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TECHNISCHE UNIVERSITÄT
CHEMNITZ

Universitätsverlag Chemnitz

2016

Impressum

Bibliografische Information der Deutschen Nationalbibliothek

Die Deutsche Nationalbibliothek verzeichnet diese Publikation in der Deutschen Nationalbibliografie; detaillierte bibliografische Angaben sind im Internet über http://dnb.d-nb.de abrufbar.

Diese Arbeit wurde von der Fakultät für Mathematik der Technischen Universität Chemnitz als Dissertation zur Erlangung des akademischen Grades Dr. rer. nat. genehmigt.

Die Arbeit wurde in englischer Sprache verfasst.

Tag der Einreichung: 2. Dezember 2015

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3. Gutachter: Prof. Dr. Frank Kutzschebauch, Universität Bern

Tag der öffentlichen Prüfung: 18. März 2016

Technische Universität Chemnitz/Universitätsbibliothek
Universitätsverlag Chemnitz
09107 Chemnitz
http://www.tu-chemnitz.de/ub/univerlag

Herstellung und Auslieferung
Verlagshaus Monsenstein und Vannerdat OHG
Am Hawerkamp 31
48155 Münster
http://www.mv-verlag.de

ISBN 978-3-944640-84-6

http://nbn-resolving.de/urn:nbn:de:bsz:ch1-qucosa-200815

Acknowledgments

I am deeply grateful to my advisor Albrecht Böttcher for drawing my attention to the problems considered here, for the valuable input and discussions, as well as for his support in providing a stable environment for my research.

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Chapter 1

Introduction

Contents

This work is devoted to Markov-type inequalities, which give upper bounds on the norm of the derivative of an algebraic polynomial in terms of the norm of the polynomial itself. Such an inequality is

$$\|f^{(\nu)}\|_\beta \leq C_n^{(\nu)}(\alpha,\beta)\|f\|_\alpha \quad \text{for all } f \in \mathcal{P}_n. \tag{1.1}$$

Here and in the following, $\mathcal{P}_n$ denotes the space of algebraic polynomials with complex coefficients of degree at most n, and $\|\cdot\|_\alpha$ is one of the norms

$$\|f\|_\alpha^2 = \int_0^\infty |f(t)|^2 t^\alpha \mathrm{e}^{-t}\,\mathrm{d}t \qquad \text{(Laguerre)},$$
$$\|f\|_\alpha^2 = \int_{-1}^1 |f(t)|^2 (1-t^2)^\alpha\,\mathrm{d}t \qquad \text{(Gegenbauer)},$$
$$\|f\|_\alpha^2 = \int_{-\infty}^\infty |f(t)|^2 |t|^{2\alpha} \mathrm{e}^{-t^2}\,\mathrm{d}t \qquad \text{(Hermite)},$$

and $C_n^{(\nu)}(\alpha,\beta)$ is a constant depending on n,ν,α, and β, but not on f. We are interested in finding the smallest constant such that inequality (1.1) holds for every polynomial f of degree at most n. Let $D^\nu : \mathcal{P}_n(\alpha) \to \mathcal{P}_n(\beta)$ be the operator that sends a polynomial of degree at most n to its νth derivative, where $\mathcal{P}_n(\alpha)$ and $\mathcal{P}_n(\beta)$ are the spaces $\mathcal{P}_n$ equipped with the norm

$\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$, respectively. Every functional analysis course tells us that the operator norm of this operator is defined as

$$\|D^\nu\|_{\alpha\to\beta} := \inf\{c \geq 0 \mid \forall f \in \mathcal{P}_n(\alpha) : \|D^\nu f\|_\beta \leq c\|f\|_\alpha\}.$$

This is exactly what we are looking for. Hence, determining the smallest constant in (1.1) comes down to determining the operator norm of the differential operator between the appropriate spaces. We denote this smallest constant with $\lambda_n^{(\nu)}(\alpha,\beta)$ for the Laguerre norms, with $\gamma_n^{(\nu)}(\alpha,\beta)$ for the Gegenbauer norms, and with $\eta_n^{(\nu)}(\alpha,\beta)$ for the Hermite norms.

In general, it is not possible to evaluate the exact constants. However, we can ask for the asymptotic behavior of these. This is the main goal of the present work.

1.1 State of the art

Inequalities of the considered type go back to the chemist Dimitri Ivanovich Mendeleev, best known for the periodic table of elements. In the 1880s, he studied the specific gravity of a solution as a function of the percentage of the dissolved substance. He observed that these functions can be approximated with quadratic polynomials. He raised the question how bad the transition from one point to another can be when they are in parameter ranges belonging to different polynomials. After getting some results, he told this to Andrei Andreevich Markov, who subsequently investigated the corresponding problem for polynomials of degree n [3, 5].

Markov proved that if $p(t)$ is a real polynomial of degree n with $|p(t)| \leq 1$ on $[-1,1]$, then $|p'(t)| \leq n^2$, or equivalently,

$$\|Df\|_\infty \leq n^2\|f\|_\infty \quad \text{for all real } f \in \mathcal{P}_n,$$

where D is the differential operator and $\|\cdot\|_\infty$ denotes the maximum norm on $[-1,1]$. He also showed that the constant n^2 is best-possible. One might come to the conclusion that repeated utilization of this formula leads to optimal results for higher derivatives. However, these are not sharp, e. g., this would give $\|D^2 f\|_\infty \leq n^2(n-1)^2\|f\|_\infty$, which is not the best possible constant. Markov's younger brother, Vladimir Andreevich, showed that

$$\|D^2 f\|_\infty \leq \frac{n^2(n^2-1)}{3}\|f\|_\infty \quad \text{for all } f \in \mathcal{P}_n,$$

and more generally,

$$\|D^\nu f\|_\infty \leq \frac{n^2(n^2-1)(n^2-2^2)\cdots(n^2-(\nu-1)^2)}{(2\nu-1)!!}\|f\|_\infty \quad \text{for all } f \in \mathcal{P}_n.$$

These constants are best-possible.

Erhard Schmidt [22] was the first to consider inequalities of this type with the maximum norm replaced by a Hilbert space norm. More specifically, he investigated the norms

$$\|f\|^2 = \int_0^\infty |f(t)|^2 e^{-t}\,dt \qquad \text{(Laguerre)},$$
$$\|f\|^2 = \int_{-1}^1 |f(t)|^2\,dt \qquad \text{(Legendre)},$$
$$\|f\|^2 = \int_{-\infty}^\infty |f(t)|^2 e^{-t^2}\,dt \qquad \text{(Hermite)},$$

i. e., he assumed that $\alpha = \beta = 0$. He studied the case $\nu = 1$ and proved

$$\lambda_n^{(1)}(0,0) \sim \frac{2}{\pi}n, \quad \gamma_n^{(1)}(0,0) \sim \frac{1}{\pi}n^2, \quad \eta_n^{(1)}(0,0) = \sqrt{2n},$$

The expression $a_n \sim b_n$ means that the quotient $\frac{a_n}{b_n}$ converges to 1 as n goes to infinity. So, he could determine the exact value in the Hermite case and find an asymptotic value in the Laguerre and Legendre cases. He even gave two more terms in the asymptotic expansion.

Later, Pál Turán [27] presented the exact value for $\lambda_n^{(1)}(0,0)$, namely

$$\lambda_n^{(1)}(0,0) = \left(2\sin\frac{\pi}{4n+2}\right)^{-1}.$$

More recently, András Króo [18] also found an exact value for $\gamma_n^{(1)}(0,0)$, which he identified to be the largest positive solution of the equation

$$\sum_{k=0}^{\lfloor (n+1)/2 \rfloor} (-1)^k x^{-2k} \frac{(n+1+2k)!}{2^{2k}(2k)!(n+1-2k)!} = 0.$$

Both, Turán and Króo, gave the extremal polynomials for obtaining these constants.

Lawrence F. Shampine [23, 24] later began to investigate second order derivatives for the Laguerre and Legendre norms. He found

$$\lambda_n^{(2)}(0,0) \sim \frac{n^2}{\mu_0^2}, \quad \gamma_n^{(2)}(0,0) \sim \frac{n^4}{4\mu_0^2},$$

where $\mu_0 \approx 1.8751041$ is the smallest positive solution of the equation $1 + \cos\mu\cosh\mu = 0$.

Finally, Peter Dörfler [14, 15] bounded $\lambda_n^{(\nu)}(0,0)$ for larger values of ν by

$$\frac{1}{2\nu!}\sqrt{\frac{4}{2\nu+1}} \le \liminf_{n\to\infty} \frac{\lambda_n^{(\nu)}(0,0)}{n^\nu} \le \limsup_{n\to\infty} \frac{\lambda_n^{(\nu)}(0,0)}{n^\nu} \le \frac{1}{2\nu!}\sqrt{\frac{2\nu}{2\nu-1}}. \qquad (1.2)$$

Shampine and Dörfler took a basis of orthonormal polynomials and determined the matrix representation of the differential operator in this basis. This led to a special Toeplitz matrix, and the problem then was to determine the spectral norm (which coincides with the operator norm) or to find at least good estimates. In this way, they obtained the results cited above. After Dörfler's papers [14, 15], the development paused for almost two decades. In 2008, Dörfler wrote a letter to Böttcher and asked him whether he had an idea of how to determine the spectral norm of the Toeplitz matrices in question and thus to make some progress after many years of stagnation. Böttcher immediately understood that the problem can be solved by having recourse to an old (and then already forgotten) trick used by Harold Widom in the 1960s in another context [28, 29]. This trick consists in considering the integral operator K_n on $L^2(0,1)$ with the piecewise constant kernel $k_n(x,y) = a_{\lfloor nx \rfloor, \lfloor ny \rfloor}$ for an $(n \times n)$-matrix $A_n = (a_{jk})_{j,k=0}^{n-1}$. Then, one has the identity

$$\|A_n\|_\infty = n\|K_n\|_\infty.$$

Here and in what follows, we denote by $\|A\|_\infty$ the spectral norm if A is a matrix, and the operator norm if A is an operator. If, after appropriate scaling, the operators K_n can be shown to converge uniformly to some operator K (i. e., $\|n^{-\mu}K_n - K\|_\infty \to 0$ for $n \to \infty$), then $\|A_n\|_\infty \sim \|K\|_\infty n^{\mu+1}$. By employing this trick, Böttcher and Dörfler settled a whole series of problems that had then been open. In [6], they found in particular asymptotic expressions for $\lambda_n^{(\nu)}(0,0)$ as $n \to \infty$ in the case of arbitrary $\nu \geq 3$. To their surprise, they later discovered that Shampine [23, 24] also made use of the trick consisting in passing from matrices to integral operators. Thus, this trick was discovered twice, independently by Shampine and Widom, it fell into oblivion for over more than 40 years, and received a kind of renaissance in the work of Böttcher and Dörfler. We will exploit this trick later in Chapter 5, too.

The limitation of Shampine's original approach was that he considered the operator $(D^\nu)^* D^\nu$, which gets really complex for higher values of ν. But, as shown by Böttcher and Dörfler, one can go further quite a lot. The first expansion is to substitute the norms by their weighted analogues. In the case of the Laguerre and Legendre norms, i. e., for the Laguerre and Gegenbauer norms, Böttcher and Dörfler [8] have shown

$$\lambda_n^{(\nu)}(\alpha,\alpha) \sim \|L_{\nu,\alpha,\alpha}\|_\infty n^\nu,$$
$$\gamma_n^{(\nu)}(\alpha,\alpha) \sim \|G_{\nu,\alpha,\alpha}\|_\infty n^{2\nu},$$

where $L_{\nu,\alpha,\alpha}$ and $G_{\nu,\alpha,\alpha}$ are the integral operators on $L^2(0,1)$ given by

$$(L_{\nu,\alpha,\alpha}f)(x) = \frac{1}{\Gamma(\nu)} \int_x^1 x^{\alpha/2} y^{-\alpha/2} (y-x)^{\nu-1} f(y)\,\mathrm{d}y,$$
$$(G_{\nu,\alpha,\alpha}f)(x) = \frac{1}{2^{\nu-1}\Gamma(\nu)} \int_x^1 x^{1/2+\alpha} y^{1/2-\alpha} (y^2-x^2)^{\nu-1} f(y)\,\mathrm{d}y.$$

These results first of all proved the existence of the limits $\lambda_n^{(\nu)}(\alpha,\alpha)n^{-\nu}$ and $\gamma_n^{(\nu)}(\alpha,\alpha)n^{-2\nu}$ as $n \to \infty$. Note that the existence of these limits was previously not even known for $\alpha = 0$; in that case one had only the bounds (1.2).

At first glance, it does not seem that we gained much by replacing the spectral norm of some matrix with the operator norm of an integral operator. However, this replacement benefits of good two-sided estimates for the norms of the integral operators. (I learned from my advisor that working with integrals is often easier than working with sums.)

In 2009, Jürgen Prestin drew the attention of Böttcher and Dörfler to the problem of using two different weights in the inequality. This concerns two different norms, and as changes of norms may improve error estimates, this case could be useful in approximation theory and numerical analysis. A particularly simple case is $\beta = \alpha + \nu$. Then, the matrix representation has a single diagonal in the Laguerre and Gegenbauer settings. So, we arrive at

$$\lambda_n^{(\nu)}(\alpha, \alpha + \nu) = \sqrt{\frac{n!}{(n-\nu)!}} \sim n^{\nu/2},$$
$$\gamma_n^{(\nu)}(\alpha, \alpha + \nu) = \sqrt{\frac{n!}{(n-\nu)!} \frac{\Gamma(n + 2\alpha + \nu + 1)}{\Gamma(n + 2\alpha + 1)}} \sim n^{\nu}.$$

The first identity has been observed before by Ravi Agarwal, Gradimir Milovanović, and Allal Guessab in [1, 16]. The second one was established by Guessab and Milovanović; see also [16].

In [7], Böttcher and Dörfler raised the conjecture

$$\lambda_n^{(\nu)}(\alpha, \beta) \sim C_\nu(\alpha, \beta) n^{(\nu + |\beta - \alpha - \nu|)/2} \quad \text{as } n \to \infty$$

with some nonzero constant $C_\nu(\alpha, \beta)$ depending only on ν, α, β. They partially proved this: if $\beta - \alpha \geq \nu$ and $\beta - \alpha$ is an integer, then this is true with $C_\nu(\alpha, \beta) = 2^{\beta - \alpha - \nu}$, while for $\beta - \alpha < \nu - 1/2$, it is true with $C_\nu(\alpha, \beta) = \|L^*_{\nu,\alpha,\beta}\|_\infty$. Again, an integral operator occurs, this time

$$(L^*_{\nu,\alpha,\beta} f)(x) = \frac{1}{\Gamma(\nu - \beta + \alpha)} \int_0^x x^{-\alpha/2} y^{\beta/2} (x - y)^{\nu - \beta + \alpha - 1} f(y) \mathrm{d}y \quad \text{on} \quad L^2(0,1).$$

Actually, the induced operator is given by

$$(L_{\nu,\alpha,\beta} f)(x) = \frac{1}{\Gamma(\nu - \beta + \alpha)} \int_x^1 y^{-\alpha/2} x^{\beta/2} (y - x)^{\nu - \beta + \alpha - 1} f(y) \mathrm{d}y,$$

but we prefer to work with its adjoint $L^*_{\nu,\alpha,\beta}$. Since the norms of both operators are the same, this is no problem. In the special case $\beta - \alpha = \nu - 1$, the chosen operator has a much simpler structure. Then,

$$(L^*_{\nu,\alpha,\alpha+\nu-1} f)(x) = \int_0^x x^{-\alpha/2} y^{(\alpha+\nu-1)/2} f(y) \mathrm{d}y.$$

Together with a corollary from [8], this implies that its operator norm is $2/(\nu + 1)$ times the inverse of the smallest positive zero of the Bessel function $J_{(\alpha-1)/(\nu+1)}$. For the very special values $\upsilon = -1/2$ and $\upsilon = 1/2$, the Bessel functions J_υ take the elementary form

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x, \quad J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x.$$

We repeat some of the examples given in [7], which we obtain from these equalities. We have

$$\lambda_n^{(2)}\left(-\tfrac{1}{2},\tfrac{1}{2}\right) \sim \frac{4}{3\pi}n^{3/2}, \quad \lambda_n^{(2)}\left(\tfrac{5}{2},\tfrac{7}{2}\right) \sim \frac{2}{3\pi}n^{3/2}, \quad \lambda_n^{(3)}(3,5) \sim \frac{1}{2\pi}n^2,$$
$$\lambda_n^{(4)}\left(\tfrac{7}{2},\tfrac{13}{2}\right) \sim \frac{2}{5\pi}n^{5/2}, \quad \lambda_n^{(5)}(4,8) \sim \frac{1}{3\pi}n^3.$$

Similarly, in [10], Böttcher and Dörfler showed that if $\beta - \alpha$ is an integer, then

$$\gamma_n^{(\nu)}(\alpha,\beta) \sim n^\nu \qquad \text{for } \beta - \alpha - \nu \geq 0$$

and

$$\gamma_n^{(\nu)}(\alpha,\beta) \sim \frac{1}{2^{\nu+\alpha-\beta}}\|L^*_{\nu,\alpha,\beta}\|_\infty n^{2\nu+\alpha-\beta} \qquad \text{for } \beta - \alpha - \nu < -\frac{1}{2}.$$

Here, we meet the operator $L^*_{\nu,\alpha,\beta}$ from above, again. In the same way, for $\beta = \alpha + \nu - 1$, this implies that the constant is $1/(\nu+1)$ times the reciprocal of the smallest positive zero of the Bessel function $J_{(\alpha-1)/(\nu+1)}$. Further examples given there include

$$\gamma_n^{(1)}(0,0) \sim \frac{1}{\pi}n^2, \quad \gamma_n^{(1)}(2,2) \sim \frac{1}{2\pi}n^2.$$

In [19], the author treated the Laguerre case for arbitrary $\beta - \alpha \geq \nu$. The results are explained in Chapter 4. Thus, the Laguerre case is almost completely treated. It turns out that the method of [19] can easily be applied to the Gegenbauer case, as well. However, some more cases still have to be considered.

The present work shows that Böttcher and Dörfler's restriction to the integral differences in the Laguerre and Gegenbauer cases may be dropped. Moreover, we generalize the classical Hermite norm to a weighted version. This increases the complexity drastically. However, along the line of treatment for the other two norms, this can be dealt with.

Unfortunately, in each of these problems, there remains an unhandled interval of differences, namely $\beta - \alpha - \nu \in [-1/2, 0)$ in the Laguerre and Gegenbauer cases and $\beta - \alpha \in [-1/2, 0)$ in the Hermite case. The joint paper by Böttcher, Widom, and the author [11] attempts to tackle this problem for the Laguerre case. Although an overall answer is yet to be given, we provide some tools for handling the topic. Moreover, the proofs in [11] have some beauty on their own, so that we do not want to withhold them from the reader.

1.2 Objective of the thesis and main results

The existing results for the constants in the Laguerre and Gegenbauer cases anticipate that the study of the matrices clearly depends on the number $\omega = \beta - \alpha - \nu$. Indeed, if ω is a nonnegative integer, the corresponding matrices are banded and allow a relatively simple treatment. If ω is nonnegative and an arbitrary real number, not equal to an integer, we may use our knowledge

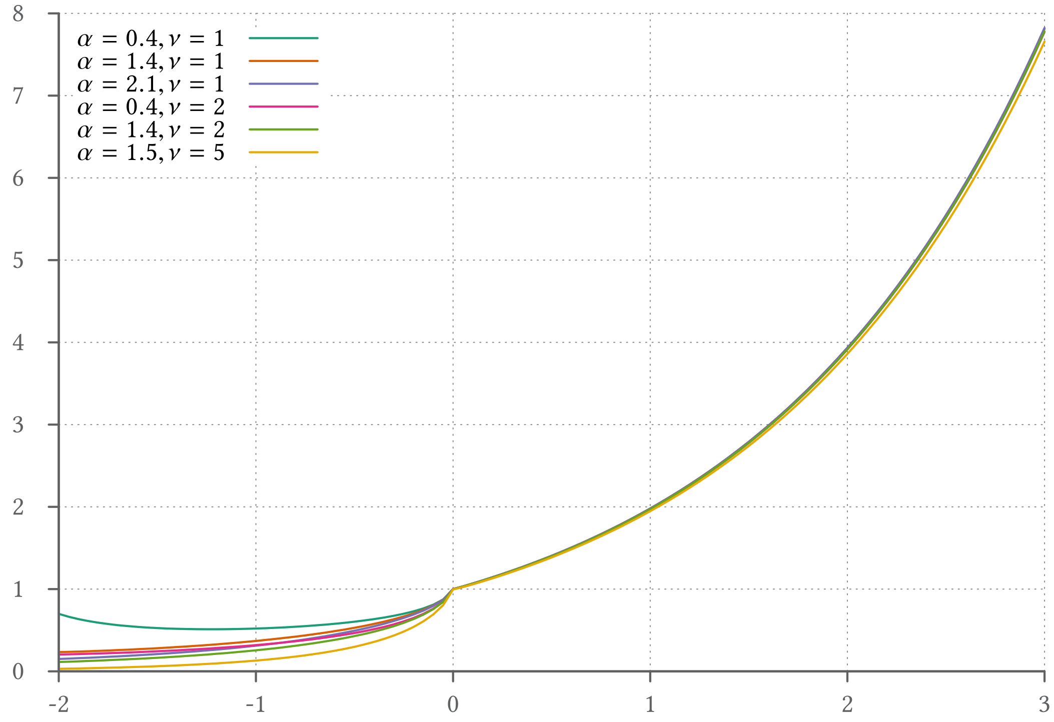

Figure 1.1: Dependence of the norm in the Laguerre setting on ω, scaled by $n^{-(\nu+|\omega|)/2}$, for different values of α and ν, here for $n = 1023$.

from the integer case to determine bounds for these values. Finally, if ω is negative, the methods from the nonnegative case fail to work, but we can find some integral operator, which in that case is bounded (contrary to the nonnegative case). Keeping this in mind almost dictates the structure of the thesis. Figure 1.1 already gives an idea that the worlds left and right from $\omega = 0$ are different.

Chapter 2 will deal with the matrix representation of the differential operator in the bases belonging to the types of the norms under consideration. We introduce the systems of orthonormal polynomials derived from each norm. The Laguerre and Gegenbauer representations are merely for reference and the proofs are given for the sake of completeness. It seems that the generalized Hermite case has not been considered before, so this is a new result.

In Chapter 3, we bring back to mind the treatment of the Laguerre and Gegenbauer norms for ω a nonnegative integer. These ideas will also be employed for the Hermite norm. However, in that case, we have to deal with one more special case, namely $\beta = \alpha$. The matrix in question is not banded. We cannot use an interpolation theorem since we are at the end of the parameter space and an integral operator is not applicable. The good news is that the matrix is "almost diagonal." In other words, the diagonal elements are significantly bigger than the remaining entries. Because generalized Hermite norms have not been considered before, this part is entirely new to this thesis.

The next chapter uses the results established in the preceding one and those already known to derive upper bounds for all parameter differences in between. This is possible due to a very helpful theorem by Elias Stein [25]. Lower bounds can be derived by the right choice of some vector, for which the norm of its image under the effect of the matrix can be estimated. This concludes the investigation for the case $\omega \geq 0$ in the Laguerre and Gegenbauer settings, and $\beta - \alpha \geq 0$ in the Hermite setting. Apart from the author's own work [19] this is completely new and an original result of this thesis.

Chapter 5 is then devoted to the case $\omega < 0$ in the Laguerre and Gegenbauer settings, and $\beta - \alpha < 0$ in the Hermite setting. This chapter almost entirely deals with the associated integral operators and with how to show that the operators derived from the matrices converge uniformly to them. A proof for the convergence in the Laguerre setting is given because it turned out to be not as straightforward as claimed in [7]. However, the method of the proof also works in the Gegenbauer case, where we drop the necessity of integral differences, and in the Hermite setting. Therefore, the last two cases are presented the first time. Moreover, we prove two theorems from [11] that show that the integral operator in the Laguerre setting is a compact operator and even belongs to some Schatten class for any $\omega < 0$.

Finally, Chapter 6 wraps up everything and gives hints on what can be done in the future.

In the following, we present the main results of this thesis. The first three theorems will be proved step by step according to the aforementioned outline.

First, in the Laguerre setting, we have the following.

Theorem 1.1. *Let $\alpha, \beta > -1$ be real numbers, and let ν be a positive integer. Put $\omega = \beta - \alpha - \nu$. Then,*

$$\lambda_n^{(\nu)}(\alpha, \beta) \sim C_\nu(\alpha, \beta) n^{(\nu + |\omega|)/2}$$

with

$$C_\nu(\alpha, \beta) = \begin{cases} 2^\omega & : \omega \geq 0, \\ \|L^*_{\nu,\alpha,\beta}\|_\infty & : \omega < -\frac{1}{2}, \end{cases}$$

where $L^*_{\nu,\alpha,\beta}$ is the Volterra integral operator on $L^2(0,1)$ given by

$$(L^*_{\nu,\alpha,\beta} f)(x) = \frac{1}{\Gamma(-\omega)} \int_0^x x^{-\alpha/2} y^{\beta/2} (x - y)^{-\omega - 1} f(y) \mathrm{d}y. \tag{1.3}$$

As remarked before, the case $\omega \geq 0$, $\omega \in \mathbb{Z}$ as well as the case $\omega < -1/2$ have been disposed of already in [7].

The next theorem concerns the Gegenbauer setting.

Theorem 1.2. *Let $\alpha, \beta > -1$ be real numbers, and let ν be a positive integer. Put $\omega = \beta - \alpha - \nu$. Then,*

$$\gamma_n^{(\nu)}(\alpha,\beta) \sim \begin{cases} n^\nu & : \omega \geq 0, \\ 2^\omega \|L^*_{\nu,\alpha,\beta}\|_\infty n^{2\nu-\beta+\alpha} & : \omega < -\frac{1}{2}, \end{cases}$$

where $L^*_{\nu,\alpha,\beta}$ is the Volterra integral operator on $L^2(0,1)$ given by (1.3).

Again, $\omega \in \mathbb{Z}$ was known before thanks to [10]. Note that the integral operator coming into play here is the same as in the Laguerre case.

The following result on the general Hermite setting is completely new.

Theorem 1.3. *Let $\alpha, \beta > -1/2$ be real numbers and let ν be a positive integer. Then,*

$$\eta_n^{(\nu)}(\alpha,\beta) \sim C_\nu(\alpha,\beta) n^{(|\beta-\alpha|+\nu)/2}$$

with

$$C_\nu(\alpha,\beta) = \begin{cases} 2^{(\beta-\alpha+\nu)/2} & : \beta - \alpha \geq 0, \\ 2^{(\beta-\alpha-\nu)/2} \cdot \max\left\{\|H^{(0)}_{\nu,\alpha,\beta}\|_\infty, \|H^{(1)}_{\nu,\alpha,\beta}\|_\infty\right\} & : \beta - \alpha < -\frac{1}{2}, \end{cases}$$

where $H^{(0)}_{\nu,\alpha,\beta}$ and $H^{(1)}_{\nu,\alpha,\beta}$ are the integral operators on $L^2(0,1)$ defined by

$$(H^{(0)}_{\nu,\alpha,\beta} f)(x) = \frac{2^\nu \Gamma(\lceil \nu/2 \rceil + 1)}{\Gamma(\alpha - \beta + \lceil \nu/2 \rceil)} \int_x^1 x^{\beta/2-1/4} y^{-\alpha/2+1/4+(\lfloor \nu/2 \rfloor - \lceil \nu/2 \rceil)/2} (y-x)^{\alpha-\beta+\lceil \nu/2 \rceil - 1} \times \sum_{\ell=0}^{\lceil \nu/2 \rceil} \binom{\beta}{\ell} \binom{\beta-\alpha-\ell}{\lceil \nu/2 \rceil - \ell} \left(\frac{x}{y-x}\right)^{\lceil \nu/2 \rceil - \ell} f(y) \mathrm{d}y$$

and

$$(H^{(1)}_{\nu,\alpha,\beta} f)(x) = \frac{2^\nu \Gamma(\lfloor \nu/2 \rfloor + 1)}{\Gamma(\alpha - \beta + \lfloor \nu/2 \rfloor)} \int_x^1 x^{\beta/2+1/4} y^{-\alpha/2-1/4+(\lceil \nu/2 \rceil - \lfloor \nu/2 \rfloor)/2} (y-x)^{\alpha-\beta+\lfloor \nu/2 \rfloor - 1} \times \sum_{\ell=0}^{\lfloor \nu/2 \rfloor} \binom{\beta}{\ell} \binom{\beta-\alpha-\ell}{\lfloor \nu/2 \rfloor - \ell} \left(\frac{x}{y-x}\right)^{\lfloor \nu/2 \rfloor - \ell} f(y) \mathrm{d}y.$$

Although the operators here look a lot more complex than the operator (1.3), a resemblance should not be questioned. However, nice formulas as in the Laguerre and Gegenbauer cases cannot be obtained. The reason for this is that the operator $H^{(0)}_{\nu,\alpha,\beta}$ has at least two summands. Even in the simplest case $\nu = 1$, where the norm of the operator $H^{(1)}_{\nu,\alpha,\beta}$ can be given explicitly for a few values of α and β, the other operator does not play along.

To get a better feeling how these operators evolve for higher derivatives, we will present the explicit formulas for $\nu = 1, 2, 3$.

$$\begin{aligned}
(H^{(0)}_{1,\alpha,\beta} f)(x) &= \frac{2}{\Gamma(\alpha-\beta+1)} \int_x^1 \Big((\beta-\alpha) x^{\beta/2+3/4} y^{-\alpha/2-1/4} (y-x)^{\alpha-\beta-1} \\
&\qquad + \beta x^{\beta/2-1/4} y^{-\alpha/2-1/4} (y-x)^{\alpha-\beta} \Big) f(y) \mathrm{d}y, \\
(H^{(0)}_{2,\alpha,\beta} f)(x) &= \frac{8}{\Gamma(\alpha-\beta+1)} \int_x^1 \Big((\beta-\alpha) x^{\beta/2+3/4} y^{-\alpha/2+1/4} (y-x)^{\alpha-\beta-1} \\
&\qquad + \beta x^{\beta/2-1/4} y^{-\alpha/2+1/4} (y-x)^{\alpha-\beta} \Big) f(y) \mathrm{d}y, \\
(H^{(0)}_{3,\alpha,\beta} f)(x) &= \frac{48}{\Gamma(\alpha-\beta+2)} \int_x^1 \Big(\tfrac{(\beta-\alpha)(\beta-\alpha-1)}{2} x^{\beta/2+7/4} y^{-\alpha/2-1/4} (y-x)^{\alpha-\beta-1} \\
&\qquad + \beta(\beta-\alpha-1) x^{\beta/2+3/4} y^{-\alpha/2-1/4} (y-x)^{\alpha-\beta} \\
&\qquad + \tfrac{\beta(\beta-1)}{2} x^{\beta/2-1/4} y^{-\alpha/2-1/4} (y-x)^{\alpha-\beta+1} \Big) f(y) \mathrm{d}y, \\
(H^{(1)}_{1,\alpha,\beta} f)(x) &= \frac{2}{\Gamma(\alpha-\beta)} \int_x^1 x^{\beta/2+1/4} y^{-\alpha/2+1/4} (y-x)^{\alpha-\beta-1} f(y) \mathrm{d}y, \\
(H^{(1)}_{2,\alpha,\beta} f)(x) &= \frac{4}{\Gamma(\alpha-\beta+1)} \int_x^1 \Big((\beta-\alpha) x^{\beta/2+5/4} y^{-\alpha/2-1/4} (y-x)^{\alpha-\beta-1} \\
&\qquad + \beta x^{\beta/2+1/4} y^{-\alpha/2-1/4} (y-x)^{\alpha-\beta} \Big) f(y) \mathrm{d}y, \\
(H^{(1)}_{3,\alpha,\beta} f)(x) &= \frac{8}{\Gamma(\alpha-\beta+1)} \int_x^1 \Big(x^{\beta/2+5/4} y^{-\alpha/2+1/4} (y-x)^{\alpha-\beta-1} \\
&\qquad + x^{\beta/2+1/4} y^{-\alpha/2+1/4} (y-x)^{\alpha-\beta} \Big) f(y) \mathrm{d}y.
\end{aligned}$$

Finally, we present two theorems that were obtained in collaboration with Böttcher and Widom [11]. Since the notion of Schatten classes does not play any role in the rest of the thesis, we refer to Section 5.4 for the preliminaries. In the following, we will abbreviate $L = L^*_{\nu,\alpha,\beta}$. The technically most difficult part is to prove the convergence of $N^{1-(\nu+|\omega|)/2} L_N$ to L in the operator norm, where L_N is the integral operator with piecewise constant kernel derived from the matrix representation of the differential operator in the Laguerre setting (see Theorem 5.4). Fortunately, L can be shown to be a Hilbert-Schmidt operator if $\omega < -1/2$, and it can also be shown that $N^{1-(\nu+|\omega|)/2} L_N$ converges to L in the Hilbert-Schmidt norm for $\omega < -1/2$. This has been presented in [7] and will be done, in more detail, in Theorem 5.4 of this work.

If $\omega \geq -1/2$, the operator L is no longer Hilbert-Schmidt. However, in [11] we raised the conjecture that the restriction in the second part of Theorem 1.1 is merely an effect of the method rather than being inherent to the problem. This conjecture is given here again as Conjecture 5.8. One result in that direction, stated below as Theorem 1.5, tells us that L is still a Schatten class operator for $\omega < 0$. This is not of immediate help for proving Conjecture 5.8 but could be of use for further attempts towards accomplishing that goal. In particular, it follows that L is compact and therefore $P_N L P_N$ converges to L in the operator norm whenever $\{P_N\}$ is a sequence of operators such that P_N and the adjoints P_N^* converge strongly (i. e., pointwise) to the identity operator. Our hope is that one can find a fitting sequence $\{P_N\}$ which enables one to prove

$$\|N^{1-(\nu+|\omega|)/2}L_N - P_N L P_N\|_\infty \to 0,$$

Together with the fact that $\|P_N L P_N - L\|_\infty \to 0$, this would imply the desired uniform convergence of $N^{1-(\nu+|\omega|)/2}L_N$ to L.

Theorem 1.4 (Theorem 1.2 in [11]). *Let α, β, ω be real numbers. Suppose $\beta > -1$, $\omega < 0$, and $\omega < (\beta - \alpha)/2$. Then, the operator given by* (1.3) *is compact.*

Theorem 1.5 (Theorem 1.3 in [11]). *Let $\alpha > -1$, $\beta > -1$, $\nu \geq 1$ be real numbers and put $\omega = \beta - \alpha - \nu$. If n is a positive integer and $\omega < -1/2^n$, then the operator* (1.3) *belongs to the 2^nth Schatten class.*

1.3 Preliminaries

In this section, we want to collect some well-known results and notions. Some of them may seem trivial for the reader while others are not immediately at hand. Since we use them quite often and without further notice, we present them here.

First, in the whole work the variables $n, \alpha, \beta, \nu, \omega$, and N are reserved unless otherwise stated. The variable n stands for the maximal degree of the polynomials and thus the dimension of the matrix minus one, while N is a shorthand for $n - \nu + 1$. The constant ν will then denote the order of the derivative. Also fixed in their meaning, α denotes the parameter for the norm in the area of definition of the differential operator, and β is the corresponding parameter in the image space. Then, $\omega = \beta - \alpha - \nu$ is the abbreviation for the parameter difference in the Laguerre and Gegenbauer setting. Since we always work with matrices, we have to deal with their entries. These are denoted by $c_{jk}^{(\nu)}(\alpha, \beta)$, no matter which norm we currently consider. There is no risk of mixing them up because we treat each case separately.

The matrices we examine will all be upper triangular and their first ν diagonals are also zero. Due to this circumstance, we will just investigate the upper-right block where each row and column contains at least one nonzero entry. The norm of this block is the same as the norm of the whole matrix, so this restriction is justified. The original matrices are of order $n + 1$, the upper-right block mentioned is of order $N = n - \nu + 1$.

To fix notation, let $\mathbb{Z}$ denote the set of all integers and $\mathbb{N}$ the set of all natural numbers, i. e., all positive integers, $\{1, 2, \ldots\}$. Should the need arise to account for the number 0, we denote this set by $\mathbb{N}_0$. Moreover, we make an important distinction between the integer parts of a real number. Namely, $\lfloor x \rfloor$ denotes the largest integer not greater than x, while $\lceil x \rceil$ stands for the smallest integer not less than x. In particular, we have for $x \in \mathbb{R} \setminus \mathbb{Z}$ that $\lceil x \rceil = \lfloor x \rfloor + 1$ and for x in $\mathbb{Z}$, obviously, $\lfloor x \rfloor = \lceil x \rceil$. This is especially important in the Hermite setting where we will also make use of the fact $\lceil \nu/2 \rceil + \lfloor \nu/2 \rfloor = \nu$ for $\nu \in \mathbb{N}$.

Recall that the expression $a_n \sim b_n$ means that the quotient a_n/b_n converges to 1 as n goes to infinity. We write $f(n) = O(g(n))$ if there is a nonnegative constant C so that $|f(n)| \leq C|g(n)|$ as n goes to infinity. With this notation, we can for example write

$$\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} = n^{\alpha}\left(1 + O\left(\tfrac{1}{n}\right)\right).$$

We will use this result frequently.

The function $\Gamma(x)$ is defined by

$$\Gamma(x) := \int_0^\infty t^{x-1} \mathrm{e}^{-t} \mathrm{d}t \quad \text{for } x > 0,$$

and by analytic continuation for all complex numbers except the nonpositive integers. One of the most important properties is the identity $x\Gamma(x) = \Gamma(x+1)$. The Legendre duplication formula

$$\Gamma(2m) = \Gamma(m)\Gamma(m+1/2)\frac{2^{2m-1}}{\sqrt{\pi}} \tag{1.4}$$

is another important relation (see [2, page 22]). Closely connected to the gamma function is the beta function

$$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1} \mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

which is well-defined for all $x, y > 0$ and is also known as the Euler integral of the first kind, whereas the gamma function is sometimes called Euler integral of the second kind.

A very useful tool we will encounter is the hypergeometric series

$${}_pF_q\left(\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}; x\right) := \sum_{\tau=0}^{\infty} \frac{(a_1)_\tau (a_2)_\tau \cdots (a_p)_\tau}{(b_1)_\tau (b_2)_\tau \cdots (b_q)_\tau \tau!} x^\tau.$$

The symbol $(a)_\tau$, the rising factorial or Pochhammer symbol, is given by

$$(a)_\tau = a(a+1)(a+2)\cdots(a+\tau-1),$$

or, if applicable,

$$(a)_\tau = \frac{\Gamma(a+\tau)}{\Gamma(a)}.$$

Here, we will only have to deal with $(p,q) = (3,2)$ and $(p,q) = (2,1)$. In these cases, the series converges absolutely for all x with $|x| < 1$ as well as for $|x| = 1$ if the sum of the lower parameters is larger than the sum of the upper parameters. If one of the upper parameters is a negative integer, the series terminates naturally [2, page 62].

In the special case where $p = 2,\ q = 1,\ x = 1$ and one of the upper parameters is a negative integer, we have the Chu-Vandermonde identity [2, page 67]

$$ {}_2F_1\left(\begin{matrix} -n,\ a \\ c \end{matrix} ; 1 \right) = \frac{(c-a)_n}{(c)_n}. \tag{1.5} $$

In another form, this is the well-known Vandermonde identity

$$ \binom{a+b}{n} = \sum_{k=0}^{n} \binom{a}{k} \binom{b}{n-k} \tag{1.6} $$

with arbitrary complex numbers a and b.

We will use several different norms for an operator $T : X \to Y$. Most importantly, we are interested in the operator norm

$$ \|T\|_\infty = \inf\{c \geq 0 : \|Tx\|_Y \leq c\|x\|_X \text{ for all } x \in X\}, $$

or, equivalently,

$$ \|T\|_\infty = \sup_{\|x\|_X \neq 0} \frac{\|Tx\|_Y}{\|x\|_X}, $$

where $\|\cdot\|_X$ and $\|\cdot\|_Y$ are the norms in the spaces X and Y, respectively. If X and Y are separable Hilbert spaces – which they are in our setting – the norm $\|T\|_\infty$ coincides with the largest singular value, hence it is also called spectral norm.

Next, we will need the Hilbert-Schmidt norm, which in turn is a special type of Schatten norm (see Section 5.4) and given by

$$ \|T\|_2 = \left(\sum_{\ell=0}^{\infty} \sigma_\ell^2(T) \right)^{1/2}, $$

where $\sigma_\ell(T)$ is the ℓth singular value of T (in nonincreasing order). From this, it is immediately clear that $\|T\|_2 \geq \|T\|_\infty$. In the special case that T is an integral operator, for the sake of convenience on $L^2(0,1)$, with the kernel $\rho : [0,1]^2 \to \mathbb{R}$,

$$ (Tf)(x) = \int_0^1 \rho(x,y) f(y) \mathrm{d}y, $$

its Hilbert-Schmidt norm can easily be determined as

$$ \|T\|_2 = \left(\int_0^1 \int_0^1 |\rho(x,y)|^2 \mathrm{d}y \mathrm{d}x \right)^{1/2}. $$

The last norm we will make use of is for vectors. Here, we use the usual Euclidean or ℓ_2 norm. For a vector $v = (v_j)_{j=0}^n \in \mathbb{R}^{n+1}$, this is given as

$$ \|v\|_2 = \left(\sum_{j=0}^{n} v_j^2 \right)^{1/2}. $$

Chapter 2

Matrix representation of the operators

Contents

The problem of finding the best constant C in the inequality

$$\|D^\nu f\|_\beta \le C\|f\|_\alpha \quad \text{for all } f \in \mathcal{P}_n$$

comes down to determining the operator norm of the operator of differentiation D^ν acting on $\mathcal{P}_n$. Here, $\mathcal{P}_n$ is the linear space of all algebraic polynomials in one variable with complex coefficients of degree at most n. The space $\mathcal{P}_n$ is endowed with an appropriate inner product. The norm of D^ν is the same as the spectral norm of the matrix representation in a pair of orthonormal bases with respect to the according norm. We denote by $\mathcal{P}_n(\alpha)$ the space $\mathcal{P}_n$ equipped with the norm $\|\cdot\|_\alpha$, where in our setting $\|\cdot\|_\alpha$ is one of the following

$$\|f\|_\alpha^2 = \int_0^\infty |f(t)|^2 t^\alpha \mathrm{e}^{-t}\,\mathrm{d}t \qquad \text{(Laguerre)}, \tag{2.1}$$

$$\|f\|_\alpha^2 = \int_{-1}^1 |f(t)|^2 (1-t^2)^\alpha\,\mathrm{d}t \qquad \text{(Gegenbauer)}, \tag{2.2}$$

$$\|f\|_\alpha^2 = \int_{-\infty}^\infty |f(t)|^2 |t|^{2\alpha} \mathrm{e}^{-t^2}\,\mathrm{d}t \qquad \text{(Hermite)}. \tag{2.3}$$

These norms are well-defined for $\alpha > -1$ in the Laguerre and Gegenbauer cases and for $\alpha > -1/2$ in the Hermite case. The operator D^ν then maps $\mathcal{P}_n(\alpha)$ to $\mathcal{P}_n(\beta)$. The orthonormal bases we choose are the normalized Laguerre, Gegenbauer, and Hermite polynomials, respectively.

