|K|,$$

which contradicts Inequality (A.2). Therefore, we find $c \in S$ such that (A.1) holds. ∎

The following two lemmas are preliminaries for our main tiling theorems, namely Theorem 5.20 and Theorem 5.22. The first of these two lemmas says that that given $\varepsilon > 0$ and given some (KK^{-1}, δ)-invariant set $T \subseteq G$ (and under certain additional assumptions), we can cover a portion of T, which lies between $\varepsilon - \delta$ and $\varepsilon + \delta$ by ε-disjoint translates of K. The lower bound was already known in [OW87], whereas the idea to put an upper bound is due to Felix Pogorzelski. A detailed look in the proofs shows that A.3 is a major ingredient for Lemma A.4. In the second lemma we show that under additional assumptions we can achieve that the part of T which remains to be covered by translates still obeys a certain invariance property. This makes it possible to apply this result inductively in order to obtain the announced Theorem 5.20. Another feature which is developed within these Lemmas is that the tiles keep certain invariance properties with respect to a given set, cf. property (iv) of Definition 5.17. The implementation of this idea was supported by private communication with Benjamin Weiss.

We we will use the notion of maximal ε-disjointness. Let (P) be a property which a subset of group G can obey, let I be some index set, $J \subseteq I$ and $(K_i)_{i\in I}$ a family of subsets of G. The family $(K_i)_{i\in J}$ is called *maximal ε-disjoint* with property (P), if $(K_i)_{i\in J}$ is ε-disjoint and each K_i satisfies (P), however for each $j \in I \setminus J$ such that K_j satisfies (P), the family $(K_i)_{i\in J\cup j}$ is no longer ε-disjoint. A family of *maximal disjoint* sets with property (P) is defined analogously. In our examples the property (P) will be "being a translate of a certain set" or/and "being a subset of a certain set". We use for instance the term maximal ε-disjoint family of translates of K contained in T, where $K, T \subseteq G$.

For the next Lemma, recall the notion of a (B,ζ)-good small ε-quasi tiling in Definition 5.17.

Lemma A.3. *Let G be a finitely generated group, $0 < \varepsilon, \delta < 1/2$ and $0 < \zeta \le \delta/2$. Furthermore let $T, K, B \in \mathcal{F}(G)$ be sets such that T is (KK^{-1}, δ)-invariant, K is (B, ζ^2)-invariant and* $\mathrm{id} \in K \cap B$. *Then there exists a center set $C \in \mathcal{F}(G)$ such that K together with C is a $(B, 4\zeta)$-good small ε-quasi tiling of T with accuracy δ.*

Proof. We start the proof with a simple calculation to estimate the fraction $|K|/|T|$. For each $g \in \partial_K(T)$ and $t \in K$ we have $tg \in \partial_{KK^{-1}}(T)$, which immediately gives $|K| \le |\partial_{KK^{-1}}(T)|$. This implies

$$\frac{|K|}{|T|} \le \frac{|\partial_{KK^{-1}}(T)|}{|T|} < \delta, \tag{A.3}$$

as T is (KK^{-1}, δ)-invariant.

Now we formulate the following claim: If $C \in \mathcal{F}(G)$ is a set such that the conditions (i), (ii) and (iv) of Definition 5.17 are satisfied and additionally

$$\left|\bigcup_{c\in C} Kc\right| < \varepsilon(1 - 2\delta)|T|,$$

then there exists some $\tilde{c} \in T$ such that conditions (i), (ii) and (iv) of Definition 5.17 still hold for $C \cup \{\tilde{c}\}$.

We postpone the proof of the claim and assume for the moment that it holds. We proceed here inductively starting with a set C such that only (iii) of Definition 5.17 is violated. Then we use the claim to add elements to C, such that in the end all conditions are satisfied with this new set. To this end, we start with some maximal disjoint family $(Kc)_{c\in C}$ of translates of K contained in T with $|C||K| \leq (\varepsilon+\delta)|T|$, which is possible by (A.3). Besides this, set $K^{(c)} := K$, $c \in C$. Then obviously (i), (ii) and (iv) of Definition 5.17 hold. If

$$\left|\bigcup_{c\in C} Kc\right| = |C||K| \geq \varepsilon(1-2\delta)|T|,$$

then we are done with the proof since $\varepsilon \leq 1/2$. Otherwise we apply the claim and get some $\tilde{c} \in T$ such that conditions (i), (ii) and (iv) of the Definition are still fulfilled for $C \cup \{\tilde{c}\}$. By (A.3) we obtain

$$\left|\bigcup_{c\in C\cup\tilde{c}} Kc\right| \leq \varepsilon(1-2\delta)|T| + \delta|T| \leq (\varepsilon+\delta)|T|.$$

If now the first inequality in condition (iii) of Definition 5.17 is satisfied for $C \cup \tilde{c}$ as well, then we are done, if not, we apply the claim again. This procedure will end after finitely many steps since T contains only finitely many elements and after each iteration we cover at least $(1-\varepsilon)|K|$ more than before. Thus, it remains to prove the claim.

Let $C \in \mathcal{F}$ be such that conditions (i), (ii) and (iv) of Definition 5.17 are satisfied and additionally $|KC| < \varepsilon(1-2\delta)|T|$. We define the sets

$$S := \{g \in T \mid Kg \subseteq T\}$$

and

$$U := \{g \in S \mid |Kg \cap \partial_B(KC)| \leq \zeta|K|\}.$$

As T is (KK^{-1}, δ)-invariant, it is also (K, δ)-invariant, which gives together with Lemma A.1 that $|S| \geq (1-\delta)|T|$. We use this to obtain

$$\begin{aligned}|T \setminus U| \leq |T \setminus S| + |S \setminus U| \leq \delta|T| + \sum_{g\in S} \mathbf{1}_{S\setminus U}(g) \\ \leq \delta|T| + \sum_{g\in S} \frac{|Kg \cap \partial_B(KC)|}{\zeta|K|}.\end{aligned}$$

A closer look on the last term gives with the application of Lemma A.1 part (ii)

$$\sum_{g\in S} |Kg \cap \partial_B(KC)| \le \sum_{g\in S} \sum_{h\in\partial_B(KC)} \mathbf{1}_{Kg}(h) \le |\partial_B(KC)||K|.$$