Except for the generalized Hermite polynomials, a good overview for the various definitions and normalizations is given in [4].

In the following sections we want to derive the matrix representation of the operators in the respective bases.

2.1 Laguerre weights

The Laguerre polynomials with respect to some parameter $\alpha > -1$ form an orthogonal system with respect to the norm

$$\|f\|_\alpha^2 = \int_0^\infty |f(t)|^2 t^\alpha \mathrm{e}^{-t}\,\mathrm{d}t.$$

The nth Laguerre polynomial for this norm is given by

$$L_n(t,\alpha) = \frac{1}{\Gamma(n+1)} t^{-\alpha} \mathrm{e}^t \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(t^{n+\alpha}\mathrm{e}^{-t}\right) = \sum_{\ell=0}^{n} (-1)^\ell \binom{n+\alpha}{n-\ell} \frac{t^\ell}{\ell!}. \tag{2.4}$$

We define the nth normalized Laguerre polynomial by

$$\widehat{L}_n(t,\alpha) = w_n(\alpha) L_n(t,\alpha), \quad w_n(\alpha) := \sqrt{\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}}.$$

The following lemma concerns the matrix representation of the operator of differentiation in a pair of bases consisting of normalized Laguerre polynomials. It is taken from [7]. The proof there assumed that $\beta - \alpha$ is an integer. However, as already stated in [19], we can drop this assumption and employ the same arguments. For the sake of completeness, we reproduce the proof here.

Lemma 2.1. *Let $C_n = (c_{jk}^{(\nu)}(\alpha,\beta))_{j,k=0}^n$ be the matrix representation for the differential operator $D^\nu : \mathcal{P}_n(\alpha) \to \mathcal{P}_n(\beta)$ with respect to the orthonormal bases $\{\widehat{L}_0(\cdot,\alpha),\dots,\widehat{L}_n(\cdot,\alpha)\}$ and $\{\widehat{L}_0(\cdot,\beta),\dots,\widehat{L}_n(\cdot,\beta)\}$. Then, C_n satisfies*

$$(-1)^\nu C_n = \Delta_n^{-1}(\beta) \begin{pmatrix} 0 & T^*_{n-\nu+1}((1-z)^{\beta-\alpha-\nu}) \\ 0 & 0 \end{pmatrix} \Delta_n(\alpha),$$

where

$$\Delta_n(\gamma) = \mathrm{diag}(w_0(\gamma),\dots,w_n(\gamma))$$

and $T^*_{n-\nu+1}((1-z)^{\beta-\alpha-\nu})$ is the adjoint (transposed) of the $(n-\nu+1)\times(n-\nu+1)$ Toeplitz matrix generated by the Taylor coefficients of $(1-z)^{\beta-\alpha-\nu}$ at $z=0$. Thus,

$$c_{jk}^{(\nu)}(\alpha,\beta) = (-1)^{k-j} \frac{w_k(\alpha)}{w_j(\beta)} \binom{\beta-\alpha-\nu}{k-\nu-j}$$

for $0 \le j \le k-\nu$ and it is zero otherwise.

Proof. We have to show that

$$D^\nu \widehat{L}_k(t,\alpha) = \sum_{j=0}^{k-\nu} c_{jk}^{(\nu)}(\alpha,\beta)\widehat{L}_j(t,\beta) = \sum_{j=0}^{k-\nu}(-1)^{k-j}\frac{w_k(\alpha)}{w_j(\beta)}\binom{\beta-\alpha-\nu}{k-\nu-j}\widehat{L}_j(t,\beta).$$

Since $\widehat{L}_k(t,\alpha) = w_k(\alpha)L_k(t,\alpha)$ and $\widehat{L}_j(t,\beta) = w_j(\beta)L_j(t,\beta)$, this is the same as

$$D^\nu L_k(\alpha) = \sum_{j=0}^{k-\nu}(-1)^{k-j}\binom{\beta-\alpha-\nu}{k-\nu-j}L_j(\beta).$$

The well-known identity $L_k'(t,\alpha) = -L_{k-1}(\alpha+1)$ can be obtained by differentiation of (2.4). Repeating this process gives $D^\nu L_k(t,\alpha) = (-1)^\nu L_{k-\nu}(\alpha+\nu)$. Inserting this in our statement, we have

$$D^\nu L_k(t,\alpha) = (-1)^\nu \sum_{\ell}^{k-\nu}(-1)^\ell\binom{k+\alpha}{k-\nu-\ell}\frac{t^\ell}{\ell!}.$$

We now compare the coefficients for t^m, $m = 0,\dots,k-\nu$. That is, we have to verify

$$(-1)^\nu(-1)^m\binom{k+\alpha}{k-\nu-m}\frac{1}{m!} = \sum_{\ell=m}^{k-\nu}\binom{\beta-\alpha-\nu}{k-\nu-\ell}(-1)^{k-\ell}(-1)^m\binom{\ell+\beta}{\ell-m}\frac{1}{m!}$$

$$\Updownarrow$$

$$\binom{k+\alpha}{k-\nu-m} = \sum_{\ell=m}^{k-\nu}\binom{\beta-\alpha-\nu}{k-\nu-\ell}(-1)^{k-\nu-\ell}\binom{\ell+\beta}{\ell-m}$$

for each $m = 0,\dots,k-\nu$. Shifting the sum and substituting $n = k-\nu-m$, this becomes

$$\begin{aligned}\binom{n+m+\nu+\alpha}{n} &= \sum_{\ell=0}^{n}\binom{\beta-\alpha-\nu}{n-\ell}(-1)^{n-\ell}\binom{\ell+m+\beta}{\ell}\\ &= (-1)^n\sum_{\ell=0}^{n}\binom{\beta-\alpha-\nu}{n-\ell}\binom{-\beta-m-1}{\ell},\end{aligned}$$

which is true due to Vandermonde's identity (1.6) and proves the lemma. □

2.2 Gegenbauer weights

The Gegenbauer (or ultraspherical) polynomials are a special case of the Jacobi polynomials. They form an orthogonal system with respect to the norm

$$\|f\|_\alpha^2 = \int_{-1}^{1}|f(t)|^2(1-t^2)^\alpha \mathrm{d}t.$$

The nth Gegenbauer polynomial for this norm is given by

$$G_n(t,\alpha) = \frac{1}{2^n} \sum_{\ell=0}^{\lfloor n/2 \rfloor} \binom{2\alpha + 2n - 2\ell}{n - 2\ell} \binom{\alpha + n}{\ell} (-1)^\ell t^{n-2\ell},$$

which can also be represented by a Rodrigues formula as

$$G_n(t,\alpha) = \frac{(-1)^n}{2^n n!} (1 - t^2)^{-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}t^n} (1 - t^2)^{\alpha+n}.$$

The nth normalized Gegenbauer polynomial then takes the form

$$\widehat{G}_n(t,\alpha) = w_n(\alpha) G_n(t,\alpha), \qquad w_n(\alpha) := \sqrt{\frac{n!(2n + 2\alpha + 1)\Gamma(n + 2\alpha + 1)}{2^{2\alpha+1}\Gamma^2(n + \alpha + 1)}}.$$

Lemma 2.2. *Let $C_n = (c_{jk}^{(\nu)}(\alpha,\beta))_{j,k=0}^n$ be the matrix representation for the differential operator $D^\nu : \mathcal{P}_n(\alpha) \to \mathcal{P}_n(\beta)$ with respect to the orthonormal bases $\{\widehat{G}_0(\cdot,\alpha), \dots, \widehat{G}_n(\cdot,\alpha)\}$ and $\{\widehat{G}_0(\cdot,\beta), \dots, \widehat{G}_n(\cdot,\beta)\}$. Then the entries $c_{jk}^{(\nu)}(\alpha,\beta)$ are given by*

$$\begin{aligned} c_{jk}^{(\nu)}(\alpha,\beta) = {} & \frac{w_k(\alpha)}{w_j(\beta)} \frac{(2\alpha + k + 1)_\nu}{2^\nu} \frac{(\alpha + \nu + 1)_{k-\nu} (2\beta + 1)_j}{(2\alpha + 2\nu + 1)_{k-\nu} (\beta + 1)_j} \\ & \times \frac{j + \beta + 1/2}{(k + j - \nu)/2 + \beta + 1/2} \frac{(\alpha + \nu - \beta)_{(k-j-\nu)/2} (\alpha + \nu + 1/2)_{(k+j-\nu)/2}}{((k - j - \nu)/2)! (\beta + 1/2)_{(k+j-\nu)/2}} \end{aligned} \tag{2.5}$$

if $k - \nu - j \geq 0$ is even, or zero otherwise.

There are several possibilities to prove this. As stated in [10], this reduces to the connection problem for Gegenbauer polynomials. The result is known and can be found for example in [2]. However, these polynomials are defined slightly different in [2]. The form we use here coincides with the Jacobi polynomials $P_n^{(\alpha,\alpha)}(t)$ given in [2]. The connection coefficients for the Jacobi polynomials are then as in Lemma 7.1.1 of [2]. The proof is a little simpler when not performed for general Jacobi polynomials but for our Gegenbauer polynomials instead. The following is inspired by the proof given in [2]. We present it modified to our needs.

Proof. The numbers $c_{jk}^{(\nu)}(\alpha,\beta)$ are determined by the equation

$$\widehat{G}_k^{(\nu)}(t,\alpha) = \sum_{j=0}^{k-\nu} c_{jk}^{(\nu)}(\alpha,\beta) \widehat{G}_j(\beta).$$

It is easy to show that

$$\frac{\mathrm{d}}{\mathrm{d}t} G_k(t,\alpha) = \frac{2\alpha + k + 1}{2} G_{k-1}(t,\alpha + 1).$$

By induction, we conclude that

$$\frac{\mathrm{d}^\nu}{\mathrm{d}t^\nu} G_k(t,\alpha) = \frac{(2\alpha+k+1)_\nu}{2^\nu} G_{k-\nu}(t,\alpha+\nu).$$

Therefore, the above equation can be written as

$$w_k(\alpha)\frac{(2\alpha+k+1)_\nu}{2^\nu} G_{k-\nu}(t,\alpha+\nu) = \sum_{j=0}^{k-\nu} c_{jk}^{(\nu)}(\alpha,\beta) w_j(\beta) G_j(t,\beta).$$

Since the polynomials $w_j(\beta)G_j(t,\beta)$ form an orthonormal system with respect to the inner product $(f,g)_\beta := \int_{-1}^1 f(t)\overline{g(t)}(1-t^2)^\beta \mathrm{d}t$, the $c_{jk}^{(\nu)}(\alpha,\beta)$ are given by

$$c_{jk}^{(\nu)}(\alpha,\beta) = w_k(\alpha)\frac{(2\alpha+k+1)_\nu}{2^\nu} w_j(\beta)(G_{k-\nu}(\cdot,\alpha+\nu), G_j(\cdot,\beta))_\beta.$$

We first evaluate the inner product. Employing the Rodrigues formula for the Gegenbauer polynomials, we get

$$\int_{-1}^1 G_{k-\nu}(t,\alpha+\nu)G_j(t,\beta)(1-t^2)^\beta \mathrm{d}t = \frac{(-1)^j}{2^j j!}\int_{-1}^1 G_{k-\nu}(t,\alpha+\nu)\frac{\mathrm{d}^j}{\mathrm{d}t^j}(1-t^2)^{\beta+j}\mathrm{d}t.$$

Using integration by parts, this is the same as

$$\begin{aligned}\frac{1}{2^j j!}\int_{-1}^1 (1-t^2)^{\beta+j}\frac{\mathrm{d}^j}{\mathrm{d}t^j}G_{k-\nu}(\alpha+\nu)\mathrm{d}t \\ = \frac{(2\alpha+k+\nu+1)_j}{2^{2j}j!}\int_{-1}^1 G_{k-j-\nu}(t,\alpha+\nu+j)(1-t^2)^{\beta+j}\mathrm{d}t.\end{aligned}$$

Inserting the explicit formula for the Gegenbauer polynomial $G_{k-j-\nu}(t,\alpha+\nu+j)$, this becomes

$$\frac{(2\alpha+k+\nu+1)_j}{2^{k+j-\nu}j!}\sum_{\ell=0}^{\lfloor (k-j-\nu)/2\rfloor}\binom{2\alpha+2k-2\ell}{k-j-\nu-2\ell}\binom{\alpha+k}{\ell}(-1)^\ell\int_{-1}^1 t^{k-j-\nu-2\ell}(1-t^2)^{\beta+j}\mathrm{d}t.$$

The integral vanishes if $k-j-\nu$ is an odd number and evaluates to

$$\frac{\Gamma((k-j-\nu)/2-\ell+1/2)\Gamma(\beta+j+1)}{\Gamma((k-j-\nu)/2-\ell+\beta+j+3/2)}$$

whenever $k-j-\nu \geq 0$ is an even number, which we will assume in the following. We now get

$$\begin{aligned}&\frac{(2\alpha+k+\nu+1)_j}{2^{k+j-\nu}j!}\sum_{\ell=0}^{(k-j-\nu)/2}\frac{(2\alpha+k+j+\nu+1)_{k-j-\nu-2\ell}}{(1)_{k-j-\nu-2\ell}}\frac{(\alpha+k-\ell+1)_\ell}{\ell!}(-1)^\ell \\ &\qquad\times\frac{\Gamma((k-j-\nu)/2-\ell+1/2)\Gamma(\beta+j+1)}{\Gamma((k-j-\nu)/2-\ell+\beta+j+3/2)} \\ &= \frac{(2\alpha+k+\nu+1)_{k-\nu}\Gamma(\beta+j+1)\Gamma((k-j-\nu)/2+1/2)}{2^{k+j-\nu}j!(k-j-\nu)!\Gamma((k-j-\nu)/2+\beta+j+3/2)} \\ &\quad\times\sum_{\ell=0}^{(k-j-\nu)/2}\frac{(k-j-\nu-2\ell+1)_{2\ell}}{(2\alpha+2k-2\ell+1)_{2\ell}}\frac{(-\alpha-k)_\ell}{\ell!}\frac{((k-j-\nu)/2-\ell+\beta+j+3/2)_\ell}{((k-j-\nu)/2-\ell+1/2)_\ell}.\end{aligned}$$

The sum can be transformed to

$${}_2F_1\left(\begin{matrix}-(k-j-\nu)/2,\ -\beta-(k+j-\nu)/2-1/2\\ -\alpha-k+1/2\end{matrix};1\right)=\frac{(\beta-\alpha-\nu-(k-j-\nu)/2+1)_{(k-j-\nu)/2}}{(-\alpha-k+1/2)_{(k-j-\nu)/2}}$$

by the Chu-Vandermonde identity (1.5). Next, we evaluate $w_j^2(\beta)(G_{k-\nu}(\cdot,\alpha+\nu),G_j(\cdot,\beta))_\beta$:

$$\frac{(\beta+j+1/2)\Gamma(2\beta+j+1)}{2^{2\beta}\Gamma(\beta+j+1)} \times\frac{(2\alpha+k+\nu+1)_{k-\nu}\Gamma((k-j-\nu)/2+1/2)(\alpha+\nu-\beta)_{(k-j-\nu)/2}}{2^{k+j-\nu}(1)_{k-j-\nu}\Gamma(\beta+(k+j-\nu)/2+3/2)(\alpha+\nu+(k+j-\nu)/2+1/2)_{(k-j-\nu)/2}}.$$

After writing the rising factorials in terms of the gamma function, multiplying by the factor $\frac{(\alpha+\nu+1)_{k-\nu}}{(2\alpha+2\nu+1)_{k-\nu}}\frac{(2\beta+1)_j}{(\beta+1)_j}$ and its reciprocal, and canceling of some terms, this transforms to

$$\frac{(\alpha+\nu+1)_{k-\nu}}{(2\alpha+2\nu+1)_{k-\nu}}\frac{(2\beta+1)_j}{(\beta+1)_j}\frac{(\beta+j+1/2)\Gamma(2\alpha+2k+1)\Gamma((k-j-\nu)/2+1/2)}{2^{2\beta+2k-2\nu}(1/2)_{(k-j-\nu)/2}(1)_{(k-j-\nu)/2}} \times\frac{(\alpha+\nu-\beta)_{(k-j-\nu)/2}\Gamma(\alpha+\nu+(k+j-\nu)/2+1/2)\Gamma(\alpha+\nu+1)\Gamma(2\beta+1)}{\Gamma(\alpha+k+1/2)\Gamma(\beta+(k+j-\nu)/2+3/2)\Gamma(2\alpha+2\nu+1)\Gamma(\alpha+k+1)\Gamma(\beta+1)}.$$

Finally, applying the Legendre duplication formula (1.4), a lot more cancels out, and after writing the gamma functions again as rising factorials, we arrive at

$$\frac{(\alpha+\nu+1)_{k-\nu}}{(2\alpha+2\nu+1)_{k-\nu}}\frac{(2\beta+1)_j}{(\beta+1)_j}\frac{\beta+j+1/2}{\beta+(k+j-\nu)/2+1/2} \times\frac{(\alpha+\nu-\beta)_{(k-j-\nu)/2}}{(1)_{(k-j-\nu)/2}}\frac{(\alpha+\nu+1/2)_{(k+j-\nu)/2}}{(\beta+1/2)_{(k+j-\nu)/2}}.$$

Putting this in the original term for $c_{jk}^{(\nu)}(\alpha,\beta)$, the claim follows. □

We can write (2.5) in the slightly shorter and more symmetric form as

$$c_{jk}^{(\nu)}(\alpha,\beta)=\frac{1}{2^{\beta-\alpha-\nu}}\sqrt{\frac{\Gamma(k+1)(k+\alpha+1/2)}{\Gamma(k+2\alpha+1)}}\sqrt{\frac{\Gamma(j+2\beta+1)(j+\beta+1/2)}{\Gamma(j+1)}} \times\binom{\alpha+\nu-\beta+(k-j-\nu)/2-1}{(k-j-\nu)/2}\frac{\Gamma(\alpha+\nu+(k+j-\nu)/2+1/2)}{\Gamma(\beta+(k+j-\nu)/2+3/2)}. \quad (2.6)$$

2.3 Hermite weights

The last weights we want to consider are Hermite weights. For the classical Hermite weights, the results are already known. Schmidt already called this “trivial” [22]. Indeed, due to the relation

$$H_n^{(\nu)}(t,0)=2^\nu(n-\nu+1)_\nu H_{n-\nu}(t,0)$$

for the classical Hermite polynomials, the matrix representation of the operator of differentiation is just a diagonal matrix. Therefore, the norm and thus the smallest constant is just the maximal absolute value on the diagonal. The corresponding entry is

$$\sqrt{2^{\nu}\frac{\Gamma(n+1)}{\Gamma(n-\nu+1)}} \sim (2n)^{\nu/2}.$$

So, in that case, the constant

$$\eta_n^{(\nu)}(0,0) \sim (2n)^{\nu/2}$$

is fully identified. Since we will treat weighted versions of the Laguerre and Legendre (i. e., Gegenbauer) weights, it is just natural to look for weighted versions of the Hermite weight. The generalized Hermite weights that we will use here have been introduced first by Szegő [26]. They have been studied in more detail by Chihara [13]. As we will see in the following, the matrix representation is in general no longer simple.

We now consider orthogonal polynomials for the norm

$$\|f\|_{\alpha}^2 = \int_{-\infty}^{\infty} |f(t)|^2 |t|^{2\alpha} \mathrm{e}^{-t^2} \mathrm{d}t. \tag{2.7}$$

The nth generalized Hermite polynomial is given by

$$H_n(t,\alpha) = 2^n\,\Gamma(\lfloor n/2\rfloor+1)\sum_{j=0}^{\lfloor n/2\rfloor}\binom{\alpha+\lceil n/2\rceil-1/2}{j}\frac{(-1)^j}{\Gamma(\lfloor n/2\rfloor-j+1)}t^{n-2j}.$$

Normalizing this, we arrive at

$$\begin{aligned}\widehat{H}_n(t,\alpha) &= w_n(\alpha)H_n(t,\alpha)\\ &= \sqrt{\frac{\Gamma(\lfloor n/2\rfloor+1)}{\Gamma(\lceil n/2\rceil+\alpha+1/2)}}\sum_{j=0}^{\lfloor n/2\rfloor}\binom{\alpha+\lceil n/2\rceil-1/2}{j}\frac{(-1)^j}{\Gamma(\lfloor n/2\rfloor-j+1)}t^{n-2j},\end{aligned}$$

where

$$w_n(\alpha) = \left(2^n\sqrt{\Gamma(\lfloor n/2\rfloor+1)\Gamma(\lceil n/2\rceil+\alpha+1/2)}\right)^{-1}.$$

We get the classical Hermite polynomials for $\alpha = 0$.

The matrix representation for the differential operator $D^{\nu} : \mathcal{P}_n(\alpha) \to \mathcal{P}_n(\beta)$ in these bases can be obtained by first transforming the polynomials to the basis consisting of monomials, then taking the νth derivative, and transforming this back to a representation in terms of $\{\widehat{H}_k(\cdot,\beta)\}_{k=0}^{n}$.

Theorem 2.3. *Let* $C_n = (c_{jk}^{(\nu)}(\alpha,\beta))_{j,k=0}^{n}$ *be the matrix representation of the differential operator* D^ν *with respect to the orthonormal bases given by the generalized Hermite polynomials with the corresponding weight. The entries* $c_{jk}^{(\nu)}(\alpha,\beta)$ *are given by*

$$\begin{aligned} c_{jk}^{(\nu)}(\alpha,\beta) = 2^\nu\,\Gamma(\lfloor \nu/2 \rfloor + \nu_k + 1)&\sqrt{\frac{\Gamma(\lceil j/2\rceil + \beta + 1/2)}{\Gamma(\lfloor j/2\rfloor + 1)}\frac{\Gamma(\lfloor k/2\rfloor + 1)}{\Gamma(\lceil k/2\rceil + \alpha + 1/2)}} \\ &\times \binom{\lceil (j+\nu)/2\rceil - 1/2}{\lfloor \nu/2\rfloor + \nu_k}\binom{\beta - \alpha - \lfloor \nu/2\rfloor - \nu_k}{(k-j-\nu)/2} \\ &\times {}_3F_2\left(\begin{matrix} -\lfloor \nu/2\rfloor - \nu_k,\ -(k-j-\nu)/2,\ \beta + \lceil j/2\rceil + 1/2 \\ \beta - \alpha - \lfloor \nu/2\rfloor - \nu_k - (k-j-\nu)/2 + 1,\ \lceil j/2\rceil + 1/2\end{matrix};1\right) \end{aligned} \tag{2.8}$$

if $0 \le k - \nu - j$ *is even, and zero otherwise. Here,* $\nu_k = 1$ *if* k *and* ν *are odd, and* $\nu_k = 0$ *if* k *or* ν *is even (i. e.,* $\nu_k = k\nu \mod 2$*).*

Note that the hypergeometric series occurring here is not defined if $\beta - \alpha - \lfloor \nu/2\rfloor - \nu_k$ is a nonnegative integer smaller than $(k-j-\nu)/2$. But then, the coefficient before this term is zero, and therefore, we define the whole term to be zero. In the other cases, the series terminates naturally before we would come to dividing by zero.

Proof. We have to verify that

$$\widehat{H}_k^{(\nu)}(t,\alpha) = \sum_{j=0}^{k-\nu} c_{jk}^{(\nu)}(\alpha,\beta)\widehat{H}_j(t,\beta) \tag{2.9}$$

holds for all $k \ge \nu$. We do this by a comparison of coefficients. For each $k \ge \nu$,

$$\widehat{H}_k^{(\nu)}(t,\alpha) = \sum_{m=0}^{\lfloor (k-\nu)/2\rfloor} \sqrt{\frac{\Gamma(\lfloor k/2\rfloor + 1)}{\Gamma(\lceil k/2\rceil + \alpha + 1)}}\binom{\alpha + \lceil k/2\rceil - 1/2}{m}\frac{(-1)^m(k-\nu-2m+1)_\nu}{\Gamma(\lfloor k/2\rfloor - m + 1)}t^{k-\nu-2m}.$$

Let a_ℓ^j denote the coefficient of $t^{j-2\ell}$ in $\widehat{H}_j(t,\beta)$. Utilizing the fact that the $c_{jk}^{(\nu)}(\alpha,\beta)$ are zero whenever $k-j-\nu$ is an odd number, we transform the sum in (2.9) to

$$\sum_{m=0}^{\lfloor (k-\nu)/2\rfloor}\sum_{\ell=0}^{m} c_{k-\nu-2m+2\ell,k}^{(\nu)} a_\ell^{k-\nu-2m+2\ell} t^{k-\nu-2m}.$$

For a fixed m, we put in the definition of the coefficients and get

$$\sqrt{\frac{\Gamma(\lfloor k/2\rfloor + 1)}{\Gamma(\lceil k/2\rceil + \alpha + 1)}}\binom{\alpha + \lceil k/2\rceil - 1/2}{m}\frac{(-1)^m(k-\nu-2m+1)_\nu}{\Gamma(\lfloor k/2\rfloor - m + 1)}$$

$$= \sum_{\ell=0}^{m} \frac{(-1)^\ell 2^\nu\, \Gamma(\lfloor \nu/2 \rfloor + \nu_k + 1)}{\Gamma(\lfloor (k-\nu)/2 \rfloor - m + 1)} \sqrt{\frac{\Gamma(\lceil (k-\nu)/2 \rceil + \ell - m + \beta + 1/2)}{\Gamma(\lfloor (k-\nu)/2 \rfloor + \ell - m + 1)}}$$
$$\times \sqrt{\frac{\Gamma(\lfloor k/2 \rfloor + 1)}{\Gamma(\lceil k/2 \rceil + \alpha + 1/2)}} \binom{\lceil k/2 \rceil - m + \ell - 1/2}{\lfloor \nu/2 \rfloor + \nu_k} \binom{\beta - \alpha - \lfloor \nu/2 \rfloor - \nu_k}{m - \ell}$$
$$\times {}_3F_2\left({-\lfloor \nu/2 \rfloor - \nu_k,\ \ell - m,\ \beta - m + \ell + \lceil (k-\nu)/2 \rceil + 1/2 \atop \beta - \alpha - \lfloor \nu/2 \rfloor - \nu_k + \ell - m + 1,\ \lceil (k-\nu)/2 \rceil - m + \ell + 1/2}; 1 \right)$$
$$\times \sqrt{\frac{\Gamma(\lfloor (k-\nu)/2 \rfloor - m + \ell + 1)}{\Gamma(\lceil (k-\nu)/2 \rceil - m + \ell + \beta + 1/2)}} \binom{\beta + \lceil (k-\nu)/2 \rceil - m + \ell - 1/2}{\ell}.$$

When we cancel out the obvious terms, we are left with showing

$$\binom{\alpha + \lceil k/2 \rceil - 1/2}{m} \frac{(-1)^m (k - \nu + 2m + 1)_\nu}{\Gamma(\lfloor k/2 \rfloor - m + 1)} = \frac{2^\nu\, \Gamma(\lfloor \nu/2 \rfloor + \nu_k + 1)}{\Gamma(\lfloor (k-\nu)/2 \rfloor - m + 1)}$$
$$\times \sum_{\ell=0}^{m} \binom{\lceil k/2 \rceil - m + \ell - 1/2}{\lfloor \lfloor \nu/2 \rfloor + \nu_k} \binom{\beta - \alpha - \lfloor \nu/2 \rfloor - \nu_k}{m - \ell} \binom{\beta + \lceil (k-\nu)/2 \rceil - m + \ell - 1/2}{\ell}$$
$$\times (-1)^\ell\, {}_3F_2\left({-\lfloor \nu/2 \rfloor - \nu_k,\ \ell - m,\ \beta - m + \ell + \lceil (k-\nu)/2 \rceil + 1/2 \atop \beta - \alpha - \lfloor \nu/2 \rfloor - \nu_k + \ell - m + 1,\ \lceil (k-\nu)/2 \rceil - m + \ell + 1/2}; 1 \right).$$

We can now write this sum as

$$\sum_{\ell=0}^{m} \sum_{\tau=0}^{\min\{\lfloor \nu/2 \rfloor + \nu_k, m-\ell\}} \binom{\lceil k/2 \rceil - m + \ell - 1/2}{\lfloor \nu/2 \rfloor + \nu_k - \tau} \binom{\beta - \alpha - \lfloor \nu/2 \rfloor - \nu_k}{m - \ell - \tau}$$
$$\times \binom{\beta + \lceil (k-\nu)/2 \rceil - m + \ell - 1/2 + \tau}{\ell + \tau} \binom{\ell + \tau}{\ell} (-1)^\ell$$
$$= \sum_{s=0}^{\lfloor \nu/2 \rfloor + \nu_k - 1} \sum_{\tau=0}^{s} \binom{\lceil k/2 \rceil - m + s - \tau - 1/2}{\lfloor \nu/2 \rfloor + \nu_k - \tau}$$
$$\times \binom{\beta - \alpha - \lfloor \nu/2 \rfloor - \nu_k}{m - s} \binom{\beta + \lceil (k-\nu)/2 \rceil - m + s - 1/2}{s} \binom{s}{\tau} (-1)^{s-\tau}$$
$$+ \sum_{s=\lfloor \nu/2 \rfloor + \nu_k}^{m} \sum_{\tau=0}^{\lfloor \nu/2 \rfloor + \nu_k} \binom{\lceil k/2 \rceil - m + s - \tau - 1/2}{\lfloor \nu/2 \rfloor + \nu_k - \tau}$$
$$\times \binom{\beta - \alpha - \lfloor \nu/2 \rfloor - \nu_k}{m - s} \binom{\beta + \lceil (k-\nu)/2 \rceil - m + s - 1/2}{s} \binom{s}{\tau} (-1)^{s-\tau}.$$

The inner sums are actually only taken for

$$\binom{\lceil k/2 \rceil - m + s - \tau - 1/2}{\lfloor \nu/2 \rfloor + \nu_k - \tau} \binom{s}{\tau} (-1)^\tau$$
$$= \binom{-\lceil k/2 \rceil + m - s + \lfloor \nu/2 \rfloor + \nu_k - 1/2}{\lfloor \nu/2 \rfloor + \nu_k - \tau} \binom{s}{\tau} (-1)^{\lfloor \nu/2 \rfloor + \nu_k}.$$

For $s < \lfloor \nu/2 \rfloor + \nu_k$, this can be written as

$$\binom{-\lceil k/2 \rceil + m - s + \lfloor \nu/2 \rfloor + \nu_k - 1/2}{\lfloor \nu/2 \rfloor + \nu_k - s}\binom{-\lceil k/2 \rceil + m - 1/2}{s - \tau} \\ \times \binom{\lfloor \nu/2 \rfloor + \nu_k}{\tau}\frac{\Gamma(s+1)\Gamma(\lfloor \nu/2 \rfloor + \nu_k - s + 1)}{\Gamma(\lfloor \nu/2 \rfloor + \nu_k + 1)}(-1)^{\lfloor \nu/2 \rfloor + \nu_k}.$$

Now we can apply Vandermonde's identity (1.6) to both inner sums, which then both evaluate to

$$\binom{\lceil k/2 \rceil - m - 1/2}{\lfloor \nu/2 \rfloor + \nu_k}.$$

Since this is independent of s, we can combine the two outer sums again and also apply Vandermonde's identity. So, we arrive at

$$(-1)^m \frac{2^\nu\,\Gamma(\lfloor \nu/2 \rfloor + \nu_k + 1)}{\Gamma(\lfloor (k-\nu)/2 \rfloor - m + 1)}\binom{\lceil k/2 \rceil - m - 1/2}{\lfloor \nu/2 \rfloor + \nu_k}\binom{\alpha + \lfloor \nu/2 \rfloor + \nu_k + \lceil (k-\nu)/2 \rceil - 1/2}{m}.$$

Using $\lceil (k-\nu)/2 \rceil + \lfloor \nu/2 \rfloor + \nu_k = \lceil k/2 \rceil$ and the equality

$$\frac{2^\nu\,\Gamma(\lceil k/2 \rceil - m + 1/2)}{\Gamma(\lfloor (k-\nu)/2 \rfloor - m + 1)\Gamma(\lfloor (k-\nu)/2 \rfloor - m + 1/2)} = \frac{(k-\nu-2m+1)_\nu}{\Gamma(\lfloor k/2 \rfloor - m + 1)},$$

we have shown the theorem. □

Since the hypergeometric series occurring in (2.8) doesn't always satisfy our needs, we will express this in a slightly different form. This is subject of the next corollary.

Corollary 2.4. *Under the assumptions of Theorem 2.3,*

$$c_{jk}^{(\nu)}(\alpha,\beta) = 2^\nu\,\Gamma(\lfloor \nu/2 \rfloor + \nu_k + 1)\sqrt{\frac{\Gamma(\lceil j/2 \rceil + \beta + 1/2)}{\Gamma(\lfloor j/2 \rfloor + 1)}\frac{\Gamma(\lfloor k/2 \rfloor + 1)}{\Gamma(\lceil k/2 \rceil + \alpha + 1/2)}} \\ \times \sum_{\tau=0}^{\min\{\lfloor \nu/2 \rfloor + \nu_k, (k-\nu-j)/2\}} \binom{\lceil (j+\nu)/2 \rceil - 1/2}{\lfloor \nu/2 \rfloor + \nu_k - \tau}\binom{\beta + \lceil j/2 \rceil + \tau - 1/2}{\tau}\binom{\beta - \alpha - \lfloor \nu/2 \rfloor - \nu_k}{(k-\nu-j)/2 - \tau}$$

if $k - \nu - j$ is even and $k - \nu \geq j$, and zero otherwise.

Proof. The sums occurring in this representation are hypergeometric series. We see this by determining the quotient of the terms belonging to $\tau + 1$ and τ and separating the common factor by setting $\tau = 0$ (see also [20, page 16]). The quotient is

$$\frac{(-\lfloor \nu/2 \rfloor - \nu_k + \tau)(-(k-\nu-j)/2 + \tau)(\beta + \lceil j/2 \rceil + \tau)}{(\beta - \alpha - \lfloor \nu/2 \rfloor - \nu_k - (k-\nu-j)/2 + 1 + \tau)(\lceil j/2 \rceil + 1/2 + \tau)(\tau + 1)},$$

and the term for $\tau = 0$ is simply

$$\binom{\lceil (j+\nu)/2\rceil - 1/2}{\lfloor \nu/2\rfloor + \nu_k}\binom{\beta-\alpha-\lfloor \nu/2\rfloor - \nu_k}{(k-\nu-j)/2}.$$

Putting all this together, the sum over τ is the same as

$$\binom{\lceil (j+\nu)/2\rceil - 1/2}{\lfloor \nu/2\rfloor + \nu_k}\binom{\beta-\alpha-\lfloor \nu/2\rfloor - \nu_k}{(k-\nu-j)/2} \times {}_3F_2\left(\begin{matrix} -\lfloor \nu/2\rfloor - \nu_k,\ -(k-\nu-j)/2,\ \beta+\lceil j/2\rceil + 1/2 \\ \beta-\alpha-\lfloor \nu/2\rfloor - \nu_k - (k-\nu-j)/2+1,\ \lceil j/2\rceil + 1/2 \end{matrix};1\right).$$

The assumption follows from Theorem 2.3. □

Since the matrix has a chessboard structure above the diagonal and the first ν columns are zero, we are going to consider the odd and even parts separately. These are given by

$$\begin{aligned} c^{(\nu)}_{2j,2k+\nu}(\alpha,\beta) &= 2^\nu\,\Gamma(\lceil \nu/2\rceil + 1)\sqrt{\frac{\Gamma(j+\beta+1/2)}{\Gamma(j+1)}\frac{\Gamma(k+\lfloor \nu/2\rfloor+1)}{\Gamma(k+\lceil \nu/2\rceil+\alpha+1/2)}} \\ &\quad\times \sum_{\tau=0}^{\min\{\lceil \nu/2\rceil, k-j\}} \binom{j+\lceil \nu/2\rceil - 1/2}{\lceil \nu/2\rceil - \tau}\binom{\beta+j+\tau-1/2}{\tau}\binom{\beta-\alpha-\lceil \nu/2\rceil}{k-j-\tau}, \\ c^{(\nu)}_{2j+1,2k+1+\nu}(\alpha,\beta) &= 2^\nu\,\Gamma(\lfloor \nu/2\rfloor + 1)\sqrt{\frac{\Gamma(j+\beta+3/2)}{\Gamma(j+1)}\frac{\Gamma(k+\lceil \nu/2\rceil+1)}{\Gamma(k+\lfloor \nu/2\rfloor+\alpha+3/2)}} \\ &\quad\times \sum_{\tau=0}^{\min\{\lfloor \nu/2\rfloor, k-j\}} \binom{j+\lfloor \nu/2\rfloor + 1/2}{\lfloor \nu/2\rfloor - \tau}\binom{\beta+j+\tau+1/2}{\tau}\binom{\beta-\alpha-\lfloor \nu/2\rfloor}{k-j-\tau}. \end{aligned}$$

Although it is not of immediate use in our analysis, it is worth mentioning that these restricted matrices have some special structure. This is kept in the next corollary.

Corollary 2.5. *Let $E_N = (e_{jk})_{j,k=0}^N$ and $F_M = (f_{jk})_{j,k=0}^M$ be the matrices with the entries*

$$\begin{aligned} e_{jk} &= c^{(\nu)}_{2j,2k+\nu}(\alpha,\beta), && 0 \le j,k \le N, \\ f_{jk} &= c^{(\nu)}_{2j+1,2k+\nu+1}(\alpha,\beta), && 0 \le j,k \le M \end{aligned}$$

for appropriate choices of N and M and $c^{(\nu)}_{jk}(\alpha,\beta)$ from above. Then,

$$\begin{aligned} E_N &= 2^\nu\Gamma(\lceil \nu/2\rceil + 1)\,\mathrm{diag}\left(\sqrt{\frac{\Gamma(j+\beta+1/2)}{\Gamma(j+1)}}\right)_{j=0}^N \\ &\quad\times R_N T_N^*((1+z)^{\beta-\alpha-\lceil \nu/2\rceil})\,\mathrm{diag}\left(\sqrt{\frac{\Gamma(k+\lfloor \nu/2\rfloor+1)}{\Gamma(k+\lceil \nu/2\rceil+\alpha+1/2)}}\right)_{j=0}^N, \end{aligned}$$

and

$$F_M = 2^\nu \Gamma(\lfloor \nu/2 \rfloor + 1) \operatorname{diag}\left(\sqrt{\frac{\Gamma(j+\beta+3/2)}{\Gamma(j+1)}} \right)_{j=0}^{M} \times R_M T_M^* \left((1+z)^{\beta-\alpha-\lfloor \nu/2 \rfloor}\right) \operatorname{diag}\left(\sqrt{\frac{\Gamma(k+\lceil \nu/2 \rceil+1)}{\Gamma(k+\lfloor \nu/2 \rfloor+\alpha+3/2)}} \right)_{j=0}^{M},$$

where $T_N\left((1+z)^{\beta-\alpha-\lceil \nu/2 \rceil}\right)$ *and* $T_N\left((1+z)^{\beta-\alpha-\lfloor \nu/2 \rfloor}\right)$ *are Toeplitz matrices generated by the Taylor coefficients of their corresponding symbol.* $R_N = (r_{jk}^e)_{j,k=0}^N$ *and* $R_M = (r_{jk}^f)_{j,k=0}^M$ *are banded matrices of bandwidth* $\lceil \nu/2 \rceil + 1$ *and* $\lfloor \nu/2 \rfloor + 1$, *respectively, with*

$$r_{jk}^e = \binom{j+\lceil \nu/2 \rceil - 1/2}{\lceil \nu/2 \rceil - k}\binom{\beta+j+k-1/2}{k},$$
$$r_{jk}^f = \binom{j+\lfloor \nu/2 \rfloor + 1/2}{\lfloor \nu/2 \rfloor - k}\binom{\beta+j+k+1/2}{k}.$$

Proof. This follows immediately from the aforementioned representations. □

Chapter 3

Best constants for integral differences

Contents

This chapter is devoted to the cases in which the parameter differences are nonnegative integers. These are relatively simple compared to the more general setting lying ahead of us. The matrix entries $c_{jk}^{(\nu)}(\alpha, \beta)$ for each of the Laguerre, Gegenbauer, and Hermite norms contain a factor that looks similar to $\binom{\omega}{k-j-\nu}$. Now, if ω is a nonnegative integer, this binomial coefficient is zero whenever $k - j - \nu > \omega$. Therefore, all diagonals that are far away enough from the main diagonal (well, actually the νth diagonal) are zero. This eases things up tremendously. We can and will employ the very simple estimate $\|B\|_\infty \le \sum_{\ell=0}^{n} \|B_\ell\|_\infty$, where B denotes said block, and the matrices B_ℓ contain the ℓth diagonal and are zero otherwise. Since we have only a few nonzero diagonals, say m, the sum is actually independent on n and ends at $m - 1$. The norms of the B_ℓ can be given explicitly. They are just the maximal absolute value among all entries on the ℓth diagonal.

Moreover, due to the banded structure it is easy to show that the matrices converge to a special Toeplitz matrix the symbol of which is at hand and turns out to be bounded. One important result about such matrices (see, e. g., [12, page 10]) is that the spectral norm is determined by the maximal absolute value of the symbol over the unit circle. While this might also work for nonintegral differences, it would be harder to show the convergence. On the other hand, for $\omega < 0$ the symbol is not bounded anymore, and this approach fails.

In the next section, we will recap the results for the Laguerre and Gegenbauer cases found in [7, 10]. After this, we turn to the Hermite case, which is a new result.

3.1 Integral differences in the Laguerre and Gegenbauer setting

In this section, we will only repeat the basic ideas from [7] and [10] to get the direction for the upcoming treatment of the Hermite case and to have the results available readily.