The ε-disjointness yields

$$(1-\varepsilon)|C||K| \le |KC| \le \varepsilon(1-2\delta)|T|,$$

which immediately implies, using the upper bounds for ε and δ, that $|K||C| \le 2|T|$. With the above estimates and property (vi) of Lemma 2.1 we end up with

$$|T \setminus U| \le \delta|T| + \frac{|\partial_B(K)||C|}{\zeta} \le \delta|T| + \frac{2|T||\partial_B(K)|}{\zeta|K|} \le 2\delta|T|,$$

where the last inequality follows from the fact that $\zeta \le \delta/2$. In other words, we obtained $|U| \ge (1-2\delta)|T|$, which puts us into the position to apply Lemma A.2. Thus, we find some $\tilde{c} \in U$ such that

$$|K\tilde{c} \cap KC| \le \frac{|KC||K|}{|T|(1-2\delta)} < \varepsilon|K|,$$

which proves that property (ii) of Definition 5.17 holds for $C \cup \{\tilde{c}\}$. As $\tilde{c} \in U \subseteq S$ we have $K\tilde{c} \subseteq T$, which gives that also (i) holds with the new set $C \cup \{\tilde{c}\}$. We set $K^{(\tilde{c})} := (K\tilde{c} \setminus KC)\,\tilde{c}^{-1}$ then by the above inequality we get $|K^{(\tilde{c})}| \ge (1-\varepsilon)|K|$. Hence, with the statements (v) and (vii) of Lemma 2.1 and with $\tilde{c} \in U$ and the definition of U, we have

$$\begin{aligned}|\partial_B(K^{(\tilde{c})})| \le |\partial_B(K\tilde{c} \setminus KC)| &\le |K\tilde{c} \cap \partial_B(KC)| + |\partial_B(K)| \\ &\le \zeta|K| + |\partial_B(K)|\end{aligned}$$

and using $0 < \varepsilon < 1/2$, one obtains

$$\frac{|\partial_B(K^{(\tilde{c})})|}{|K^{(\tilde{c})}|} \le \frac{\zeta|K|}{(1-\varepsilon)|K|} + \frac{|\partial_B(K)|}{(1-\varepsilon)|K|} \le 2\zeta + 2\zeta^2 \le 4\zeta.$$

Thus (iii) holds as well and the claim is proven. ■

In the following we refine the result of Lemma A.3. We show that under additional assumptions we can ensure that the part of T which is not yet covered by tiles, still satisfies an invariance property.

Lemma A.4. *Let G be a finitely generated group, $0 < \varepsilon, \delta < 1/6$, $0 < \zeta < \delta/4$ and $\eta > 0$. Furthermore let $T, K, L, B \in \mathcal{F}(G)$ with $\mathrm{id} \in L \subseteq K$, $\mathrm{id} \in B$ and let T be (KK^{-1}, δ)-invariant and K be (LL^{-1}, η)-invariant, as well as (B, ζ^2)-invariant. Then there is a set $C \in \mathcal{F}(G)$ such that $T \setminus KC$ is $(LL^{-1}, 2\delta + \eta)$-invariant and K together with C is a $(B, 4\zeta)$-good small ε-quasi tiling of T with accuracy δ.*

Proof. As the assumptions of Lemma A.3 are satisfied, we get a set C such that the properties (i) to (iv) of Definition 5.17 are fulfilled, i.e. we obtain a $(B, 4\zeta)$-good small ε-quasi tiling of T with accuracy δ. We show that with this set C, the set $T \setminus KC$ is $(LL^{-1}, 2\delta + \eta)$-invariant. Therefore, first note that by properties (i) and (iii) in Definition 5.17 and we have

$$|T \setminus KC| = |T| - |KC| \geq (1 - (\varepsilon + \delta))|T| \geq \frac{2}{3}|T|. \tag{A.4}$$

Besides this, we obtain by another application of (i) and (iii) in Definition 5.17

$$\frac{|C|}{|T|} \leq \frac{(\varepsilon + \delta)|C|}{|KC|} = \frac{(\varepsilon + \delta)|C|}{\sum_{c \in C}|K^{(c)}|} \leq \frac{(\varepsilon + \delta)|C|}{(1 - \varepsilon)|C||K|} \leq \frac{2}{5|K|}. \tag{A.5}$$

Now, we use properties (iii) and (vi) of Lemma 2.1 and put the estimates (A.4) and (A.5) together to end up with

$$\begin{aligned} \frac{|\partial_{LL^{-1}}(T \setminus KC)|}{|T \setminus KC|} &\leq \frac{3|\partial_{LL^{-1}}(T)|}{2|T|} + \frac{3|C||\partial_{LL^{-1}}(K)|}{2|T|} \\ &\leq \frac{3}{2}\delta + \frac{3|\partial_{LL^{-1}}(K)|}{5|K|} \leq 2\delta + \eta. \end{aligned}$$

Note that here, we used that T is (LL^{-1}, δ)-invariant since $L \subseteq K$ and T is (KK^{-1}, δ)-invariant, cf. property (iv) in Lemma 2.1. This finishes the proof. ∎

Now we are in the position to prove the first tiling theorem. It improves results from [OW87] and it is joint work with Felix Pogorzelski.

Proof of Theorem 5.20. Let ε and β with $0 < \beta < \varepsilon \leq 1/10$ be given. Set $\delta := \delta(\varepsilon, \beta) := \beta 6^{-N(\varepsilon)}$. We start choosing the sets $K_i \in \{Q_n \mid n \in \mathbb{N}\}$, $i = 1, \ldots, N(\varepsilon)$ inductively in the following way: set $K_1 := Q_1$ and if $K_i = Q_k$ then take $K_{i+1} \in \{Q_n \mid n \geq k+1\}$, which is $(K_i K_i^{-1}, \delta)$-invariant. Then obviously $K_i \in \{Q_n \mid n \geq i\}$ for all $i = 1, \ldots, N(\varepsilon)$. Furthermore, as (Q_n) is nested each K_i contains the unit element.

Now let some $\zeta > 0$ and $B \in \mathcal{F}(G)$ with $\mathrm{id} \in B$ be given. Without loss of generality we assume that each element of the sequence (Q_n) is $(B, \zeta^2/16)$-invariant and that $\zeta < \delta$. If the first assumption would not hold, take the K_i from a subsequence of (Q_n) containing only $(B, \zeta^2/16)$-invariant elements. If ζ is not chosen to be smaller than δ, then we can take some $\tilde{\zeta} < \delta$ and repeat all the steps of the proof. Hence, all claimed statements will hold for the original ζ as well. We will use the notation $N := N(\varepsilon)$.