The matrix entries in the Laguerre setting are particularly simple. Assume $\beta = \alpha + m$ with some integer $m \geq \nu$. Furthermore, let B_N denote the upper right $(N \times N)$-block of the matrix representation of the differential operator D^ν, let $N = n - \nu + 1$, and let $B_{N,\ell}$ be the matrix consisting only of the ℓth diagonal of B_N. Then,

$$\|B_{N,\ell}\|_\infty = \max_{0 \leq j \leq N-\ell} \binom{m-\nu}{\ell} \frac{w_{n-j}(\alpha)}{w_{n-\nu-j-\ell}(\alpha+m)}.$$

The last quotient can be shown to be $n^{m/2}(1 + O(1/n))$, which is independent of j. Similarly, the binomial coefficient is independent of j. Summing over all diagonals, we get

$$\|B_N\|_\infty \leq \sum_{\ell=0}^{m-\nu} \|B_{N,\ell}\|_\infty \leq \sum_{\ell=0}^{m-\nu} \binom{m-\nu}{\ell} n^{m/2}(1 + O(1/n)) = 2^{m-\nu} n^{m/2}(1 + O(1/n)).$$

To derive a lower bound, we consider the scaled operators $n^{-m/2} B_N \pi_N$ on ℓ^2. Here, π_N is the projection onto the first N coordinates. Because these operators are uniformly bounded and the bandwidth is independent of N, it suffices to show that they converge entrywise to the Toeplitz operator $T^*((1-z)^{m-\nu})$, which is given by the semi-infinite matrix consisting of the Taylor coefficients of the function $(1-z)^{m-\nu}$ at $z = 0$. From the Banach-Steinhaus theorem it follows that

$$\liminf_{n\to\infty} \|n^{-m/2} B_N\|_\infty \geq \|T^*((1-z)^{m-\nu})\|_\infty.$$

But

$$\|T^*((1-z)^{m-\nu})\|_\infty = \max_{|z|=1} |1-z|^{m-\nu} = 2^{m-\nu}.$$

Putting together the upper and lower bound, Böttcher and Dörfler have indeed shown that

$$\lambda_n^{(\nu)}(\alpha, \alpha + m) = \|B_N\|_\infty \sim 2^{m-\nu} n^{m/2} = 2^{\beta-\alpha-\nu} n^{(\beta-\alpha)/2}.$$

The matrix representation in the Gegenbauer setting is a little bit more involved. Here we have a chessboard structure, i. e., the entries of the matrix are zero whenever $k - j - \nu$ is an odd number. Therefore, at most the $\nu + 2\ell$th diagonals, $\ell \in \mathbb{N}_0$, will contain nonzero entries. Due to the special case they try to handle, Böttcher and Dörfler [10] first stated that the kth entry of the 2ℓth diagonal behaves like

$$\omega_{k+\nu}^{(\nu)}(\alpha)(-1)^\ell \binom{m-\nu}{\ell} \frac{1}{2^{m-\nu}} + O(1/k) \quad \text{as } k \to \infty,$$

where

$$\omega_k^{(\nu)}(\alpha) = \sqrt{\frac{k!}{(k-\nu)!}\frac{\Gamma(k+2\alpha+\nu+1)}{\Gamma(k+2\alpha+1)}}.$$

With this the corresponding norm can be estimated by

$$\|B_{N,\ell}\|_\infty \leq n^\nu \binom{m-\nu}{\ell}\frac{1}{2^{m-\nu}}(1+O(1/n)).$$

Here, $B_{N,\ell}$ is the matrix consisting only of the 2ℓth diagonal of B_N – the same upper right block as in the Laguerre case. Therefore, the norm can be estimated by

$$\|B_N\|_\infty \leq n^\nu(1+O(1/n)).$$

For the lower bound, exactly the same course of action is taken as in the Laguerre case, this time with the symbol $2^{-(m-\nu)}(1-z^2)^{m-\nu}$. Combining the estimates, they have shown that

$$\gamma_n^{(\nu)}(\alpha,\alpha+m) \sim n^\nu.$$

3.2 The Hermite case

The matrix representation of the operator D^ν has a chessboard structure above the main diagonal, again. Since $\eta_{n-1}^{(\nu)}(\alpha,\beta) \leq \eta_n^{(\nu)}(\alpha,\beta) \leq \eta_{n+1}^{(\nu)}(\alpha,\beta)$, we assume that $N = n-\nu+1$ is an even number. Then there is some permutation matrix U_N with

$$A_N = U_N \begin{pmatrix} E_N & 0 \\ 0 & F_N \end{pmatrix} U_N,$$

where $E_N = (e_{jk})_{j,k=0}^{N/2-1}$, $F_N = (f_{jk})_{j,k=0}^{N/2-1}$ are built from the entries

$$e_{jk} = c_{2j,2k+\nu}^{(\nu)}(\alpha,\beta), \quad f_{jk} = c_{2j+1,2k+\nu+1}^{(\nu)}(\alpha,\beta).$$

We confine ourselves to the investigation of the matrix E_N, and we point out that the matrix F_N can be treated likewise. In the following, we will consider two distinct cases of integral differences. First, we will restrict ourselves to $\beta-\alpha \geq 0$. We can then exploit the much simpler structure of the matrix. Later, we will handle one more case, namely $\beta = \alpha$, in order to prepare the proofs of the general situation.

Assume now that $\beta-\alpha$ is an integer, not smaller than $\lceil \nu/2 \rceil$. We can see from Theorem 2.3 that the matrix under investigation is banded. This is due to the term $\binom{\beta-\alpha-\lfloor \nu/2 \rfloor-\nu_k}{(k-j-\nu)/2}$ occurring in the matrix representation. We will employ the same idea that was applied for the Laguerre case in [7] and the Gegenbauer case in [10]: consider the matrix as a sum of (shifted) diagonal matrices and use that the norm of the sum is less than the sum of the norms of these diagonals. To derive a lower estimate, we show that some scaled version of the matrix E_N converges in the norm to a given Toeplitz operator.

Let $m = \beta - \alpha - \lceil \nu/2 \rceil \in \mathbb{N}_0$, i. e., consider the banded case. Then the entries $d_j^{(\ell)}$, $\ell = 0, \dots, m$, of the ℓth diagonal in row j are given by

$$d_j^{(\ell)} = 2^\nu \Gamma(\lceil \nu/2 \rceil + 1) \sqrt{\frac{\Gamma(j+\beta+1/2)}{\Gamma(j+1)} \frac{\Gamma(j+\ell+\lfloor \nu/2 \rfloor + 1)}{\Gamma(j+\ell+\lceil \nu/2 \rceil + \alpha + 1/2)}} \\ \times \sum_{\tau=0}^{\min\{\lceil \nu/2 \rceil, \ell\}} \binom{j+\lceil \nu/2 \rceil - 1/2}{\lceil \nu/2 \rceil - \tau} \binom{\beta + j + \tau - 1/2}{\tau} \binom{m}{\ell - \tau}.$$

We now take a closer look on the particular terms. First, for the terms under the square root, the ratio of the term under the square root for $j+1$ and j satisfies

$$\frac{j+\beta+1/2}{j+1} \cdot \frac{j+\ell+\lfloor \nu/2 \rfloor + 1}{j+\ell+\lceil \nu/2 \rceil + \alpha + 1/2} = \frac{j+m+1/2+\alpha+\lceil \nu/2 \rceil}{j+\ell+1/2+\alpha+\lceil \nu/2 \rceil} \cdot \frac{j+\ell+\lfloor \nu/2 \rfloor + 1}{j+1} \geq 1,$$

since $\ell \leq m$. Thus, these factors are increasing with respect to j. The same is true for the first two binomial coefficients in the sum. The ratio for every single summand of the sum is

$$\frac{j+\lceil \nu/2 \rceil + 1/2}{j+\tau+1/2} \cdot \frac{\beta + j + \tau + 1/2}{\beta + j + 1/2} \geq 1.$$

The third binomial coefficient is constant along the diagonal, independently of j. So, the maximum (and with this the norm of this diagonal matrix) is attained for $j = N/2 - 1 - \ell$. We get the following upper estimate for the norm of E_N:

$$\|E_N\|_\infty \leq 2^\nu \Gamma(\lceil \nu/2 \rceil + 1) \sum_{\ell=0}^{m} \sqrt{\frac{\Gamma(N/2-\ell+\beta-1/2)\Gamma(N/2+\lfloor \nu/2 \rfloor)}{\Gamma(N/2-\ell)\Gamma(N/2+\lceil \nu/2 \rceil + \alpha - 1/2)}} \\ \times \sum_{\tau=0}^{\min\{\lceil \nu/2 \rceil, \ell\}} \binom{N/2-\ell+\lceil \nu/2 \rceil - 3/2}{\lceil \nu/2 \rceil - \tau} \binom{\beta + N/2 - \ell + \tau - 3/2}{\tau} \binom{m}{\ell - \tau}.$$

We ignore the constant factor for the moment and replace the square root terms by the maximum over $0 \leq \ell \leq m$, ignoring its value for the moment, too. The sum now reduces to

$$\begin{aligned} &\sum_{\ell=0}^{m} \sum_{\tau=0}^{\min\{\lceil \nu/2 \rceil, \ell\}} \binom{N/2-\ell+\lceil \nu/2 \rceil - 3/2}{\lceil \nu/2 \rceil - \tau} \binom{\beta + N/2 - \ell + \tau - 3/2}{\tau} \binom{m}{\ell - \tau} \\ &\quad \leq \sum_{\ell=0}^{m} \sum_{\tau=0}^{\min\{\lceil \nu/2 \rceil, \ell\}} \binom{N/2+\lceil \nu/2 \rceil - 1/2}{\lceil \nu/2 \rceil - \tau} \binom{\beta + N/2 + \lceil \nu/2 \rceil - 1/2}{\tau} \binom{m}{\ell - \tau} \\ &\quad = \sum_{\tau=0}^{\lceil \nu/2 \rceil} \binom{N/2+\lceil \nu/2 \rceil - 1/2}{\lceil \nu/2 \rceil - \tau} \binom{\beta + N/2 + \lceil \nu/2 \rceil - 1/2}{\tau} \sum_{\ell=0}^{m-\tau} \binom{m}{\ell} \\ &\quad \leq \sum_{\tau=0}^{\lceil \nu/2 \rceil} \binom{N/2+\lceil \nu/2 \rceil - 1/2}{\lceil \nu/2 \rceil - \tau} \binom{\beta + N/2 + \lceil \nu/2 \rceil - 1/2}{\tau} \sum_{\ell=0}^{m} \binom{m}{\ell} \\ &\quad = \frac{\Gamma(\beta + N + 2\lceil \nu/2 \rceil)}{\Gamma(\beta + N + \lceil \nu/2 \rceil)\Gamma(\lceil \nu/2 \rceil + 1)} 2^m. \end{aligned}$$

Applying the standard asymptotic formulas for all terms, we get to the following upper bound for $\|E_N\|_\infty$:

$$\begin{aligned}\|E_N\|_\infty &\le 2^\nu \left(\tfrac{N}{2}\right)^{(\beta-1/2)/2}\left(\tfrac{N}{2}\right)^{(-\alpha+1/2+\lfloor\nu/2\rfloor-\lceil\nu/2\rceil)/2}\cdot 2^{\beta-\alpha-\lceil\nu/2\rceil}N^{\lceil\nu/2\rceil}(1+O(1/N))\\ &= 2^{(\beta-\alpha+\nu)/2}N^{(\beta-\alpha+\nu)/2}(1+O(1/N)).\end{aligned} \tag{3.1}$$

To derive a lower bound on the norm of E_N, we use an approach analogous to the one in [7]. Let J_N denote the $(N\times N)$-matrix with ones on the counterdiagonal and zeros elsewhere. We set $B_N = J_N E_N J_N$. Obviously, $\|E_N\|_\infty = \|E_N^*\|_\infty = \|B_N\|_\infty$. It is easily seen that the entry at position jk of B_N equals the entry at the position $N/2-1-j, N/2-1-k$ from E_N. Now, let π_N be the projection

$$\pi_N : \ell^2 \to \ell^2, \quad \{x_0,x_1,x_2,\dots\}\mapsto\{x_0,x_1,\dots,x_{N/2-1},0,\dots\}$$

and consider the operators $T_N = 2^{(\beta-\alpha-\nu)/2}N^{(\alpha-\beta-\nu)/2}B_N\pi_N$ on ℓ^2. We will show that these operators converge strongly to the Toeplitz operator $T^*((1+z)^{\beta-\alpha})$ on ℓ^2 that is given by the infinite Toeplitz matrix $(\varphi_{jk})_{j,k=0}^\infty$ with $\varphi_{jk}=0$ for $k>j$ and

$$\varphi_{jk} = \binom{\beta-\alpha}{j-k} \quad \text{for } k\le j. \tag{3.2}$$

First, we infer from (3.1) that $\|T_N\|_\infty \le 2^{\beta-\alpha}(1+O(1/N))$. Thus, the operators T_N are uniformly bounded. To prove that $T_N \to T((1+z)^{\beta-\alpha})$ strongly, it is therefore enough to show $T_N e_k$ converges to $T((1+z)^{\beta-\alpha})e_k$ for every $k\ge 0$, where $e_k\in\ell^2$ has 1 at the kth position and zeros elsewhere. As all involved operators are banded with bandwidth $m+1$ independent of N, it suffices to verify that the jk entry of T_N converges to the jk entry of the matrix $T((1+z)^{\beta-\alpha})$. But, the jk entry of T_N is zero for $k>j$ and for $k<j-m$, and if $k\le j\le k+m$, it equals

$$\begin{aligned}&2^{(\beta-\alpha+\nu)/2}N^{(\alpha-\beta-\nu)/2}\Gamma(\lceil\nu/2\rceil+1)\\ &\quad\times\sqrt{\frac{\Gamma(N/2-j+\beta-1/2)}{\Gamma(N/2-j)}}\sqrt{\frac{\Gamma(N/2-k+\lfloor\nu/2\rfloor)}{\Gamma(N/2-k+\lceil\nu/2\rceil+\alpha-1/2)}}\\ &\quad\times\sum_{\tau=0}^{\min\{\lceil\nu/2\rceil,j-k\}}\binom{N/2-j+\lceil\nu/2\rceil-3/2}{\lceil\nu/2\rceil-\tau}\binom{\beta+N/2-j+\tau-3/2}{\tau}\binom{\beta-\alpha-\lceil\nu/2\rceil}{j-k-\tau}\\ &= 2^{(\beta-\alpha+\nu)/2}N^{-(\beta-\alpha+\nu)/2}\sqrt{\frac{\Gamma(N/2-j+\beta-1/2)}{\Gamma(N/2-j)}}\sqrt{\frac{\Gamma(N/2-k+\lfloor\nu/2\rfloor)}{\Gamma(N/2-k+\lceil\nu/2\rceil+\alpha-1/2)}}\\ &\quad\times\sum_{\tau=0}^{\min\{\lceil\nu/2\rceil,j-k\}}\frac{\Gamma(N/2-j+\lceil\nu/2\rceil-1/2)}{\Gamma(N/2-j+\tau-1/2)}\frac{\Gamma(\beta+N/2-j+\tau-1/2)}{\Gamma(\beta+N/2-j-1/2)}\\ &\quad\times\frac{\Gamma(\lceil\nu/2\rceil+1)}{\Gamma(\lceil\nu/2\rceil-\tau+1)\Gamma(\tau+1)}\binom{\beta-\alpha-\lceil\nu/2\rceil}{j-k-\tau}\\ &\sim\left(\tfrac{N}{2}\right)^{-(\beta-\alpha+\nu)/2}\left(\tfrac{N}{2}\right)^{(\beta-1/2)/2}\left(\tfrac{N}{2}\right)^{(-\alpha-\lceil\nu/2\rceil+\lfloor\nu/2\rfloor+1/2)/2}\\ &\quad\times\sum_{\tau=0}^{\min\{\lceil\nu/2\rceil,j-k\}}\left(\tfrac{N}{2}\right)^{\lceil\nu/2\rceil-\tau}\left(\tfrac{N}{2}\right)^{\tau}\binom{\lceil\nu/2\rceil}{\tau}\binom{\beta-\alpha-\lceil\nu/2\rceil}{j-k-\tau}\end{aligned}$$

$$= \binom{\beta - \alpha}{j-k}.$$

We assumed that the last sum actually runs to $j-k$ in order to use Vandermonde's identity (1.6). For $j-k \leq \lceil \nu/2 \rceil$, this is clear. For $j-k > \lceil \nu/2 \rceil$, this can be justified by the fact that all of the terms $\binom{\lceil \nu/2 \rceil}{\tau}$ are zero for $\tau > \lceil \nu/2 \rceil$. Comparing this result with (3.2), we arrive at the conclusion that T_N converges strongly to $T^*((1+z)^{\beta-\alpha})$, as asserted. From the Banach-Steinhaus theorem we therefore deduce

$$\liminf_{N\to\infty} \|2^{(\beta-\alpha-\nu)/2} N^{(\alpha-\beta-\nu)/2} B_N\|_\infty \geq \|T((1+z)^{\beta-\alpha})\|_\infty.$$

But, by a well-known result on the norm of Toeplitz operators (see, e.g., [12, page 10]), the latter is

$$\|T((1+z)^{\beta-\alpha})\|_\infty = \max_{|z|=1} |1+z|^{\beta-\alpha} = 2^{\beta-\alpha}.$$

Thus,

$$\liminf_{N\to\infty} N^{(\alpha-\beta-\nu)/2} \|E_N\|_\infty \geq 2^{(\beta-\alpha+\nu)/2}. \tag{3.3}$$

Combining (3.1) and (3.3), we obtain that

$$\|E_N\|_\infty \sim (2N)^{(\beta-\alpha+\nu)/2}.$$

As above, one can show that

$$\|F_N\|_\infty \sim (2N)^{(\beta-\alpha+\nu)/2}.$$

Note that the latter is even true for $\beta - \alpha = \lfloor \nu/2 \rfloor$. However, when ν is an odd number, the matrix C_n has some weird structure, because F_N is banded and E_N is a full triangular matrix. Since $\eta_n^{(\nu)}(\alpha,\beta)$ depends on both, $\|E_N\|_\infty$ and $\|F_N\|_\infty$, this result on $\|F_N\|_\infty$ alone is not of substantial value. Anyway, we will later prove the same asymptotics for $\|E_N\|_\infty$ by more sophisticated means. It will then be a consequence of the investigation of the nonintegral case.

Since $\|C_n\|_\infty = \max\{\|E_N\|_\infty, \|F_N\|_\infty\}$, we obtain for $\beta - \alpha \geq \lceil \nu/2 \rceil$, $\beta - \alpha$ an integer, the following asymptotic behavior for $n \to \infty$:

$$\eta_n^{(\nu)}(\alpha,\beta) \sim (2n)^{(\beta-\alpha+\nu)/2}.$$

We will prove one more integer case. If $\beta - \alpha = 0$ the matrix is in general not banded anymore. The special case $\alpha = \beta = 0$ has been disposed of before. Then, the matrix is indeed a diagonal matrix and it is known that

$$\eta_n^{(\nu)}(0,0) \sim (2n)^{\nu/2}.$$

Assume for the rest of this section that $\alpha = \beta \neq 0$. We will show that the asymptotic expressions obtained above, with the restriction $\beta - \alpha - \lceil \nu/2 \rceil \geq 0$, also hold for $\beta = \alpha$. Taking a closer look

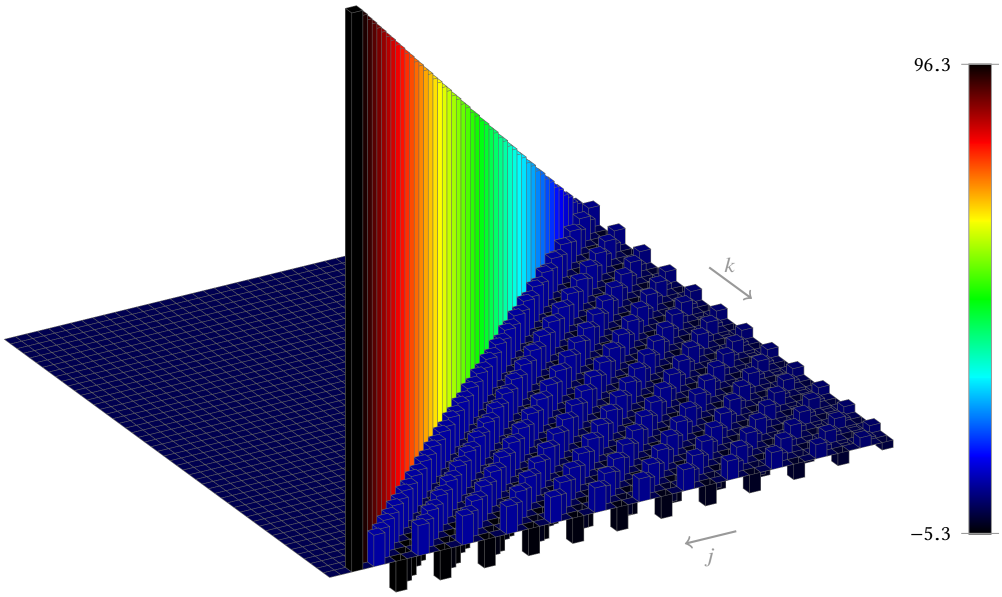

Figure 3.1: Matrix plot for $n = 50$, $\nu = 2$, $\alpha = \beta = 1.4$ in the Hermite setting. Each bar over a square corresponds to an entry in a matrix $(a_{jk})_{j,k=0}^{n}$. The height and color of the bar at the jth row and kth column are determined by the value a_{jk} of the matrix.

at the matrix, we see that, although the matrix is not banded anymore, it is close to a diagonal matrix in the sense that the entries along the diagonal are significantly bigger in their absolute values than the off-diagonal entries. An example is shown in Figure 3.1. Indeed, for $\beta = \alpha$, the last entry on the diagonal of the matrix E_N is given by

$$\begin{aligned} e_{N/2-1,N/2-1} &= 2^{\nu}\Gamma(\lceil \nu/2 \rceil + 1)\sqrt{\frac{\Gamma(N/2+\alpha-1/2)\Gamma(N/2+\lfloor \nu/2 \rfloor)}{\Gamma(N/2)\Gamma(N/2+\lceil \nu/2 \rceil+\alpha-1/2)}}\binom{N/2+\lceil \nu/2 \rceil-3/2}{\lceil \nu/2 \rceil} \\ &= 2^{\nu}\sqrt{\frac{\Gamma(N/2+\alpha-1/2)\Gamma(N/2+\lfloor \nu/2 \rfloor)}{\Gamma(N/2)\Gamma(N/2+\lceil \nu/2 \rceil+\alpha-1/2)}}\frac{\Gamma(N/2+\lceil \nu/2 \rceil-1/2)}{\Gamma(N/2-1/2)} \\ &= 2^{\nu}\left(\frac{N}{2}\right)^{(\alpha-1/2+\lfloor \nu/2 \rfloor-\lceil \nu/2 \rceil-\alpha+1/2)/2+\lceil \nu/2 \rceil}(1+O(1/N)) \\ &= 2^{\nu}\left(\frac{N}{2}\right)^{\nu/2}(1+O(1/N)). \end{aligned}$$

This is exactly what we want. Since $\|E_N e_{N/2-1}\|_2 = \sqrt{\sum_{j=0}^{N/2-1} e_{j,N/2-1}^2} \geq e_{N/2-1,N/2-1}$, this already provides a lower bound. An upper bound is harder to show. The approach we used for the banded matrices does not work here anymore. What we do instead is to use a corollary of the Geršgorin theorem [17, page 344]. The Geršgorin theorem provides discs in the complex plane containing the eigenvalues of a matrix. The closer such a matrix is to a diagonal matrix, the more precise the location can be given. Since we look for the singular values, we could apply the theorem to $E_N^* E_N$. Its eigenvalues all are nonnegative real numbers, so the discs are in fact intervals. However, the matrix representation of $E_N^* E_N$ is not easy to work with. The paper [21] uses the ideas of the Geršgorin theorem directly with the matrix E_N to provide intervals for the location of the singular values. Even more, it also provides a scaled version of the theorem. Since the eigenvalues do not change if we multiply a matrix by an invertible matrix from the right and its inverse from the left, we may modify the matrix entries slightly to get better bounds.

Since we are interested only in the largest singular value of the matrix E_N, the combination of Theorem 2 and Theorem 4 of [21] yields

$$\|E_N\|_\infty \le \max_{0\le i\le N/2-1} \left\{ \sum_{j=0}^{N/2-1} \frac{d_j}{d_i}|e_{ij}|, \sum_{j=0}^{N/2-1} \frac{d_j}{d_i}|e_{ji}| \right\}, \tag{3.4}$$

where $d_0, \dots, d_{N/2-1}$ are positive real numbers. We will later see that the maximum is attained for $i = N/2-1$. So, assume this is already shown. Then, the first expression, which is associated to the row sums, contains just the diagonal entry. This term is also a part of the second expression. Therefore, we only have to investigate this entry.

We set $d_j = \left(\frac{\sqrt{j+1}}{N}\right)^\varepsilon$ for $0 \le j \le N/2-2$ and $d_{N/2-1} = 1$, where $\varepsilon > 0$ is a small positive number. We have already shown that the entry $e_{N/2-1,N/2-1}$ provides the desired bound, and that it is of order $N^{\nu/2}$. We will now show that the sum over the remaining terms in the above maximum is of lower order. Although the theorems from [21] do not immediately yield such good bounds, they suffice for our asymptotical statements. We have

$$\begin{aligned}\sum_{j=0}^{N/2-2} \frac{d_j}{d_{N/2-1}}|e_{j,N/2-1}| &= 2^\nu \Gamma(\lceil \nu/2\rceil + 1)\sqrt{\frac{\Gamma(N/2+\lfloor \nu/2\rfloor)}{\Gamma(N/2+\lceil \nu/2\rceil+\alpha-1/2)}} N^{-\varepsilon} \\ &\quad\times \sum_{j=0}^{N/2-2} (j+1)^{\varepsilon/2} \sqrt{\frac{\Gamma(j+\alpha+1/2)}{\Gamma(j+1)}} \binom{j+\lceil \nu/2\rceil - 1/2}{\lceil \nu/2\rceil} \binom{N/2-j+\lceil \nu/2\rceil-2}{N/2-1-j} \\ &\quad\times \left| {}_3F_2\left(\begin{matrix} -\lceil \nu/2\rceil,\ j-N/2+1,\ \alpha+j+1/2 \\ -\lceil \nu/2\rceil + j - N/2+2,\ j+1/2 \end{matrix}; 1\right)\right|. \end{aligned} \tag{3.5}$$

We need to closer investigate the hypergeometric series. For readability we set $m = N/2-1$. Assume $m - j \ge 1$ and $\alpha \ne 0$. We employ the Chu-Vandermonde identity (1.5) for the term

$$\frac{(\alpha+j+1/2)_\tau}{(j+1/2)_\tau} = {}_2F_1\left(\begin{matrix}-\tau,\ -\alpha \\ j+1/2\end{matrix}; 1\right) = \sum_{\sigma=0}^{\tau} \frac{(-\tau)_\sigma(-\alpha)_\sigma}{(j+1/2)_\sigma \sigma!}$$

occurring in the hypergeometric series over the summation variable τ. Therefore,

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix}-\lceil\nu/2\rceil,\ -m+j,\ \alpha+j+1/2\\ -\lceil\nu/2\rceil-m+j+1,\ j+1/2\end{matrix};1\right)\\
&\quad=\sum_{\tau=0}^{m-j}\frac{(-\lceil\nu/2\rceil)_\tau(-m+j)_\tau}{(-\lceil\nu/2\rceil-m+j+1)_\tau\tau!}\sum_{\sigma=0}^{\tau}\frac{(-\tau)_\sigma(-\alpha)_\sigma}{(j+1/2)_\sigma\sigma!}\\
&\quad=\sum_{\sigma=0}^{m-j}\frac{(-\lceil\nu/2\rceil)_\sigma(-m+j)_\sigma(-\alpha)_\sigma(-1)^\sigma}{(j+1/2)_\sigma\sigma!(-\lceil\nu/2\rceil-m+j+1)_\sigma}\sum_{\tau=0}^{m-j-\sigma}\frac{(-\lceil\nu/2\rceil+\sigma)_\tau(-m+j+\sigma)_\tau}{(-\lceil\nu/2\rceil-m+j+\sigma+1)_\tau\tau!}\\
&\quad=\sum_{\sigma=0}^{m-j}\frac{(-\lceil\nu/2\rceil)_\sigma(-m+j)_\sigma(-\alpha)_\sigma(-1)^\sigma(-\lceil\nu/2\rceil+1)_{\lceil\nu/2\rceil-\sigma}}{(j+1/2)_\sigma\sigma!(-\lceil\nu/2\rceil-m+j+1)_{\lceil\nu/2\rceil}}.
\end{aligned}
$$

The last identity is again an application of the Chu-Vandermonde identity (1.5). Observe that the term $(-\lceil\nu/2\rceil+1)_{\lceil\nu/2\rceil-\sigma}$ vanishes for $\sigma=0$. Therefore, the sum starts at $\sigma=1$. Since we assumed $m-j\geq 1$, we can now write the sum as

$$
\begin{aligned}
&-\frac{(-\lceil\nu/2\rceil)(-m+j)(-\alpha)(-\lceil\nu/2\rceil+1)_{\lceil\nu/2\rceil-1}}{(j+1/2)(-\lceil\nu/2\rceil-m+j+1)_{\lceil\nu/2\rceil}}\sum_{\sigma=0}^{m-j-1}\frac{(-\lceil\nu/2\rceil+1)_\sigma(-m+j+1)_\sigma(1-\alpha)_\sigma}{(j+3/2)_\sigma(2)_\sigma\sigma!}\\
&\quad=\frac{\Gamma(\lceil\nu/2\rceil+1)(-\alpha)}{(j+1/2)}\frac{\Gamma(m-j+1)}{\Gamma(m-j+\lceil\nu/2\rceil)}\,{}_3F_2\left(\begin{matrix}-\lceil\nu/2\rceil+1,\ -m+j+1,\ 1-\alpha\\ 2,\ j+3/2\end{matrix};1\right). \qquad (3.6)
\end{aligned}
$$

We note that this transformation also holds in the case $\lceil\nu/2\rceil=1$. To confess, we might have proved this a lot simpler. For $\lceil\nu/2\rceil=1$, we get

$$
{}_3F_2\left(\begin{matrix}-\lceil\nu/2\rceil,\ -m+j,\ \alpha+j+1/2\\ -\lceil\nu/2\rceil-m+j+1,\ j+1/2\end{matrix};1\right)={}_2F_1\left(\begin{matrix}-1,\ \alpha+j+1/2\\ j+1/2\end{matrix};1\right)=\frac{-\alpha}{j+1/2},
$$

which is just the above term.

We still have to investigate the hypergeometric series from (3.6). Take a closer look at the term

$$
\frac{(-m+j+1)_\tau}{(j+3/2)_\tau}=\left(\frac{-m-1/2}{j+3/2}+1\right)\cdots\left(\frac{-m-1/2}{j+\tau+1/2}+1\right).
$$

This implies that the absolute value of the whole series is at most a constant times a polynomial in m/j of degree at most $\lceil\nu/2\rceil-1$.

We now go back to the original problem (3.5). First, treat the term for $j=0$ separately. It is

$$
\begin{aligned}
&2^\nu\sqrt{\frac{\Gamma(N/2+\lfloor\nu/2\rfloor)}{\Gamma(N/2+\lceil\nu/2\rceil+\alpha-1/2)}}N^{-\varepsilon}\sqrt{\Gamma(\alpha+1/2)}\frac{\Gamma(\lceil\nu/2\rceil+1/2)}{\Gamma(1/2)}\frac{\Gamma(N/2+\lceil\nu/2\rceil-1)}{\Gamma(N/2)}\\
&\qquad\times\left|{}_3F_2\left(\begin{matrix}-\lceil\nu/2\rceil,\ -N/2+1,\ \alpha+1/2\\ -\lceil\nu/2\rceil-N/2+2,\ 1/2\end{matrix};1\right)\right|.
\end{aligned}
$$

The hypergeometric series is bounded by a constant. This can be shown by examining the two factors containing an N. The whole term is at most a constant times

$$
N^{\nu/2-\varepsilon-1-(\alpha-1/2)/2}=O(N^{\nu/2-\varepsilon}).
$$

So, for $\varepsilon > 0$, this is of smaller order than the entry $e_{N/2-1,N/2-1}$.

After applying the above identities, the rest of the sum now is

$$2^\nu \sqrt{\frac{\Gamma(N/2+\lfloor \nu/2 \rfloor)}{\Gamma(N/2+\lceil \nu/2 \rceil+\alpha-1/2)}} N^{-\varepsilon} |\lceil \nu/2 \rceil \alpha| \\ \times \sum_{j=1}^{N/2-2} (j+1)^{\varepsilon/2} \sqrt{\frac{\Gamma(j+\alpha+1/2)}{\Gamma(j+1)}} \frac{\Gamma(j+\lceil \nu/2 \rceil+1/2)}{\Gamma(j+3/2)} O\big((N/j)^{\lceil \nu/2 \rceil - 1}\big),$$

a term in $O(N^{\nu/2-\varepsilon/2})$ and thus also of smaller order than the entry $e_{N/2-1,N/2-1}$.

What is left to show is that the maximum is really attained for $i = N/2 - 1$. What we have seen so far is that the off-diagonal elements do not really matter in comparison to the diagonal element. Therefore, the maximum of the sum in (3.4) is determined by the elements on the main diagonal. It is easy to compare these elements since the sum occurring inside e_{jj} is actually just a single term. It can be seen that the terms are strictly increasing, starting with a small index. Indeed, we have discussed before that these terms behave like $2^\nu (j/2)^{\nu/2}$, which is clearly growing in j.

The same estimate can be done for F_N. In conclusion, we have shown that for $\alpha = \beta$,

$$\eta_n^{(\nu)}(\alpha,\alpha) \sim (2n)^{\nu/2}$$

gives the asymptotic behavior as n goes to infinity.

Chapter 4

The nonintegral case

Contents

We now switch to the nonintegral case. The main difficulty here is that the involved matrices are not banded anymore, but they are full upper triangular matrices. All three norms considered here share the same main idea. To get an upper bound, we employ the results from the integral case and a theorem by Stein to interpolate between these. For the lower bound, we construct a special vector and estimate the norm of its image under the effect of the operator. Letting the dimension of the matrix go to infinity, this lower bound will tend to the upper bound.

Before constraining ourselves to the details of the particular cases, we take a look at some results that will be used several times.

4.1 General considerations

The following lemma is an application of Stein's interpolation theorem [25] in the special cases of the norms considered here. To this end, let $u(\cdot,\alpha)$ denote one of the weight functions

$$u_L(t,\alpha) = (t^\alpha \mathrm{e}^{-t})^{1/2} \qquad \text{(Laguerre)}, \tag{4.1}$$

$$u_G(t,\alpha) = (1-t^2)^{\alpha/2} \qquad \text{(Gegenbauer)}, \tag{4.2}$$

$$u_H(t,\alpha) = \left(|t|^{2\alpha} \mathrm{e}^{-t^2}\right)^{1/2} \qquad \text{(Hermite)}, \tag{4.3}$$

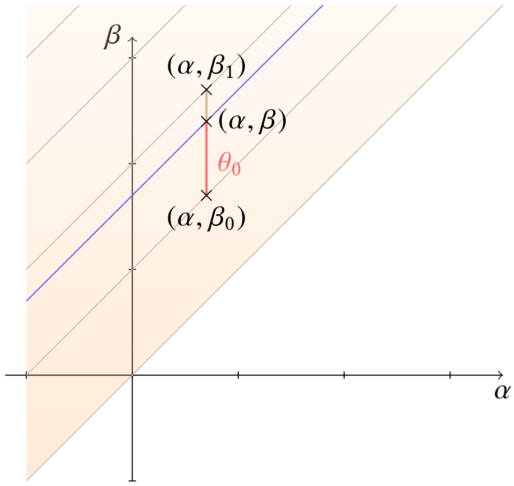

Figure 4.1: The possible parameter set for $\alpha, \beta > -1,\ \beta - \alpha \geq 0$.

and Ω one of the corresponding domains $(0, \infty)$, $(-1, 1)$, $(-\infty, \infty)$. Note that we can now write the respective norms in terms of the usual unweighted L^2-norm $\|\cdot\|_{L^2(\Omega)}$ as follows:

$$\|f\|_\alpha^2 = \int_\Omega |f(t)|^2 u^2(t, \alpha) \mathrm{d}t = \int_\Omega |f(t) u(t, \alpha)|^2 \mathrm{d}t = \|f u(\cdot, \alpha)\|_{L^2(\Omega)}^2. \tag{4.4}$$

Before we go further, we will illustrate the targeted idea of what we want to achieve. We assume $\alpha, \beta > -1$ and $\beta - \alpha \geq 0$. The set of possible parameters is illustrated in Figure 4.1. The diagonal gray lines indicate the pairs (α, β) that have an integral difference. As can be seen in this picture, we can find for any valid pair (α, β) with nonintegral difference two neighboring pairs (α, β_0) and (α, β_1) that satisfy $\beta_0 - \alpha \in \mathbb{Z}$ and $\beta_1 - \alpha \in \mathbb{Z}$. This is true for any $\alpha > -1$. Since we have good upper estimates in these cases, the hope is that we can exploit the knowledge to get good estimates for the cases in between. The following lemma tells us that this is indeed possible and also provides information about the constants.

Lemma 4.1. *Fix α and let $\gamma > -1$ (or $\gamma > -1/2$ in the Hermite case) be arbitrary. Let $u(\cdot, \alpha)$ and Ω be as above. Define an operator*

$$T : L^2(\Omega, u(\cdot, \alpha)) \to L^2(\Omega, u(\cdot, \gamma))$$

via

$$Tf = D^\nu f \quad \text{for all } f \in \mathcal{P}_n(\alpha),$$
$$Tg = 0 \quad \text{for all } g \in \mathcal{P}_n(\alpha)^\perp,$$

where $D^\nu : \mathcal{P}_n \to \mathcal{P}_n$ is the operator that maps a polynomial of degree at most n to its νth derivative, and $\mathcal{P}_n(\alpha)$ is the space of all algebraic polynomials of degree at most n equipped with the norm $\|f\|_\alpha$. Then, $\|T\|_{\alpha \to \gamma} = \|D^\nu\|_{\alpha \to \gamma}$ for any $\gamma > -1$ ($\gamma > -1/2$). Furthermore, if

$$\|D^\nu f\|_\beta \leq C_n^{(\nu)}(\alpha, \beta) \|f\|_\alpha \quad \text{for all } f \in \mathcal{P}_n, \tag{4.5}$$

for all $\beta = \beta' + k,\ k \in \mathbb{N}_0$ with some β' satisfying $\beta' - \alpha \in \mathbb{Z}$, and if the coefficients $C_n^{(\nu)}(\alpha, \beta)$ satisfy

$$C_n^{(\nu)}(\alpha, \beta'(1-\theta) + (\beta'+1)\theta) = (C_n^{(\nu)}(\alpha, \beta'))^{1-\theta}(C_n^{(\nu)}(\alpha, \beta'+1))^{\theta}, \quad \theta \in [0,1],$$

then (4.5) *holds for all $\beta \in [\beta', \infty)$.*

Proof. First, we observe that

$$\begin{aligned}\|T\|^2_{\alpha\to\gamma} &= \sup_{f\in\mathcal{P}_n(\alpha),\ g\in\mathcal{P}_n(\alpha)^\perp} \frac{\|T(f+g)\|^2_\gamma}{\|f+g\|^2_\alpha} = \sup_{f\in\mathcal{P}_n(\alpha),\ g\in\mathcal{P}_n(\alpha)^\perp} \frac{\|D^\nu f\|^2_\gamma}{\|f\|^2_\alpha + \|g\|^2_\alpha} \\ &\le \sup_{f\in\mathcal{P}_n(\alpha)} \frac{\|D^\nu f\|^2_\gamma}{\|f\|^2_\alpha} = \|D^\nu\|^2_{\alpha\to\gamma}.\end{aligned}$$

There exists an $f_0 \in \mathcal{P}_n(\alpha)$ such that $\|D^\nu f_0\|_\gamma = \|D^\nu\|_{\alpha\to\gamma}\|f_0\|_\alpha$. Hence,

$$\|T\|_{\alpha\to\gamma} = \sup_{f\in\mathcal{P}_n(\alpha),\ g\in\mathcal{P}_n(\alpha)^\perp} \frac{\|T(f+g)\|_\gamma}{\|f+g\|_\alpha} \ge \frac{\|Tf_0\|_\gamma}{\|f_0\|_\alpha} = \frac{\|D^\nu f_0\|_\gamma}{\|f_0\|_\alpha} = \|D^\nu\|_{\alpha\to\gamma}.$$

Consequently, $\|T\|_{\alpha\to\gamma} = \|D^\nu\|_{\alpha\to\gamma}$ for arbitrary γ. We now employ the interpolation theorem of Stein [25]. Given any $\beta \ge \beta'$, define

$$\theta_0 := \beta - \alpha - \lfloor \beta - \alpha \rfloor \in (0,1), \quad \beta_0 := \beta - \theta_0, \quad \beta_1 := \beta + (1-\theta_0).$$

Then, $\beta_0 - \alpha = \lfloor \beta - \alpha \rfloor \in \mathbb{Z}$ and $\beta_1 - \alpha = \lceil \beta - \alpha \rceil \in \mathbb{Z}$. Since $\beta_1 - 1 = \beta_0 \ge \beta'$, (4.5) provides an upper bound on the norms

$$\|(Tf)u(\cdot, \beta_i)\|_2 = \|Tf\|_{\beta_i} \le C_n^{(\nu)}(\alpha, \beta_i)\|f\|_\alpha \quad \text{for all } f \in \mathcal{P}_n,\ i = 0,1.$$

Since $u(t,\beta) = u^{1-\theta_0}(t,\beta_0) \cdot u^{\theta_0}(t,\beta_1)$, we can apply Theorem 2 of [25], which leads us to

$$\begin{aligned}\|Tf\|_{\beta_0(1-\theta)+\beta_1\theta} &= \|(Tf)u(\cdot, \beta_0(1-\theta) + \beta_1\theta)\|_2 \\ &\le (C_n^{(\nu)}(\alpha, \beta_0))^{1-\theta}(C_n^{(\nu)}(\alpha, \beta_1))^{\theta}\|f\|_\alpha\end{aligned}$$

for all $f \in \mathcal{P}_n$ and all $\theta \in [0,1]$. As we have $\beta = (1-\theta_0)\beta_0 + \theta_0\beta_1$, we conclude that

$$\|Tf\|_\beta \le C_n^{(\nu)}(\alpha, \beta)\|f\|_\alpha \quad \text{for all } f \in \mathcal{P}_n. \qquad \square$$

The next lemma provides us with an inequality which is used in deriving a lower bound in the Laguerre and Gegenbauer cases. It contains the essence of the proof presented in [19].

Lemma 4.2. *Let $\omega \in (0,\infty) \setminus \mathbb{N}$, $n, \mu \in \mathbb{N}$, $n \ge \mu > \lceil \omega \rceil$, and $\rho_i > 0$, $i = 0, \ldots, n$. Then,*

$$\sum_{i=0}^{n} \rho_i \cdot \left(\sum_{k=\max\{0, n-i-\mu+1\}}^{n-i} \binom{\omega}{k} \right)^2 \ge 2 \left\lfloor \frac{\mu - \lceil \omega \rceil}{2} \right\rfloor 2^{2\omega} \cdot \min_{n-\mu+1 \le i \le n} \rho_i.$$

In order to prove this, we first collect some results on general binomial coefficients for reference in the following lemma. It can also be found as Lemma 1 in [19].

Lemma 4.3. *Let* $\omega \in (0,\infty) \setminus \mathbb{N}$. *Then for any* $k,\ell,m \in \mathbb{N} \cup \{0\}$ *the following statements are valid:*

$$\operatorname{sgn}\binom{\omega}{k} = \begin{cases} +1 & : k \leq \lceil\omega\rceil \\ (-1)^{k-\lceil\omega\rceil} & : k \geq \lceil\omega\rceil + 1, \end{cases} \tag{4.6}$$

$$\left|\binom{\omega}{k}\right| > \left|\binom{\omega}{k+1}\right| \quad \textit{for } k \geq \lfloor\omega\rfloor, \quad \textit{and } 0 < \binom{\omega}{\lceil\omega\rceil} < 1, \tag{4.7}$$

$$\sum_{j=0}^{\lceil\omega\rceil+2\ell} \binom{\omega}{j} \geq \sum_{j=0}^{\lceil\omega\rceil+2(\ell+m)} \binom{\omega}{j} \geq 2^{\omega}, \tag{4.8}$$

$$\sum_{j=0}^{\lceil\omega\rceil+2\ell+1} \binom{\omega}{j} \leq \sum_{j=0}^{\lceil\omega\rceil+2(\ell+m)+1} \binom{\omega}{j} \leq 2^{\omega}, \tag{4.9}$$

$$\sum_{j=0}^{\lceil\omega\rceil+\ell+1} \binom{\omega}{j}(-1)^j \begin{cases} \geq 0 & : \lfloor\omega\rfloor \equiv 0 \mod 2 \\ \leq 0 & : \lfloor\omega\rfloor \equiv 1 \mod 2. \end{cases} \tag{4.10}$$

Proof. We write the binomial coefficient as

$$\binom{\omega}{k} = \prod_{j=1}^{k} \frac{\omega - j + 1}{j}.$$

All factors with $j < \omega + 1$ are positive, and all factors with $j > \omega + 1$ are negative. So, for $k \leq \lfloor\omega\rfloor + 1 = \lceil\omega\rceil$, the product is only over positive factors and thus also positive. Otherwise, there are $k - \lceil\omega\rceil$ factors in the product that are negative, proving (4.6).

To show (4.7), we consider $\binom{\omega}{k+1} = \frac{\omega-(k+1)+1}{k+1}\binom{\omega}{k}$. If $k > \omega$, the absolute value of the first factor is strictly smaller than 1. For $k = \lceil\omega\rceil$, we have

$$\binom{\omega}{\lceil\omega\rceil} = \frac{\omega}{\lceil\omega\rceil} \cdot \frac{\omega - 1}{\lceil\omega\rceil - 1} \cdots \frac{\omega - \lceil\omega\rceil + 1}{1}.$$

Every factor of this product is positive and smaller than 1. This completes the proof of (4.7).

Now let $s_k = \sum_{j=0}^{\lceil\omega\rceil+k} \binom{\omega}{j}$. With (4.6) and (4.7), we conclude that

$$s_{2(i+1)} - s_{2i} = \binom{\omega}{\lceil\omega\rceil + 2i + 1} + \binom{\omega}{\lceil\omega\rceil + 2i + 2} < 0,$$

and

$$s_{2(i+1)+1} - s_{2i+1} = \binom{\omega}{\lceil\omega\rceil + 2i + 2} + \binom{\omega}{\lceil\omega\rceil + 2i + 3} > 0.$$

Thus, the sequence $\{s_{2i}\}_{i=0}^{\infty}$ is decreasing and the sequence $\{s_{2i+1}\}_{i=0}^{\infty}$ is increasing. But s_k is a partial sum of the power series for $(1+x)^{\omega}$, evaluated at $x = 1$. This converges absolutely for $\omega > 0$ and has the sum 2^{ω}, implying (4.8) and (4.9).

Again, from (4.6), we conclude that for $j \geq \lceil\omega\rceil + 1$,

$$\operatorname{sgn}\left(\binom{\omega}{j}(-1)^j\right) = (-1)^{j-\lceil\omega\rceil}(-1)^j = (-1)^{\lceil\omega\rceil}.$$

So, for even or odd $\lceil\omega\rceil$, all summands from $\lceil\omega\rceil + 1$ onwards stay positive or negative, respectively. Again, the left-hand side of (4.10) is a partial sum of the power series for $(1+x)^{\omega}$ evaluated at $x = -1$. At this point, it converges absolutely with sum 0. From this, (4.10) follows. □

Another important lemma used for proving Lemma 4.2 already appeared as Lemma 2 in [19].

Lemma 4.4. *If* $\omega \geq 1$ *and* $\mu \in \mathbb{N} \cup \{0\}$, *then*

$$\sum_{m=0}^{\lfloor\omega\rfloor}\left(\sum_{j=0}^{m}\binom{\omega}{j}\right)^2 \geq 4 \cdot \sum_{k=0}^{\left\lfloor\frac{\mu-\lceil\omega\rceil}{2}\right\rfloor-1}\left[\binom{\omega-1}{\lceil\omega\rceil+2k+1} \cdot \sum_{j=0}^{\lceil\omega\rceil+2k+1}\binom{\omega-1}{j}\right]. \tag{4.11}$$

Proof. First, we observe that if $\omega \in \mathbb{N}$, then the right-hand side of (4.11) equals 0 since all the terms $\binom{\omega-1}{\lceil\omega\rceil+2k+1}$ are 0. On the left-hand side, we sum over squares of real numbers, so this sum is nonnegative and hence the inequality holds in this case. Thus, we only need to investigate the case $\omega \in (1,\infty) \setminus \mathbb{N}$.

We first assume $\omega > 2$. Clearly, $\sum_{m=0}^{\lfloor\omega\rfloor}\left(\sum_{j=0}^{m}\binom{\omega}{j}\right)^2 \geq \left(\sum_{j=0}^{\lfloor\omega\rfloor}\binom{\omega}{j}\right)^2$. With (4.6) and (4.8), we obtain

$$\sum_{j=0}^{\lfloor\omega\rfloor}\binom{\omega}{j} = \sum_{j=0}^{\lfloor\omega\rfloor}\left[\binom{\omega-1}{j} + \binom{\omega-1}{j-1}\right] \geq \sum_{j=0}^{\lfloor\omega\rfloor}\binom{\omega-1}{j} \geq 2^{\omega-1}.$$

So, we have

$$\sum_{m=0}^{\lfloor\omega\rfloor}\left(\sum_{j=0}^{m}\binom{\omega}{j}\right)^2 \geq 2^{2\omega-2}.$$

Next, we want to point out that both $\binom{\omega-1}{\lceil\omega\rceil+2k+1}$ and $\sum_{j=0}^{\lceil\omega\rceil+2k}\binom{\omega-1}{j}$ are positive for every nonnegative integer k. We have

$$\begin{aligned}
&\sum_{k=0}^{\left\lfloor\frac{\mu-\lceil\omega\rceil}{2}\right\rfloor-1}\left[\binom{\omega-1}{\lceil\omega\rceil+2k+1} \cdot \sum_{j=0}^{\lceil\omega\rceil+2k+1}\binom{\omega-1}{j}\right] \\
&\qquad = \sum_{k=0}^{\left\lfloor\frac{\mu-\lceil\omega\rceil}{2}\right\rfloor-1}\left[\binom{\omega-1}{\lceil\omega\rceil+2k+1} \cdot \left[\sum_{j=0}^{\lceil\omega\rceil+2k}\binom{\omega-1}{j} + \binom{\omega-1}{\lceil\omega\rceil+2k+1}\right]\right].
\end{aligned}$$

With (4.7) and (4.9), we get the following upper bound for the right hand side of (4.11):

$$4(2^{\omega-1}+1)\cdot\sum_{k=0}^{\lfloor\frac{\mu-\lceil\omega\rceil}{2}\rfloor-1}\binom{\omega-1}{\lceil\omega\rceil+2k+1}.$$

From (4.6) and (4.7), we infer

$$\begin{aligned}\sum_{k=0}^{\lfloor\frac{\mu-\lceil\omega\rceil}{2}\rfloor-1}\binom{\omega-1}{\lceil\omega\rceil+2k+1}&=\sum_{k=0}^{\lfloor\frac{\mu-\lceil\omega\rceil}{2}\rfloor-1}\left[\binom{\omega-2}{\lceil\omega\rceil+2k+1}+\binom{\omega-2}{\lceil\omega\rceil+2k}\right]\\&=\sum_{k=0}^{2\lfloor\frac{\mu-\lceil\omega\rceil}{2}\rfloor-1}\binom{\omega-2}{\lceil\omega\rceil+k}\le\binom{\omega-2}{\lceil\omega\rceil}.\end{aligned}$$

Set $\delta=\omega-\lfloor\omega\rfloor\in(0,1)$. By simple calculation, we can show that

$$\binom{\omega-2}{\lceil\omega\rceil}\le\frac{\delta(\delta-1)(\delta-2)}{6}\le\frac{1}{9\sqrt{3}}<\frac{1}{8}.$$

So, for the right-hand side of (4.11), we get

$$\begin{aligned}4\cdot\sum_{k=0}^{\lfloor\frac{\mu-\lceil\omega\rceil}{2}\rfloor-1}\left[\binom{\omega-1}{\lceil\omega\rceil+2k+1}\cdot\sum_{j=0}^{\lceil\omega\rceil+2k+1}\binom{\omega-1}{j}\right]&<4(2^{\omega-1}+1)\cdot\tfrac{1}{8}=2^{\omega-2}+\tfrac{1}{2}\\&<2^{\omega-2}+2^{\omega-2}=2\cdot2^{\omega-2}\\&<2^{\omega-1}(2\cdot2^{\omega-2})=2^{2\omega-2},\end{aligned}$$

which completes the proof for $\omega>2$.

Now, assume $1<\omega<2$. The left-hand side of (4.11) simplifies to

$$\binom{\omega}{0}^2+\left[\binom{\omega}{0}+\binom{\omega}{1}\right]^2=2+2\omega+\omega^2\in(5,10).$$

For the inner sum of the right-hand side of (4.11), we have

$$\sum_{j=0}^{2k+3}\binom{\omega-1}{j}\le1+(\omega-1)+\frac{(\omega-1)(\omega-2)}{2}+\frac{(\omega-1)(\omega-2)(\omega-3)}{6}\le\omega.$$

From (4.6) and (4.7), we obtain

$$2\binom{\omega-1}{2k+3}\le\binom{\omega-1}{2k+3}-\binom{\omega-1}{2k+2}.$$

Taking all this into account, we arrive at the upper estimate

$$\begin{aligned}
4 \cdot \sum_{k=0}^{\lfloor \mu/2 \rfloor - 2} \binom{\omega-1}{2k+3} \sum_{j=0}^{2k+3} \binom{\omega-1}{j} &\le 2\omega \cdot \sum_{k=0}^{\lfloor \mu/2 \rfloor - 2} 2\binom{\omega-1}{2k+3} \\
&\le 2\omega \cdot \sum_{k=0}^{\lfloor \mu/2 \rfloor - 2} \left[\binom{\omega-1}{2k+3} - \binom{\omega-1}{2k+2} \right] = 2\omega \cdot \sum_{k=0}^{2\lfloor \mu/2 \rfloor - 3} \binom{\omega-1}{k+2} (-1)^{k+1} \\
&= 2\omega \cdot \left(\sum_{k=0}^{2\lfloor \mu/2 \rfloor - 3} \binom{\omega-1}{k+2} (-1)^{k+1} + \binom{\omega-1}{1} - \binom{\omega-1}{0} + \binom{\omega-1}{0} - \binom{\omega-1}{1} \right) \\
&= 2\omega \cdot \left(- \sum_{k=0}^{2\lfloor \mu/2 \rfloor - 3} \binom{\omega-1}{k} (-1)^{k} + 2 - \omega \right) \le 2\omega(2-\omega) \le 2.
\end{aligned}$$

In the third estimate, we made use of (4.10). □

Proof of Lemma 4.2. We partition the sum on the left-hand side into four parts $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$ as follows:

- Σ_1 contains all terms whose inner sums have only summands with $k < \omega$, i. e.,

$$\Sigma_1 = \sum_{i=n-\lceil \omega \rceil + 1}^{n} \rho_i \left(\sum_{k=0}^{n-i} \binom{\omega}{k} \right)^2 .$$

- Σ_2 contains all terms with $n - i > \omega$ whose inner sums start at 0 and where the last term in the inner sum is positive, i. e.,

$$\Sigma_2 = \sum_{\ell=0}^{\left\lfloor \frac{\mu - \lceil \omega \rceil - 1}{2} \right\rfloor} \rho_{n-\lceil \omega \rceil - 2\ell} \left(\sum_{k=0}^{\lceil \omega \rceil + 2\ell} \binom{\omega}{k} \right)^2 .$$

- Σ_3 contains all terms with $n - i > \omega$ whose inner sums start at 0 and where the last term in the inner sum is negative, i. e.,

$$\Sigma_3 = \sum_{\ell=0}^{\left\lfloor \frac{\mu - \lceil \omega \rceil - 2}{2} \right\rfloor} \rho_{n-\lceil \omega \rceil - 2\ell - 1} \left(\sum_{k=0}^{\lceil \omega \rceil + 2\ell + 1} \binom{\omega}{k} \right)^2 .$$

- Σ_4 is made up of the rest, i. e., the inner sum does not start at 0,

$$\Sigma_4 = \sum_{i=0}^{n-\mu} \rho_i \left(\sum_{k=n-i-\mu+1}^{n-i} \binom{\omega}{k} \right)^2 .$$

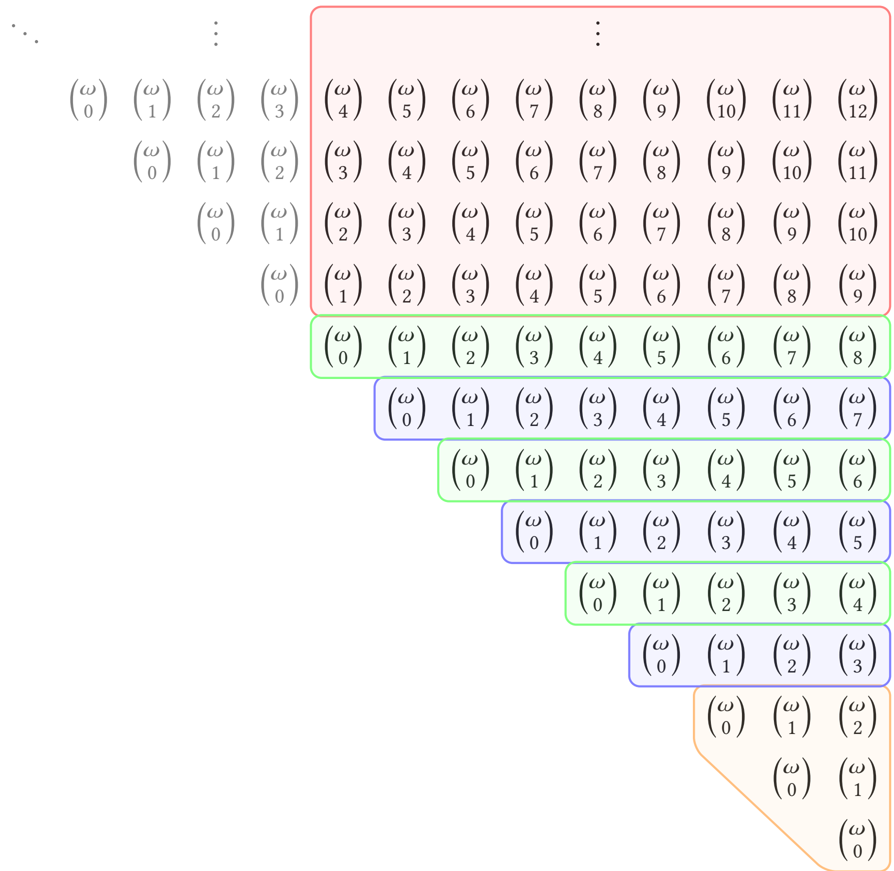

Figure 4.2: Illustration of the partitioning of the sum.

To get a better understanding of the way this partition works, we want to illustrate this by a small example: Let $\lceil\omega\rceil = 3$ and $\mu = 9$. Then, the elements of the inner sum may be arranged in a way that Figure 4.2 suggests (omitting the "+" between the elements, as well as the factors ρ_i).

Here, each line corresponds to a fixed i. The lower orange box of Figure 4.2 is the part for Σ_1. The parts highlighted belong to Σ_2, and the ones highlighted belong to Σ_3. The upper block is the part for Σ_4. Everything left from the highlighted area is not considered, since it is absent in the overall sum.

We will see that Σ_2, in fact, exceeds the desired estimate for the lower bound. In contrast to that, Σ_3 is below the bound we want to show. We will prove that this can be repaired by adding Σ_1 and Σ_3.

Since all involved summands are nonnegative numbers, we may drop Σ_4 and retain a lower estimate on the whole sum. Now, we only have to consider the indices i with $n - \mu + 1 \le i \le n$

for the values ρ_i and thus replace all occurrences by the minimum over this set. Then, let Σ_j' denote the sum Σ_j ($j = 1,2,3$) without the ρ_i part. With (4.8), we arrive at $\Sigma_2' \geq \left\lfloor \frac{\mu - \lceil\omega\rceil}{2} \right\rfloor \cdot 2^{2\omega}$. Our first goal is to show that

$$\Sigma_1' + \Sigma_3' \geq \left\lfloor \frac{\mu - \lceil\omega\rceil}{2} \right\rfloor \cdot 2^{2\omega} \qquad \text{for } \omega \in (0,\infty) \setminus \mathbb{N}. \tag{4.12}$$

We rewrite the inner sums of Σ_3' for $\omega > 1$ as

$$\begin{aligned} \sum_{k=0}^{\lceil\omega\rceil+2\ell+1} \binom{\omega}{k} &= \sum_{k=0}^{\lceil\omega\rceil+2\ell+1} \binom{\omega-1}{k} + \sum_{k=0}^{\lceil\omega\rceil+2\ell} \binom{\omega-1}{k} \\ &= 2 \cdot \sum_{k=0}^{\lceil\omega\rceil+2\ell+1} \binom{\omega-1}{k} - \binom{\omega-1}{\lceil\omega\rceil+2\ell+1} \end{aligned} \tag{4.13}$$

and Σ_1' as

$$\sum_{m=0}^{\lfloor\omega\rfloor} \left[\sum_{j=0}^{m} \binom{\omega}{j} \right]^2. \tag{4.14}$$

Putting Σ_1' and Σ_3' back together, we accomplish by (4.13) and (4.14) that

$$\begin{aligned} \Sigma_1' + \Sigma_3' &= \sum_{m=0}^{\lfloor\omega\rfloor} \left[\sum_{j=0}^{m} \binom{\omega}{j} \right]^2 + \sum_{\ell=0}^{\left\lfloor \frac{\mu-\lceil\omega\rceil}{2} \right\rfloor - 1} \left[\left(2 \cdot \sum_{k=0}^{\lceil\omega\rceil+2\ell+1} \binom{\omega-1}{k} \right) \right. \\ &\quad \left. - 4 \binom{\omega-1}{\lceil\omega\rceil+2\ell+1} \sum_{k=0}^{\lceil\omega\rceil+2\ell+1} \binom{\omega-1}{k} + \binom{\omega-1}{\lceil\omega\rceil+2\ell+1}^2 \right] \\ &\geq \sum_{m=0}^{\lfloor\omega\rfloor} \left[\sum_{j=0}^{m} \binom{\omega}{j} \right]^2 + \sum_{\ell=0}^{\left\lfloor \frac{\mu-\lceil\omega\rceil}{2} \right\rfloor - 1} \left[2^{2\omega} - 4 \binom{\omega-1}{\lceil\omega\rceil+2\ell+1} \sum_{k=0}^{\lceil\omega\rceil+2\ell+1} \binom{\omega-1}{k} \right] \\ &\geq \left\lfloor \frac{\mu - \lceil\omega\rceil}{2} \right\rfloor \cdot 2^{2\omega}. \end{aligned}$$

The first inequality follows from (4.8), while the last inequality is a direct consequence of Lemma 4.2. Thus, estimate (4.12) is proved for $\omega > 1$.

Now, assume $0 < \omega < 1$. Then, Σ_1' is just $\binom{\omega}{0}^2$, and we write Σ_3' as

$$\sum_{\ell=0}^{\left\lfloor \frac{\mu-\lceil\omega\rceil}{2} \right\rfloor - 1} \sum_{k=0}^{2\ell+2} \binom{\omega}{k} = \sum_{\ell=1}^{\left\lfloor \frac{\mu-\lceil\omega\rceil}{2} \right\rfloor} \sum_{k=0}^{2\ell} \binom{\omega}{k}.$$

The desired estimate (4.12) will follow as above, provided we can show the inequality

$$\binom{\omega}{0}^2 + \sum_{k=1}^{M} \left[\sum_{j=0}^{2k} \binom{\omega}{j} \right]^2 \geq \sum_{k=1}^{M} \left[\sum_{j=0}^{2k+1} \binom{\omega}{j} \right]^2 \tag{4.15}$$

for $M = \left\lfloor \frac{\mu - \lceil \omega \rceil}{2} \right\rfloor$. For the right-hand side of (4.15), we now have

$$\sum_{k=1}^{M} \left[\sum_{j=0}^{2k+1} \binom{\omega}{j} \right]^2 \leq \sum_{k=1}^{M} \left[\sum_{j=0}^{2k} \binom{\omega}{j} \right]^2 + 2 \sum_{k=1}^{M} \left[\binom{\omega}{2k+1} \sum_{j=0}^{2k+1} \binom{\omega}{j} \right].$$

With analogous arguments as above,

$$\begin{aligned} 2 \sum_{k=1}^{M} \binom{\omega}{2k+1} \sum_{j=0}^{2k+1} \binom{\omega}{j} &\leq 2^{\omega} \cdot \sum_{k=1}^{M} 2 \binom{\omega}{2k+1} \leq 2^{\omega} \cdot \sum_{k=1}^{M} \left[\binom{\omega}{2k+1} - \binom{\omega}{2k} \right] \\ &= 2^{\omega} \cdot \sum_{k=2}^{2M+1} \binom{\omega}{k} (-1)^{k+1} = 2^{\omega} \left[- \sum_{k=0}^{2M+1} \binom{\omega}{k} (-1)^{k} + 1 - \omega \right] \\ &\leq 2^{\omega} (1 - \omega) \leq 1 = \binom{\omega}{0}^2, \end{aligned}$$

from which inequality (4.15) follows. Thus, we have shown (4.12) for all $\omega \in (0, \infty) \setminus \mathbb{N}$. Together with the estimate for Σ_2', the lemma follows. □

In the following sections, we will work out the specific details for each of the regarded norms.

4.2 The Laguerre case

This section deals with the Laguerre case when $\beta - \alpha \geq \nu$ is not an integer. We will apply Lemma 4.1 of the previous section to derive an upper bound from the results already known for the integral case. For the lower bound, we construct a special vector in such a way that Lemma 4.2 can be employed. The proof already appeared in [19].

From [7] (see also Chapter 3), we already know that

$$\lambda_n^{(\nu)}(\alpha, \beta) \leq 2^{\beta - \alpha - \nu} n^{(\beta - \alpha)/2} (1 + O(1/n))$$

as n goes to infinity in case $\beta - \alpha \geq \nu$ is an integer. Set

$$\theta_0 \coloneqq \beta - \alpha - \lfloor \beta - \alpha \rfloor, \quad \beta_0 \coloneqq \beta - \theta_0, \quad \beta_1 \coloneqq \beta_1 + (1 - \theta_0).$$

Obviously, $\beta = (1 - \theta_0)\beta_0 + \theta \beta_1$ and $\beta_1 = \beta_0 + 1$. With $u(\cdot, \alpha)$ according to (4.1), Lemma 4.1 now tells us that

$$\begin{aligned} \lambda_n^{(\nu)}(\alpha, \beta) &= \lambda_n^{(\nu)}(\alpha, (1 - \theta_0)\beta_0 + \theta_0 \beta_1) \\ &\leq \left(\lambda_n^{(\nu)}(\alpha, \beta_0)\right)^{1 - \theta_0} \left(\lambda_n^{(\nu)}(\alpha, \beta_1)\right)^{\theta_0} \\ &= \left(2^{\beta_0 - \alpha - \nu} n^{(\beta_0 - \alpha)/2}\right)^{1 - \theta_0} \left(2^{\beta_1 - \alpha - \nu} n^{(\beta_1 - \alpha)/2}\right)^{\theta_0} (1 + O(1/n)) \\ &= 2^{(1 - \theta_0)\beta_0 + \theta_0 \beta_1 - \alpha - \nu} n^{((1 - \theta_0)\beta_0 + \theta_0 \beta_1 - \alpha)/2} (1 + O(1/n)) \\ &= 2^{\beta - \alpha - \nu} n^{(\beta - \alpha)/2} (1 + O(1/n)), \end{aligned}$$

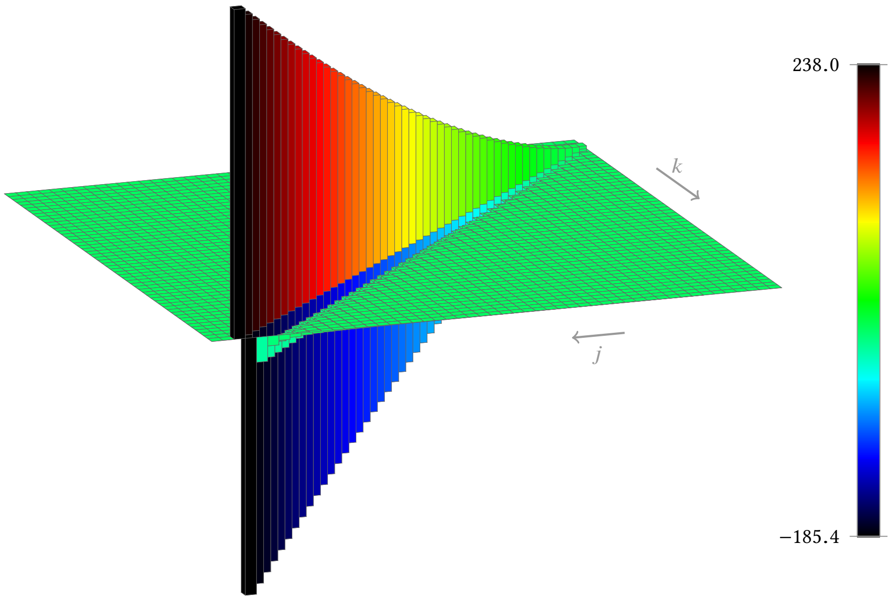

Figure 4.3: Matrix plot for $n = 50,\ \beta = 2.6,\ \alpha = -0.2,\ \nu = 2$ in the Laguerre setting.

which is exactly the constant we wanted to derive.

Now, we go over to showing that the lower bound to the norm has a similar form. The main idea to prove this is to choose some unit vector v and apply the matrix A_N ($N = n - \nu + 1$), which is just the upper nonzero block of the matrix representation of the operator of differentiation in the corresponding Laguerre bases (see Section 2.1 and [7]). Next, we estimate the norm of the image of that vector. We arrange the involved summands in an appropriate way to apply Lemma 4.2. To anticipate the choice of the vector, look at the matrix plot in Figure 4.3. Here, we get the image that the main portion of the matrix is concentrated along the diagonal and elements farther off almost don't matter. This is indeed the case and can easily be seen by a closer look on the matrix entries, together with (4.7).

In the following, let $\omega = \beta - \alpha - \nu$. To get the lower estimate on the norm of the matrix A_N, we introduce vectors $v^+ = (v_j^+)_{j=0}^{n-\nu}$, $v^- = (v_j^-)_{j=0}^{n-\nu} \in \mathbb{R}^{n-\nu+1}$ for $\alpha \geq 0$ and $\alpha < 0$, respectively, as follows:

$$v_j^+ = \begin{cases} (-1)^j \displaystyle\prod_{k=j+1}^{n} \sqrt{\frac{k+\nu}{\alpha+k+\nu}} & : j \geq n-\nu-\mu+1 \\ 0 & : \text{otherwise} \end{cases}$$

$$v_j^- = \begin{cases} (-1)^j \displaystyle\prod_{k=n-\nu-\mu+2}^{j} \sqrt{\frac{\alpha+k+\nu}{k+\nu}} & : j \geq n-\nu-\mu+1 \\ 0 & : \text{otherwise} \end{cases}$$

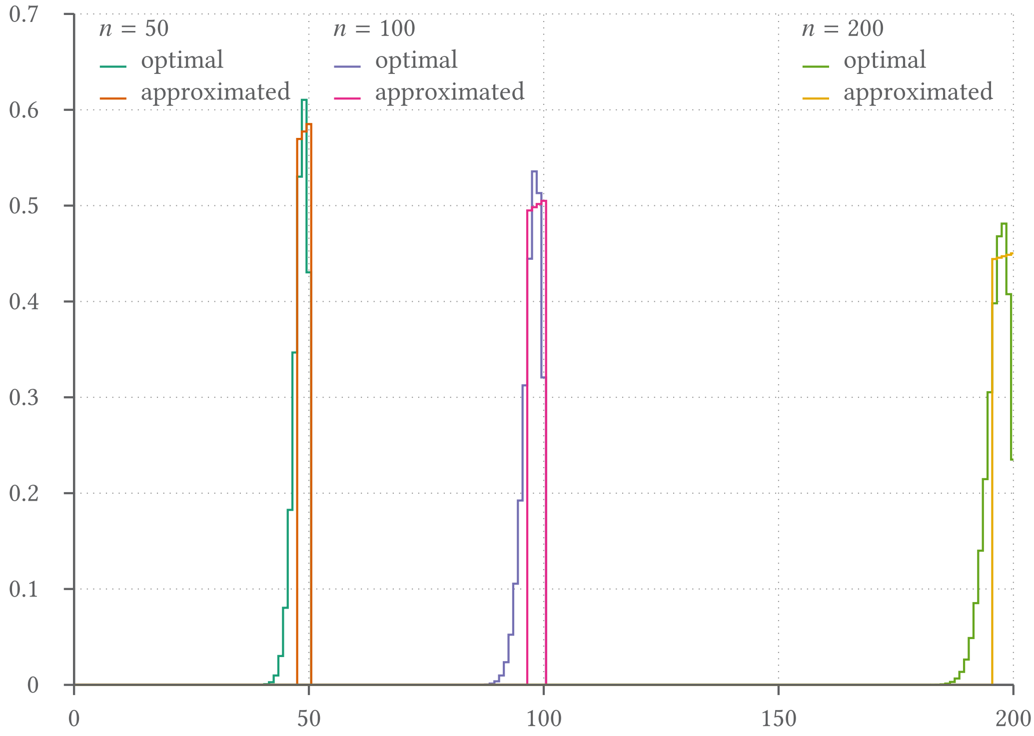

Figure 4.4: Comparison of the (normalized) vectors v^+ with the optimal, norm-realizing solution, in the Laguerre case. Pictured are the magnitudes of the oscillating entries for sizes $n = 50,\ 100$, and 200, with $\alpha = 1.3,\ \beta = 4.2,\ \nu = 2$, and $\mu = \lfloor \log n \rfloor$.

where $0 \le j \le n - \nu$, with some $\mu := \mu(n) \in \mathbb{N},\ \mu \ll n - \nu$. Figure 4.4 gives an impression that the major parts of the actual solution are covered. The entries of A_N are given by (see Section 2.1 and [7])

$$(A_N)_{ij} = (-1)^{j-i+\nu} \frac{w_{j+\nu}(\alpha)}{w_i(\beta)} \binom{\beta - \alpha - \nu}{j - i}$$

for $0 \le i \le j \le n - \nu$, where

$$w_k(\alpha) = \sqrt{\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)}}.$$

The vectors v^+ and v^- are chosen so that all entries in the last μ columns share the factor $w_n(\alpha)$ or $w_{n-\mu+1}(\alpha)$ and the modulus of each entry of the vector is smaller than one, depending on the sign of α. With this representation, we can now write the jth entry of $A_N v^+$ as

$$\begin{aligned}
(A_N v^+)_j &= \sum_{k=0}^{n-\nu} b_{jk} v_k^+ = \sum_{\substack{k=n-\nu-\mu+1 \\ k\geq j}}^{n-\nu} b_{jk} v_k^+ \\
&= \sum_{\substack{k=n-\nu-\mu+1 \\ k\geq j}}^{n-\nu} (-1)^{\nu+k-j} \binom{\omega}{k-j} \frac{w_{k+\nu}(\alpha)}{w_j(\beta)} \cdot (-1)^k \prod_{\ell=k+1}^{n} \sqrt{\frac{\ell+\nu}{\alpha+\ell+\nu}} \\
&= (-1)^{\nu-j} \frac{w_n(\alpha)}{w_j(\beta)} \sum_{\substack{k=n-\nu-\mu+1 \\ k\geq j}}^{n-\nu} \binom{\omega}{k-j} \\
&= (-1)^{\nu-j} \frac{w_n(\alpha)}{w_j(\beta)} \sum_{k=\max\{0,n-\nu-\mu+1-j\}}^{n-\nu-j} \binom{\omega}{k},
\end{aligned}$$

so that a lower estimate for the norm of A_N in the case $\alpha \geq 0$ reads

$$\|A_N\|_\infty^2 \geq \frac{\|A_N v^+\|_2^2}{\|v^+\|_2^2} = \frac{w_n^2(\alpha) \sum_{j=0}^{n-\nu} w_j^{-2}(\beta) \left(\sum_{k=\max\{0,n-\nu-\mu+1-j\}}^{n-\nu-j} \binom{\omega}{k} \right)^2}{\sum_{k=n-\nu-\mu+1}^{n-\nu} \prod_{\ell=k+1}^{n} \frac{\ell+\nu}{\alpha+\ell+\nu}}. \tag{4.16}$$

Analogously, we derive for $\alpha < 0$ that

$$\|A_N\|_\infty^2 \geq \frac{\|A_N v^-\|_2^2}{\|v^-\|_2^2} = \frac{w_{n-\mu+1}^2(\alpha) \sum_{j=0}^{n-\nu} w_j^{-2}(\beta) \left(\sum_{k=\max\{0,n-\nu-\mu+1-j\}}^{n-\nu-j} \binom{\omega}{k} \right)^2}{\sum_{k=n-\nu-\mu+1}^{n-\nu} \prod_{\ell=n-\nu-\mu+2}^{k} \frac{\alpha+\ell+\nu}{\ell+\nu}}. \tag{4.17}$$

We may assume that μ is sufficiently large, at least $\mu > \lceil \omega \rceil$.

Since $w_j^{-2}(\beta) > 0$ for all $j \in \mathbb{N}_0$, we can apply Lemma 4.2 to the upper sum of both (4.16) and (4.17). The sequence of the $w_j^{-2}(\beta)$ is increasing with respect to j for $\beta \geq 0$, and decreasing for $\beta < 0$. For $\nu \geq 1$, it follows from $\beta - \alpha - \nu > 0$ that $\beta > 0$. In this case,

$$\min\{w_j^{-2}(\beta) : n-\nu-\mu+1 \leq j \leq n-\nu\} = w_{n-\nu-\mu+1}^{-2}(\beta).$$

If we would allow $\nu = 0$, a negative value of β would be possible. In that case, the minimum would take the value $w_{n-\nu}^{-2}(\beta)$. We will confine ourselves to $\nu \geq 1$, but note that the proof for $\nu = 0$ requires only small modifications and uses the same arguments.

Since both, v^+ and v^-, have exactly μ nonzero entries with an absolute value smaller than one, the squared norms of both can be estimated from above by μ. Putting all this together, we arrive at

$$\begin{aligned}
\frac{\|A_N v^+\|_2^2}{\|v^+\|_2^2} &\geq 2^{2\omega} \cdot 2\mu^{-1} \cdot \left\lfloor \frac{\mu - \lceil \omega \rceil}{2} \right\rfloor \cdot \frac{w_n^2(\alpha)}{w_{n-\nu-\mu+1}^2(\beta)}, \\
\frac{\|A_N v^-\|_2^2}{\|v^-\|_2^2} &\geq 2^{2\omega} \cdot 2\mu^{-1} \cdot \left\lfloor \frac{\mu - \lceil \omega \rceil}{2} \right\rfloor \cdot \frac{w_{n-\mu+1}^2(\alpha)}{w_{n-\nu-\mu+1}^2(\beta)}.
\end{aligned}$$

So, if we let μ grow slowly with n (e. g., $\mu(n) = \lfloor \log n \rfloor$), so that $n - \nu - \mu + 1 \sim n$ and $2\left\lfloor \frac{\mu - \lceil \omega \rceil}{2} \right\rfloor \sim \mu$, we obtain the asymptotic lower bound $2^{\beta-\alpha-\nu} n^{(\beta-\alpha)/2}$ for the norm of the matrix, and thus for the smallest constant. We therefore have shown

$$\lambda_n^{(\nu)}(\alpha,\beta) \sim 2^{\beta-\alpha-\nu} n^{(\beta-\alpha)/2}$$

for $\beta - \alpha - \nu \geq 0$.

4.3 The Gegenbauer case

This section is devoted to the details in the Gegenbauer case for $\beta - \alpha \geq \nu$ where $\beta - \alpha$ is not an integer. From [10], we know that the constant in the integral case has an asymptotic value of n^ν. The methods used here are very similar to the Laguerre case, but differ in some details. First, we will again employ Stein's interpolation theorem to derive an upper bound. Then, using the same idea as above, we construct vectors and determine the norm of their images under the effect of the matrix.

As in the Laguerre case, set

$$\theta_0 := \beta - \alpha - \lfloor \beta - \alpha \rfloor, \quad \beta_0 := \beta - \theta_0, \quad \beta_1 := \beta_0 + (1 - \theta_0).$$

For $u(\cdot, \alpha)$ as in (4.2), Lemma 4.1 tells us that

$$\begin{aligned} \gamma_n^{(\nu)}(\alpha,\beta) &= \gamma_n^{(\nu)}(\alpha, (1-\theta_0)\beta_0 + \theta_0\beta_1) \\ &\leq \left(\gamma_n^{(\nu)}(\alpha,\beta_0)\right)^{1-\theta_0} \left(\gamma_n^{(\nu)}(\alpha,\beta_1)\right)^{\theta} \\ &= n^{\nu((1-\theta_0)+\theta_0)}(1 + O(1/n)) \\ &= n^\nu(1 + O(1/n)). \end{aligned}$$

As we can see from (2.5), the matrix has a chessboard structure. Without loss of generality, we can assume that $N = n - \nu + 1$ is an even number, since $\gamma_{n-1}^{(\nu)}(\alpha,\beta) \leq \gamma_n^{(\nu)}(\alpha,\beta) \leq \gamma_{n+1}^{(\nu)}(\alpha,\beta)$. Then, there is a permutation matrix U_n such that

$$A_n = U_n \begin{pmatrix} E_n & 0 \\ 0 & F_n \end{pmatrix} U_n,$$

where $E_n = (e_{jk})_{j,k=0}^{N/2-1}$ and $F_n = (f_{jk})_{j,k=0}^{N/2-1}$ with

$$e_{jk} = c_{2j,2k+\nu}^{(\nu)}(\alpha,\beta), \quad f_{jk} = c_{2j+1,2k+\nu+1}^{(\nu)}(\alpha,\beta). \tag{4.18}$$

Clearly, $\|A_n\|_\infty = \max\{\|E_n\|_\infty, \|F_n\|_\infty\}$. Again, we set $\omega = \beta - \alpha - \nu$. Taking a closer look on

$$\begin{aligned} e_{jk} = 2^{-\omega} &\sqrt{\frac{\Gamma(2k+\nu+1)(2k+\alpha+\nu+1/2)}{\Gamma(2k+2\alpha+\nu+1)}} \sqrt{\frac{\Gamma(2j+2\beta+1)(2j+\beta+1/2)}{\Gamma(2j+1)}} \\ &\times \frac{\Gamma(\alpha+\nu+k+j+1/2)}{\Gamma(\beta+1+k+j+1/2)} (-1)^{k-j} \binom{\omega}{k-j}, \end{aligned} \tag{4.19}$$

we notice a structure similar to the matrix in the Laguerre case. Since

$$\frac{\Gamma(\alpha+\nu+k+j+1+1/2)}{\Gamma(\beta+1+k+j+1+1/2)} = \frac{\alpha+\nu+k+j+1/2}{\omega+1+\alpha+\nu+k+j+1/2}\frac{\Gamma(\alpha+\nu+k+j+1/2)}{\Gamma(\beta+1+k+j+1/2)},$$

these terms are always decreasing for growing k or j, independently of α and β, provided that $\beta-\alpha \geq \nu$. Hence, we estimate from below each occurrence by

$$\frac{\Gamma(\alpha+\nu+N-3/2)}{\Gamma(\beta+1+N-3/2)}.$$

Define the vectors $v^+ = (v_j^+)_{j=0}^{N/2-1}$, $v^- = (v_j^-)_{j=0}^{N/2-1} \in \mathbb{R}^{N/2}$ for $\alpha \geq 1/2$ and $\alpha < 1/2$, respectively, as follows:

$$v_j^+ = \begin{cases} (-1)^j \sqrt{\dfrac{N+\alpha+\nu-3/2}{2j+\alpha+\nu+1/2}} \\ \qquad \times \displaystyle\prod_{\ell=j+1}^{N/2-1} \sqrt{\frac{(2\ell+\nu-1)(2\ell+\nu)}{(2\ell+2\alpha+\nu-1)(2\ell+2\alpha+\nu)}} & : j \geq N/2-\mu \\ 0 & : \text{otherwise} \end{cases}$$

$$v_j^- = \begin{cases} (-1)^j \sqrt{\dfrac{N-2\mu+\alpha+\nu+1/2}{2j+\alpha+\nu+1/2}} \\ \qquad \times \displaystyle\prod_{\ell=N/2-\mu+1}^{j} \sqrt{\frac{(2\ell+2\alpha+\nu-1)(2\ell+2\alpha+\nu)}{(2\ell+\nu-1)(2\ell+\nu)}} & : j \geq N/2-\mu \\ 0 & : \text{otherwise} \end{cases}$$

with some $\mu := \mu(N) \in \mathbb{N}$, $\mu \ll N/2-1$. To get an impression how these vectors look like, we refer to Figure 4.4, which, up to a permutation, imparts a similar picture. The first factor in v_j^+ can be written as

$$\sqrt{\frac{N+\alpha+\nu-3/2}{2j+\alpha+\nu+1/2}} = \prod_{\ell=j+1}^{N/2-1} \sqrt{\frac{2\ell+\alpha+\nu+1/2}{2\ell+\alpha+\nu-3/2}}.$$

Joining these products and putting $\alpha = \delta + 1/2$, $\delta > -3/2$ into the factors in the entries of v^+, we see that the term under the square root becomes

$$\frac{2\ell+\nu-1}{2\ell+\nu-1+\delta}\cdot\frac{2\ell+\nu}{2\ell+\nu+2\delta}\cdot\frac{2\ell+\nu+1}{2\ell+\nu+1+2\delta}.$$

Here, each fraction is smaller than one for $\delta \geq 0$, i. e., for $\alpha \geq 1/2$, and strictly larger than one for $\delta < 0$, i. e., for $\alpha < 1/2$. Thus, for the values of α where these vectors are applied, the estimates $\|v^+\|_2^2 \leq \mu$ and $\|v^-\|_2^2 \leq \mu$ hold true.