Now assume that $T \in \mathcal{F}(G)$ is $(K_N K_N^{-1}, \delta)$-invariant. We apply Lemma A.4 with "$T = T$", "$K = K_N$", "$L = K_{N-1}$", "$B = B$" and "$\zeta = \zeta/4$" to obtain a finite set C_N^T such that K_N with center set C_N^T is a (B, ζ)-good small ε-quasi tiling of T with accuracy δ. And we have that $D_1 := T \setminus K_N C_N^T$ is $(K_{N-1} K_{N-1}^{-1}, \delta_1)$-invariant, where $\delta_1 = 3\delta$.

Now we use Lemma A.4 inductively. If for some $l \in \{1, \ldots, N-1\}$ the set D_l is chosen as a $(K_{N-l} K_{N-l}^{-1}, \delta_l)$-invariant set, we apply the Lemma with "$T = D_l$", "$K = K_{N-l}$", "$L = K_{N-l-1}$", "$B = B$", "$\delta = \delta_l$", "$\eta = \delta$" and "$\zeta = \zeta/4$". Note that here it is important that δ_l is small enough, which we will ensure afterwards. This gives an appropriate set $C_{N-l}^T \in \mathcal{F}(G)$ such that $D_{l+1} := D_l \setminus K_{N-l} C_{N-l}^T$ is $(K_{N-l-1} K_{N-l-1}^{-1}, \delta_{l+1})$-invariant, where $\delta_{l+1} := 2\delta_l + \delta$. Again we obtain a (B, ζ)-good small ε-quasi tiling with accuracy δ_l.

We set $\delta_0 = \delta$ and obtain the closed formula $\delta_l = (2^{l+1} - 1)\delta$ for all $l = 1, \ldots, N-1$. Therefore, for arbitrary $l \in \{1, \ldots, N-1\}$ we have $\delta_l \leq \delta_{N-1} = (2^N - 1)\delta \leq 1/6$, which shows that all δ_l are small enough to apply Lemma A.4. Furthermore the Lemma implies the inequalities

$$\varepsilon - \delta_l \leq \frac{|K_{N-l} C_{N-l}^T|}{|D_l|} \leq \varepsilon + \delta_l \tag{A.6}$$

for all $l = 0, \ldots, N-1$, where $D_0 := T$. We claim that for all $l = 0, \ldots, N-1$ we have

$$\varepsilon(1-\varepsilon)^l - 3^l \delta_l \leq \frac{|K_{N-l} C_{N-l}^T|}{|T|} \leq \varepsilon(1-\varepsilon)^l + 3^l \delta_l. \tag{A.7}$$

We proceed by induction on l. Note that the case $l = 0$ follows from inequality (A.6). Now let $l \in \mathbb{N}$ with $l \leq N-1$ and assume that (A.7) holds for all $k = 0 \ldots, l-1$. By the induction hypothesis, we can sum up the resulting inequalities and arrive at

$$\begin{aligned} \varepsilon \sum_{k=0}^{l-1} (1-\varepsilon)^k - \sum_{k=0}^{l-1} 3^k \delta_k \leq \frac{|\bigcup_{k=0}^{l-1} K_{N-k} C_{N-k}^T|}{|T|} \\ \leq \varepsilon \sum_{k=0}^{l-1} (1-\varepsilon)^k + \sum_{k=0}^{l-1} 3^k \delta_k. \end{aligned}$$

The inductive definition of the set D_l gives the equality

$$T \setminus D_l = \bigcup_{k=0}^{l-1} K_{N-k} C_{N-k}^T$$

and hence

$$1 - \varepsilon \sum_{k=0}^{l-1} (1-\varepsilon)^k - \sum_{k=0}^{l-1} 3^k \delta_k \leq \frac{|D_l|}{|T|} \leq 1 - \varepsilon \sum_{k=0}^{l-1} (1-\varepsilon)^k + \sum_{k=0}^{l-1} 3^k \delta_k.$$

This simplifies, using the sum formula for the geometric series and $\delta_l \geq \delta_k$ for $k \leq l$, to

$$(1-\varepsilon)^l - 3^l \delta_l \leq \frac{|D_l|}{|T|} \leq (1-\varepsilon)^l + 3^l \delta_l. \tag{A.8}$$

A combination of the inequalities (A.8) and (A.6) gives

$$(\varepsilon - \delta_l) \left((1-\varepsilon)^l - 3^l \delta_l \right) \leq \frac{|K_{N-l} C_{N-l}^T|}{|T|} \leq (\varepsilon + \delta_l) \left((1-\varepsilon)^l + 3^l \delta_l \right).$$

We use this to obtain

$$\begin{aligned}\left|\frac{|K_{N-l}C_{N-l}^T|}{|T|} - \varepsilon(1-\varepsilon)^l\right| &\leq \delta_l(1-\varepsilon)^l + 3^l\delta_l^2 + 3^l\varepsilon\delta_l \\ &\leq \delta_l(1 + 3^l\delta_l + 3^l\varepsilon).\end{aligned}$$

In order to prove (A.7) it is sufficient to show $1 + 3^l\delta_l + 3^l\varepsilon \leq 3^l$. This is true since $\varepsilon \leq 1/4$ and $\delta_l < 2^N\delta \leq 1/4$ by the choice of δ. Now we use that for all $l \in \{0, \ldots, N-1\}$ we have $3^l \leq 3^N$ and $\delta_l < 2^N\delta = \beta 3^{-N}$ to obtain for all $i \in \{1, \ldots, N\}$:

$$\left|\frac{|K_iC_i^T|}{|T|} - \varepsilon(1-\varepsilon)^{N-i}\right| < 3^N 2^N \delta = \beta.$$

This proves (iii) of Definition 5.17. Properties (i), (ii) and (iv) of Definition 5.17 follow from the construction of the sets C_k^T, $k = 1, \ldots, N$. ■

Now we have everything together to prove the uniform tiling theorem, namely Theorem 5.22. This theorem substantially refines results from [OW87]. The main ideas and the concept of the proof are due to Felix Pogorzelski. Before starting the proof, let us roughly describe the idea. In Theorem 5.20 we have seen that one can, for fixed ε and β, find sets K_i, $i = 1, \ldots, N$ such that each sufficiently invariant set T can be ε-quasi tiled by these sets. In Theorem 5.22 we show that this result holds even in a certain uniformity. We claim that for each sufficiently invariant T we can find several tilings with these (fixed) sets K_i, such that on average each of these tiles K_i appears at any position in T with the same "frequency".