Putting all of the above together and having recourse to Lemma 4.2, for $\alpha \geq 1/2$, we arrive at

$$\begin{aligned}
\|E_n\|_\infty^2 &\geq \frac{\|E_n v^+\|_2^2}{\|v^+\|_2^2} \\
&\geq 2^{-2\omega} \frac{\Gamma(N+\nu-1)(N+\alpha+\nu-3/2)}{\Gamma(N+2\alpha+\nu-1)} \frac{\Gamma^2(\alpha+\nu+N-3/2)}{\Gamma^2(\beta+1+N-3/2)} \\
&\quad \times \mu^{-1} \sum_{j=0}^{N/2-1} \frac{\Gamma(2j+2\beta+1)(2j+\beta+1/2)}{\Gamma(2j+1)} \left(\sum_{\ell=\max\{0,N/2-j-\mu\}}^{N/2-1-j} \binom{\omega}{\ell} \right)^2 \\
&\geq 2^{-2\omega} \frac{\Gamma(N+\nu-1)(N+\alpha+\nu-3/2)}{\Gamma(N+2\alpha+\nu-1)} \cdot \frac{\Gamma^2(\alpha+\nu+N-3/2)}{\Gamma^2(\beta+1+N-3/2)} \\
&\quad \times \mu^{-1} \cdot 2 \cdot \left\lfloor \frac{\mu - \lceil \omega \rceil}{2} \right\rfloor \min_{N/2-\mu \leq j \leq N/2-1} \frac{\Gamma(2j+2\beta+1)(2j+\beta+1/2)}{\Gamma(2j+1)} \cdot 2^{2\omega}
\end{aligned}$$

while, for $\alpha < 1/2$, we get

$$\begin{aligned}
\|E_n\|_\infty^2 &\geq \frac{\|E_n v^-\|_2^2}{\|v^-\|_2^2} \\
&\geq 2^{-2\omega} \frac{\Gamma(N-2\mu+\nu+1)(N-2\mu+\alpha+\nu+1/2)}{\Gamma(N-2\mu+2\alpha+\nu+1)} \frac{\Gamma^2(\alpha+\nu+N-3/2)}{\Gamma^2(\beta+1+N-3/2)} \\
&\quad \times \mu^{-1} \sum_{j=0}^{N/2-1} \frac{\Gamma(2j+2\beta+1)(2j+\beta+1/2)}{\Gamma(2j+1)} \left(\sum_{\ell=\max\{0,N/2-j-\mu\}}^{N/2-1-j} \binom{\omega}{\ell} \right)^2 \\
&\geq 2^{-2\omega} \frac{\Gamma(N-2\mu+\nu+1)(N-2\mu+\alpha+\nu+1/2)}{\Gamma(N-2\mu+2\alpha+\nu+1)} \cdot \frac{\Gamma^2(\alpha+\nu+N-3/2)}{\Gamma^2(\beta+1+N-3/2)} \\
&\quad \times \mu^{-1} \cdot 2 \cdot \left\lfloor \frac{\mu - \lceil \omega \rceil}{2} \right\rfloor \min_{N/2-\mu \leq j \leq N/2-1} \frac{\Gamma(2j+2\beta+1)(2j+\beta+1/2)}{\Gamma(2j+1)} \cdot 2^{2\omega}.
\end{aligned}$$

Because of

$$\frac{\Gamma(2j+2\beta+1)(2j+\beta+1/2)}{\Gamma(2j+1)} = \frac{\Gamma(2j+2\beta+1)}{\Gamma(2j)} \frac{(2j+\beta+1/2)}{(2j)},$$

the term in the minimum is increasing for $\beta > -1/2$ with respect to j, and because β is positive due to the assumption $\beta - \alpha - \nu \geq 0$, the minimum is given by

$$\frac{\Gamma(N-2\mu+2\beta+1)(N-2\mu+\beta+1/2)}{\Gamma(N-2\mu+1)}.$$

As before, we let μ tend to infinity in a lower order than N, in such a way that $2\left\lfloor \frac{\mu-\lceil\omega\rceil}{2} \right\rfloor \sim \mu$ and $N - 2\mu \sim N$. Following this, we derive in both estimates the asymptotic lower bound of N^ν for the norm of the matrix E_n. For F_n, we use the same approach and get the same bound. Thus, the lower bound of $\|A_N\|_\infty$ and with it the lower bound on $\gamma_n^{(\nu)}(\alpha,\beta)$ is asymptotically equal to the upper bound shown before. We have, consequently,

$$\gamma_n^{(\nu)}(\alpha,\beta) \sim n^\nu,$$

whenever $\beta - \alpha - \nu \geq 0$ also if $\beta - \alpha$ is not an integer.

4.4 The Hermite case

In the current section, we will show that the best constant $\eta_n^{(\nu)}(\alpha,\beta)$ in the Hermite case when $\beta-\alpha-\lceil\nu/2\rceil \geq 0$ and $\beta-\alpha$ not an integer, as well as for $0 < \beta-\alpha < \lceil\nu/2\rceil$ with arbitrary $\beta-\alpha$, has the asymptotic behavior

$$\eta_n^{(\nu)}(\alpha,\beta) \sim (2n)^{(\beta-\alpha+\nu)/2}$$

as n goes to infinity. In order to determine this constant, we want to apply the same ideas as in the Laguerre and Gegenbauer cases. The estimate for the upper bound works exactly like it was done there. However, the approach for the lower bound is not that simple anymore. We still apply the operator to some cut-off vector and estimate the norm of its image, and we partition the sum after some initial simplifications into parts. Now, the entries are not given in a closed form but as a sum. Even worse, this sum is alternating in the last column. But, the sums Σ_2 and Σ_3 in the proof of Lemma 4.2 were characterized by the sign of the last entry. With a more direct approach, we could just provide estimates for $\beta-\alpha \geq \lceil\nu/2\rceil$, and it was even more technical than the proof of Lemma 4.2. But it turns out that a closer examination of the hypergeometric term leads to the desired results in a relatively simple fashion.

As in the proof of Lemma 4.1, set

$$\theta_0 := \beta-\alpha-\lfloor\beta-\alpha\rfloor, \quad \beta_0 := \beta-\theta_0, \quad \beta_1 := \beta_0+(1-\theta_0).$$

From Section 3.2, we know that

$$\eta_n^{(\nu)}(\alpha,\beta) = (2n)^{(\beta-\alpha+\nu)/2}(1+O(1/n))$$

whenever $\beta-\alpha-\lceil\nu/2\rceil \in \mathbb{N}_0$, $\alpha,\beta > -1/2$. Then, for $u(\cdot,\alpha)$ as in (4.3), Lemma 4.1 tells us that

$$\begin{aligned}\eta_n^{(\nu)}(\alpha,\beta) &= \eta_n^{(\nu)}(\alpha,(1-\theta_0)\beta_0+\theta_0\beta_1) &\quad (4.20)\\ &\leq \left(\eta_n^{(\nu)}(\alpha,\beta_0)\right)^{1-\theta_0}\left(\eta_n^{(\nu)}(\alpha,\beta_1)\right)^{\theta} \\ &= (2n)^{(\beta_0(1-\theta_0)+\beta_1\theta_0-\alpha+\nu)/2}(1+O(1/n)) \\ &= (2n)^{(\beta-\alpha+\nu)/2}(1+O(1/n)) &\quad (4.21)\end{aligned}$$

holds for all $\beta-\alpha-\lceil\nu/2\rceil \geq 0$. Even more is true. In Section 3.2, we have moreover shown that

$$\eta_n^{(\nu)}(\alpha,\alpha) \leq (2n)^{\nu/2}(1+O(1/n))$$

for arbitrary $\alpha > -1/2$. However, we did not provide statements for $\beta-\alpha \in \{1,2,\ldots,\lceil\nu/2\rceil-1\}$. Therefore, Lemma 4.1 does not immediately give the necessary bounds. We weaken the assumptions of Lemma 4.1, so that we do not need the statement for all integer differences. Note that we could even start with an arbitrary (countable) set of differences for which we know that the bound is valid. However, this generalization is not necessary here. In our case, we set $\theta_0 = (\beta-\alpha)/\lceil\nu/2\rceil$ in the proof of Lemma 4.1. With $\beta_0 = \alpha$ and $\beta_1 = \alpha+\lceil\nu/2\rceil$, all necessary conditions for the interpolation theorem of Stein are fulfilled. Employing this idea in the proof, we can show the following lemma.

Lemma 4.5. *Under the assumptions of Lemma 4.1, for any $\gamma > -1$ ($\gamma > -1/2$), we have $\|T\|_{\alpha\to\gamma} = \|D^\nu\|_{\alpha\to\gamma}$. If*

$$\|D^\nu f\|_\beta \le C_n^{(\nu)}(\alpha,\beta)\|f\|_\alpha \quad \text{for all } f \in \mathcal{P}_n \tag{4.22}$$

is true for some $\beta = \beta'$ satisfying $\beta' - \alpha \in \mathbb{Z}$, and for all $\beta = \beta' + k$, $k \in K \subseteq \mathbb{N}$, K containing infinitely many numbers, and if $C_n^{(\nu)}(\alpha,\beta)$ satisfies

$$C_n^{(\nu)}(\alpha, \beta'(1-\theta) + (\beta'+1)\theta) = \left(C_n^{(\nu)}(\alpha,\beta')\right)^{1-\theta}\left(C_n^{(\nu)}(\alpha,\beta'+k)\right)^{\theta}, \quad \theta \in [0,1],$$

for all $k \in K$, then (4.22) *holds for all $\beta \in [\beta',\infty)$.*

Applying this lemma to the above considerations, we have shown that (4.20) holds whenever $\beta - \alpha \ge 0$.

With the same argumentation as before, we investigate submatrices derived by a permutation and restrict our inquiry to the matrix E_N (see Section 3.2). However, we will not work with the matrix directly, but flip it like we did before. Thus, we actually investigate $B_N = J_N E_N J_N$. The entries b_{jk} of this matrix then can be written as

$$\begin{aligned} b_{jk} = 2^\nu \Gamma(\lceil \nu/2 \rceil + 1) &\sqrt{\frac{\Gamma(N/2 - j + \beta - 1/2)}{\Gamma(N/2 - j)}} \sqrt{\frac{\Gamma(N/2 - k + \lfloor \nu/2 \rfloor)}{\Gamma(N/2 - k + \lceil \nu/2 \rceil + \alpha - 1/2)}} \\ &\times \binom{N/2 - j + \lceil \nu/2 \rceil - 3/2}{\lceil \nu/2 \rceil}\binom{\beta - \alpha - \lceil \nu/2 \rceil}{j-k} \\ &\times {}_3F_2\left(\begin{matrix} -\lceil \nu/2 \rceil,\ k-j,\ \beta + N/2 - j - 1/2 \\ \beta - \alpha - \lceil \nu/2 \rceil + k - j + 1,\ N/2 - j - 1/2 \end{matrix}; 1\right). \end{aligned}$$

As in the Laguerre and Gegenbauer cases, we define vectors $v^+ = (v_j^+)_{j=0}^{N/2-1}$ and $v^- = (v_j^-)_{j=0}^{N/2-1}$ for $\alpha + \lceil \nu/2 \rceil - \lfloor \nu/2 \rfloor \ge 1/2$ and $\alpha + \lceil \nu/2 \rceil - \lfloor \nu/2 \rfloor < 1/2$, respectively, by

$$v_j^+ = \begin{cases} \displaystyle\prod_{\ell=1}^{j} \sqrt{\frac{N/2 - \ell + \lfloor \nu/2 \rfloor}{N/2 - \ell + \lceil \nu/2 \rceil + \alpha + 1/2}} & : 0 \le j \le \mu - 1 \\ 0 & : \text{otherwise}, \end{cases}$$

$$v_j^- = \begin{cases} \displaystyle\prod_{\ell=j}^{j+\mu-1} \sqrt{\frac{N/2 - \ell + \lceil \nu/2 \rceil + \alpha - 3/2}{N/2 - \ell + \lceil \nu/2 \rceil - 1}} & : 0 \le j \le \mu - 1 \\ 0 & : \text{otherwise}. \end{cases}$$

The parameter μ is chosen as in the Laguerre and Gegenbauer cases. Up to a permutation, the overall picture is similar to the one in Figure 4.4. We confine ourselves to the detailed treatment of $\|B_N v^+\|_2$ and note that the norm $\|B_N v^-\|_2$ can be estimated similarly. As before, we want to employ Lemma 4.2. Anticipating some of the arguments used in the proof and translating the lemma to our situation, we have

$$\sum_{j=0}^{\mu-1} \rho_j \left(\sum_{k=0}^{j}\binom{\beta-\alpha}{k}\right)^2 \ge 2\left\lfloor \frac{\mu - \lceil \beta - \alpha \rceil}{2} \right\rfloor 2^{2\beta - 2\alpha} \cdot \min_{0 \le j \le \mu - 1} \rho_j,$$

for $\beta - \alpha \in (0, \infty) \setminus \mathbb{N}$, $\mu \in \mathbb{N}$, $\mu > \lceil \beta - \alpha \rceil$, and $\rho_j > 0$, $j = 0, \dots, \mu - 1$. In fact, this is even true for $\beta - \alpha \in (0, \infty)$ without the restriction not to be an integer. Then, the sums terminate naturally, and the corresponding lines each add up to $2^{\beta-\alpha}$. Now, consider the jth entry of $B_N v^+$,

$$(B_N v^+)_j = 2^\nu \Gamma(\lceil \nu/2 \rceil + 1) \sqrt{\frac{\Gamma(N/2 - j + \beta - 1/2)}{\Gamma(N/2 - j)}} \sqrt{\frac{\Gamma(N/2 + \lfloor \nu/2 \rfloor + 1)}{\Gamma(N/2 + \lceil \nu/2 \rceil + \alpha + 1/2)}} \times \binom{N/2 - j + \lceil \nu/2 \rceil - 3/2}{\lceil \nu/2 \rceil} \binom{\beta - \alpha - \lceil \nu/2 \rceil}{j - k} \times {}_3F_2 \left(\begin{matrix} -\lceil \nu/2 \rceil, \; k - j, \; \beta + N/2 - j - 1/2 \\ \beta - \alpha - \lceil \nu/2 \rceil + k - j + 1, \; N/2 - j - 1/2 \end{matrix} ; 1 \right).$$

We are interested in statements for large N only. Then, the last upper argument and the last lower argument in the hypergeometric series are almost the same and cancel out. Hence, the ${}_3F_2$ transforms to ${}_2F_1$, and

$${}_2F_1 \left(\begin{matrix} -\lceil \nu/2 \rceil, \; k - j \\ \beta - \alpha - \lceil \nu/2 \rceil + k - j + 1 \end{matrix} ; 1 \right) = \frac{(\beta - \alpha - \lceil \nu/2 \rceil + 1)_{\lceil \nu/2 \rceil}}{(\beta - \alpha - \lceil \nu/2 \rceil + k - j + 1)_{\lceil \nu/2 \rceil}}.$$

Together with the coefficient $\binom{\beta-\alpha-\lceil \nu/2 \rceil}{j-k}$, this turns into $\binom{\beta-\alpha}{j-k}$.

The vectors v^+ and v^- were chosen in such a way that $\|v^+\|_2^2 \leq \mu$ and $\|v^-\|_2^2 \leq \mu$ is granted for the values of α where they will be applied. Therefore, we can estimate the spectral norm of B_N with help of the estimate for the hypergeometric series by

$$\|B_N\|_\infty^2 \geq \frac{\|B_N v^+\|_2^2}{\|v^+\|_2^2} \geq 2^{2\nu} \frac{\Gamma(N/2 + \lfloor \nu/2 \rfloor + 1)}{\Gamma(N/2 + \lceil \nu/2 \rceil + \alpha + 1/2)} \sum_{j=0}^{\mu-1} \frac{\Gamma(N/2 - j + \beta - 1/2)}{\Gamma(N/2 - j)} \times \left[\binom{N/2 - j + \lceil \nu/2 \rceil - 3/2}{\lceil \nu/2 \rceil} \right]^2 \left(\sum_{k=0}^{j} \binom{\beta - \alpha}{j - k} \right)^2.$$

Employing Lemma 4.2 in the aforementioned form, this is not greater than

$$2^{2\nu} \frac{\Gamma(N/2 + \lfloor \nu/2 \rfloor + 1)}{\Gamma(N/2 + \lceil \nu/2 \rceil + \alpha + 1/2)} \cdot \mu^{-1} \cdot 2 \left\lfloor \frac{\mu - \lceil \beta - \alpha \rceil}{2} \right\rfloor 2^{2\beta - 2\alpha} \times \min_{0 \leq j \leq \mu - 1} \left\{ \frac{\Gamma(N/2 - j + \beta - 1/2)}{\Gamma(N/2 - j)} \frac{\Gamma^2(N/2 - j + \lceil \nu/2 \rceil - 1/2)}{\Gamma^2(N/2 - j - 1/2)} \right\}.$$

Again, letting $\mu(N)$ go to infinity controlled by $N - 2\mu \sim N$ and $\mu - \lceil \omega \rceil - \lceil \nu/2 \rceil \sim \mu$, this is asymptotically equal to

$$2^{2\beta - 2\alpha + 2\nu} \left(\frac{N}{2} \right)^{\lfloor \nu/2 \rfloor - \lceil \nu/2 \rceil - \alpha + \beta + 2\lceil \nu/2 \rceil} = (2N)^{\beta - \alpha + \nu}.$$

Thus, we get the asymptotic lower bound

$$\|E_N\|_\infty \geq \frac{\|E_N v^+\|_2}{\|v^+\|_2} \geq (2n)^{(\beta-\alpha+\nu)/2},$$

exactly as we wanted. Similarly, we can prove the estimate

$$\|F_N\|_\infty \geq (2n)^{(\beta-\alpha+\nu)/2},$$

and since $\eta_n^{(\nu)}(\alpha,\beta) = \max\{\|E_N\|_\infty, \|F_N\|_\infty\}$, we get $\eta_n^{(\nu)}(\alpha,\beta) \geq (2n)^{(\beta-\alpha+\nu)/2}$. Together with (4.20), we arrive at

$$\eta_n^{(\nu)}(\alpha,\beta) \sim (2n)^{(\beta-\alpha+\nu)/2}$$

as n goes to infinity, for arbitrary $\beta - \alpha \geq 0$.

Chapter 5

Integral operators

Contents

We now turn our focus to the "negative case." The previous chapters heavily depend on the fact that the symbol of the underlying Toeplitz operator is bounded, either explicitly or implicitly when interpolating between the integral cases. However, this is not the case anymore if $\omega = \beta - \alpha - \nu < 0$, or $\beta - \alpha < 0$ in the Hermite case. What we are going to show in the following sections is that the smallest constants for the Laguerre, Gegenbauer, and Hermite cases can be expressed in terms of the operator norm of some integral operator. It might seem strange at first to replace something as simple as a matrix by something as complicated as an integral operator, but by the means of a very handy result by Widom [28, 29], which was independently also rediscovered by Shampine [23, 24], this indeed simplifies things.

The result of Widom and Shampine has been used and proved before. Although we only need the result in the $L^2(0,1)$ case afterwards, we will give the more general form for an operator on $L^p(0,1)$, $p \geq 1$, which may be of use for tackling similar problems. The proof is very close to the proofs given in [8] and [9] with the main difference being the operators R_p and S_p occurring inside.

Lemma 5.1 (Widom and Shampine)**.** *Let A_N be an $(N \times N)$-matrix and define the simple function $k_N(x,y) = (A_N)_{\lfloor Nx \rfloor, \lfloor Ny \rfloor}$. Let K_N be the integral operator on $L^p(0,1)$ $(1 \leq p \leq \infty)$ that is* *given by*

$$(K_N f)(x) = \int_0^1 k_N(x,y) f(y) \mathrm{d}y.$$

Then,

$$\|A_N\|_\infty = N\|K_N\|_\infty.$$

Proof. We define I_k as the interval $\left[\frac{k}{N}, \frac{k+1}{N}\right)$ and χ_k as its characteristic function. Furthermore, let $\ell_p^N := (\mathbb{C}^N, \|\cdot\|_p)$ denote the vector space of all N-tupels with complex entries equipped with the ℓ_p norm. To keep the notation simple, we will assume in the following $p < \infty$, but we point out that the same can be done for $p = \infty$ by just replacing any occurrence of $1/p$ with a 0. We define the operators

$$R_p : \ell_p^N \to L^p(0,1), \qquad \{x_k\}_{k=0}^{N-1} \mapsto N^{1/p} \sum_{k=0}^{N-1} x_k \chi_k,$$

$$S_p : L^p(0,1) \to \ell_p^N, \qquad f \mapsto \left\{ N^{1-1/p} \int_{I_k} f(t) \mathrm{d}t \right\}_{k=0}^{N-1}.$$

For R_p, we determine the operator norm as

$$\begin{aligned}
\|R_p\|_\infty^p &= \sup \frac{\int_0^1 \left| N^{1/p} \sum_{k=0}^{N-1} x_k \chi_k(t) \right|^p \mathrm{d}t}{\sum_{k=0}^{N-1} |x_k|^p} = \sup \frac{N \sum_{j=0}^{N-1} \int_{I_j} |x_j|^p \mathrm{d}t}{\sum_{k=0}^{N-1} |x_k|^p} \\
&= \sup \frac{N \sum_{j=0}^{N-1} \frac{|x_j|^p}{N}}{\sum_{k=0}^{N-1} |x_k|^p} = 1,
\end{aligned}$$

the supremum over all vectors $x = \{x_k\}_{k=0}^{N-1}$ in ℓ_p^N with $\|x\|_p \neq 0$. On the other hand, we get for S_p that

$$\begin{aligned}
\|S_p\|_\infty^p &= \sup \frac{\sum_{k=0}^{N-1} \left| N^{1-1/p} \int_{I_k} f(t) \mathrm{d}t \right|^p}{\|f\|_p^p} \leq \sup \frac{N^{p-1} \sum_{k=0}^{N-1} \left(\int_{I_k} |f(t) \cdot 1| \mathrm{d}t \right)^p}{\|f\|_p^p} \\
&\leq \sup \frac{N^{p-1} \sum_{k=0}^{N-1} \int_{I_k} |f(t)|^p \mathrm{d}t \cdot \left(\int_{I_k} \mathrm{d}t \right)^{p-1}}{\|f\|_p^p} = \sup \frac{N^{p-1} \|f\|_p^p N^{1-p}}{\|f\|_p^p} = 1,
\end{aligned}$$

the supremum taken over all $f \in L^p(0,1)$ with $\|f\|_p \neq 0$. The first inequality is the triangle inequality first for the sum and then once more for the integral, while the second one is Hölder's inequality. Taking $f \equiv 1$, we see immediately that $\|S_p\|_\infty = 1$.

Taking a vector $\{x_k\}_{k=0}^{N-1} \in \ell_p^N$, we verify that

$$\begin{aligned}
S_p R_p \{x_k\}_k &= S_p \left(N^{1/p} \sum_k x_k \chi_x \right) = \left\{ N^{1-1/p} N^{1/p} \int_{I_j} \sum_k x_k \chi_k(t) \mathrm{d}t \right\}_j \\
&= \left\{ N \int_{I_j} x_j \mathrm{d}t \right\}_j = \{x_k\}_k.
\end{aligned}$$

That is, $S_p R_p$ is the identity operator on ℓ_p^N.

Let $A_N = (a_{ij})_{i,j=0}^{N-1}$ and set $f_k := \int_{I_k} f(t)\mathrm{d}t$. A simple calculation yields

$$(R_p A_N S_p f)(x) = N \sum_{k=0}^{N-1} a_{jk} f_k \qquad \text{for } x \in \left[\tfrac{k}{N}, \tfrac{k+1}{N}\right).$$

This is just $N(K_N f)(x)$ for every x. Therefore, $R_p A_N S_p = N K_N$ and with $S_p R_p = I$, we conclude $A_N = N S_p K_N R_p$.

Collecting all of the above, we arrive at

$$\begin{aligned}\|K_N\|_\infty &= \|N^{-1} R_p A_N S_p\|_\infty \le N^{-1}\|R_p\|_\infty \|A_N\|_\infty \|S_p\|_\infty = N^{-1}\|A_N\|_\infty \\ &= \|N^{-1} N S_p K_N R_p\|_\infty \le \|K_N\|_\infty,\end{aligned}$$

which is what we wanted to show. □

The integral operator for the Laguerre and Gegenbauer cases is known to be the Volterra integral operator $L^*_{\nu,\alpha,\beta}$ on $L^2(0,1)$. It is given by

$$(L^*_{\nu,\alpha,\beta} f)(x) = \frac{1}{\Gamma(\nu-\beta+\alpha)} \int_0^x x^{-\alpha/2} y^{\beta/2} (x-y)^{\nu-\beta+\alpha-1} f(y)\mathrm{d}y. \qquad \text{(1.3 revisited)}$$

Reducing the study of the smallest constant to determining the norm of such an integral operator introduces new problems. However, tight estimates for $\|L_{\nu,\alpha,\beta}\|_\infty$ are available. Moreover, when $\beta = \alpha + \nu - 1$, it is known [7, 8] that the norm equals $2/(\nu+1)$ times the inverse of the smallest positive zero of the Bessel function $J_{(\alpha-1)/(\nu+1)}$.

The main difficulty, however, is that the operators have piecewise constant kernels obtained from the matrix representations of the differential operator in the appropriate bases. Do they converge in the operator norm to the operator (1.3)? Things get a lot easier when $\beta-\alpha-\nu < -1/2$. Because the operator then is Hilbert-Schmidt, it suffices to show the convergence in the Hilbert-Schmidt norm.

Still open is the problem for $-1/2 \le \beta - \alpha - \nu < 0$, since then the operator is no longer Hilbert-Schmidt. Nevertheless, it can be shown that the operator belongs to some Schatten class and is thus compact. While this is not of immediate use in determining the best constant, it could be of help in further attempts towards closing the gap $-1/2 \le \beta - \alpha - \nu < 0$.

A similar restriction holds in the Hermite case, where the operator in question is Hilbert-Schmidt only for $\beta - \alpha < -1/2$.

Before we dive deeper into the details of each of these operators, we will embark on two small lemmas that play an important role in proving the convergence in every case. The results are more or less folklore. Nevertheless, we restate them here for reference and give a proof for the sake of completeness.

This first lemma provides an alternate representation for the incomplete beta integral occurring in some of the proofs.

Lemma 5.2. *Let* $0 < z < 1$, $a, b > 0$. *Then, for the incomplete beta integral,*

$$B(z; a, b) := \int_0^z u^{a-1}(1-u)^{b-1} \mathrm{d}u = \frac{z^a}{a} \, {}_2F_1\left(\begin{matrix} a, \; 1-b \\ a+1 \end{matrix}; z\right).$$

Proof. First, we expand $(1-u)^{b-1}$ by the binomial theorem:

$$\int_0^z u^{a-1}(1-u)^{b-1} \mathrm{d}u = \int_0^z u^{a-1} \sum_{n=0}^{\infty} \binom{b-1}{n} (-u)^n \mathrm{d}u.$$

The sum is absolutely convergent for $|u| < 1$, and the integrals $\int_0^z u^{n+a-1} \mathrm{d}u$ are bounded for $a > 0$, $n \geq 0$. So, we can exchange the sum and the integral. Furthermore, the identity

$$\binom{b-1}{n}(-1)^n = \binom{n-b}{n} = \frac{(1-b)_n}{n!}$$

holds. In consequence,

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(1-b)_n}{n!} \int_0^z u^{n+a-1} \mathrm{d}u &= \sum_{n=0}^{\infty} \frac{(1-b)_n}{n+a} \frac{z^{n+a}}{n!} = \frac{z^a}{a} \sum_{n=0}^{\infty} \frac{(a)_n (1-b)_n}{(a+1)_n} \frac{z^n}{n!} \\ &= \frac{z^a}{a} \, {}_2F_1\left(\begin{matrix} a, \; 1-b \\ a+1 \end{matrix}; z\right).\end{aligned}$$

□

The next lemma uses the previous one to show that integrals of the occurring kernels are sufficiently small when taken over a small stripe.

Lemma 5.3. *Let* α, β, γ *be real numbers,* $\beta, \gamma > -1$, $\alpha + \beta + \gamma > -2$. *The two integrals*

$$\int_0^{\varepsilon} z^{\beta}(1-z)^{\gamma} \mathrm{d}z \qquad \text{and} \qquad \int_{\varepsilon}^1 x^{\alpha} \int_{x-\varepsilon}^{x} y^{\beta}(x-y)^{\gamma} \mathrm{d}y \mathrm{d}x$$

converge to zero when $\varepsilon > 0$ *tends to zero.*

Proof. We estimate the term $(1-z)^{\gamma}$ from above. For $\gamma \geq 0$, clearly $(1-z)^{\gamma} \leq 1$. For $\gamma \in (-1, 0)$, we have $(1-z)^{\gamma} \leq (1-\varepsilon)^{\gamma} \leq (1-\varepsilon)^{-1} = 1 + \frac{\varepsilon}{1-\varepsilon}$. Therefore,

$$\int_0^{\varepsilon} z^{\beta}(1-z)^{\gamma} \mathrm{d}z \leq \int_0^{\varepsilon} z^{\beta}\left(1 + \tfrac{\varepsilon}{1-\varepsilon}\right) \mathrm{d}z = \left(1 + \tfrac{\varepsilon}{1-\varepsilon}\right) \frac{\varepsilon^{\alpha+1}}{\alpha+1} \xrightarrow[\varepsilon \to 0]{} 0,$$

since $\alpha + 1 > 0$.

For the second integral, we first substitute $z = y/x$ and then pass from z to $1-z$, i. e.,

$$\begin{aligned}\int_{\varepsilon}^1 x^{\alpha} \int_{x-\varepsilon}^{x} y^{\beta}(x-y)^{\gamma} \mathrm{d}y \mathrm{d}x &= \int_{\varepsilon}^1 x^{\alpha+\beta+\gamma+1} \int_{1-\varepsilon/x}^{1} z^{\beta}(1-z)^{\gamma} \mathrm{d}z \mathrm{d}x \\ &= \int_{\varepsilon}^1 x^{\alpha+\beta+\gamma+1} \int_0^{\varepsilon/x} z^{\gamma}(1-z)^{\beta} \mathrm{d}z \mathrm{d}x.\end{aligned}$$

By virtue of Lemma 5.2, this is the same as

$$\int_\varepsilon^1 x^{\alpha+\beta+\gamma+1} \frac{(\varepsilon/x)^{\gamma+1}}{\gamma+1} {}_2F_1\left(\begin{matrix}\gamma+1,\ -\beta\\ \gamma+2\end{matrix};\frac{\varepsilon}{x}\right)\mathrm{d}x \\ = \frac{\varepsilon^{\gamma+1}}{\gamma+1}\int_\varepsilon^1 \sum_{\tau=0}^\infty \frac{(\gamma+1)_\tau(-\beta)_\tau}{(\gamma+2)_\tau \tau!}\varepsilon^\tau x^{\alpha+\beta-\tau}\mathrm{d}x.$$

For the moment, assume that $\alpha+\beta$ is not in $\{-1,0,1,2,\dots\}$. Since the hypergeometric function converges absolutely for $x \geq \varepsilon$ (i. e., for $\varepsilon/x \leq 1$; see, e. g., Theorem 2.1.2 of [2]), and the integral $\int_\varepsilon^1 x^{\alpha+\beta-\tau}\mathrm{d}x$ is bounded, we can exchange the integral and the sum. Now, we can write this as

$$\frac{\varepsilon^{\gamma+1}}{(\alpha+\beta+1)(\gamma+1)}\sum_{\tau=0}^\infty \frac{(\gamma+1)_\tau(-\beta-\alpha-1)_\tau(-\beta)_\tau}{(\gamma+2)_\tau(-\beta-\alpha)_\tau \tau!}(\varepsilon^\tau - \varepsilon^{\alpha+\beta+1}) \\ = \frac{\varepsilon^{\gamma+1}}{(\alpha+\beta+1)(\gamma+1)} {}_3F_2\left(\begin{matrix}\gamma+1,\ -\beta-\alpha-1,\ -\beta\\ \gamma+2,\ -\beta-\alpha\end{matrix};\varepsilon\right) \\ - \frac{\varepsilon^{\alpha+\beta+\gamma+2}}{(\alpha+\beta+1)(\gamma+1)} {}_3F_2\left(\begin{matrix}\gamma+1,\ -\beta-\alpha-1,\ -\beta\\ \gamma+2,\ -\beta-\alpha\end{matrix};1\right).$$

The two appearing hypergeometric series converge absolutely. Since both exponents, $\gamma+1$ and $\alpha+\beta+\gamma+2$, are greater than zero, the whole sum goes to zero as ε does.

In case $\alpha+\beta \in \{-1,0,1,2,\dots\}$, for exactly one value of τ, the above argumentation is not working anymore. Consider $\alpha+\beta = m \in \mathbb{N}\cup\{-1,0\}$. We may still exchange the sum and the integral, but then, for $\tau = m+1$, we cannot express the integral as above. The sum of the first $m+1$ terms is finite, and the factor $\varepsilon^{\gamma+1}$ ensures that this goes to zero. A similar result holds for the absolute convergent sum starting with $m+2$. The corresponding term for $\tau = m+1$ is

$$\frac{\varepsilon^{\gamma+1}}{\gamma+1}\frac{(\gamma+1)_{m+1}(-\beta)_{m+1}}{(\gamma+2)_{m+2}(m+1)!}\varepsilon^{m+1}\log(1/\varepsilon).$$

So, this is a constant times

$$\varepsilon^{\gamma+m+2}\log(1/\varepsilon) = \frac{\log(1/\varepsilon)}{(1/\varepsilon)^{\alpha+\beta+\gamma+2}},$$

due to our choice of m. Since $\alpha+\beta+\gamma+2 > 0$ by assumption, this tends to zero as $1/\varepsilon \to \infty$, i. e., for $\varepsilon \to 0$. Thus, the integral vanishes also in that case. □

5.1 The Laguerre case

In [7], it has already been stated that the integral operator (built from the matrix representation of the operator of differentiation) with respect to the Laguerre bases converges in the norm to the operator

$$(L^*_{\nu,\alpha,\beta}f)(x) = \frac{1}{\Gamma(\nu-\beta+\alpha)}\int_0^x x^{-\alpha/2}y^{\beta/2}(x-y)^{\nu-\beta+\alpha-1}f(y)\mathrm{d}y. \qquad \text{(1.3 revisited)}$$

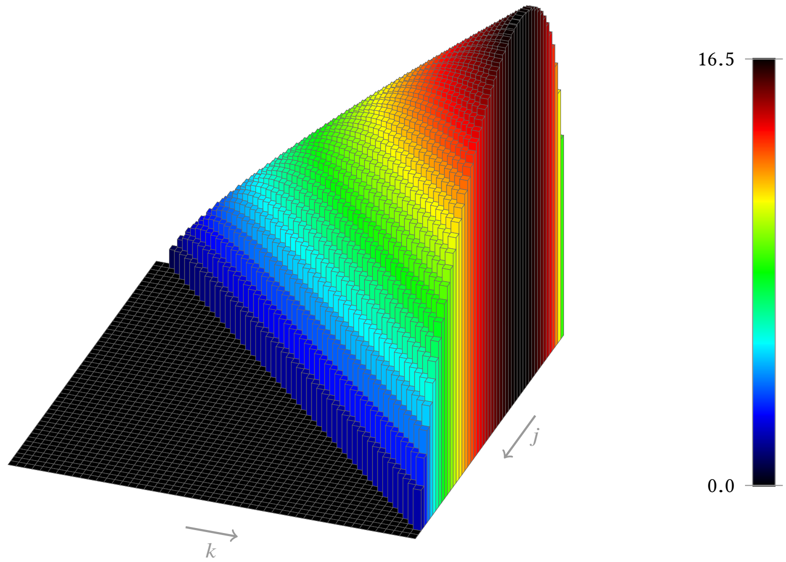

Figure 5.1: Matrix plot for $n = 50,\ \alpha = 0.3,\ \beta = 0.6,\ \nu = 2$ in the Laguerre setting.

However, the statement is left without a proof and only mentions that this can be made precise. This is indeed true, but not at all trivial, although elementary. In the following section, we want to deliver the missing details. Figure 5.1 gives a rough idea of what the matrix in this case may look like.

As before, we use the abbreviation $\omega = \beta - \alpha - \nu$ and assume $\omega < -1/2$. In that case, the operator $L^*_{\nu,\alpha,\beta}$ is Hilbert-Schmidt. By a simple calculation, we end up with

$$\|L^*_{\nu,\alpha,\beta}\|_2^2 = \|L_{\nu,\alpha,\beta}\|_2^2 = \frac{1}{\Gamma^2(-\omega)} \cdot \frac{\Gamma(\beta+1)\Gamma(-2\omega-1)}{\Gamma(\beta-2\omega)} \cdot \frac{1}{\nu-\omega} < \infty,$$

which holds for $\omega < -1/2$, which is the best possible for the Hilbert-Schmidt norm.

To show what we promised above, we follow a path similar to the one in [8]. There, the square $[0,1]^2$ was split into N^2 small squares. On each of those the kernel of the operator derived from the matrix representation is constant. Then, the borders surrounding the area of integration received special treatment when not both of the kernels vanish. Here is the main difference. In contrast to just considering the border of width 1, i. e., $j = k$, we now widen the border slightly in order to get better estimates for the remaining interior of the integration area.

Theorem 5.4. *Let $\omega = \beta - \alpha - \nu < -1/2$ and let a_{jk} denote the entry jk of the upper-right nonzero matrix block of the matrix representation of the operator D^ν in the bases mentioned above. Then, the operator* (1.3) *is Hilbert-Schmidt and the integral operator $N^{-(\nu-\beta/2+\alpha/2-1)}K_N$ with the kernel $N^{-(\nu-\beta/2+\alpha/2-1)}a_{\lfloor Nx\rfloor,\lfloor Ny\rfloor}$ converges in the Hilbert-Schmidt norm, and thus in the operator norm, to the operator $L^*_{\nu,\alpha,\beta}$.*

Proof. We set

$$k_N(x,y) = N^{-(\nu-\beta/2+\alpha/2-1)} a_{\lfloor Nx\rfloor,\lfloor Ny\rfloor},$$
$$\rho(x,y) = \frac{1}{\Gamma(-\omega)} x^{-\alpha/2} y^{\beta/2} (x-y)^{-\omega-1},$$

representing the kernels of the integral operators $N^{-(\nu-\beta/2+\alpha/2-1)}K_N$ and $L^*_{\nu,\alpha,\beta}$, respectively. Divide the square $[0,1]^2$ into N^2 small squares with side length $1/N$. These squares are denoted by Q_{jk}, $0 \le j,k \le N-1$, with $Q_{jk} = \left[\frac{j}{N}, \frac{j+1}{N}\right) \times \left[\frac{k}{N}, \frac{k+1}{N}\right)$. The kernel k_N is just a constant on each of these squares.

First, we note that both, k_N and ρ, vanish on Q_{jk}, for $0 \le j < k \le N-1$. So, there is nothing to prove. We are left with verifying that

$$\sum_{j=0}^{N-1} \sum_{k=0}^{j} \iint_{Q_{jk}} |k_N(x,y) - \rho(x,y)|^2 \mathrm{d}(x,y)$$

tends to zero as N goes to infinity. Next, we will separately treat different groups of squares. This is done to overcome some computational difficulties caused by the kernel ρ possibly having poles along the borders of the integration area (namely for $\omega > -1$ along the diagonal, for $\alpha < 0$ in the origin, and for $\beta < 0$ along the x axis). Furthermore, we widen the diagonal border to derive a better estimate in the inner part.

To this end, we introduce the parameter $m \in \mathbb{N}$, the width of the border, i. e., the number of squares side-by-side on a direct line from the actual domain boundary to the non-border area. This parameter allows a more general consideration of the convergence along these borders and provides better estimates for the terms resulting in a singularity at the borders, especially in the case $j = k$. For the arguments to hold, it is necessary that $m = o(N)$. A lower bound on m has to be chosen such that $N = o(m^3)$. In other words, there should exist constants $c, C > 0$ and some $0 < \varepsilon, \delta < 2/3$ with $cN^{1/3+\delta} \le m \le CN^{1-\varepsilon}$ for all N larger than some $N_0 \in \mathbb{N}$.

We denote by $\partial\Omega$ the set of squares alongside said border. This set is further subdivided into three groups of squares, i. e., $\partial\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ with

$$\Omega_1 = Q_{00} \qquad \text{(the corner)},$$
$$\Omega_2 = \bigcup_{j=1}^{N-1} Q_{j0} \qquad \text{(the lower border)},$$
$$\Omega_3 = \bigcup_{j=1}^{N-1} \bigcup_{k=\max\{j-m,1\}}^{j} Q_{jk} \qquad \text{(the diagonal part)}.$$

Figure 5.2 illustrates these sets.

On $\partial\Omega$, we utilize

$$\iint_{\partial\Omega} |k_N(x,y) - \rho(x,y)|^2 \mathrm{d}(x,y) \le 2 \iint_{\partial\Omega} |k_N(x,y)|^2 \mathrm{d}(x,y) + 2 \iint_{\partial\Omega} |\rho(x,y)|^2 \mathrm{d}(x,y).$$

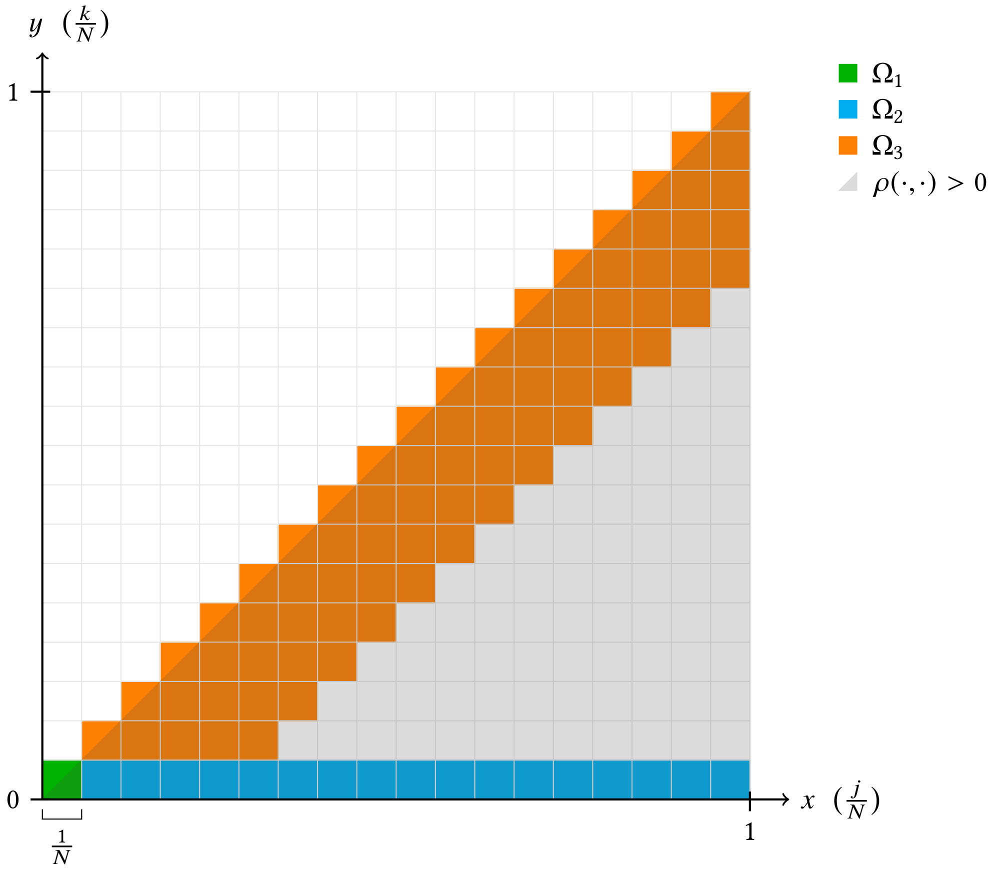

Figure 5.2: Illustration of the partition of the area of integration ($N = 18, m = 4$).

We split up both integrals even further and evaluate each on its own. For the integral over k_N, we have

$$\iint_{\partial\Omega} |k_N(x,y)|^2 \mathrm{d}(x,y) = \iint_{\Omega_1} |k_N(x,y)|^2 \mathrm{d}(x,y) + \iint_{\Omega_2} |k_N(x,y)|^2 \mathrm{d}(x,y) + \iint_{\Omega_3} |k_N(x,y)|^2 \mathrm{d}(x,y),$$

and for the integral over ρ, we get

$$\iint_{\partial\Omega} |\rho(x,y)|^2 \mathrm{d}(x,y) \leq \int_0^{\frac{m+1}{N}} \int_0^x |\rho(x,y)|^2 \mathrm{d}(x,y) + \int_{\frac{m+1}{N}}^1 \int_0^{\frac{1}{N}} |\rho(x,y)|^2 \mathrm{d}(x,y) + \int_{\frac{m+1}{N}}^1 \int_{x-\frac{m+1}{N}}^x |\rho(x,y)|^2 \mathrm{d}(x,y),$$

the estimate taking place in the last part by increasing the area of integration (see Figure 5.3 for details).

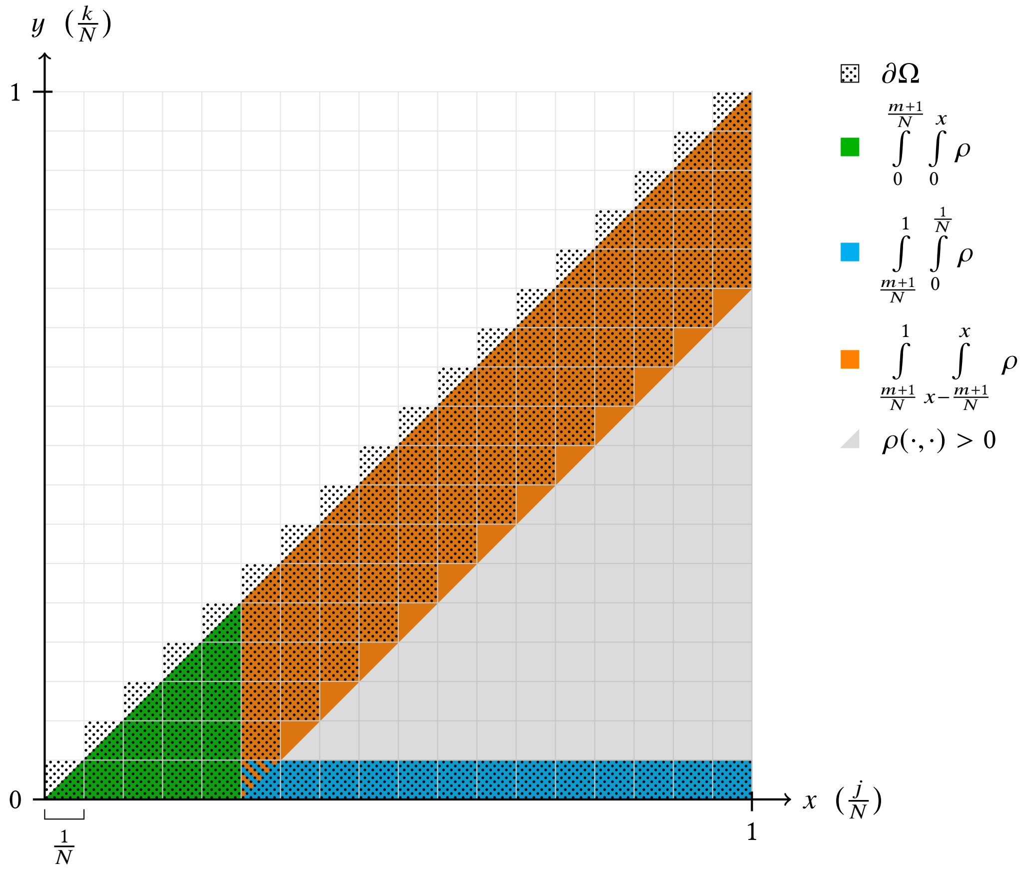

Figure 5.3: Increased area of integration. Squares are partially taken from the inner part, or counted twice ($N = 18,\ m = 4$).

We now take a closer look on each of these six integrals. First,

$$\iint_{\Omega_1} |k_N(x,y)|^2 \mathrm{d}(x,y) = \frac{1}{N^2} N^{\omega-\nu+2} \frac{1}{\Gamma^2(-\omega)} \frac{\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)} \frac{\Gamma(\beta+1)}{\Gamma(1)} \frac{\Gamma^2(-\omega)}{\Gamma^2(1)}.$$

This is just a constant times $N^{\omega-\nu}$. Since $\omega - \nu < -\nu - 1/2 < 0$, it converges to zero as N goes to infinity. Secondly, we have

$$\begin{aligned}\iint_{\Omega_2} |k_N(x,y)|^2 \mathrm{d}(x,y) &= N^{\omega-\nu} \frac{1}{\Gamma^2(-\omega)} \sum_{j=1}^{N-1} \frac{\Gamma(j+\nu+1)}{\Gamma(j+\nu+\alpha+1)} \frac{\Gamma(\beta+1)}{\Gamma(1)} \frac{\Gamma^2(j-\omega)}{\Gamma^2(j+1)} \\ &= N^{\omega-\nu} \frac{\Gamma(\beta+1)}{\Gamma^2(-\omega)} \sum_{j=1}^{N-1} j^{-\alpha-2\omega-2}(1+O(1/j)).\end{aligned}$$

For $-\alpha-2\omega-2 \neq -1$, this is $O(N^{\omega-\nu-\alpha-2\omega-1}) = O(N^{-\beta-1})$, which goes to zero because $\beta > -1$

and therefore $-\beta - 1 < 0$. On the other hand, for $-\alpha - 2\omega - 2 = -1$, this is $O(N^{\omega-\nu}\log N)$, which goes to zero because of $\omega - \nu < 0$.

The third part needs a little more elaborate analysis. Here, we have

$$\begin{aligned}\iint_{\Omega_3} |k_N(x,y)|^2 \mathrm{d}(x,y) &= N^{\omega-\nu} \sum_{j=1}^{N-1} \sum_{k=\max\{j-m,1\}}^{j} \left|k_N\left(\tfrac{j}{N}, \tfrac{k}{N}\right)\right|^2 \\ &= N^{\omega-\nu} \sum_{\ell=0}^{m} \sum_{k=1}^{N-1-\ell} \left|k_N\left(\tfrac{k+\ell}{N}, \tfrac{k}{N}\right)\right|^2 \\ &\leq \frac{N^{\omega-\nu}}{\Gamma^2(-\omega)} \sum_{\ell=0}^{m} \sum_{k=1}^{N-1} \frac{\Gamma(k+\ell+\nu+1)\Gamma(k+\beta+1)\Gamma^2(\ell-\omega)}{\Gamma(k+\ell+\nu+\alpha+1)\Gamma(k+1)\Gamma^2(\ell+1)}.\end{aligned}$$

We consider the $\ell = 0$ part separately. It equals

$$\frac{N^{\omega-\nu}}{\Gamma^2(-\omega)} \frac{\Gamma^2(-\omega)}{\Gamma^2(1)} \sum_{k=1}^{N-1} k^{\beta-\alpha}(1 + O(1/k)).$$

This is $O(N^{2\omega+1})$ for $\beta - \alpha \neq -1$, and $O(N^{\omega-\nu}\log N)$ for $\beta - \alpha = -1$. In both cases, this tends to zero. The rest of the sum equals

$$\frac{N^{\omega-\nu}}{\Gamma^2(-\omega)} \sum_{\ell=1}^{m} \sum_{k=1}^{N-1} (k+\ell)^{-\alpha} k^{\beta} \ell^{-2\omega-2}(1 + O(1/k) + O(1/\ell)). \tag{5.1}$$

For $\alpha \geq 0$, the inequality $(k+\ell)^{-\alpha} \leq k^{-\alpha}$ holds. Therefore, the sum becomes

$$\frac{N^{\omega-\nu}}{\Gamma^2(-\omega)} \sum_{\ell=1}^{m} \ell^{-2\omega-2} \sum_{k=1}^{N-1} k^{\beta-\alpha}(1 + O(1/k) + O(1/\ell)).$$

For $\beta - \alpha \neq -1$, this is $O(N^{\omega-\nu+\beta-\alpha+1} m^{-2\omega-1}) = O((N/m)^{2\omega+1})$, and thanks to $2\omega + 1 < 0$ and $m = o(N)$, this goes to zero. If $\beta - \alpha = -1$, the sum is $O((N/m)^{-2\nu-1}\log N)$, also approaching zero.

Finally, for $-1 < \alpha < 0$, we have $(k+\ell)^{-\alpha} \leq k^{-\alpha} + \ell^{-\alpha}$. So, we can split (5.1) into two parts, ignoring the O terms and the leading constant:

$$N^{\omega-\nu} \sum_{\ell=1}^{m} \ell^{-2\omega-2} \sum_{k=1}^{m} k^{\beta-\alpha} + N^{\omega-\nu} \sum_{\ell=1}^{m} \ell^{-2\omega-2-\alpha} \sum_{k=1}^{N-1} k^{\beta}.$$

We have just shown that the first sum converges to zero. The distinction between $\alpha > 0$ and $\alpha < 0$ was not essential in the final argument. Note that, since $-2\omega - 2 > -1$ and $-\alpha > 0$, the exponent for ℓ is never equal to -1 so we do not have to consider this. Therefore, the second sum is $O((N/m)^{2\omega+\alpha+1})$, and since $2\omega + 1 < 0$ and $\alpha < 0$, this goes to zero.

To summarize, we have shown that $\iint_{\partial\Omega} |k_N(x,y)|^2 \mathrm{d}(x,y) \to 0$ as N goes to infinity and m increases with N.

We now turn to the integrals over the kernel ρ. The first integral is particularly easy to evaluate. Ignoring the constant factor $\Gamma^{-2}(-\omega)$, we here have

$$\begin{aligned}\int_0^{\frac{m+1}{N}} x^{-\alpha} \int_0^x y^{\beta}(x-y)^{-2\omega-2}\mathrm{d}y\mathrm{d}x &= \frac{\Gamma(\beta+1)\Gamma(-2\omega-1)}{\Gamma(\beta-2\omega)} \int_0^{\frac{m+1}{N}} x^{\nu-\omega-1}\mathrm{d}x \\ &= \frac{\Gamma(\beta+1)\Gamma(-2\omega-1)}{\Gamma(\beta-2\omega)(\nu-\omega)}\left(\frac{m+1}{N}\right)^{\nu-\omega}.\end{aligned}$$

Clearly, $\nu-\omega>0$, and thus the value of the integral goes to zero as N increases.

Again, ignoring the constant factor $\Gamma^{-2}(-\omega)$, we get the following for the second integral:

$$\begin{aligned}\int_{\frac{m+1}{N}}^1 x^{-\alpha} \int_0^{\frac{1}{N}} y^{\beta}(x-y)^{-2\omega-2}\mathrm{d}y\mathrm{d}x &= \int_{\frac{m+1}{N}}^1 x^{\beta-\alpha-2\omega-1} \int_0^{\frac{1}{Nx}} z^{\beta}(1-z)^{-2\omega-2}\mathrm{d}z\mathrm{d}x \\ &\le \int_{\frac{m+1}{N}}^1 x^{\nu-\omega-1}\mathrm{d}x \cdot \int_0^{\frac{1}{m+1}} z^{\beta}(1-z)^{-2\omega-2}\mathrm{d}z \\ &\le \int_0^1 x^{\nu-\omega-1}\mathrm{d}x \cdot \int_0^{\frac{1}{m+1}} z^{\beta}(1-z)^{-2\omega-2}\mathrm{d}z \\ &= \frac{1}{\nu-\omega}\int_0^{\frac{1}{m+1}} z^{\beta}(1-z)^{-2\omega-2}\mathrm{d}z.\end{aligned}$$

Clearly, the last integral converges to zero as N (and with that m) goes to infinity. This follows from Lemma 5.3 with $\varepsilon = 1/(m+1)$, together with $\beta > -1$ and $-2\omega - 2 > -1$. Therefore, the above expression goes to zero.

For the last integral, we have

$$\int_{\frac{m+1}{N}}^1 \int_{x-\frac{m+1}{N}}^x |\rho(x,y)|^2\mathrm{d}y\mathrm{d}x = \int_{\frac{m+1}{N}}^1 x^{-\alpha} \int_{x-\frac{m+1}{N}}^x y^{\beta}(x-y)^{-2\omega-2}\mathrm{d}y\mathrm{d}x.$$

This converges for the same reasons as above, and by Lemma 5.3 it tends to zero with N going to infinity.

So, we have shown that $\iint_{\partial\Omega} |\rho(x,y)|^2\mathrm{d}(x,y)$ goes to zero as N goes to infinity. Putting all of the pieces together, it follows that $\iint_{\partial\Omega} |k_N(x,y)-\rho(x,y)|^2\mathrm{d}(x,y) \to 0$. What remains to show is that the difference in the inner area also becomes arbitrarily small.

By choosing m large enough, for the border along $j = k$, we have $j - k \ge m$ and therefore $1/(j-k) \le 1/m$, for all remaining squares. So,

$$(j-k)^{-\omega-1}(1+O(1/(j-k))) = (j-k)^{-\omega-1}(1+O(1/m)).$$

Now, both, a_{jk} and $N^{(\nu-\omega-2)/2}\rho\left(\frac{j+\xi}{N}, \frac{k+\eta}{N}\right)$ with $\xi,\eta \in [0,1)$, are equal to

$$j^{-\alpha/2}k^{\beta/2}(j-k)^{-\omega-1}(1+O(1/j)+O(1/k)+O(1/m)).$$

Therefore,

$$\iint_{Q_{jk}} \left| \frac{a_{jk}}{N^{\nu-\beta/2+\alpha/2-1}} - \rho(x,y) \right|^2 \mathrm{d}(x,y) \\ = \frac{1}{N^{\nu-\omega}} j^{-\alpha} k^{\beta} (j-k)^{-2\omega-2} (O(1/j^2) + O(1/k^2) + O(1/m^2)).$$

Next, by the above estimate and the restriction $\omega < -1/2$, we get

$$(j-k)^{-2\omega-2} = (j-k)^{-2\omega-1}(j-k)^{-1} \le j^{-2\omega-1} m^{-1},$$

since $-2\omega - 1 > 0$. Summing up and employing this estimate, we get that the difference is at most

$$N^{\omega-\nu} \sum_{j=m+2}^{N-1} \sum_{k=1}^{j-m-1} j^{-\alpha-2\omega-1} k^{\beta} m^{-1} (O(1/j^2) + O(1/k^2) + O(1/m^2)) \\ \le N^{\omega-\nu} \sum_{j=1}^{N-1} \sum_{k=1}^{N-1} j^{-\alpha-2\omega-1} k^{\beta} m^{-1} (O(1/j^2) + O(1/k^2) + O(1/m^2)).$$

We split this into the three parts associated with the O terms. For each part, we have to consider several corner cases. Keep in mind that $\alpha, \beta > -1$ and $\nu \ge 1$.

The part belonging to $O(1/j^2)$ is smaller than a constant times

$$m^{-1} N^{\omega-\nu} \sum_{j=1}^{N-1} j^{-\alpha-2\omega-3} \sum_{k=1}^{N-1} k^{\beta}.$$

For $2\omega + \alpha + 2 \ne 0$, this is $O(m^{-1}N^{-1})$. Otherwise, this is $O(m^{-1}N^{-1}\log N)$. In both cases, the part goes to zero as N and therefore m increase.

Similarly, we can treat the sum associated with $O(1/k^2)$. It is not greater than a constant times

$$m^{-1} N^{\omega-\nu} \sum_{j=1}^{N-1} j^{-\alpha-2\omega-1} \sum_{k=1}^{N-1} k^{\beta-2}.$$

Under the assumption that $\beta \ne 1$ and $2\beta - \alpha - 2\nu \ne 0$, this is $O(m^{-1}N^{-1})$. For both, $\beta = 1$ with $2\beta - \alpha - 2\nu \ne 0$ and $\beta \ne 1$ with $2\beta - \alpha - 2\nu = 0$, it is $O(m^{-1}N^{-1}\log N)$. Finally, for $\beta = 1$ together with $2\beta - \alpha - 2\nu = 0$, it is $O(m^{-1}N^{\omega-\nu}\log^2 N)$. In either case, this tends to zero.

Last, we look into the part belonging to $O(1/m^2)$. This is at most a constant times

$$m^{-3} N^{\omega-\nu} \sum_{j=1}^{N-1} j^{-\alpha-2\omega-1} \sum_{k=1}^{N-1} k^{\beta}.$$

If $2\beta - \alpha - 2\nu \neq 0$, this term is $O(m^{-3}N)$ and, for $2\beta - \alpha - 2\nu = 0$, it is $O(m^{-3}N\log N)$. This is the part where we need $N = o(m^3)$. Given that, the sum goes to zero, too, as N, m go to infinity.

Finally, we have shown that the Hilbert-Schmidt norm of the difference of the scaled operator K_N and the operator $L^*_{\nu,\alpha,\beta}$ converges to zero as N goes to infinity, thus proving our claim. □

5.2 The Gegenbauer case

In this section, we are concerned with the Gegenbauer case. The problem for $\beta - \alpha < \nu$ has already been treated in [10], where $\beta-\alpha$ was assumed to be an integer. The restriction simplified things a lot, since the term $(y^2 - x^2)^{\alpha+\nu-\beta-1}$, which plays an important role in the investigation, is just a polynomial. Then, a result from [8] can be applied immediately. Assuming $\beta - \alpha$ is not an integer, this is not the case anymore. As mentioned in [10], the result can be extended to this situation, requiring a more elaborate analysis. In what follows, we will show that this is indeed true.

As before, since $\gamma^{(\nu)}_{n-1} \leq \gamma^{(\nu)}_{n} \leq \gamma^{(\nu)}_{n+1}$, we may assume that $N = n - \nu + 1$ is an even number. We use the notation from (4.18) and employ (4.19). Then, we look at the four parts separately, inserting $k = \lfloor Ny/2 \rfloor$ and $j = \lfloor Nx/2 \rfloor$. Thereafter,

$$\sqrt{\frac{\Gamma(2\lfloor Ny/2\rfloor)(2\lfloor Ny/2\rfloor + \alpha + \nu + 1/2)}{\Gamma(2\lfloor Ny/2\rfloor)}} = (Ny)^{-\alpha+1/2}(1 + O(1/N))$$

and

$$\sqrt{\frac{\Gamma(2\lfloor Nx/2\rfloor + 2\beta + 1)(2\lfloor Nx/2\rfloor + \beta + 1/2)}{\Gamma(2\lfloor Nx/2\rfloor + 1)}} = (Nx)^{\beta+1/2}(1 + O(1/N)).$$

For the other two terms, we set $\lfloor Ny/2\rfloor - \lfloor Nx/2\rfloor = N(y - x)/2 + \delta_N(x, y)$, with some $|\delta_N(x, y)| \leq 2$ and $\lfloor Ny/2\rfloor + \lfloor Nx/2\rfloor = N(y + x)/2 + \delta_N(x, y)$, with some $|\delta_N(x, y)| \leq 2$, not necessarily the same. Then, we obtain

$$\begin{aligned}\frac{\Gamma(\alpha + \nu + \lfloor Ny/2\rfloor + \lfloor Nx/2\rfloor + 1/2)}{\Gamma(\beta + 1 + \lfloor Ny/2\rfloor + \lfloor Nx/2\rfloor + 1/2)} &= \left(\left\lfloor \tfrac{Ny}{2}\right\rfloor + \left\lfloor \tfrac{Nx}{2}\right\rfloor\right)^{-\omega-1}\left(1 + O\left(\tfrac{1}{N(y+x)}\right)\right) \\ &= \left(\tfrac{N}{2}(y + x) + \delta_N(x, y)\right)^{-\omega-1}\left(1 + O\left(\tfrac{1}{N}\right)\right) \\ &= \left(\tfrac{N}{2}\right)^{-\omega-1}(y + x)^{-\omega-1}\left(1 + \tfrac{2\delta_N(x,y)}{N(y+x)}\right)^{-\omega-1} \\ &\qquad \times \left(1 + O\left(\tfrac{1}{N}\right)\right) \\ &= \left(\tfrac{N}{2}\right)^{-\omega-1}(y + x)^{-\omega-1}\left(1 + O\left(\tfrac{1}{N}\right)\right).\end{aligned}$$

Analogously, we derive

$$(-1)^{\lfloor Ny/2\rfloor - \lfloor Nx/2\rfloor}\binom{\omega}{\lfloor Ny/2\rfloor - \lfloor Nx/2\rfloor} = (N/2)^{-\omega-1}(\Gamma(-\omega))^{-1}(y - x)^{-\omega-1}(1 + O(1/N)).$$

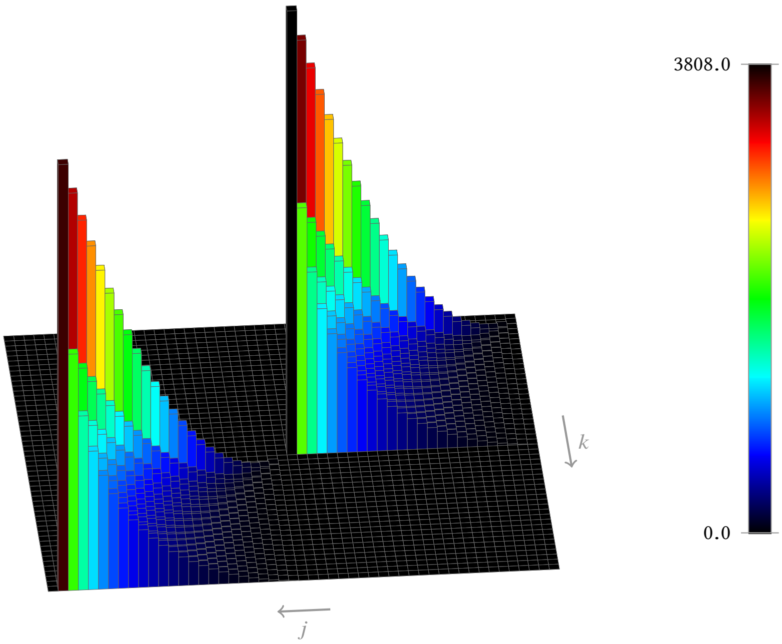

Figure 5.4: Matrix plot in the Gegenbauer case for $n = 50$, $\alpha = 0.2$, $\beta = 1.6$, $\nu = 2$, already modified by a permutation matrix.

Putting all of the above together, we see that, for large N, the entries $e_{\lfloor Nx/2\rfloor,\lfloor Ny/2\rfloor}$ of E_n behave like

$$N^{-\omega+\nu-1}\frac{2^{\omega+2}}{\Gamma(-\omega)}x^{\beta+1/2}y^{-\alpha+1/2}(y^2-x^2)^{-\omega-1}(1+O(1/N)).$$

In slightly different form, this has already been stated in [10]. Defining K_n as the integral operator on $L^2(0,1)$ with kernel $e_{\lfloor Nx/2\rfloor,\lfloor Ny/2\rfloor}$, this indicates that $N^{\omega-\nu+1}K_N$ should converge to the operator $G_{\nu,\alpha,\beta}$ on $L^2(0,1)$ given by

$$(G_{\nu,\alpha,\beta}f)(x)=\frac{2^{\omega+2}}{\Gamma(-\omega)}\int_x^1 x^{\beta+1/2}y^{-\alpha+1/2}(y^2-x^2)^{-\omega-1}f(y)\mathrm{d}y. \tag{5.2}$$

Employing Lemma 5.1, we therefore should have

$$\|E_n\|_\infty\sim\tfrac{N}{2}N^{-\omega+\nu-1}\|G_{\nu,\alpha,\beta}\|_\infty=\tfrac{N^{-\omega+\nu}}{2}\|G_{\nu,\alpha,\beta}\|_\infty.$$

We confine ourselves to $\beta-\alpha<\nu-1/2$, for the same reasons as before: the operator in question then is Hilbert-Schmidt and it suffices to show the convergence in the corresponding, easy accessible norm, which in turn implies convergence in the operator norm. This will be done in the following theorem. The matrix plot in Figure 5.4 gives an idea why we have to take special care when showing convergence close to the diagonal.

Theorem 5.5. *Let $\omega = \beta - \alpha - \nu < -1/2$. Then, the operator* (5.2) *is Hilbert-Schmidt and the integral operator $N^{\omega-\nu+1}K_N$ with the kernel $N^{\omega-\nu+1}e_{\lfloor Nx/2\rfloor,\lfloor Ny/2\rfloor}$ converges in the Hilbert-Schmidt norm and thus in the operator norm to the operator $G_{\nu,\alpha,\beta}$.*

Proof. Instead of the operators given in the statement, we consider the convergence for the adjoint operators. The kernel $k_N(x,y)$ of $N^{\omega-\nu+1}K_N^*$ is given by

$$\begin{aligned} k_N(x,y) = N^{\omega-\nu+1}\frac{2^{-\omega}}{\Gamma(-\omega)} & \sqrt{\frac{\Gamma(2\lfloor Nx/2\rfloor+\nu+1)(2\lfloor Nx/2\rfloor+\alpha+\nu+1/2)}{\Gamma(2\lfloor Nx/2\rfloor+2\alpha+\nu+1)}} \\ &\times\sqrt{\frac{\Gamma(2\lfloor Ny/2\rfloor+2\beta+1)(2\lfloor Ny/2\rfloor+\beta+1/2)}{\Gamma(2\lfloor Ny/2\rfloor+1)}} \\ &\times\frac{\Gamma(\alpha+\nu+\lfloor Nx/2\rfloor+\lfloor Ny/2\rfloor+1/2)}{\Gamma(\beta+1+\lfloor Nx/2\rfloor+\lfloor Ny/2\rfloor+1/2)}\frac{\Gamma(-\omega+\lfloor Nx/2\rfloor-\lfloor Ny/2\rfloor)}{\Gamma(\lfloor Nx/2\rfloor-\lfloor Ny/2\rfloor+1)}, \end{aligned}$$

for $\lfloor Nx/2\rfloor \geq \lfloor Ny/2\rfloor$, and is zero otherwise. We denote the kernel of $G^*_{\nu,\alpha,\beta}$ by $\rho(x,y)$, which is given by

$$\rho(x,y) = \frac{2^{\omega+2}}{\Gamma(-\omega)}y^{\beta+1/2}x^{-\alpha+1/2}(x^2-y^2)^{-\omega-1}$$

for $x > y$, and is zero otherwise. First, by a simple calculation,

$$\begin{aligned} \|G^*_{\nu,\alpha,\beta}\|_2^2 &= \int_0^1\int_0^1 |\rho(x,y)|^2\mathrm{d}y\mathrm{d}y \\ &= \frac{2^{2\omega+4}}{\Gamma^2(-\omega)}\int_0^1 x^{-2\alpha+1}\int_0^x y^{2\beta+1}(x^2-y^2)^{-2\omega-2}\mathrm{d}y\mathrm{d}x \\ &= \frac{2^{2\omega+3}}{\Gamma^2(-\omega)}\int_0^1 x^{-2\alpha+1+2\beta+1-4\omega-4+1}\mathrm{d}x\int_0^1 z^{\beta}(1-z)^{-2\omega-2}\mathrm{d}z \\ &= \frac{2^{2\omega+3}\Gamma(\beta+1)\Gamma(-2\omega-1)}{\Gamma^2(-\omega)(2\nu-2\omega)\Gamma(\beta-2\omega)} < \infty, \end{aligned}$$

which holds for $\omega < -1/2$. Thus, the operator is indeed Hilbert-Schmidt.

We now have to show that

$$\int_0^1\int_0^1 |k_N(x,y)-\rho(x,y)|^2\mathrm{d}y\mathrm{d}x \tag{5.3}$$

goes to zero as N goes to infinity. The proof is very similar to the proof of Theorem 5.4, where we have shown this for the Laguerre case.

We divide the area of integration $[0,1]^2$ into $N/2\times N/2$ squares Q_{jk}, $0 \leq j,k \leq N/2-1$ of side length $2/N$, given by

$$Q_{jk} = \left[\tfrac{2j}{N},\tfrac{2(j+1)}{N}\right)\times\left[\tfrac{2k}{N},\tfrac{2(k+1)}{N}\right).$$

Since the kernel k_N is constant on each of these squares, things are eased up. The double integral (5.3) then equals

$$\sum_{j,k=0}^{N/2-1} \iint_{Q_{jk}} |k_N(x,y) - \rho(x,y)|^2 \mathrm{d}(x,y).$$

Since both kernels are zero on the squares Q_{jk}, for $k \geq j+1$, we do not need to consider these terms.

As in the Laguerre case, divide the main area of integration $\Omega = \bigcup_{0\leq k\leq j\leq N/2-1} Q_{jk}$ into a border set $\partial\Omega$ and an interior set $\mathring{\Omega}$, and split $\partial\Omega$ even further into the union $\Omega_1 \cup \Omega_2 \cup \Omega_3$ with Ω_1, Ω_2, and Ω_3 given by

$$\begin{aligned}
\Omega_1 &= Q_{00}, \\
\Omega_2 &= \bigcup_{j=1}^{N/2-1} Q_{j0}, \\
\Omega_3 &= \bigcup_{j=1}^{N/2-1} \bigcup_{k=\max\{j-m,1\}}^{j} Q_{jk}.
\end{aligned}$$

Here, we make use of the parameter m again. As before, $m \in \mathbb{N}$ is dependent on N in such a way that it is $o(N)$ and $N = o(m^3)$. Recall Figure 5.2 to get an idea of the partition. Indeed, the partition is done in the same way as in the Laguerre case with the only difference that the side length of the small squares is $2/N$ instead of $1/N$.

Again, we employ the estimate

$$\iint_{\partial\Omega} |k_N(x,y) - \rho(x,y)|^2 \mathrm{d}(x,y) \leq 2\iint_{\partial\Omega} |k_N(x,y)|^2 \mathrm{d}(x,y) + 2\iint_{\partial\Omega} |\rho(x,y)|^2 \mathrm{d}(x,y),$$

as well as the equality

$$\iint_{\partial\Omega} |k_N(x,y)|^2 \mathrm{d}(x,y) = \iint_{\Omega_1} |k_N(x,y)|^2 \mathrm{d}(x,y) + \iint_{\Omega_2} |k_N(x,y)|^2 \mathrm{d}(x,y) + \iint_{\Omega_3} |k_N(x,y)|^2 \mathrm{d}(x,y)$$

and the estimate

$$\iint_{\partial\Omega} |\rho(x,y)|^2 \mathrm{d}(x,y) \leq \int_0^{\frac{2(m+1)}{N}} \int_0^x |\rho(x,y)|^2 \mathrm{d}y\,\mathrm{d}x + \int_{\frac{2(m+1)}{N}}^1 \int_0^{\frac{2}{N}} |\rho(x,y)|^2 \mathrm{d}y\,\mathrm{d}x + \int_{\frac{2(m+1)}{N}}^x \int_{x-\frac{2(m+1)}{N}}^x |\rho(x,y)|^2 \mathrm{d}y\,\mathrm{d}x.$$

As one may anticipate from the preparations, the rest of the proof is done in the same way as for Theorem 5.4. The main difference lies in the details.

First, we have

$$\iint_{\Omega_1} |k_N(x,y)|^2 \mathrm{d}(x,y) = \\ N^{2\omega-2\nu} \frac{2^{-2\omega}}{\Gamma^2(-\omega)} \frac{\Gamma(\nu+1)(\alpha+\nu+1/2)}{\Gamma(2\alpha+\nu+1)} \frac{\Gamma(2\beta+1)(\beta+1/2)}{\Gamma(1)} \frac{\Gamma^2(\alpha+\nu+1/2)\Gamma^2(-\omega)}{\Gamma^2(\beta+3/2)\Gamma^2(1)}.$$

This is a constant times $N^{2\omega-2\nu}$, which goes to zero for N to infinity, because of $2\omega - 2\nu < 0$. Secondly, for k_N on Ω_2,

$$\begin{aligned} \iint_{\Omega_2} |k_N(x,y)|^2 \mathrm{d}(x,y) &= N^{2\omega-2\nu} \frac{2^{-2\omega}}{\Gamma^2(-\omega)} \frac{\Gamma(2\beta+1)(\beta+1/2)\Gamma^2(j-\omega)}{\Gamma^2(j+1)} \\ &\quad \times \sum_{j=1}^{N/2-1} \frac{\Gamma(2j+\nu+1)(2j+\alpha+\nu+1/2)}{\Gamma(2j+2\alpha+\nu+1)} \frac{\Gamma^2(\alpha+\nu+j+1/2)}{\Gamma^2(\beta+1+j+1/2)} \\ &= CN^{2\omega-2\nu} \sum_{j=1}^{N/2-1} j^{-4\omega-2\alpha-3}(1+O(1/j)) \end{aligned}$$

with some constant C not depending on N. For $-4\omega - 2\nu - 3 \neq -1$, this is $O(N^{-2\beta-2})$ and, for $-4\omega - 2\alpha - 3 = -1$, it is $O(N^{2\omega-2\nu} \log N)$. In both cases, this tends to zero with growing N.

Finally, for k_N on Ω_3, we get

$$\begin{aligned} &\iint_{\Omega_3} |k_N(x,y)|^2 \mathrm{d}(x,y) = \\ &\qquad \times N^{2\omega-2\nu} \frac{2^{-2\omega}}{\Gamma^2(-2\omega)} \sum_{j=1}^{N/2-1} \sum_{k=\max\{j-m,1\}}^{j} \frac{\Gamma(2j+\nu+1)(2j+\alpha+\nu+1/2)}{\Gamma(2j+2\alpha+\nu+1)} \\ &\qquad \times \frac{\Gamma(2k+2\beta+1)(2k+\beta+1/2)}{\Gamma(2k+1)} \frac{\Gamma^2(\alpha+\nu+j-k+1/2)}{\Gamma^2(\beta+1+j-k+1/2)} \frac{\Gamma^2(-\omega+j+k)}{\Gamma^2(j+k+1)}. \end{aligned}$$

Set $\ell = j - k$. Then, this is the same as

$$\begin{aligned} &N^{2\omega-2\nu} \frac{2^{-2\omega}}{\Gamma^2(-\omega)} \sum_{\ell=0}^{m} \sum_{k=1}^{N/2-1-\ell} \frac{\Gamma(2k+2\ell+\nu+1)(2k+2\ell+\alpha+\nu+1/2)}{\Gamma(2k+2\ell+2\alpha+\nu+1)} \\ &\qquad \times \frac{\Gamma(2k+2\beta+1)(2k+\beta+1/2)}{\Gamma(2k+1)} \frac{\Gamma^2(\alpha+\nu+\ell+1/2)}{\Gamma^2(\beta+1+\ell+1/2)} \frac{\Gamma^2(-\omega+2k+\ell)}{\Gamma^2(2k+\ell+1)}. \end{aligned}$$

Additionally, we let the inner sum run even further, up to $N/2 - 1$. By doing so, the sum can be made only larger.

We consider the part for $\ell = 0$ separately. This term is just a constant times

$$N^{2\omega-2\nu} \sum_{k=1}^{N/2-1} k^{-2\alpha+1+2\beta+1-2\omega-2}\big(1 + O(1/k)\big) = N^{2\omega-2\nu} \sum_{k=1}^{N/2-1} k^{2\nu}\big(1 + O(1/k)\big).$$

So, this is $O(N^{2\omega+1})$ and because of our assumption $\omega < -1/2$, this vanishes as N goes to infinity.

For the rest, the sum is at most a constant times

$$N^{2\omega-2\nu} \sum_{\ell=1}^{m} \sum_{k=1}^{N/2-1} (2k + 2\ell)^{-2\alpha+1}(2k)^{2\beta+1}\ell^{-2\omega-2}(2k + \ell)^{-2\omega-2}\big(1 + O(1/k) + O(1/\ell)\big).$$

For $-2\alpha + 1 < 0$, we employ the estimate $(2k + 2\ell)^{-2\alpha+1} \le (2k)^{-2\alpha+1}$ and, for $1 > -2\alpha + 1 \ge 0$, the estimate $(2k + 2\ell)^{-2\alpha+1} \le (2k)^{-2\alpha+1} + (2\ell)^{-2\alpha+1}$. We apply a similar estimate on the term $(2k+\ell)^{-2\omega-2}$. For $-2\omega-2 < 0$, we again get $(2k+\ell)^{-2\omega-2} \le (2k)^{-2\omega-2}$. For $-2\omega-2 \ge 0$, we get $(2k + \ell)^{-2\omega-2} \le c(\omega)\big((2k)^{-2\omega-2} + \ell^{-2\omega-2}\big)$ with $c(\omega) = 1$ for $-2\omega - 2 < 1$, and $c(\omega) = 2^{-2\omega-3}$ otherwise. So, the resulting sum is a constant times

$$N^{2\omega-2\nu} \sum_{\ell=1}^{m} \ell^{-2\omega-2} \sum_{k=1}^{N/2-1} k^{2\beta+1}\big[\big(k^{-2\alpha+1} + \ell^{-2\alpha+1}\big)\big(k^{-2\omega-2} + \ell^{-2\omega-2}\big)\big]$$

plus the O terms. Note that the terms $\ell^{-2\alpha+1}$ and $\ell^{-2\omega-2}$ only occur if the above conditions are satisfied. We could write this more precise, but this would be even more confusing. In the following, we will show that any of these actually four sums goes to zero, provided said conditions hold.

In any case, we have the sum

$$N^{2\omega-2\nu} \sum_{\ell=1}^{m} \ell^{-2\omega-2} \sum_{k=1}^{N/2-1} k^{2\beta-2\alpha-2\omega}.$$

This is $O\big((N/m)^{2\omega}m^{-1}\big)$, which goes to zero. Next, if $-2\omega - 2 \ge 0$, we additionally have the sum

$$N^{2\omega-2\nu} \sum_{\ell=1}^{m} \ell^{-4\omega-4} \sum_{k=1}^{N/2-1} k^{2\beta-2\alpha+2}.$$

This becomes $O\big((N/m)^{4\omega+3}\big)$ if $2\omega + 2\nu + 2 \ne -1$, and since we assumed $-2\omega - 2 \ge 0$, it follows $4\omega + 3 \le -1$. On the other hand, if $2\omega + 2\nu + 2 = -1$, this is $O\big(N^{2\omega-2\nu}m^{-4\omega-3}\log N\big)$, which can be written as $O\big((N/m)^{-4\nu-3}\log N\big)$. Therefore, this also goes to zero. Analogously, for $-2\alpha + 1 \ge 0$, we get the sum

$$N^{2\omega-2\nu} \sum_{\ell=1}^{m} \ell^{-2\omega-2\alpha-1} \sum_{k=1}^{N/2-1} k^{2\beta-2\omega-1}.$$

This is $O((N/m)^{2\omega+1}N^{2\alpha-1})$, which obviously tends to zero, too. The last part is only present if both, $-2\alpha+1\geq 0$ and $-2\omega-2\geq 0$, are fulfilled. Then, the sum

$$N^{2\omega-2\nu}\sum_{\ell=1}^{m}\ell^{-4\omega-2\alpha-3}\sum_{k=1}^{N/2-1}k^{2\beta+1}$$

is $O((N/m)^{4\omega+2\alpha+2})$. The exponent of N (and m^{-1}) may be written as $4\omega+4+2\alpha-2\leq 0-1$. So, the last part also goes to zero. Note that none of the occurring exponents in the single sums was -1.