In order to find this family of tilings, we make use of Theorem 5.20 on different levels. First we apply Theorem 5.20 to obtain for the given ε and β the sets K_i. Then, using the same theorem, we obtain a collection of much more invariant tiles $\bar{K}_l$, in a way such that each quasi tiling with the sets $\bar{K}_l$ can be made disjoint using property (v) of Definition 5.18. Then, the resulting disjoint sets are still invariant enough, such that one can ε-quasi tile them with the sets K_i.

Having these different levels of tilings at hand, one chooses T so invariant, such that it can be tiled with both sets of tiles. Furthermore a set $\hat{T}$ is chosen even more invariant, in a way that it can be tiled with the $\bar{K}_l$ and that the TT^{-1}-boundary of $\hat{T}$ is very small.

This brings us in the position to choose the appropriate center sets. First we choose these center set for a tiling of $\hat{T}$ with elements of $\bar{K}_l$. Then we make these translates disjoint, and choose for each such set a center set for a tiling with the sets K_i. If we now consider translates Ta of T, which are completely contained in $\hat{T}$, then also these translates Ta are tiled by the $\bar{K}_l$ and therefore as well tiled by the sets K_i. Shifting these tilings with a^{-1} back to T then gives a family of tilings for T.

Proof of Theorem 5.22. First realize that the assumptions of Theorem 5.20 are satisfied and let

$$\mathrm{id} \in K_1 \subseteq \cdots \subseteq K_{N(\varepsilon)}$$

be the sets given by this theorem as elements of the nested Følner sequence (Q_n).

Define $\zeta := 6^{-N(\varepsilon)}\beta$. Furthermore, we introduce a new parameter, namely $\delta := \beta^2/(100|K_{N(\varepsilon)}|^2)$. During the proof we will need, that this choice implies

$$\delta \leq \frac{\beta}{2}, \quad \delta \leq \frac{\varepsilon^2}{4}, \quad \frac{1}{1-5\sqrt{\delta}} - 1 \leq \frac{\beta}{|K_{N(\varepsilon)}|} \quad \text{and} \quad 5\sqrt{\delta} \leq \frac{\beta}{|K_{N(\varepsilon)}|}. \tag{A.9}$$

We proceed in nine steps.

(1) Let (Q'_n) be a subsequence of (Q_n) consisting of sets, which are $(K_{N(\varepsilon)}K_{N(\varepsilon)}^{-1}, \zeta^2)$-invariant and which satisfies the property $K_{N(\varepsilon)} \subseteq Q'_1$. Furthermore, let $\bar{\beta} := \delta/N(\delta)$ be given. We apply Theorem 5.20 with the nested Følner sequence (Q'_n), $0 < \bar{\beta} < \delta$, $B = K_{N(\varepsilon)}K_{N(\varepsilon)}^{-1}$ and $\zeta = \zeta$ and obtain sets

$$\bar{K}_1 \subseteq \cdots \subseteq \bar{K}_{N(\delta)}.$$

As (U_j) is assumed to be a Følner sequence, we find $j_0 \in \mathbb{N}$ such that for each $j \geq j_0$ the set U_j is $(\bar{K}_{N(\delta)}\bar{K}_{N(\delta)}^{-1}, \delta/N(\delta))$-invariant. Note that j_0 does only depend on ε, β and the chosen sets $\bar{K}_l$, $l = 1, \dots, N(\delta)$. We choose an arbitrary $j \geq j_0$ and set $T := U_j$. Now we choose another very invariant set: let $\hat{T}$ be (TT^{-1}, δ)-invariant and $(\bar{K}_{N(\delta)}\bar{K}_{N(\delta)}^{-1}, \bar{\beta}6^{-N(\delta)})$-invariant. One can for instance take $\hat{T}$ as an element of the Følner sequence (U_j) for j large enough. We define

$$\overline{A} := \{a \in \hat{T} \,|\, TT^{-1}a \subseteq \hat{T}\} \quad \text{and} \quad A := \{g \in G \,|\, Tg \subseteq \hat{T}\}$$

and obtain with Lemma A.1 that $|\overline{A}| \geq (1-\delta)|\hat{T}|$. Furthermore, since for every $s \in T$ we have $\overline{A} \subseteq s\,A$ the inequality $|A| \geq (1-\delta)|\hat{T}|$ holds as well.

(2) As $\hat{T}$ is chosen invariant enough, we find by Theorem 5.20 sets $\bar{C}_l$ such that the $\bar{K}_l$ and $\bar{C}_l$, $l = 1, \dots, N(\delta)$ are a $(K_{N(\varepsilon)}K_{N(\varepsilon)}^{-1}, \zeta)$-good δ-quasi tiling of $\hat{T}$ with accuracy $\bar{\beta}$ and densities $\eta_l(\delta)$, $l = 1, \dots, N(\delta)$. As in Definition 5.18 we use the notation $\bar{K}_l^{(c)} \subseteq \bar{K}_l$, $c \in \bar{C}_l$ for the pairwise disjoint $(K_{N(\varepsilon)}K_{N(\varepsilon)}^{-1}, \zeta)$-invariant sets which also fulfill the rest of the properties in (v) of Definition 5.18. By a calculation in Remark 5.19 and the fact that $\bar{\beta} = \delta/N(\delta)$ we get

$$|B(\delta)| = \sum_{l=1}^{N(\delta)} \sum_{c \in \bar{C}_l} |\bar{K}_l^{(c)}c| \geq (1-2\delta)|\hat{T}|, \tag{A.10}$$

where

$$B(\delta) := \bigcup_{l=1}^{N(\delta)} \bigcup_{c \in \bar{C}_l} \bar{K}_l^{(c)}c, \tag{A.11}$$

which shows that these disjoint translates $(1-2\delta)$-cover the set $\hat{T}$.

(3) By Theorem 5.20 and since each $\bar{K}_l^{(c)}$ is $(K_{N(\varepsilon)}K_{N(\varepsilon)}^{-1}, \zeta)$-invariant and $\zeta = \beta 6^{-N(\varepsilon)}$ we find for each $l \in \{1, \dots, N(\delta)\}$ and $c \in \bar{C}_l$ a set $C_i(l,c)$, such that K_i with center sets $C_i(l,c)$, $i = 1, \dots, N(\varepsilon)$ are an ε-quasi tiling of $\bar{K}_l^{(c)}c$ with accuracy β and densities $\eta_i(\varepsilon)$, $i = 1, \dots, N(\varepsilon)$. Using again the last item in Remark 5.19 and

the assumption $\beta \le \varepsilon/N(\varepsilon)$ we obtain

$$\left|\bigcup_{i=1}^{N(\varepsilon)} K_i C_i(l,c)\right| \ge (1-2\varepsilon)\left|\bar{K}_l^{(c)}\right|. \tag{A.12}$$

Now, define for $i \in \{1,\ldots,N(\varepsilon\}$ the set

$$\hat{C}_i := \bigcup_{l=1}^{N(\delta)} \bigcup_{c\in\bar{C}_l} C_i(l,c).$$

This set $\hat{C}_i$ can be seen as a center set for the sets K_i in the set $\hat{T}$. In fact we have that $K_i c$, $c \in \hat{C}_i$ are ε-disjoint and for $i \neq j$ it follows from the disjointness of the $\bar{K}_l^{(c)} c$, $l \in \{1,\ldots,N(\delta)\}$, $c \in \bar{C}_l$ that $K_i\hat{C}_i \cap K_j\hat{C}_j = \emptyset$. Let us investigate the covering properties of this tiling.