Summarizing, we ensured that $\iint_{\partial\Omega}|k_N(x,y)|^2\mathrm{d}(x,y)$ goes to zero as N goes to infinity. Next, we investigate the three parts of $\iint_{\partial\Omega}|\rho(x,y)|^2\mathrm{d}(x,y)$. First, we get

$$\begin{aligned}\int_0^{\frac{2(m+1)}{N}}\int_0^x|\rho(x,y)|^2\mathrm{d}y\mathrm{d}x &= \frac{2^{2\omega+4}}{\Gamma^2(-\omega)}\int_0^{\frac{2(m+1)}{N}}x^{-2\alpha+1}\int_0^x y^{2\beta+1}(x^2-y^2)^{-2\omega-2}\mathrm{d}y\\ &= \frac{2^{2\omega+3}}{\Gamma^2(-\omega)}\int_0^{\frac{2(m+1)}{N}}x^{2\beta-2\alpha-4\omega-1}\mathrm{d}x\int_0^1 z^{\beta}(1-z)^{-2\omega-2}\mathrm{d}z\\ &= \frac{2^{2\omega+3}}{\Gamma^2(-\omega)}\frac{\Gamma(\beta+1)\Gamma(-2\omega-1)}{\Gamma(\beta-2\omega)}\int_0^{\frac{2(m+1)}{N}}x^{2\nu-2\omega-1}\mathrm{d}x\\ &= \frac{2^{2\omega+2}\Gamma(\beta+1)\Gamma(-2\omega-1)}{\Gamma^2(-\omega)\Gamma(\beta-2\omega)(\nu-\omega)}\left(\frac{2(m+1)}{N}\right)^{2\nu-2\omega}.\end{aligned}$$

Since $2\nu-2\omega>2\nu+1>0$ and $m=o(N)$, this clearly converges to zero. Next, we have

$$\begin{aligned}\int_{\frac{2(m+1)}{N}}^1\int_0^{\frac{2}{N}}|\rho(x,y)|^2\mathrm{d}y\mathrm{d}x &= \frac{2^{2\omega+4}}{\Gamma^2(-\omega)}\int_{\frac{2(m+1)}{N}}^1 x^{-2\alpha+1}\int_0^{\frac{2}{N}}y^{2\beta+1}(x^2-y^2)^{-2\omega-2}\mathrm{d}y\mathrm{d}x\\ &= \frac{2^{2\omega+3}}{\Gamma^2(-\omega)}\int_{\frac{2(m+1)}{N}}x^{2\beta-2\alpha-4\omega-1}\int_0^{\frac{4}{N^2x^2}}z^{\beta}(1-z)^{-2\omega-2}\mathrm{d}z\mathrm{d}x\\ &\leq \frac{2^{2\omega+3}}{\Gamma^2(-\omega)}\int_{\frac{2(m+1)}{N}}x^{2\nu-2\omega-1}\mathrm{d}x\int_0^{\frac{1}{(m+1)^2}}z^{\beta}(1-z)^{-2\omega-2}\mathrm{d}z\\ &\leq \frac{2^{2\omega+3}}{\Gamma^2(-\omega)}\int_0^1 x^{2\nu-2\omega-1}\mathrm{d}x\int_0^{\frac{1}{(m+1)^2}}z^{\beta}(1-z)^{-2\omega-2}\mathrm{d}z\\ &= \frac{2^{2\omega+2}}{\Gamma^2(-\omega)(\nu-\omega)}\int_0^{\frac{1}{(m+1)^2}}z^{\beta}(1-z)^{-2\omega-2}\mathrm{d}z.\end{aligned}$$

Since the obtained integral is over a nonnegative function and exists for all m, it goes to zero as m goes to infinity by Lemma 5.3. Last, by the same substitution, we get

$$\begin{aligned}&\int_{\frac{2(m+1)}{N}}^1\int_{x-\frac{2(m+1)}{N}}^x|\rho(x,y)|^2\mathrm{d}y\mathrm{d}x\\ &\qquad= \frac{2^{2\omega+3}}{\Gamma^2(-\omega)}\int_{\frac{2(m+1)}{N}}^1 x^{2\nu-2\omega-1}\int_{\left(1-\frac{2(m+1)}{Nx}\right)^2}^1 z^{\beta}(1-z)^{-2\omega-2}\mathrm{d}z\mathrm{d}x.\end{aligned}$$

And with the help of Lemma 5.3, we conclude that this goes to zero.

Again summarizing, we have shown that $\iint_{\partial\Omega} |\rho(x,y)|^2 \mathrm{d}(x,y)$ goes to zero as N and $m(N)$ go to infinity. Therefore, the convergence in the Hilbert-Schmidt norm along the border $\partial\Omega$ is confirmed.

It remains to prove the convergence in the interior $\mathring{\Omega}$. Here, the inequality $1/(j-k) \leq 1/m$ holds. Therefore, we have

$$\frac{\Gamma(-\omega+j-k)}{\Gamma(j-k+1)} = (j-k)^{-\omega-1}(1+O(1/m)).$$

Furthermore, since $\frac{1}{j+k} \leq \frac{1}{j}$,

$$\frac{\Gamma(\alpha+\nu+j+k+1/2)}{\Gamma(\beta+1+j+k+1/2)} = (j+k)^{-\omega-1}(1+O(1/j)).$$

We get a similar expression for the corresponding terms in $\rho(x,y)$. Therefore, the resulting term $(j^2-k^2)^{-2\omega-2}$ may be estimated from above by $j^{-4\omega-3}m^{-1}$. So, we arrive at

$$\begin{aligned} &|k_N(x,y)-\rho(x,y)|^2 = \\ &\qquad N^{2\omega-2\nu}\frac{2^{-2\omega}}{\Gamma^2(-\omega)} j^{-2\alpha-4\omega-2}k^{2\beta+1}m^{-1}(O(1/j^2)+O(1/k^2)+O(1/m^2)) \end{aligned}$$

on $Q_{jk} \subset \mathring{\Omega}$. Summing over all $Q_{jk} \subset \mathring{\Omega}$ and ignoring the constant factor, we obtain the sum

$$\begin{aligned} &N^{2\omega-2\nu}m^{-1}\sum_{j=m+2}^{N/2-1}\sum_{k=1}^{j-m-1} j^{-2\alpha-4\omega-2}k^{2\beta+1}(O(1/j^2)+O(1/k^2)+O(1/m^2)) \\ &\qquad \leq N^{2\omega-2\nu}m^{-1}\sum_{j=1}^{N/2-1}\sum_{k=1}^{N/2-1} j^{-2\alpha-4\omega-2}k^{2\beta+1}(O(1/j^2)+O(1/k^2)+O(1/m^2)). \end{aligned}$$

These are actually three sums. The first, coming from the $O(1/j^2)$, is at most a constant times

$$N^{2\omega-2\nu}m^{-1}\sum_{j=1}^{N/2-1} j^{-4\omega-2\alpha-4}\sum_{k=1}^{N/2-1} k^{2\beta+1}.$$

Given that $-4\omega-2\alpha-4 \neq -1$, this is $O(1/mN)$. Otherwise, it is $O(m^{-1}N^{-1}\log N)$. In both cases, this goes to zero as N goes to infinity. The sum attached to $O(1/k^2)$ is not greater than a constant times

$$N^{2\omega-2\nu}m^{-1}\sum_{j=1}^{N/2-1} j^{-4\omega-2\alpha-2}\sum_{k=1}^{N/2-1} k^{2\beta-1}.$$

Here, the critical exponents occur for $-4\omega-2\alpha-2=-1$ and $\beta=0$. First, if $-4\omega-2\alpha-2\neq-1$ and $\beta\neq 0$, this is $O(1/mN)$. For $-4\omega-2\alpha-2=-1$ and $\beta=0$, it is $O(m^{-1}N^{2\omega-2\nu}\log N)$.

In the other two cases, it is $O(m^{-1}N^{-1}\log N)$. The sums always go to zero as N and m go to infinity.

Again, we get the lower bound on m from the last sum belonging to $O(1/m^2)$. This sum is at most a constant times

$$N^{2\omega-2\nu}m^{-3}\sum_{j=1}^{N/2-1} j^{-4\omega-2\alpha-2}\sum_{k=1}^{N/2-1} k^{2\beta+1},$$

which, for $-4\omega-2\alpha-2\neq-1$, is $O(m^{-3}N)$, and for $-4\omega-2\alpha-2=-1$, we obtain $O(m^{-3}N\log N)$, both going to zero as N and m go to infinity.

In conclusion, we know that the Hilbert-Schmidt norm of the difference of the modified operators $N^{\omega-\nu+1}K_N^*$ and $G^*_{\nu,\alpha,\beta}$ goes to zero as N increases. Therefore, we have proved our claim. □

So far, we have verified that the scaled integral operator built from the matrix E_n converges in the Hilbert-Schmidt norm to the operator $G_{\nu,\alpha,\beta}$. The same is true if we exchange E_n with F_n.

As already stated in [10], we can define the unitary operator V on $L^2(0,1)$ by

$$(Vf)(x)=2^{1/2}x^{1/2}f(x^2)\qquad\text{with}\qquad(V^{-1}f)(x)=2^{-1/2}x^{-1/4}f(x^{1/2}).$$

With this, we get

$$\begin{aligned}(V^{-1}G^*_{\nu,\alpha,\beta}Vf)(x)&=\frac{2^{\omega+1}}{\Gamma(-\omega)}\int_0^1 x^{-\alpha/2}y^{\beta/2}(x-y)^{-\omega-1}f(y)\mathrm{d}y\\&=2^{\omega+1}(L^*_{\nu,\alpha,\beta}f)(x)\end{aligned}$$

and conclude

$$\|G_{\nu,\alpha,\beta}\|_\infty=2^{\omega+1}\|L_{\nu,\alpha,\beta}\|_\infty.$$

Putting all of the above together, we have now proved that

$$\gamma_n^{(\nu)}(\alpha,\beta)\sim 2^{\omega}n^{2\nu-\beta+\alpha}\|L^*_{\nu,\alpha,\beta}\|_\infty$$

for $\beta-\alpha<\nu-1/2$ as n goes to infinity.

5.3 The Hermite case

As in the Laguerre and Gegenbauer cases, we consider $\beta-\alpha<-1/2$. Note that we do not have the direct dependence on ν anymore. In the following, we will show that the norm of the matrix

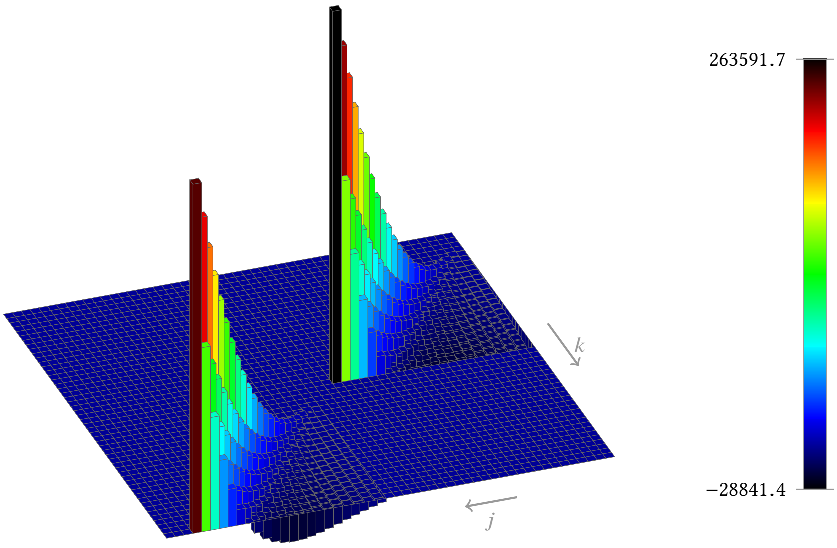

Figure 5.5: Matrix plot for $n = 50$, $\alpha = 1.3$, $\beta = 0.6$, $\nu = 6$ in the Hermite setting, already permuted and modified by alternating signs.

E_N from above is determined by the operator norm of the integral operator

$$(H^{(0)}_{\nu,\alpha,\beta}f)(x) = \frac{2^\nu \Gamma(\lceil \nu/2 \rceil + 1)}{\Gamma(\alpha - \beta + \lceil \nu/2 \rceil)} \int_x^1 x^{\beta/2-1/4} y^{-\alpha/2+1/4+(\lfloor \nu/2 \rfloor - \lceil \nu/2 \rceil)/2} (y-x)^{\alpha-\beta+\lceil \nu/2 \rceil - 1} \times \sum_{\ell=0}^{\lceil \nu/2 \rceil} \binom{\beta}{\ell} \binom{\beta - \alpha - \ell}{\lceil \nu/2 \rceil - \ell} \left(\frac{x}{y-x} \right)^{\lceil \nu/2 \rceil - \ell} f(y)\,\mathrm{d}y. \quad (5.4)$$

In contrast to the aforementioned cases, we now have a polynomial in $x/(y-x)$ in the kernel, which is not necessarily greater than zero in the interesting interval. Figure 5.5 illustrates this. To get the desired result, it does no longer suffice just to let j and k go to infinity in $c^{(\nu)}_{2j,2k+\nu}(\alpha,\beta)$, but we have to transform it first a little bit.

Since the sign of the matrix entries changes between two entries, we cannot immediately work with these. Instead, we consider the matrix $\tilde{E}_N = SE_NS$, where S is the diagonal matrix $S = \operatorname{diag}\left(\{(-1)^j\}_{j=0}^{N/2-1}\right)$. Because $\|SAS\| = \|A\|$ holds for any matrix A, this does not change our claim.

Before we can proceed, one more technical lemma is required.

Lemma 5.6. *For $x, \mu \in \mathbb{C}$, $m, \tau \in \mathbb{N}_0$, the following identity holds:*

$$\binom{\mu + m}{m - \tau} \binom{x + \mu + \tau}{\tau} = \sum_{\ell=0}^{\tau} \binom{m - \ell}{m - \tau} \binom{\mu + m}{m - \ell} \binom{x}{\ell}.$$

Proof. We begin by evaluating the sum on the right-hand side and writing the binomial coefficients in terms of rising factorials. Therefore,

$$\sum_{\ell=0}^{\tau}\binom{m-\ell}{m-\tau}\binom{\mu+m}{m-\ell}\binom{x}{\ell}=\sum_{\ell=0}^{\tau}\frac{(\mu+\ell+1)_{m-\ell}(x-\ell+1)_{\ell}}{(m-\tau)!(1)_{\tau-\ell}\ell!}.$$

This can be rewritten as

$$\frac{(-1)^m}{(m-\tau)!\tau!}\sum_{\ell=0}^{\tau}\frac{(\tau-\ell+1)_{\ell}(-\mu-m)_{m-\ell}(-x)_{\ell}}{\ell!}=\frac{(\mu+1)_m}{(m-\tau)!\tau!}\sum_{\ell=0}^{\tau}\frac{(-\tau)_{\ell}(-x)_{\ell}}{(\mu+1)_{\ell}\ell!}.$$

Applying the Chu-Vandermonde identity, this is the same as

$$\frac{(\mu+1)_m}{(\mu+1)_{\tau}(m-\tau)!}\frac{(x+\mu+1)_{\tau}}{\tau!}=\binom{\mu+m}{m-\tau}\binom{x+\mu+\tau}{\tau},$$

which is exactly what we wanted to show. □

We can now go on with deriving asymptotic expressions for the modified entries $\tilde{e}_{jk}$, given by $\tilde{e}_{jk}=(-1)^{k-j}c^{(\nu)}_{2j,2k+\nu}(\alpha,\beta)$, of the matrix $\tilde{E}_N$ introduced above, and thus derive an integral operator in a way similar to the methods used before. For making the following work, we have to assume that $k-j\geq\lceil\nu/2\rceil$. Ignore the coefficients before the sum in $c^{(\nu)}_{2j,2k+\nu}(\alpha,\beta)$ for a moment. It remains to investigate the sum

$$(-1)^{k-j}\sum_{\tau=0}^{\min\{\lceil\nu/2\rceil,k-j\}}\binom{j+\lceil\nu/2\rceil-1/2}{\lceil\nu/2\rceil-\tau}\binom{\beta+j+\lceil\nu/2\rceil+\tau-1/2}{\tau}\binom{\beta-\alpha-\lceil\nu/2\rceil}{k-j-\tau}.$$

First, we note that

$$\binom{\beta-\alpha-\lceil\nu/2\rceil}{k-j-\tau}=\binom{\beta-\alpha-\lceil\nu/2\rceil}{k-j-\lceil\nu/2\rceil}\binom{\beta-\alpha-k+j}{\lceil\nu/2\rceil-\tau}\frac{(\lceil\nu/2\rceil-\tau)!}{(k-j-\lceil\nu/2\rceil+1)_{\lceil\nu/2\rceil-\tau}}.$$

Furthermore, by Lemma 5.6 the equality

$$\binom{j+\lceil\nu/2\rceil-1/2}{\lceil\nu/2\rceil-\tau}\binom{\beta+j+\tau-1/2}{\tau}=\sum_{\ell=0}^{\tau}\binom{\lceil\nu/2\rceil-\ell}{\lceil\nu/2\rceil-\tau}\binom{j+\lceil\nu/2\rceil-1/2}{\lceil\nu/2\rceil-\ell}\binom{\beta}{\ell}$$

holds. Hence, the above sum is equal to

$$\begin{aligned}(-1)^{k-j}\binom{\beta-\alpha-\lceil\nu/2\rceil}{k-j-\lceil\nu/2\rceil}&\sum_{\tau=0}^{\lceil\nu/2\rceil}\sum_{\ell=0}^{\tau}\binom{j+\lceil\nu/2\rceil-1/2}{\lceil\nu/2\rceil-\ell}\binom{\beta}{\ell}\\&\times\binom{\lceil\nu/2\rceil-\ell}{\lceil\nu/2\rceil-\tau}\binom{\beta-\alpha-k+j}{\lceil\nu/2\rceil-\tau}\frac{(\lceil\nu/2\rceil-\tau)!}{(k-j-\lceil\nu/2\rceil+1)_{\lceil\nu/2\rceil-\tau}}.\end{aligned}\tag{5.5}$$

Ignoring the first factor temporarily, exchanging the order of summation, and performing an index shift, this becomes

$$\sum_{\ell=0}^{\lceil\nu/2\rceil}\binom{j+\lceil\nu/2\rceil-1/2}{\lceil\nu/2\rceil-\ell}\binom{\beta}{\ell}\sum_{\tau=0}^{\lceil\nu/2\rceil-\ell}\binom{\lceil\nu/2\rceil-\ell}{\tau}\binom{\beta-\alpha-k+j}{\lceil\nu/2\rceil-\ell-\tau}\frac{(\lceil\nu/2\rceil-\ell-\tau)!}{(k-j-\lceil\nu/2\rceil+1)_{\lceil\nu/2\rceil-\ell-\tau}}.$$

Writing the inner sum in terms of rising factorials, this is the same as

$$\begin{aligned}
&\sum_{\tau=0}^{\lceil\nu/2\rceil-\ell}\frac{(\lceil\nu/2\rceil-\ell-\tau+1)_\tau}{\tau!}\frac{(\beta-\alpha-k+j-\lceil\nu/2\rceil+\ell+\tau+1)_{\lceil\nu/2\rceil-\ell-\tau}}{(k-j-\lceil\nu/2\rceil+1)_{\lceil\nu/2\rceil-\ell-\tau}}\\
&\quad=(-1)^{\lceil\nu/2\rceil-\ell}\frac{(\alpha-\beta+k-j)_{\lceil\nu/2\rceil-\ell}}{(k-j-\lceil\nu/2\rceil+1)_{\lceil\nu/2\rceil-\ell}}\,{}_2F_1\left(\begin{matrix}-(\lceil\nu/2\rceil-\ell),\ j-k+\ell\\ \beta-\alpha-k+j-\lceil\nu/2\rceil+\ell+1\end{matrix};1\right)\\
&\quad=(-1)^{\lceil\nu/2\rceil-\ell}\frac{(\alpha-\beta+k-j)_{\lceil\nu/2\rceil-\ell}}{(k-j-\lceil\nu/2\rceil+1)_{\lceil\nu/2\rceil-\ell}}\frac{(\beta-\alpha-\lceil\nu/2\rceil+1)_{\lceil\nu/2\rceil-\ell}}{(\beta-\alpha-k+j-\lceil\nu/2\rceil+\ell+1)_{\lceil\nu/2\rceil-\ell}}\\
&\quad=\frac{(\beta-\alpha-\lceil\nu/2\rceil+1)_{\lceil\nu/2\rceil-\ell}}{(k-j-\lceil\nu/2\rceil+1)_{\lceil\nu/2\rceil-\ell}}.
\end{aligned}$$

Therefore, the whole sum just is

$$\begin{aligned}
&\sum_{\ell=0}^{\lceil\nu/2\rceil}\binom{j+\lceil\nu/2\rceil-1/2}{\lceil\nu/2\rceil-\ell}\binom{\beta}{\ell}\frac{(\beta-\alpha-\lceil\nu/2\rceil+1)_{\lceil\nu/2\rceil-\ell}}{(k-j-\lceil\nu/2\rceil+1)_{\lceil\nu/2\rceil-\ell}}\\
&\qquad=\sum_{\ell=0}^{\lceil\nu/2\rceil}\binom{\beta}{\ell}\binom{\beta-\alpha+\ell}{\lceil\nu/2\rceil-\ell}\frac{(j+\ell+1/2)_{\lceil\nu/2\rceil-\ell}}{(k-j-\lceil\nu/2\rceil+1)_{\lceil\nu/2\rceil-\ell}}.
\end{aligned}$$

So, (5.5) can be written as

$$(-1)^{k-j}\binom{\beta-\alpha-\lceil\nu/2\rceil}{k-j-\lceil\nu/2\rceil}\sum_{\ell=0}^{\lceil\nu/2\rceil}\binom{\beta}{\ell}\binom{\beta-\alpha-\ell}{\lceil\nu/2\rceil-\ell}\frac{(j+\ell+1/2)_{\lceil\nu/2\rceil-\ell}}{(k-j-\lceil\nu/2\rceil+1)_{\lceil\nu/2\rceil-\ell}}.$$

Finally, we set $j=\lfloor Nx/2\rfloor$ and $k=\lfloor Ny/2\rfloor$ and put everything back together. With this, the entry $(-1)^{k-j-\lceil\nu/2\rceil}c^{(\nu)}_{2j,2k+\nu}(\alpha,\beta)$ becomes

$$\begin{aligned}
&\frac{2^{(\beta-\alpha+\nu)/2+1}\Gamma(\lceil\nu/2\rceil+1)}{\Gamma(\alpha-\beta+\lceil\nu/2\rceil)}N^{-(\beta-\alpha-\nu)/2-1}x^{\beta/2-1/4}y^{-\alpha/2+1/4+(\lfloor\nu/2\rfloor-\lceil\nu/2\rceil)/2}(y-x)^{\alpha-\beta+\lceil\nu/2\rceil-1}\\
&\qquad\times\sum_{\ell=0}^{\lceil\nu/2\rceil}\binom{\beta}{\ell}\binom{\beta-\alpha-\ell}{\lceil\nu/2\rceil-\ell}\left(\frac{x}{y-x}\right)^{\lceil\nu/2\rceil-\ell}
\end{aligned}$$

times an $(1+O(1/N))$ term as N goes to infinity.

Analogously, one can show that the corresponding term for $\tilde{F}_N$ is

$$(-1)^{k-j}\binom{\beta-\alpha-\lfloor\nu/2\rfloor}{k-j-\lfloor\nu/2\rfloor}\sum_{\ell=0}^{\lfloor\nu/2\rfloor}\binom{\beta}{\ell}\binom{\beta-\alpha-\ell}{\lfloor\nu/2\rfloor-\ell}\frac{(j+\ell+3/2)_{\lfloor\nu/2\rfloor-\ell}}{(k-j-\lfloor\nu/2\rfloor+1)_{\lfloor\nu/2\rfloor-\ell}}.$$

Thus, after setting j and k as above, it is clear that the entry $(-1)^{k-j-\lfloor \nu/2 \rfloor} c^{(\nu)}_{2j+1,2k+\nu+1}(\alpha,\beta)$ is

$$\frac{2^{(\beta-\alpha+\nu)/2+1}\Gamma(\lfloor \nu/2 \rfloor + 1)}{\Gamma(\alpha-\beta+\lfloor \nu/2 \rfloor)} N^{-(\beta-\alpha-\nu)/2-1} x^{\beta/2-1/4} y^{-\alpha/2+1/4+(\lceil \nu/2 \rceil - \lfloor \nu/2 \rfloor)/2} (y-x)^{\alpha-\beta+\lfloor \nu/2 \rfloor -1} \times \sum_{\ell=0}^{\lfloor \nu/2 \rfloor} \binom{\beta}{\ell} \binom{\beta-\alpha-\ell}{\lfloor \nu/2 \rfloor - \ell} \left(\frac{x}{y-x}\right)^{\lfloor \nu/2 \rfloor - \ell}$$

times an $(1 + O(1/N))$ term as N goes to infinity.

To show that the operators are Hilbert-Schmidt, first set

$$\rho(x,y) = x^{-\alpha/2+(\lfloor \nu/2 \rfloor - \lceil \nu/2 \rceil)/2+1/4} y^{\beta/2-1/4} (x-y)^{\alpha-\beta+\lceil \nu/2 \rceil -1} \times \sum_{\ell=0}^{\lceil \nu/2 \rceil} \binom{\beta}{\ell} \binom{\beta-\alpha-\ell}{\lceil \nu/2 \rceil - \ell} \left(\frac{y}{x-y}\right)^{\lceil \nu/2 \rceil - \ell},$$

for $x > y$, and zero otherwise. Up to a constant, this is the kernel of the adjoint operator. We have

$$|\rho(x,y)|^2 = x^{-\alpha+(\lfloor \nu/2 \rfloor - \lceil \nu/2 \rceil)+1/2} y^{\beta-1/2} (x-y)^{2\alpha-2\beta+2\lceil \nu/2 \rceil -2} \sum_{\ell=0}^{2\lceil \nu/2 \rceil} A_\ell \left(\frac{y}{x-y}\right)^{\ell}.$$

with

$$A_\ell = \sum_{i=\max\{0,\ell-\lceil \nu/2 \rceil\}}^{\min\{\ell,\lceil \nu/2 \rceil\}} \binom{\beta}{\lceil \nu/2 \rceil - i} \binom{\beta}{\lceil \nu/2 \rceil - \ell + i} \binom{\beta-\alpha-\lceil \nu/2 \rceil + i}{i} \binom{\beta-\alpha-\lceil \nu/2 \rceil + \ell - i}{\ell - i} \tag{5.6}$$

Hence, $\iint_{[0,1]^2} |\rho(x,y)|^2 \mathrm{d}(x,y)$ is a finite sum of terms of the form

$$A_\ell \int_0^1 x^{-\alpha+(\lfloor \nu/2 \rfloor - \lceil \nu/2 \rceil)+1/2} \int_0^x y^{\beta+\ell-1/2} (x-y)^{2\alpha-2\beta+2\lceil \nu/2 \rceil -2-\ell} \mathrm{d}y \mathrm{d}x$$

for $\ell = 0, \dots, 2\lceil \nu/2 \rceil$. The inner integral converges as long as two conditions are fulfilled, $\beta + \ell + 1/2 > 0$ and $2\alpha - 2\beta + 2\lceil \nu/2 \rceil - \ell - 1 > 0$. The first is true for any ℓ by our standing assumption $\beta > -1/2$. The second condition holds if $\beta - \alpha - \lceil \nu/2 \rceil < -(\ell+1)/2$. Since it has to be fulfilled for every $0 \le \ell \le 2\lceil \nu/2 \rceil$, this condition becomes $\beta - \alpha - \lceil \nu/2 \rceil < -\lceil \nu/2 \rceil - 1/2$, i. e., $\beta - \alpha < -1/2$. This immediately implies $\beta - \alpha - \nu < 0$, which is just the remaining condition for the convergence of the full integral.

Therefore, the Hilbert-Schmidt norm of the integral operator derived from the matrix $\tilde{E}_N$ is a finite sum of some constants times convergent integrals and hence finite, when $\beta - \alpha < -1/2$.

Analogously, we can show that the corresponding integral operator for the matrix $\tilde{F}_N$ is Hilbert-Schmidt whenever $\beta - \alpha < -1/2$.

Theorem 5.7. *Let $\alpha, \beta > -1/2$, $\beta - \alpha < -1/2$, and $\nu \in \mathbb{N}$. Furthermore, let E_N and F_N be defined as above. Then,*

$$\|E_N\|_\infty \sim \left(\tfrac{N}{2}\right)^{(\alpha-\beta+\nu)/2} \|H^{(0)}_{\nu,\alpha,\beta}\|_\infty, \qquad \|F_N\|_\infty \sim \left(\tfrac{N}{2}\right)^{(\alpha-\beta+\nu)/2} \|H^{(1)}_{\nu,\alpha,\beta}\|_\infty,$$

where $H^{(0)}_{\nu,\alpha,\beta}$ and $H^{(1)}_{\nu,\alpha,\beta}$ are the integral operators on $L^2(0,1)$ given by

$$(H^{(0)}_{\nu,\alpha,\beta} f)(x) = \frac{2^\nu \Gamma(\lceil \nu/2 \rceil + 1)}{\Gamma(\alpha - \beta + \lceil \nu/2 \rceil)} \int_x^1 x^{\beta/2-1/4} y^{-\alpha/2+1/4+(\lfloor \nu/2 \rfloor - \lceil \nu/2 \rceil)/2} (y-x)^{\alpha-\beta+\lceil \nu/2 \rceil - 1} \times \sum_{\ell=0}^{\lceil \nu/2 \rceil} \binom{\beta}{\ell} \binom{\beta-\alpha-\ell}{\lceil \nu/2 \rceil - \ell} \left(\frac{x}{y-x}\right)^{\lceil \nu/2 \rceil - \ell} f(y)\mathrm{d}y,$$

and

$$(H^{(1)}_{\nu,\alpha,\beta} f)(x) = \frac{2^\nu \Gamma(\lfloor \nu/2 \rfloor + 1)}{\Gamma(\alpha - \beta + \lfloor \nu/2 \rfloor)} \int_x^1 x^{\beta/2+1/4} y^{-\alpha/2-1/4+(\lceil \nu/2 \rceil - \lfloor \nu/2 \rfloor)/2} (y-x)^{\alpha-\beta+\lfloor \nu/2 \rfloor - 1} \times \sum_{\ell=0}^{\lfloor \nu/2 \rfloor} \binom{\beta}{\ell} \binom{\beta-\alpha-\ell}{\lfloor \nu/2 \rfloor - \ell} \left(\frac{x}{y-x}\right)^{\lfloor \nu/2 \rfloor - \ell} f(y)\mathrm{d}y,$$

respectively.

Proof. We show the theorem for E_N. The claim for F_N can be verified analogously. First, let K_N denote the integral operator on $L^2(0,1)$ with the piecewise constant kernel determined by $k_N(x,y) = \tilde{e}_{\lfloor Ny \rfloor, \lfloor Nx \rfloor}$, where $\tilde{e}_{jk} = (-1)^{k-j} e_{jk} = (-1)^{k-j} c^{(\nu)}_{2j,2k+\nu}(\alpha,\beta)$. Furthermore, set

$$\rho(x,y) = \frac{2^\nu \Gamma(\lceil \nu/2 \rceil + 1)}{\Gamma(\alpha - \beta + \lceil \nu/2 \rceil)} x^{-\alpha/2+1/4+(\lfloor \nu/2 \rfloor - \lceil \nu/2 \rceil)/2} y^{\beta/2-1/4} (x-y)^{\alpha-\beta+\lceil \nu/2 \rceil - 1} \times \sum_{\ell=0}^{\lceil \nu/2 \rceil} \binom{\beta}{\ell} \binom{\beta-\alpha-\ell}{\lceil \nu/2 \rceil - \ell} \left(\frac{y}{x-y}\right)^{\lceil \nu/2 \rceil - \ell}$$

for $x > y$, and zero otherwise. It is the kernel of the adjoint operator of $H^{(0)}_{\nu,\alpha,\beta}$. We claim that the scaled operators $(N/2)^{(\beta-\alpha-\nu)/2+1} K_N$ converge in the Hilbert-Schmidt norm (and thus in the operator norm) to the operator $H^{(0)}_{\nu,\alpha,\beta}$, provided that $\beta - \alpha < -1/2$. Note that the operator K_N corresponds to the transposed matrix E_N instead of the matrix E_N itself.

As we have done before, split the area of integration into squares of length $2/N$ and consider the sets on the border of the area in which at least one of the kernels does not vanish. We use the notation from Theorem 5.4 and utilize the same sets Ω_1, Ω_2, Ω_3, and $\partial\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$. Again, we employ the estimate

$$\iint_{\partial\Omega} |(N/2)^{(\beta-\alpha-\nu)/2+1} k_N(x,y) - \rho(x,y)|^2 \mathrm{d}(x,y) \leq 2 \left(\tfrac{N}{2}\right)^{\beta-\alpha-\nu+2} \iint_{\partial\Omega} |k_N(x,y)|^2 \mathrm{d}(x,y) + 2 \iint_{\partial\Omega} |\rho(x,y)|^2 \mathrm{d}(x,y)$$

on the border area $\partial\Omega$ and show that both integrals go to zero as N increases.

We have

$$N^{\beta-\alpha-\nu+2}\iint_{\Omega_1}|k_N(x,y)|^2\mathrm{d}(x,y) = \\ N^{\beta-\alpha-\nu}2^{2\nu}\Gamma^2(\lceil\nu/2\rceil+1)\Gamma(\beta+1/2)\frac{\Gamma(\lfloor\nu/2\rfloor+1)}{\Gamma(\lceil\nu/2\rceil+\alpha+1/2)}\binom{\lceil\nu/2\rceil-1/2}{\lceil\nu/2\rceil},$$

which is just a constant times $N^{\beta-\alpha-\nu}$. Because of $\beta-\alpha<-1/2$ and $\nu\geq 1$, this tends to zero as N goes to infinity.

We turn our attention to the area Ω_2. Here, we sum over all j greater than or equal to 1, while the index k is zero. Therefore,

$$N^{\beta-\alpha-\nu+2}\iint_{\Omega_2}|k_N(x,y)|^2\mathrm{d}(x,y) = \\ N^{\beta-\alpha-\nu}2^{2\nu}\Gamma^2(\lceil\nu/2\rceil+1)\Gamma(\beta+1/2)\sum_{j=1}^{N/2-1}\left(\sqrt{\frac{\Gamma(j+\lfloor\nu/2\rfloor+1)}{\Gamma(j+\lceil\nu/2\rceil+\alpha+1/2)}}\right. \\ \left.\times\binom{\lceil\nu/2\rceil-1/2}{\lceil\nu/2\rceil}\binom{\beta-\alpha-\lceil\nu/2\rceil}{j}\,{}_3F_2\left(\begin{matrix}-\lceil\nu/2\rceil,\ -j,\ \beta+1/2\\ \beta-\alpha-\lceil\nu/2\rceil-j+1,\ 1/2\end{matrix};1\right)\right)^2.$$

Examining the hypergeometric series, we see that this is actually $1+O(1/j)$. Consequently, the above integral is at most a constant times

$$N^{\beta-\alpha-\nu}\sum_{j=1}^{N/2-1}j^{\lfloor\nu/2\rfloor-\lceil\nu/2\rceil-\alpha+1/2+2\alpha-2\beta+2\lceil\nu/2\rceil-2}(1+O(1/j)) \\ = N^{\beta-\alpha-\nu}\sum_{j=1}^{N/2-1}j^{\alpha-2\beta+\nu-3/2}(1+O(1/j)).$$

We split the sum in the part belonging to the factor 1 and the part belonging to the $O(1/j)$ term. The first part is $O(N^{-\beta-1/2})$ if $\alpha-2\beta+\nu-1/2\neq 0$, and $O(N^{\beta-\alpha-\nu}\log N)$ otherwise. Similarly, for $\alpha-2\beta+\nu-3/2\neq 0$, the second part is $O(N^{-\beta-3/2})$, and $O(N^{\beta-\alpha-\nu}\log N)$ in the other case. Either way, both parts of the sum go to zero as N goes to infinity.

In order to tackle possible singularities along the diagonal, we use the previously established idea. We introduce the parameter $m=m(N)$, which grows with N but not faster, and is restricted by $N=o(m^3)$. First, treat the diagonal $j=k$ on its own. Here, the integral over the square of the scaled kernel k_N evaluates to

$$N^{\beta-\alpha-\nu}2^{2\nu}\Gamma^2(\lceil \nu/2\rceil+1)\sum_{j=1}^{N/2-1}\left(\sqrt{\frac{\Gamma(j+\beta+1/2)\Gamma(j+\lfloor \nu/2\rfloor+1)}{\Gamma(j+1)\Gamma(j+\lceil \nu/2\rceil+\alpha+1/2)}}\binom{j+\lceil \nu/2\rceil-1/2}{\lceil \nu/2\rceil}\right)^2$$
$$=2^{2\nu}N^{\beta-\alpha-\nu}\sum_{j=1}^{N/2-1}j^{\beta-\alpha+\nu}(1+O(1/j)).$$

Again, we investigate the parts belonging to 1 and the O term separately. For $\beta-\alpha+\nu\neq-1$, the first part is $O(N^{2\beta-2\alpha+1})$, and it is $O(N^{\beta-\alpha-\nu}\log N)$ otherwise. In contrast, with $\beta-\alpha+\nu\neq 0$, the second part is $O(N^{2\beta-2\alpha})$, or $O(N^{-2\nu}\log N)$ for $\beta-\alpha+\nu=0$. Because of $\nu\geq 1$ and $\beta-\alpha<-1/2$, all terms vanish for large N.

For the rest of the area along the diagonal, we set $\ell=j-k$. The entries can then be written as

$$2^{\nu}\Gamma(\lceil \nu/2\rceil+1)\sqrt{\frac{\Gamma(k+\beta+1/2)\Gamma(k+\ell+\lfloor \nu/2\rfloor+1)}{\Gamma(k+1)\Gamma(k+\ell+\lceil \nu/2\rceil+\alpha+1/2)}}$$
$$\times\binom{k+\lceil \nu/2\rceil-1/2}{\lceil \nu/2\rceil}\binom{\beta-\alpha-\lceil \nu/2\rceil}{\ell}{}_3F_2\left(\begin{matrix}-\lceil \nu/2\rceil,\ -\ell,\ \beta+k+1/2\\ \beta-\alpha-\lceil \nu/2\rceil-\ell+1,\ k+1/2\end{matrix};1\right).$$

By keeping ℓ constant and letting $k\to\infty$, the hypergeometric term becomes

$${}_2F_1\left(\begin{matrix}-\lceil \nu/2\rceil,\ -\ell\\ \beta-\alpha-\lceil \nu/2\rceil-\ell+1\end{matrix};1\right)(1+O(1/k)).$$

Now, we can evaluate this with the Chu-Vandermonde identity and see that the whole series is simply a constant times $1+O(1/k)+O(1/\ell)$. Together with the other terms, the rest of the integral over Ω_2 is at most a constant times

$$N^{\beta-\alpha-\nu}\sum_{\ell=1}^{m}\sum_{k=1}^{N/2-1}k^{\beta-1/2+2\lceil \nu/2\rceil}(k+\ell)^{\lfloor \nu/2\rfloor-\lceil \nu/2\rceil-\alpha+1/2}\ell^{2\alpha-2\beta-2}(1+O(1/k)+O(1/\ell)).$$

In the event of $\lfloor \nu/2\rfloor-\lceil \nu/2\rceil-\alpha+1/2<0$, we can estimate the $(k+\ell)$ term solely by k. With this, the above sum does not exceed

$$N^{\beta-\alpha-\nu}\sum_{\ell=1}^{m}\ell^{2\alpha-2\beta-2}\sum_{k=1}^{N/2-1}k^{\beta-\alpha+\nu}(1+O(1/k)+O(1/\ell)).$$

As before, we split the sum into three parts. First, the part belonging to 1 is $O((N/m)^{2\beta-2\alpha+1})$, given that $\beta-\alpha+\nu+1\neq 0$. On the other hand, if $\beta-\alpha+\nu+1=0$ the sum becomes $O((N/m)^{-2\nu-1}\log N)$.

For the term with $O(1/k)$, with $\beta-\alpha+\nu\neq 0$, this evaluates to $O(\ N/m)^{2\beta-2\alpha+1}N^{-1})$, and to $O((N/m)^{-2\nu}m^{-1}\log N)$ for $\beta-\alpha+\nu=0$. Again, both go to zero.

Finally, the last term is $O((N/m)^{2\beta-2\alpha+1}m^{-1})$ if $2\alpha-2\beta-2\neq 0$ and $\beta-\alpha+\nu+1\neq 0$. If $2\alpha-2\beta-2=0$, it becomes $O(N^{-1}\log m)$, and for $\beta-\alpha+\nu+1=0$, it is $O((N/m)^{-2\nu-1}m^{-1}\log N)$. In any case, the term goes to zero as N goes to infinity.

If, on the other hand, $\lceil \nu/2\rceil = \lfloor \nu/2\rfloor$, i.e., ν is an even number, we can only guarantee that $\lfloor \nu/2\rfloor - \lceil \nu/2\rceil - \alpha + 1/2 < -\beta < 1/2$, because of $\beta-\alpha < -1/2$. If this expression is negative, we are done. Assume now $1/2 > -\alpha + 1/2 > 0$. Note that then necessarily $\beta < 0$. Under these conditions, the term $(k+\ell)^{\lfloor \nu/2\rfloor - \lceil \nu/2\rceil-\alpha+1/2}$ can be estimated from above by $k^{-\alpha+1/2}+\ell^{-\alpha+1/2}$. Consequently, the sum may be estimated by

$$N^{\beta-\alpha-\nu}\sum_{\ell=1}^{m}\ell^{2\alpha-2\beta-2}\sum_{k=1}^{N/2-1}k^{\beta-\alpha+\nu}\left(1+O(1/k)+O(1/\ell)\right)$$
$$+N^{\beta-\alpha-\nu}\sum_{\ell=1}^{m}\ell^{\alpha-2\beta-3/2}\sum_{k=1}^{N/2-1}k^{\beta+\nu-1/2}\left(1+O(1/k)+O(1/\ell)\right).$$

The first sum has already been handled. In the second sum, consider the term belonging to 1. Suppose $\alpha-2\beta-3/2\neq -1$, and note that always $\beta+\nu-1/2>0$. Then, this sum is just $O\Big((N/m)^{2\beta-\alpha+1/2}\Big)$. If, on the contrary, $\alpha-2\beta-3/2=-1$, then the sum is $O(N^{2\beta-\alpha+1/2}\log m)$, and because of our assumption, the exponent is strictly smaller than β, and thus negative. Therefore, the term goes to zero in either case.

Analogously, the term belonging to $O(1/k)$ goes to zero. The exponent in the sum over k is still strictly greater than -1, and for the sum over ℓ the same distinction as above has to be made. Keeping that in mind, the term is either $O(N^{-1}(N/m)^{2\beta-\alpha-1/2})$ or $O(N^{2\beta-\alpha-1/2}\log m)$. In both cases, there is the additional factor N^{-1}, so the arguments from above apply here, too.

Finally, for the $O(1/\ell)$ term, we have to treat $\alpha-2\beta-5/2=-1$ separately. Given that, the sum is $O(N^{2\beta-\alpha+1/2}\log m)$ or $O(m^{-1}(N/m)^{2\beta-\alpha+1/2})$ otherwise. Again, this goes to zero.