(4) In this step we show that the portion of $\hat{T}$ which is covered by $K_i\hat{C}_i$ is $\eta_i(\varepsilon)$ up to a (small) error of 2β. To this end we first use the disjointness of the $\bar{K}_l^{(c)} c$ for all $c \in \bar{C}_l$ and all $l \in \{1,\ldots,N(\delta)\}$ and get

$$|K_i\hat{C}_i| = \sum_{l=1}^{N(\delta)} \sum_{c\in\bar{C}_l} |K_i C_i(l,c)| \ge \sum_{l=1}^{N(\delta)} \sum_{c\in\bar{C}_l} |\bar{K}_l^{(c)}|(\eta_i(\varepsilon)-\beta).$$

Here we used property (iv) of Definition 5.18, which holds since K_i with center sets $C_i(l,c)$, $i = 1,\ldots,N(\varepsilon)$ are a ε-quasi tiling of $\bar{K}_l^{(c)} c$ with accuracy β and densities $\eta_i(\varepsilon)$, $i = 1,\ldots,N(\varepsilon)$, see step (3). Now use the estimate (A.10) and the definition of $B(\delta)$ in (A.11) to get

$$\frac{|K_i\hat{C}_i|}{|\hat{T}|} \ge \frac{\eta_i(\varepsilon)-\beta}{|\hat{T}|}\,|B(\delta)| \ge (\eta_i(\varepsilon)-\beta)(1-2\delta) \ge \eta_i(\varepsilon)-2\beta,$$

where we applied $2\delta \le \beta$, see (A.9). The upper bound is even easier. Again by property (iv) of Definition 5.18 we get

$$\frac{|K_i\hat{C}_i|}{|\hat{T}|} \leq \sum_{l=1}^{N(\delta)} \sum_{c\in\bar{C}_l} |\bar{K}_l^{(c)}| \frac{\eta_i(\varepsilon)+\beta}{|\hat{T}|} = |B(\delta)|\,\frac{\eta_i(\varepsilon)+\beta}{|\hat{T}|}$$
$$\leq \eta_i(\varepsilon)+\beta.$$

These estimates give together that for each $i \in \{1,\dots,N(\varepsilon)\}$:

$$\left|\frac{|K_i\hat{C}_i|}{|\hat{T}|} - \eta_i(\varepsilon)\right| \leq 2\beta. \tag{A.13}$$

(5) In this step we fix $i \in \{1,\dots,N(\varepsilon)\}$ and estimate the difference between the quotients $\gamma_i := |\hat{C}_i|/|\hat{T}|$ and $\eta_i(\varepsilon)/|K_i|$. We use triangle inequality, ε-disjointness and estimate (A.13) to get

$$\left|\gamma_i - \frac{\eta_i(\varepsilon)}{|K_i|}\right| \leq \frac{1}{|K_i|}\left|\frac{|\hat{C}_i|\,|K_i|}{|\hat{T}|} - \frac{|K_i\hat{C}_i|}{|\hat{T}|}\right| + \frac{1}{|K_i|}\left|\frac{|K_i\hat{C}_i|}{|\hat{T}|} - \eta_i(\varepsilon)\right|$$
$$\leq \frac{1}{|K_i|}\frac{\varepsilon|\hat{C}_i|\,|K_i|}{|\hat{T}|} + \frac{2\beta}{|K_i|} = \gamma_i\,\varepsilon + \frac{2\beta}{|K_i|}. \tag{A.14}$$

Now, let us show that with this choice of γ_i we have the estimate $\sum_{i=1}^{N(\varepsilon)} \gamma_i|K_i| \leq 2$. It follows from the ε-disjointness and the rough bound $\varepsilon \leq 1/2$ that

$$|K_i|\,|\hat{C}_i| \leq \frac{1}{1-\varepsilon}\left|K_i\hat{C}_i\right| \leq 2\left|K_i\hat{C}_i\right|.$$

We use this to estimate

$$\sum_{i=1}^{N(\varepsilon)} \gamma_i|K_i| \leq \sum_{i=1}^{N(\varepsilon)} \frac{|K_i||\hat{C}_i|}{|\hat{T}|} \leq 2\sum_{i=1}^{N(\varepsilon)} \frac{|K_i\hat{C}_i|}{|\hat{T}|} = \frac{2}{|\hat{T}|}\left|\bigcup_{i=1}^{N(\varepsilon)} K_i\hat{C}_i\right| \leq 2,$$

which proves the claimed inequality.

(6) This step of the proof is devoted to the investigation of the translates of T which lie entirely in $\hat{T}$. We will show that most of these translates are $(1-3\varepsilon)$-covered by $\bigcup_{i=1}^{N(\varepsilon)} K_i\hat{C}_i$. In step

(1) we already defined the set A consisting of all $a \in G$ such that $Ta \subseteq \hat{T}$. For each $a \in A$ we define

$$X(a) := \frac{|Ta \cap (\hat{T} \setminus B(\delta))|}{|Ta|} = \frac{|Ta \setminus B(\delta)|}{|T|}$$

to be the part of Ta, which is not covered by translates $\bar{K}_l^{(c)} c$, $l = 1, \ldots, N(\delta)$, $c \in \bar{C}_l$, see (A.11) for the definition of $B(\delta)$. Let us for a moment treat this X as a uniformly distributed random variable, with respect to the counting measure. Evidently X maps from A to $[0, 1]$. We use Tschebyscheff inequality to obtain

$$\begin{aligned}\sqrt{\delta}\,\big|\big\{a \in A \mid X(a) > \sqrt{\delta}\big\}\big| &\le \sum_{a \in A} |X(a)| \\ &= \frac{1}{|T|} \sum_{a \in A} \sum_{g \in G} \mathbf{1}_{Ta \setminus B(\delta)}(g).\end{aligned}$$

The last sum can be estimated in the following way

$$\begin{aligned}\frac{1}{|T|} \sum_{a \in A} \sum_{g \in G} \mathbf{1}_{Ta \setminus B(\delta)}(g) &\le \frac{1}{|T|} \sum_{a \in A} \sum_{g \in \hat{T} \setminus B(\delta)} \mathbf{1}_{Ta}(g) \\ &\le \frac{1}{|T|} \sum_{g \in \hat{T} \setminus B(\delta)} \sum_{a \in A} \mathbf{1}_{Ta}(g) \\ &\le \frac{1}{|T|} |\hat{T} \setminus B(\delta)||T| = |\hat{T} \setminus B(\delta)| \le 2\delta |\hat{T}|,\end{aligned}$$

where the last step uses (A.10). We use the lower bound on $|A|$, obtained in step (1) and $\delta \le 1/2$ to get

$$|\{a \in A \mid X(a) > \sqrt{\delta}\}| \le 2\sqrt{\delta}|\hat{T}| \le \frac{2\sqrt{\delta}|A|}{1 - \delta} \le 4\sqrt{\delta}|A|,$$

or in other words

$$|\Lambda| \ge (1 - 4\sqrt{\delta})|A|, \quad \text{where} \quad \Lambda := \big\{a \in A \mid X(a) \le \sqrt{\delta}\big\}. \tag{A.15}$$

We have seen that, up to a portion of $4\sqrt{\delta}$, the translates of T which lie entirely in $\hat{T}$ are $(1 - \sqrt{\delta})$-covered by our tiling. However, as for each a we are interested in a tiling of Ta with *subsets* of Ta, we need to delete elements of this covering, which have a non-empty intersection with $G \setminus Ta$. Define for $l \in \{1, \ldots, N(\delta)\}$ and $a \in A$ the sets

$$\partial(a, l) := \Big\{c \in \bar{C}_l \mid \bar{K}_l^{(c)} c \cap Ta \neq \emptyset,\ \bar{K}_l^{(c)} c \cap (G \setminus Ta) \neq \emptyset\Big\},$$
$$I(a, l) := \Big\{c \in \bar{C}_l \mid \bar{K}_l^{(c)} c \subseteq Ta\Big\}.$$

Then we have for $\tilde{c} \in \partial(a, l)$ that $\bar{K}_l^{(c)} \tilde{c} \subseteq \partial_{\bar{K}_l \bar{K}_l^{-1}}(Ta)$, which gives

$$\bigcup_{l=1}^{N(\delta)} \bigcup_{c \in \partial(a,l)} \bar{K}_l^{(c)} c \subseteq \bigcup_{l=1}^{N(\delta)} \partial_{\bar{K}_l \bar{K}_l^{-1}}(Ta). \tag{A.16}$$

Hence, using the assumed invariance properties of T in step (1), this yields

$$\frac{1}{|T|} \Bigg| \bigcup_{l=1}^{N(\delta)} \bigcup_{c \in \partial(a,l)} \bar{K}_l^{(c)} c \Bigg| \leq \frac{1}{|T|} \sum_{l=1}^{N(\delta)} \big|\partial_{\bar{K}_l \bar{K}_l^{-1}}(T)\big| \leq \sum_{l=1}^{N(\delta)} \frac{\delta}{N(\delta)} = \delta. \tag{A.17}$$

Therefore, we have for each $a \in \Lambda$ the estimate

$$\begin{aligned} \Bigg| \bigcup_{l=1}^{N(\delta)} \bigcup_{c \in I(a,l)} \bar{K}_l^{(c)} c \Bigg| &\geq |Ta \cap B(\delta)| - \Bigg| \bigcup_{l=1}^{N(\delta)} \bigcup_{c \in \partial(a,l)} \bar{K}_l^{(c)} c \Bigg| \\ &\geq (1 - \sqrt{\delta} - \delta)|T|. \end{aligned} \tag{A.18}$$

Now let us estimate the part of Ta, which is covered by translates K_i, $i = 1, \ldots, N(\varepsilon)$, which lie completely in Ta. To this end, we set for each $a \in A$

$$\tilde{C}_i(a) := \bigcup_{l=1}^{N(\delta)} \bigcup_{c \in I(a,l)} C_i(l, c), \quad \text{and} \quad D(a) := \bigcup_{i=1}^{N(\varepsilon)} K_i \tilde{C}_i(a) \subseteq Ta.$$

Then, using the disjointness of $\bar{K}_l^{(c)}c$, $l \in \{1, \dots, N(\delta)\}$, $c \in \bar{C}_l$ and the estimate (A.12) we have for $a \in \Lambda$:

$$|D(a)| = \sum_{l=1}^{N(\delta)} \sum_{c \in I(a,l)} \left| \bigcup_{i=1}^{N(\varepsilon)} K_i C_i(l,c) \right| \geq (1-2\varepsilon) \sum_{l=1}^{N(\delta)} \sum_{c \in I(a,l)} |\bar{K}_l^{(c)}|.$$

Thus, the bound in (A.18) and $\delta \leq \varepsilon^2/4$, see (A.9), give for $a \in \Lambda$

$$|D(a)| \geq (1-2\varepsilon)(1-\sqrt{\delta}-\delta)|T| \geq (1-3\varepsilon)|T|. \tag{A.19}$$

(7) This step is devoted to put things together and to show that properties (i)-(iv) of Definition 5.21 are satisfied. The idea is that we translate the covering we obtained for Ta by a^{-1}, to obtain a family of coverings for T. In the previous step we already defined the set Λ, which obviously corresponds to $T = U_j$ and we therefore sometimes add the index j, i.e. we set $\Lambda_j := \Lambda$. The combination of (A.15) and the estimate $|A| \geq (1-\delta)|\hat{T}|$ from step (1) gives

$$|\Lambda| \geq (1-4\sqrt{\delta})(1-\delta)|\hat{T}| \geq (1-5\sqrt{\delta})|\hat{T}|. \tag{A.20}$$

Furthermore, we define for each $\lambda \in \Lambda$ and $i \in \{1, \dots, N(\varepsilon)\}$ the set

$$C_i^\lambda := C_i^\lambda(j) := \tilde{C}_i(\lambda)\lambda^{-1}.$$

Then we have

$$\bigcup_{i=1}^{N(\varepsilon)} K_i C_i^\lambda = D(\lambda)\lambda^{-1} \subseteq T. \tag{A.21}$$

By construction we get that the properties (i), (ii) and (iii) are fulfilled with this choice of $\Lambda_j = \Lambda$ and C_i^λ, $\lambda \in \Lambda$, $i \in \{1, \dots, N(\varepsilon)\}$. Moreover, property (iv) is implied by the Estimate (A.19) and (A.21).

(8) We start this step by choosing $r := \operatorname{diam}(\bar{K}_{N(\delta)}\bar{K}_{N(\delta)}^{-1})$. Therefore, r depends only on ε, β and the choices of the tiles K_i,

$i = 1, \dots, N(\varepsilon)$ and $\bar{K}_l$, $l = 1 \dots, N(\delta)$. In particular, r is independent of j, the index of $U_j = T$. Furthermore, we have

$$\partial^r(T) \supseteq \partial_{\bar{K}_{N(\delta)}\bar{K}^{-1}_{N(\delta)}}(T) = \bigcup_{l=1}^{N(\delta)} \partial_{\bar{K}_l \bar{K}_l^{-1}}(T). \tag{A.22}$$

In this step we prove that for any $i \in \{1, \dots, N(\varepsilon)\}$ and $g \in T \setminus \partial^r(T) = T^{(r)}$ we have

$$\left| \frac{1}{|\Lambda|} \sum_{\lambda \in \Lambda} \mathbf{1}_{C_i^\lambda}(g) - \gamma_i \right| \leq \frac{\beta}{|K_i|}, \tag{A.23}$$

where as before we have $\gamma_i = |\hat{C}_i|/|\hat{T}|$. To this end, note that for given $i \in \{1, \dots, N(\varepsilon)\}$ and $\lambda \in \Lambda$ we have $g \in C_i^\lambda$, if and only if $\lambda \in u^{-1}\tilde{C}_i(\lambda)$. Using that for each $\lambda \in \Lambda$ we have $\tilde{C}_i(\lambda) \subseteq \hat{C}_i$ implies that for $i \in \{1, \dots, N(\varepsilon)\}$ and $g \in T$ we obtain

$$\sum_{\lambda \in \Lambda} \mathbf{1}_{C_i^\lambda}(g) = \sum_{\lambda \in \Lambda} \mathbf{1}_{g^{-1}\tilde{C}_i(\lambda)}(\lambda) \leq \sum_{\lambda \in \Lambda} \mathbf{1}_{g^{-1}\hat{C}_i}(\lambda) = |g\Lambda \cap \hat{C}_i| \leq |\hat{C}_i|, \tag{A.24}$$

such that we get with (A.20)

$$\frac{1}{|\Lambda|} \sum_{\lambda \in \Lambda} \mathbf{1}_{C_i^\lambda}(g) \leq \frac{|\hat{C}_i|}{|\Lambda|} \leq \frac{|\hat{C}_i|}{(1 - 5\sqrt{\delta})|\hat{T}|}.$$

Applying the third inequality in (A.9), this results in

$$\begin{aligned} \frac{1}{|\Lambda|} \sum_{\lambda \in \Lambda} \mathbf{1}_{C_i^\lambda}(g) - \frac{|\hat{C}_i|}{|\hat{T}|} &\leq \left(\frac{1}{1 - 5\sqrt{\delta}} - 1 \right) \frac{|\hat{C}_i|}{|\hat{T}|} \\ &\leq \frac{1}{1 - 5\sqrt{\delta}} - 1 \leq \frac{\beta}{|K_i|}. \end{aligned} \tag{A.25}$$

Now, we estimate in the other direction. To this end, we fix some $g \in T^{(r)}$. First we claim that for each $i \in \{1, \dots, N(\varepsilon)\}$ and

$\lambda \in \Lambda$:

$$\hat{C}_i \cap T\lambda \subseteq \tilde{C}_i(\lambda) \cup \bigcup_{l=1}^{N(\delta)} \bigcup_{c \in \partial(\lambda,l)} C_i(l,c)$$
$$\subseteq \tilde{C}_i(\lambda) \cup \bigcup_{l=1}^{N(\delta)} \partial_{\bar{K}_l \bar{K}_l^{-1}}(T\lambda) \subseteq C_i^\lambda \lambda \cup \partial^r(T)\lambda. \quad \text{(A.26)}$$

To see the first inclusion, let $x \in \hat{C}_i \cap T\lambda$ be given. Then there exists $l \in \{1, \ldots, N(\delta)\}$ and $c \in \bar{C}_l$ with $x \in C_i(l,c)$. If $c \in I(\lambda, l)$, we are done, since then $x \in \tilde{C}_i(\lambda)$. Therefore, let $c \notin I(\lambda, l)$. Then we have $\bar{K}_l^{(c)} c \nsubseteq T\lambda$, but as $x \in C_i(l,c)$ and $\mathrm{id} \in K_i$ we also get $x \in K_i x \subseteq \bar{K}_l^{(c)} c$. This shows together with $x \in T\lambda$ that $c \in \partial(\lambda, l)$, which proves the first inclusion in the claim. The second inclusion follows from $C_i(l,c) \subseteq \bar{K}_l^{(c)} c$ and (A.16). The last inclusion uses (A.22) and (v) in Lemma 2.1. Now with (A.26) we get

$$\hat{C}_i \lambda^{-1} \cap (T \setminus \partial^r(T)) \subseteq C_i^\lambda.$$

This implies with $g \in T \setminus \partial^r(T)$ that we have for each $\lambda \in \Lambda$:

$$\mathbf{1}_{C_i^\lambda}(g) \geq \mathbf{1}_{\hat{C}_i \lambda^{-1} \cap (T \setminus \partial^r(T))}(g) = \mathbf{1}_{\hat{C}_i \lambda^{-1}}(g) = \mathbf{1}_{g^{-1}\hat{C}_i}(\lambda).$$

This shows with the above calculations in (A.24) that

$$\sum_{\lambda \in \Lambda} \mathbf{1}_{C_i^\lambda}(g) = |g\Lambda \cap \hat{C}_i|.$$

Next, use $g\Lambda \cap \hat{C}_i \supseteq \hat{C}_i \setminus (\hat{T} \setminus g\Lambda)$ and estimate (A.20) to calculate

$$\frac{|g\Lambda \cap \hat{C}_i|}{|\Lambda|} \geq \frac{|\hat{C}_i|}{|\hat{T}|} - \frac{|\hat{T} \setminus g\Lambda|}{|\hat{T}|} \geq \frac{|\hat{C}_i|}{|\hat{T}|} - 1 + \frac{|\Lambda|}{|\hat{T}|} \geq \frac{|\hat{C}_i|}{|\hat{T}|} - 5\sqrt{\delta}.$$

With the fourth inequality in (A.9) this implies

$$\frac{|\hat{C}_i|}{|\hat{T}|} - \frac{1}{|\Lambda|} \sum_{\lambda \in \Lambda} \mathbf{1}_{C_i^\lambda}(g) \leq 5\sqrt{\delta} \leq \frac{\beta}{|K_i|}.$$

This, together with (A.25), proves (A.23).

(9) In the final step we combine the estimates from step (8) and step (5) to obtain property (v) of Definition 5.21. To be precise, we use (A.14) and (A.23) to estimate for each $g \in T^{(r)}$ and $i \in \{1, \ldots, N(\varepsilon\}$:

$$\begin{aligned}\left| \frac{1}{|\Lambda|} \sum_{\lambda \in \Lambda} \mathbf{1}_{C_i^\lambda}(g) - \frac{\eta_i(\varepsilon)}{|K_i|} \right| &\leq \left| \frac{1}{|\Lambda|} \sum_{\lambda \in \Lambda} \mathbf{1}_{C_i^\lambda}(g) - \gamma_i \right| + \left| \gamma_i - \frac{\eta_i(\varepsilon)}{|K_i|} \right| \\ &\leq \gamma_i \varepsilon + \frac{3\beta}{|K_i|}.\end{aligned}$$

This finishes the proof. ■

Theses

The present work investigates spectral properties of operators on graphs or finitely generated groups, respectively. First we present the results concerning operators on finitely generated sofic groups. Here one obtains approximating operators by an appropriate transformation of the operator in question to a finite dimensional operator on the sofic approximation graph.

(1) Let A be a (deterministic) translation invariant, self-adjoint operator on a sofic group, such that $C_c(G)$ is a core of A. Then the integrated density of states exists as a weak limit of distribution functions. Moreover, the Pastur-Shubin trace formula holds true.

(2) Let A be a random Hamiltonian on a sofic group, given as in (4.6). Then, almost surely, the normalized eigenvalue counting functions converge weakly to a function, which is independent of the specific realization. Moreover, the Pastur-Shubin trace formula holds true.

(3) The convergence results of (1) and (2) hold in particular for operators on the free group. Here we can state a specific sequence of approximating finite graphs. In the special case where the operator in question is the adjacency operator of the Cayley graph of the free group, the IDS exists uniformly.

(4) The graph Laplacian of a long-range percolation graph on a sofic group is a random Hamiltonian in the sense of (4.6). Thus, the above results apply and we obtain for almost all realizations the existence of the IDS as a weak limit of distribution functions and the validity of the Pastur-Shubin trace formula.

The following assertions concern deterministic operators on amenable groups. We assume that there is a fixed coloring $\mathcal{C}$ which maps each element of the group into a finite set $\mathcal{A}$. Furthermore, we require that the frequencies of patterns exist along the Følner sequence (U_j).

Here, the approximating operators are defined as restrictions of the operator under consideration to the elements of the Følner sequence (U_j).

(5) Let F be an almost-additive function, mapping a finite subset of an amenable group into some Banach space. Then, for the Følner sequence (U_j) the limit

$$\lim_{j\to\infty} \frac{F(U_j)}{|U_j|}$$

exists as element in the Banach space. Moreover, this limit can be expressed using a semi-explicit formula in terms of frequencies of patterns. Besides this, one can estimate the speed of convergence. These facts are stated in a Banach space-valued ergodic theorem.

(6) Let A be a deterministic, $\mathcal{C}$-invariant operator of finite hopping range. Then the integrated density of states exists as a uniform limit of the normalized eigenvalue counting functions of the approximating operators. Besides this, an estimate for the speed of convergence can be verified.

(7) Assume the setting of (6). If the frequencies are strictly positive for all patterns which occur in $\mathcal{C}$, the spectrum of A is the topological support of the measure associated to the IDS.

(8) With the same assumption as in (7) the points of discontinuity of the IDS can be characterized as the elements in the spectrum, which admit a finitely supported eigenfunction.

(9) If the coloring $\mathcal{C}$ of the group is given randomly with an underlying measure preserving and ergodic group action on the probability space, then the frequencies of all patterns exist along a given Følner sequence almost surely.

Next, we discuss random operators on finitely generated amenable groups. The approximating operators are here obtained by restricting the operator in question to elements of a Følner sequence.

(10) Let a random operator as in (6.1) be given. Then, almost surely, the normalized eigenvalue counting functions converge weakly to

a non-random distribution function and the Pastur-Shubin trace formula holds true.

(11) If A is a random operator as in (6.48) then associated integrated density of states exists uniformly and the Pastur-Shubin trace formula holds true. Note that (6.48) is slightly more restrictive than (6.1).

(12) If the operator in consideration is the graph Laplacian of a long-range percolation graph over an ST-amenable group, then uniform convergence of (11) can be obtained by methods of a Banach space-valued ergodic theorem. Moreover, under the additional assumption that for each edge the probability of existence is an number in $(0, 1)$, we have: an element $\lambda \in \mathbb{R}$ is a point of discontinuity of the IDS if and only if λ is an eigenvalue of some finite graph.

(13) The results of (11) apply also to randomly weighted Laplacians. Here the weights are independent random variables on the edges, taking values in a possibly infinite and unbounded subset of $\mathbb{R}$.

In (5) we referred to a Banach space-valued ergodic theorem. In order to prove this for *all* amenable groups, it turns out that one needs to apply the theory of ε-quasi tilings. The following assertions are related to this topic and hold true for an arbitrary finitely generated amenable group.

(14) Given positive ε and β, then one can find finitely many sets K_i, $i = 1, \ldots, N(\varepsilon)$, such that each sufficiently invariant set T can be ε-quasi tiled with these sets, accuracy β and densities $\eta_i(\varepsilon)$, $i = 1, \ldots, N(\varepsilon)$. For the precise definitions of $\eta_i(\varepsilon)$ and $N(\varepsilon)$ we refer to (5.21) and (5.20), respectively.

(15) Given positive ε and β, then one can find finitely many sets K_i, $i = 1, \ldots, N(\varepsilon)$, such that each sufficiently invariant set T can be *uniformly* ε-quasi tiled by the sets K_i, $i = 1, \ldots, N(\varepsilon)$ with respect to certain parameters $(\beta, r, \gamma, \eta(\varepsilon))$. This means that there exists a family of ε-quasi tilings such that (nearly) each element of T is covered by (nearly) the same amount of tiles among this family.

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