What we have shown by now is that the integral over the square of the scaled k_N vanishes on $\partial\Omega$. Next, we prove that this is also true for the integral over ρ in this area. Recall the square of the kernel $\rho(x,y)$:

$$|\rho(x,y)|^2 = \frac{2^{2\nu}\Gamma^2(\lceil \nu/2\rceil+1)}{\Gamma^2(\alpha-\beta+\lceil \nu/2\rceil)}$$
$$\times\sum_{\ell=0}^{2\lceil \nu/2\rceil}A_{2\lceil \nu/2\rceil-\ell}\,y^{\beta-1/2+\ell}x^{-\alpha+\lfloor \nu/2\rfloor-\lceil \nu/2\rceil+1/2}(x-y)^{2\alpha-2\beta+2\lceil \nu/2\rceil-2-\ell},$$

where $A_{2\lceil \nu/2\rceil-\ell}$ is defined in (5.6).

Here, the integral

$$\int_0^{\frac{2(m+1)}{N}}\int_0^x|\rho(x,y)|^2\mathrm{d}y\mathrm{d}x$$

is a constant times

$$\sum_{\ell=0}^{2\lceil\nu/2\rceil} A_{2\lceil\nu/2\rceil-\ell} \int_0^{\frac{2(m+1)}{N}} x^{-\alpha+\lfloor\nu/2\rfloor-\lceil\nu/2\rceil+1/2} \int_0^x y^{\beta-1/2+\ell}(x-y)^{2\alpha-2\beta+2\lceil\nu/2\rceil-\ell-2}\mathrm{d}y\mathrm{d}x.$$

Again, from $\beta > -1/2$, $\beta-\alpha < -1/2$, and $\ell \leq 2\lceil\nu/2\rceil$, we derive that $\beta+\ell-1/2 > -1$, as well as $2\alpha-2\beta+2\lceil\nu/2\rceil-\ell-2 > -1$. Therefore, the inner integral converges for any $\ell = 0,\ldots,2\lceil\nu/2\rceil$ and we get

$$\sum_{\ell=0}^{2\lceil\nu/2\rceil} A_{2\lceil\nu/2\rceil-\ell} \frac{\Gamma(\beta+\ell+1/2)\Gamma(2\alpha-2\beta+2\lceil\nu/2\rceil-\ell-1)}{\Gamma(2\alpha-\beta+2\lceil\nu/2\rceil-1/2)} \int_0^{\frac{2(m+1)}{N}} x^{-\beta+\alpha+\nu-1}\mathrm{d}x.$$

The last integral is independent of ℓ and evaluates to $\frac{1}{\nu-\beta+\alpha}\left(\frac{2(m+1)}{N}\right)^{\nu-\beta+\alpha}$. The rest of the sum is independent of N, and thus the integral goes to zero.

In the following, we will omit the sum and the factors $A_{2\lceil\nu/2\rceil-\ell}$, since the convergence to zero does not depend on them. We treat the integrals henceforth for a fixed $\ell = 0,\ldots,2\lceil\nu/2\rceil$.

At the lower border, the corresponding integral is

$$\int_{\frac{2(m+1)}{N}}^1 x^{-\alpha+\lfloor\nu/2\rfloor-\lceil\nu/2\rceil+1/2} \int_0^{2/N} y^{\beta+\ell-1/2}(x-y)^{2\alpha-2\beta+2\lceil\nu/2\rceil-\ell-2}\mathrm{d}y\mathrm{d}x.$$

Using the substitution $z = y/x$ in the inner integral, its upper bound becomes $2/Nx$. Because x is greater than $2(m+1)/N$, the integral does not get smaller if we change the upper bound to $1/(m+1)$. Following this, the inner integral does not depend on x anymore. Therefore, the double integral is in fact a plain product of two integrals. By reducing the lower bound of the integral over x to 0, we also do not make this integral smaller, and we can simply evaluate it. Putting all this together, the integral is

$$\frac{1}{\nu-\beta+\alpha} \int_0^{\frac{1}{m+1}} z^{\beta+\ell-1/2}(1-z)^{2\alpha-2\beta+2\lceil\nu/2\rceil-\ell-2}\mathrm{d}z.$$

The last integral converges for the same reasons as above, and it goes to zero by Lemma 5.3.

To finish the study of the border, we have to evaluate the integral

$$\int_{\frac{2(m+1)}{N}}^1 x^{-\alpha+\lfloor\nu/2\rfloor-\lceil\nu/2\rceil+1/2} \int_{x-\frac{2(m+1)}{N}}^x y^{\beta+\ell-1/2}(x-y)^{2\alpha-2\beta+2\lceil\nu/2\rceil-\ell-2}\mathrm{d}y\mathrm{d}x.$$

The assumptions being fulfilled, Lemma 5.3 again delivers the vanishing property.

We now discuss the inner part. Notice that here $j-k \geq m$. Therefore,

$$\begin{aligned}\left|k_N\left(\tfrac{j}{N},\tfrac{k}{N}\right)-\rho\left(\tfrac{j}{N},\tfrac{k}{N}\right)\right|^2 &= N^{\beta-\alpha-\nu}\sum_{\ell=0}^{2\lceil\nu/2\rceil} A_{2\lceil\nu/2\rceil-\ell}\frac{2^{2\nu}\Gamma^2(\lceil\nu/2\rceil+1)}{\Gamma^2(\alpha-\beta+\lceil\nu/2\rceil)}\\ &\times k^{\beta+\ell-1/2}j^{-\alpha+\lfloor\nu/2\rfloor-\lceil\nu/2\rceil+1/2}(j-k)^{2\alpha-2\beta+2\lceil\nu/2\rceil-\ell-2}\left(O(1/k^2)+O(1/j^2)+O(1/m^2)\right).\end{aligned}$$

We write

$$(j-k)^{2\alpha-2\beta+2\lceil\nu/2\rceil-\ell-2} = (j-k)^{2\alpha-2\beta+2\lceil\nu/2\rceil-\ell-1}(j-k)^{-1} \le j^{2\alpha-2\beta+2\lceil\nu/2\rceil-\ell-1}m^{-1},$$

because $2\alpha-2\beta+2\lceil\nu/2\rceil-\ell-1$ is guaranteed to be positive. With this, we can give the following upper bound on the difference in the interior of the area of integration, ignoring a constant factor:

$$\sum_{\ell=0}^{2\lceil\nu/2\rceil} A_{2\lceil\nu/2\rceil-\ell} N^{\beta-\alpha-\nu} m^{-1} \times \sum_{j=1}^{N/2-1} j^{\alpha-2\beta+\nu-\ell-1/2} \sum_{k=1}^{N/2-1} k^{\beta+\ell-1/2}(O(1/j^2)+O(1/k^2)+O(1/m^2)).$$

As mentioned before, we just investigate the summands for each ℓ, avoiding to write the sum every time. As we already got used to, treat the sum for each O term separately.

First, for $O(1/j^2)$, the sum is $O(m^{-1}N^{-1})$ if $\alpha-2\beta+\nu-\ell \neq 3/2$, and $O(m^{-1}N^{-1}\log N)$ otherwise. In any case, it goes to zero.

Next, for $O(1/k^2)$, we have to consider four cases. If neither $\alpha-2\beta+\nu-\ell+1/2$, nor $\beta+\ell-3/2$ is zero, the sum is $O(m^{-1}N^{-1})$. If one of the two is zero, the sum becomes $O(m^{-1}N^{-1}\log N)$, and if both are zero, it is $O(m^{-1}N^{-1}\log^2 N)$. All vanish for increasing N and m.

Finally, for $O(1/m^2)$, we have $O(Nm^{-3})$ if $\alpha-2\beta+\nu-\ell+1/2 \neq 0$, and $O(m^{-3}\log N)$ otherwise. Here, the condition $N = o(m^3)$ comes into play for obtaining convergence to zero. Given that, this also goes to zero.

In conclusion, we have shown that the scaled-down integral operator derived from the matrix representation of the operator of differentiation in the Hermite case converges in the Hilbert-Schmidt norm, and thus in the operator norm, to its analogue with continuous kernel. □

5.4 Schatten class operators

We are now at the point where we have completely proved Theorems 1.1, 1.2, and 1.3. Thus, the constants are fully identified for the parameter differences mentioned there. However, in each case, there is a small gap for which we still do not have a result. One particular case is addressed by the paper [11], namely the Laguerre case. Since this paper was done in close connection to the present work, we will bring the main achievements here.

One point that is particularly striking is the method of the proof for showing the convergence of the operators. It heavily relied on the integral operator being Hilbert-Schmidt. That made the analysis somewhat easier. However, one might get the idea that this assumption is a little bit to strong, and we can achieve even more if we drop this or at least replace it with less strict prerequisites. Hence, one can come up with the following conjecture, presented in [11].

Conjecture 5.8 (Conjecture 1.1 in [11]). *Let $\alpha, \beta > -1$ be real numbers, ν be a positive integer, and put $\omega = \beta - \alpha - \nu$. Then,*

$$\lambda_n^{(\nu)}(\alpha,\beta) \sim C_\nu(\alpha,\beta) n^{(\nu+|\omega|)/2}$$

with

$$C_\nu(\alpha,\beta) = \begin{cases} 2^\omega & \text{for} \quad \omega \geq 0, \\ \|L^*_{\nu,\alpha,\beta}\|_\infty & \text{for} \quad \omega < 0, \end{cases}$$

where $L^*_{\nu,\alpha,\beta}$ is the Volterra integral operator on $L^2(0,1)$ given by

$$(L^*_{\nu,\alpha,\beta} f)(x) = \frac{1}{\Gamma(-\omega)} \int_0^x x^{-\alpha/2} y^{\beta/2} (x-y)^{-\omega-1} f(y) \mathrm{d}y. \tag{1.3 revisited}$$

Here are the necessary notions and notations. Let T be a bounded operator acting on some separable Hilbert space H, and let $\{s_k(T)\}_{k\in\mathbb{N}}$ denote the sequence of singular values of T in nonincreasing order. The operator T is said to belong to the pth Schatten class if $\{s_k(T)\}_{k\in\mathbb{N}}$ belongs to $\ell^p(\mathbb{N})$. We write $\mathcal{S}_p$ for the set of these operators and define the norm by

$$\|T\|_{\mathcal{S}_p} = \|\{s_k(T)\}_{k\in\mathbb{N}}\|_{\ell^p}.$$

In the following, we only consider values of p that are powers of two and therefore just write $\|T\|_{\mathcal{S}_p} = \|T\|_{2^n}$ for $p = 2^n$. Clearly, $\|T\|_2 \geq \|T\|_{2^2} \geq \ldots \geq \|T\|_{2^n} \geq \ldots \geq \|T\|_\infty$. All we need is the equality $\|T\|_{2^n} = \|T^*T\|_{2^{n-1}}^{1/2}$ (which holds for all $n \geq 1$) and the fact that the Hilbert-Schmidt norm $\|T\|_2$ of an integral operator T is equal to the L^2 norm of the kernel of T.

In the next subsection we will prove Theorem 1.4 (see Theorem 1.2 in [11]) and in the following two subsections we will prove Theorem 1.5 (Theorem 1.3 in [11]).

Proof of Theorem 1.4

The factor $(\Gamma(-\omega))^{-1}$ is irrelevant for the compactness of the operator (1.3). Thus, we consider the operator M defined on $L^2(0,1)$ by

$$(Mf)(x) = \int_0^x x^{-\alpha/2} y^{\beta/2} (x-y)^{-\omega-1} f(y) \mathrm{d}y.$$

For $0 < r < 1$, let M_r be the operator on $L^2(0,1)$ that is given by

$$(M_r f)(x) = \int_0^{rx} x^{-\alpha/2} y^{\beta/2} (x-y)^{-\omega-1} f(y) \mathrm{d}y.$$

The squared Hilbert-Schmidt norm of M_r is

$$\int_0^1 \int_0^{rx} x^{-\alpha} y^\beta (x-y)^{-2\omega-2} \mathrm{d}y \mathrm{d}x = \int_0^1 \int_0^r x^{\beta-\alpha-2\omega-1} y^\beta (1-y)^{-2\omega-2} \mathrm{d}y \mathrm{d}x.$$

This is finite if $\beta > -1$ and $\omega < (\beta - \alpha)/2$. Consequently, these two assumptions ensure that M_r is compact. We have

$$((M - M_r)f)(x) = \int_{rx}^{x} x^{-\alpha/2} y^{\beta/2} (x-y)^{-\omega-1} \mathrm{d}y$$
$$= \int_{r}^{1} x^{(\beta-\alpha)/2-\omega} y^{\beta/2} (1-y)^{-\omega-1} f(xy) \mathrm{d}y,$$

and since $\omega < (\beta - \alpha)/2$, it follows that

$$|((M - M_r)f)(x)| \le \int_{r}^{1} y^{\beta/2}(1-y)^{-\omega-1} |f(xy)| \mathrm{d}y.$$

We therefore obtain

$$\|(M - M_r)f\|_{L^2(0,1)} = \left(\int_0^1 |((M - M_r)f)(x)|^2 \mathrm{d}x \right)^{1/2}$$
$$\le \left(\int_0^1 \left(\int_r^1 y^{\beta/2}(1-y)^{-\omega-1} |f(xy)| \mathrm{d}y \right)^2 \mathrm{d}x \right)^{1/2},$$

and by virtue of Minkowski's inequality for integrals, this is not larger than

$$\int_r^1 \left(\int_0^1 y^{\beta}(1-y)^{-2\omega-2} |f(xy)|^2 \mathrm{d}x \right)^{1/2} \mathrm{d}y$$
$$= \int_r^1 y^{\beta/2}(1-y)^{-\omega-1} \left(\int_0^1 |f(xy)|^2 \mathrm{d}x \right)^{1/2} \mathrm{d}y. \quad (5.7)$$

Taking into account that $\int_0^1 |f(xy)|^2 \mathrm{d}x = y^{-1} \int_0^y |f(t)|^2 \mathrm{d}t \le y^{-1} \|f\|^2_{L^2(0,1)}$, we see that (5.7) does not exceed

$$\int_r^1 y^{\beta/2-1/2}(1-y)^{-\omega-1} \|f\|_{L^2(0,1)} \mathrm{d}y.$$

In summary, we have shown

$$\|(M - M_r)f\|_{L^2(0,1)} \le \left(\int_r^1 y^{\beta/2-1/2}(x-y)^{-\omega-1} \mathrm{d}y \right) \|f\|_{L^2(0,1)}. \quad (5.8)$$

The assumption $\omega < 0$ guarantees that the integral appearing in (5.8) goes to zero as $r \to 1$. This implies that $\|M - M_r\|_\infty \to 0$ as $r \to 1$, which proves M to be compact. □

Auxiliary results and an example

Let T be an integral operator on $L^2(0,1)$ with a real-valued kernel $k(\cdot,\cdot)$ and T^* its adjoint. These are then given by

$$(Tf)(x) = \int_0^1 k(x,y) f(y) \mathrm{d}y, \quad (T^* f)(x) = \int_0^1 k(y,x) f(y) \mathrm{d}y,$$

and thus,

$$((T^*T)f)(x) = \int_0^1 \left(\int_0^1 k(z,x)k(z,y)\mathrm{d}z \right) f(y)\mathrm{d}y.$$

We recursively define a sequence of kernel functions $\{k_{2^n}\}_{n\geq 0}$ that are associated with integral operators K_{2^n}. Set

$$k_1(x,y) = \begin{cases} y^{-\alpha/2}x^{\beta/2}(y-x)^{-\omega-1} & \text{for } x < y, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, K_1 is just $\Gamma(-\omega)$ times the operator (1.3). Next, set

$$k_{2^n}(x,y) = \int_0^1 k_{2^{n-1}}(z,x)k_{2^{n-1}}(z,y)\mathrm{d}z.$$

It follows that $K_{2^n} = K^*_{2^{n-1}}K_{2^{n-1}}$, and, to prove Theorem 1.5, we are left to show $\|K_1\|_{2^n} < \infty$. This is the same as $\|(K_1^*K_1)^{n-1}\|_2 = \|K_{2^{n-1}}\|_2 < \infty$. So, we reduce the estimation of the 2^nth Schatten norm of the operator K_1 to the estimation of the Hilbert-Schmidt norm of the operator $K_{2^{n-1}}$, which is given by

$$\|K_{2^{n-1}}\|_2^2 = \int_0^1 \int_0^1 k_{2^{n-1}}(x,y)k_{2^{n-1}}(x,y)\mathrm{d}x\mathrm{d}y.$$

To anticipate the arguments that will be used in the proof of the general case, we start with considering the case $n = 2$. Thus, suppose $-1/2 \leq \omega < -1/4$. Our aim is to show that K_2 is a Hilbert-Schmidt operator. Since $k_2(x,y) = k_2(y,x)$, we have

$$\begin{aligned} \|K_2\|_2^2 &= \int_0^1 \int_0^1 k_2(x_2,x_0)k_2(x_0,x_2)\mathrm{d}x_0\mathrm{d}x_2 \\ &= \int_0^1 \int_0^1 \int_0^1 \int_0^1 k_1(x_1,x_2)k_1(x_1,x_0)k_1(x_3,x_0)k_1(x_3,x_2)\mathrm{d}x_0\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3. \end{aligned}$$

The indexing of the variables might seem strange at the first glance, but it will turn out to be perfect when treating the general case. Notice also that all these kernels are nonnegative, which implies that the integral over the cube is equal to the iterated integrals, and that we can change the order of integration.

We have to distinguish between the cases $x_i < x_j$ and $x_i > x_j$. To this end, we split the area of integration, i. e., the cube $[0,1]^4$, into 4! disjoint simplices

$$\Omega_\pi = \{(x_0,x_1,x_2,x_3) \in [0,1]^4 : x_{\pi(0)} < x_{\pi(1)} < x_{\pi(2)} < x_{\pi(3)}\},$$

where π is a permutation of the numbers $0,1,2,3$. The integral for $\|K_2\|_2^2$ then splits into 4! integrals over Ω_π. In all but four cases one of the kernels $k_1(x_i,x_j)$ is zero. These four cases are the permutations which send $(0,1,2,3)$ to $(1,3,0,2)$, $(1,3,2,0)$, $(3,1,0,2)$, or $(3,1,2,0)$. We are therefore left with showing that each of these four integrals is finite. Let us consider the integral corresponding to the last permutation, i. e., the simplex given by $x_3 < x_1 < x_2 < x_0$:

$$I_4 := \int_0^1 \int_0^{x_0} \int_0^{x_2} \int_0^{x_1} \varphi_4(x) \mathrm{d}x_3 \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_0$$

with

$$\varphi_4(x) = x_0^{-\alpha} x_2^{-\alpha} x_1^{\beta} x_3^{\beta} (x_2 - x_1)^{-\sigma} (x_0 - x_1)^{-\sigma} (x_2 - x_3)^{-\sigma} (x_0 - x_3)^{-\sigma},$$

where, here and in the following, $\sigma := \omega + 1$. The inner integration in I_4 gives

$$\int_0^{x_1} \varphi_4(x) \mathrm{d}x_3 \\ = x_0^{-\alpha} x_2^{-\alpha} x_1^{\beta} (x_2 - x_1)^{-\sigma} (x_0 - x_1)^{-\sigma} \int_0^{x_1} x_3^{\beta} (x_2 - x_3)^{-\sigma} (x_0 - x_3)^{-\sigma} \mathrm{d}x_3.$$

Now, a first lemma comes into the game. Recall that $0 < \sigma = \omega + 1 < 1$.

Lemma 5.9. *Let $a > -1$, $\tau > 0$, $\sigma > 0$ be real numbers, and let $k \geq 0$ and $\ell \geq 0$ be integers. Suppose $(k + \ell + 1)\tau < 1$ and $(1 + \tau)\sigma < 1$. Assume further that $0 < s \leq y < x$. Then,*

$$\int_0^s t^a (x - t)^{-(1-k\tau)\sigma} (y - t)^{-(1-\ell\tau)\sigma} \mathrm{d}t \leq C(x - y)^{-(1-(k+\ell+1)\tau)\sigma} s^{a-(1+\tau)\sigma+1}$$

with some constant $C < \infty$.

Proof. We write $(x - t)^{-(1-k\tau)\sigma} = (x - t)^{-(1-(k+\ell+1)\tau)\sigma} (x - t)^{-(\ell+1)\tau\sigma}$, and since

$$(x - t)^{-(1-(k+\ell+1)\tau)\sigma} \leq (x - y)^{-(1-(k+\ell+1)\tau)\sigma},$$
$$(x - t)^{-(\ell+1)\tau\sigma} \leq (s - t)^{-(\ell+1)\tau\sigma},$$
$$(y - t)^{-(1-\ell\tau)\sigma} \leq (s - t)^{-(1-\ell\tau)\sigma},$$

we obtain that the integral does not exceed

$$(x - y)^{-(1-(k+\ell+1)\tau)\sigma} \int_0^s t^a (s - t)^{-(1+\tau)\sigma} \mathrm{d}t.$$

The last integral equals

$$s^{a-(1+\tau)\sigma+1} \int_0^1 t^a (1 - t)^{-(1+\tau)\sigma} \mathrm{d}t = s^{a-(1+\tau)\sigma+1} \cdot C,$$

where $C := \Gamma(a + 1)\Gamma(1 - (1 + \tau)\sigma)/\Gamma(a + 2 - (1 + \tau)\sigma) < \infty$. □

Now, choose $\tau = 1/3$. Since $\sigma = 1 + \omega < 1 - 1/4$, we have $(1 + \tau)\sigma < 1$. Applying the lemma with $k = \ell = 0$ to the above integral $\int_0^{x_1} \varphi_4(x)\mathrm{d}x_3$, we get

$$\int_0^{x_1} \varphi_4(x)\mathrm{d}x_3 \\ \leq C x_0^{-\alpha} x_2^{-\alpha} x_1^{2\beta-(1+\tau)\sigma+1}(x_2 - x_1)^{-\sigma}(x_0 - x_1)^{-\sigma}(x_0 - x_2)^{-(1-\tau)\sigma} =: \varphi_3(x).$$

Next, we evaluate the inner integration in

$$I_4 \leq \int_0^1 \int_0^{x_0} \int_0^{x_2} \varphi_3(x)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_0$$

and obtain

$$\int_0^{x_2} \varphi_3(x)\mathrm{d}x_1 \\ = C x_0^{-\alpha} x_2^{-\alpha}(x_0 - x_2)^{-(1-\tau)\sigma} \int_0^{x_2} x_1^{2\beta-(1+\tau)\sigma+1}(x_2 - x_1)^{-\sigma}(x_0 - x_1)^{-\sigma}\mathrm{d}x_1.$$

Again, use Lemma 5.9 with $k = \ell = 0$. The only question is whether $a = 2\beta - (1 + \tau)\sigma + 1 > -1$. This problem is disposed of by the following lemma.

Lemma 5.10. *Let $\alpha > -1$, $\beta > -1$, $\nu \geq 1$ be real numbers. Put $\omega = \beta - \alpha - \nu$ and suppose $-1/2^{n-1} \leq \omega < -1/2^n$. If k and ℓ are integers satisfying $0 \leq \ell \leq k \leq 2^{n-1}$ and τ is defined as $\tau = 1/(2^n - 1)$, then*

$$k\beta - \ell\alpha - (k + \ell - 1)(1 + \tau)(\omega + 1) + (k + \ell - 1) > \ell - 1.$$

Proof. Since $(1 + \tau)(\omega + 1) < 1$, we have $-(k + \ell - 1)(1 + \tau)(\omega + 1) + (k + \ell - 1) > 0$. Hence,

$$\begin{aligned} k\beta - \ell\alpha - (k + \ell - 1)(1 + \tau)(\omega + 1) + (k + \ell - 1) &> k\beta - \ell\alpha \\ = k(\beta - \alpha) + (k - \ell)\alpha = k(\omega + \nu) + (k - \ell)\alpha &> k(\omega + 1) - (k - \ell) \\ = k\omega + \ell \geq \ell - k/2^{n-1} &\geq \ell - 1. \end{aligned}$$ □

In the present case, $n = 2$ and accordingly $\tau = 1/3$, as above. Lemma 5.10 with $k = 2$ and $\ell = 0$ yields that indeed $a = 2\beta - (1 + \tau)\sigma + 1 > -1$. We may therefore use Lemma 5.9 with $k = \ell = 0$ to conclude that

$$\int_0^{x_2} \varphi_3(x)\mathrm{d}x_1 \leq C x_0^{-\alpha} x_2^{2\beta-\alpha-2(1+\tau)\sigma+2}(x_0 - x_2)^{-(1-\tau)\sigma}(x_0 - x_2)^{-(1-\tau)\sigma} =: \varphi_2(x),$$

where, here and throughout what follows, C denotes a finite constant, but not necessarily the same at each occurrence. Thus,

$$I_4 \leq \int_0^1 \int_0^{x_0} \varphi_2(x)\mathrm{d}x_2\mathrm{d}x_0.$$

We have

$$\int_0^{x_0} \varphi_2(x)\mathrm{d}x_2 = Cx_0^{-\alpha} \int_0^{x_0} x_2^{2\beta-\alpha-2(1+\tau)\sigma+2}(x_0 - x_2)^{-2(1-\tau)\sigma}\mathrm{d}x_2$$
$$= x_0^b \int_0^1 t^c(1-t)^{-2(1-\tau)\sigma}\mathrm{d}t =: x_0^b \cdot \tilde{C}$$

with $b = 2\beta - 2\alpha - 2(1+\tau)\sigma + 2 - 2(1-\tau)\sigma + 1$ and $c = 2\beta - \alpha - 2(1+\tau)\sigma + 2$. Clearly, $2(1-\tau)\sigma < 2(1-1/3)(1-1/4) = 1$, Lemma 5.10 with $k = 2$ and $\ell = 1$ gives $c > -1$, and finally,

$$\begin{aligned} b &= 2\beta - 2\alpha - 2(1+\tau)\sigma + 2 - 2(1-\tau)\sigma + 1 \\ &\geq 2(\omega+1) - 2(1+\tau)(\omega+1) - 2(1-\tau)(\omega+1) + 3 = 3 - 2(\omega+1) > 1. \end{aligned}$$

This proves that $I_4 \leq \tilde{C}\int_0^1 x_0^b \mathrm{d}x_0 < \infty$.

Proof of Theorem 1.5

We now turn to the general case. The case $n = 1$ is a simple computation. So, suppose $n \geq 2$ and $-1/2^{n-1} \leq \omega < -1/2^n$. Put $\sigma = 1 + \omega$. We have to show that

$$\|K_{2^{n-1}}\|_2^2 = \int_0^1 \int_0^1 k_{2^{n-1}}(x_0, x_{2^{n-1}})k_{2^{n-1}}(x_{2^{n-1}}, x_{2^n})\mathrm{d}x_{2^{n-1}}\mathrm{d}x_0 \quad (x_{2^n} := x_0)$$

is finite; notice that $k_{2^{n-1}}(x_{2^{n-1}}, x_0) = k_{2^{n-1}}(x_0, x_{2^{n-1}})$ for $n \geq 2$. Write

$$k_{2^{n-1}}(x_i, x_j) = \int_0^1 k_{2^{n-2}}(x_\ell, x_i)k_{2^{n-2}}(x_\ell, x_j)\mathrm{d}x_\ell,$$

where $\ell = (i+j)/2$. Continue this process until only the kernels $k_1(\cdot,\cdot)$ remain. For example, if $n = 4$, then

$$\|K_8\|_2^2 = \int_0^1 \int_0^1 k_8(x_0, x_8)k_8(x_8, x_{16})\mathrm{d}x_8\mathrm{d}x_0 \quad (x_{16} := x_0)$$

with

$$\begin{aligned} k_8(x_0, x_8) &= \int_0^1 k_4(x_4, x_0)k_4(x_4, x_8)\mathrm{d}x_4 \\ &= \int_0^1\int_0^1\int_0^1 k_2(x_2, x_4)k_2(x_2, x_0)k_2(x_6, x_4)k_2(x_6, x_8)\mathrm{d}x_2\mathrm{d}x_6\mathrm{d}x_4 \\ &= \int_0^1 \cdots \int_0^1 k_1(x_3, x_2)k_1(x_3, x_4)k_1(x_1, x_2)k_1(x_1, x_0) \\ &\qquad \times\ k_1(x_5, x_6)k_1(x_5, x_4)k_1(x_7, x_6)k_1(x_7, x_8)\mathrm{d}x_1 \cdots \mathrm{d}x_7 \end{aligned}$$

and a similar expression for $k_8(x_8, x_{16})$. In this way, the integral for $\|K_{2^{n-1}}\|_2^2$ becomes an integral over $\Omega = [0,1]^{2^n}$. We divide Ω into $(2^n)!$ disjoint simplices

$$\Omega_\pi = \{(x_0, \dots, x_{2^n-1}) \in [0,1]^{2^n} : x_{\pi(0)} < x_{\pi(1)} < \dots < x_{\pi(2^n-1)}\},$$

labeled by the permutations π of the numbers $0, 1, \dots, 2^{n-1}$. The result is

$$\|K_{2^{n-1}}\|_2^2 = \sum_\pi \int_0^1 \int_0^{x_{\pi(2^n-1)}} \int_0^{x_{\pi(2^n-2)}} \cdots \int_0^{x_{\pi(1)}} \left(\prod_{j=0}^{2^{n-1}-1} k_1(x_{2j+1}, x_{2j}) k_1(x_{2j+1}, x_{2j+2}) \right) \mathrm{d}x_{\pi(0)} \dots \mathrm{d}x_{\pi(2^n-1)}.$$

We perform the integrations from the inside to the outside and may restrict ourselves to the permutations π for which we never meet a kernel whose first variable is greater than the second. Thus, take such a permutation and consider

$$I_{2^n} = \int_0^1 \int_0^{x_{\pi(2^n-1)}} \int_0^{x_{\pi(2^n-2)}} \cdots \int_0^{x_{\pi(1)}} \varphi_{2^n}(x) \mathrm{d}x_{\pi(0)} \dots \mathrm{d}x_{\pi(2^n-1)}$$

with

$$\begin{aligned} \varphi_{2^n}(x) &= \prod_{j=0}^{2^{n-1}-1} k_1(x_{2j+1}, x_{2j}) k_1(x_{2j+1}, x_{2j+2}) \\ &= \prod_{j=0}^{2^{n-1}-1} x_{2j}^{-\alpha} x_{2j+1}^{\beta} [(x_{2j} - x_{2j+1})(x_{2j+2} - x_{2j+1})]^{-\sigma}. \end{aligned}$$

We put $\tau = 1/(2^n - 1)$. Then,

$$(1+\tau)\sigma < \left(1 + \frac{1}{2^n - 1}\right)\left(1 - \frac{1}{2^n}\right) = 1.$$

The first integral is an integral like in Lemma 5.9 with $a = \beta$ and $k = \ell = 0$. We estimate this integral from above exactly as in this lemma and obtain a function $\varphi_{2^n-1}(x)$. Integrating this function, we get an integral as in Lemma 5.9 with $k = 1$ and $\ell = 0$, and we estimate again to get a function $\varphi_{2^n-2}(x)$. In this way, we perform $2^n - 2$ integrations and estimates. In the end, we have a function $\varphi_2(x)$.

In each step, we use Lemma 5.9 with some a, and some k and ℓ. Let us first describe the evolution of the exponents a. After the first integration, it equals $2\beta - (1+\tau)\sigma + 1$. Each further integration adds $-(1+\tau)\sigma + 1$ to the exponent, and from outside the integral we still have to add the values β or $-\alpha$ in dependence on whether the j in the integral $\int_0^{x_j}$ is odd or even. Thus, each time we add $\beta - (1+\tau)\sigma + 1$ or $-\alpha - (1+\tau)\sigma + 1$, and after $k + \ell$ integrations the exponent is $(k+1)\beta - \ell\alpha - (k+\ell)(1+\tau) + (k+\ell)$. Since we do not meet kernels which are identically zero, at each place in the sequence $\pi(0) < \cdots < \pi(2^n - 1)$, the number of predecessors with odd subscript is at least as large as the number of predecessors with even subscript. This implies that always $k + 1 \geq \ell$. The first integration is over a variable with odd subscript. It follows that the number of integrals $\int_0^{x_j} *$ with odd j is at most $2^{n-1} - 1$, so that always $k + 1 \leq 2^{n-1}$. We therefore obtain from Lemma 5.10 (with k replaced by $k + 1$) that the exponent a is greater than $\ell - 1 \geq -1$.

Our next objective is the evolution of the numbers k and ℓ occurring in Lemma 5.9. For this purpose, we associate weighted graphs $G_{2^n}, \ldots, G_2$ with the functions $\varphi_{2^n}(x), \ldots, \varphi_2(x)$. The graph G_{2^n} has 2^n vertices, which are labeled from x_0 to x_{2^n-1}, and 2^n edges, which join x_j and x_{j+1} and will be denoted by $[x_j, x_{j+1}]$. Each edge gets the weight 0. This is because in $\varphi_{2^n}(x)$ each $|x_j - x_{j+1}|$ has the exponent $-\sigma$, which may be written as $-(1 - m\tau)\sigma$ with $m = 0$. The function $\varphi_{2^n-1}(x)$ results from $\varphi_{2^n}(x)$ via an estimate of the form

$$\int_0^{x_j} x_i^a (x_{i-1} - x_i)^{-\sigma} (x_{i+1} - x_i)^{-\sigma} \mathrm{d}x_i \leq C x_j^{a-(1+\tau)\sigma+1} |x_{i-1} - x_{i+1}|^{-(1-\tau)\sigma};$$

we write the differences in absolute values, since this dispenses us from distinguishing the cases $x_{i-1} < x_{i+1}$ and $x_{i+1} < x_{i-1}$. Thus, the differences $x_{i-1} - x_i$ and $x_{i+1} - x_i$ are no longer present in $\varphi_{2^n-1}(x)$. Instead, $\varphi_{2^n-1}(x)$ contains $|x_{i-1} - x_{i+1}|$ with the exponent $-(1 - \tau)\sigma$, which is $-(1 - m\tau)\sigma$ with $m = 1$. Accordingly, G_{2^n-1} results from G_{2^n} by deleting the edges $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$, and introducing a new edge $[x_{i-1}, x_{i+1}]$ with the weight $m = 1$. We proceed in this way. If $\varphi_{h-1}(x)$ is obtained from $\varphi_h(x)$ by an estimate

$$\int_0^{x_j} x_i^a (x_p - x_i)^{-(1-k\tau)\sigma} (x_q - x_i)^{-(1-\ell\tau)\sigma} \mathrm{d}x_i \leq C x_j^{a-(1+\tau)\sigma+1} |x_p - x_q|^{-(1-(k+\ell+1)\tau)\sigma}, \quad (5.9)$$

then G_h contained the edge $[x_p, x_i]$ with the weight k and the edge $[x_i, x_q]$ with the weight ℓ. We delete these two edges, and replace them by the edge $[x_p, x_q]$ with the weight $k + \ell + 1$ to obtain G_{h-1}.

The graph G_2 consists of two edges, both joining $x_{\pi(2^n-2)}$ and $x_{\pi(2^n-1)}$. Let r and s be the weights of these edges. The sum of all weights in G_{2^n} is zero, and in each step the sum of the weights increases by $-k - \ell + (k + \ell + 1) = 1$. As we made $2^n - 2$ steps, it follows that $r + s = 2^n - 2$. We see in particular that, in (5.9), we always have $k + \ell < 2^n - 2$, whence $(k + \ell + 1)\tau < (2^n - 1)/(2^n - 1) = 1$. This (together with the inequality $a > -1$ shown before) justifies the application of Lemma 5.9 in each step.

Figure 5.6 depicts the graphs for the introductory example considered in the preceding subsection, while Figure 5.7 presents the sequence of graphs for $n = 3$ and the simplex associated with the permutation $x_5 < x_1 < x_3 < x_2 < x_4 < x_7 < x_6 < x_0$.

We abbreviate $x_{\pi(2^n-2)}$ and $x_{\pi(2^n-1)}$ to x_p and x_q. What we are left with is to prove

$$\int_0^1 \int_0^{x_q} \varphi_2(x) \mathrm{d}x_p \mathrm{d}x_q < \infty$$

with

$$\varphi_2(x) = C x_q^{-\alpha} x_p^a (x_q - x_p)^{-(1-r\tau)\sigma} (x_q - x_p)^{-(1-s\tau)\sigma}.$$

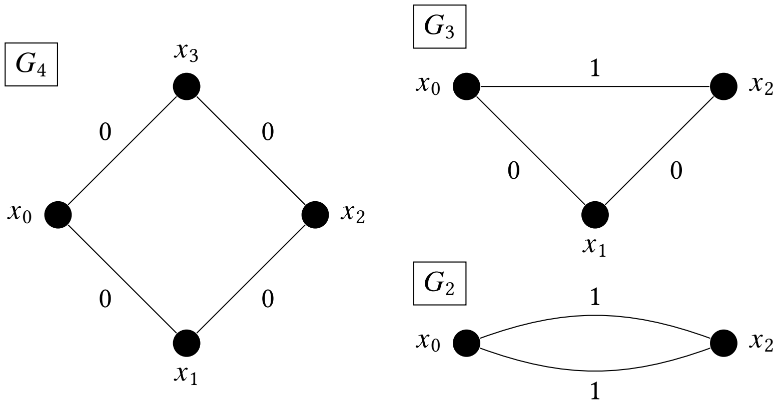

Figure 5.6: The sequence of graphs for $n = 2$ and $x_3 < x_1 < x_2 < x_0$.

The exponent a comes from $k = 2^{n-1} - 1$ integrals $\int_0^{x_j} *$ with odd subscript j and $\ell = 2^{n-1} - 1$ integrals $\int_0^{x_j} *$ with even j. (Notice that p and q are necessarily even.) Hence, the exponent a equals $(k+1)\beta - \ell\alpha - (k+\ell)(1+\tau)\sigma + (k+\ell)$, and from Lemma 5.10 we infer $a > -1$. It follows that

$$\begin{aligned}\int_0^{x_q} \varphi_2(x)\mathrm{d}x_p &= Cx_q^{-\alpha} \int_0^{x_q} x_p^a (x_q - x_p)^{-(2-(r+s)\tau)\sigma} \mathrm{d}x_p \\ &= Cx_q^{-\alpha} x_q^{a-(2-(r+s)\tau)\sigma+1} \int_0^1 t^a (1-t)^{-(2-(r+s)\tau)\sigma} \mathrm{d}t. \qquad (5.10)\end{aligned}$$

Obviously,

$$(2-(r+s)\tau)\sigma = \left(2 - \frac{2^n-2}{2^n-1}\right)(1+\omega) = \frac{2^n}{2^n-1}(1+\omega) < \frac{2^n}{2^n-1}\left(1 - \frac{1}{2^n}\right) = 1,$$

and hence (5.10) is finite. It remains to consider the integral $\int_0^1 x_q^b \mathrm{d}x_q$ with the exponent b equal to $-\alpha + a - (2-(r+s)\tau)\sigma + 1$. We just proved $1 - (2-(r+s)\tau)\sigma > 0$ and also have

$$\begin{aligned}-\alpha + a &= (k+1)\beta - (k+1)\alpha - 2k(1+\tau)\sigma + 2k \\ &= (k+1)\beta - (k+1)\alpha - (2k+1)(1+\tau)\sigma + (2k+1) + (1+\tau)\sigma - 1 \\ &> k + 1 - 1 + (1+\tau)\sigma - 1 \qquad \text{(Lemma 5.10)} \\ &= k - 1 + (1+\tau)\sigma > k - 1 = 2^{n-1} - 2 \geq 0.\end{aligned}$$

This shows that $b > 0$ and thus that $\int_0^1 x_q^b \mathrm{d}x_q < \infty$. The proof of Theorem 1.5 is complete. □

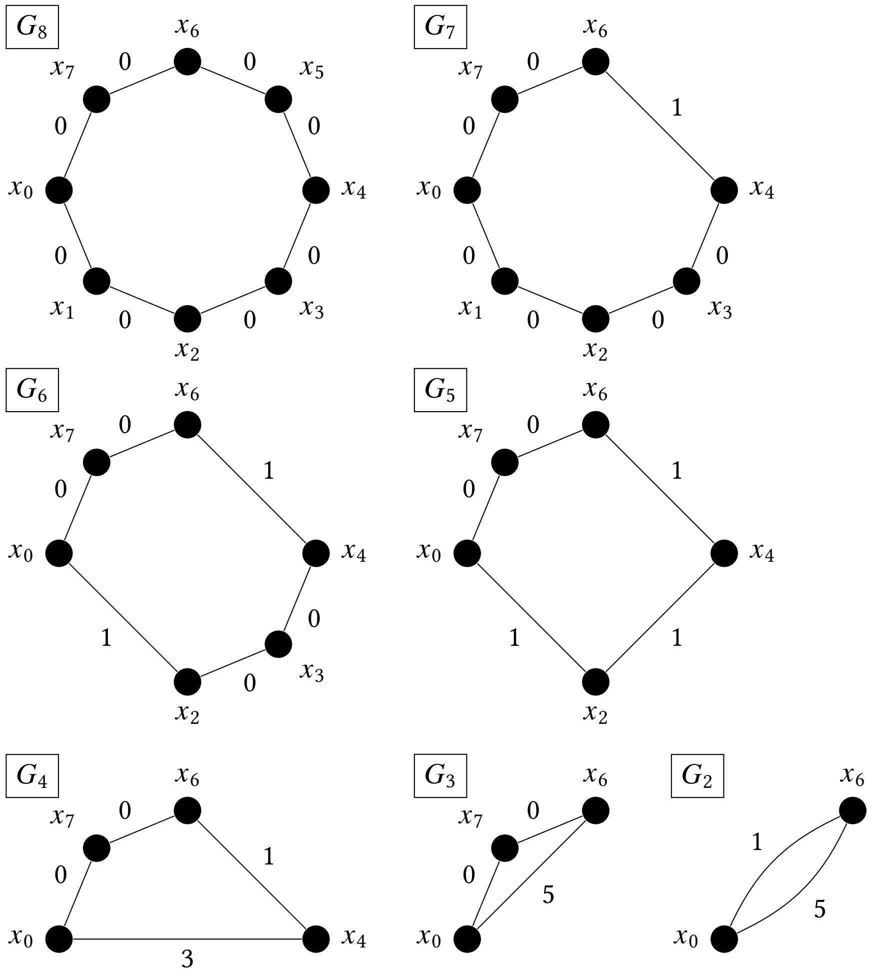

Figure 5.7: The sequence of graphs obtained for $n = 3$ with one of the most demanding permutations $x_5 < x_1 < x_3 < x_2 < x_4 < x_7 < x_6 < x_0$.

Chapter 6

Conclusion and outlook

In the preceding chapters, we gave complete proofs of the Theorems 1.1, 1.2, and 1.3. These theorems almost completely answer the question on the nature of the best constant in the Markov-type inequality

$$\|f^{(\nu)}\|_\beta \le C_n^{(\nu)}(\alpha,\beta)\|f\|_\alpha \quad \text{for all } f \in \mathcal{P}_n.$$

We identified them (asymptotically as $n \to \infty$) as

$$\lambda_n^{(\nu)}(\alpha,\beta) \sim n^{(\nu+|\beta-\alpha-\nu|)/2} \begin{cases} 2^{\beta-\alpha-\nu} & : \beta-\alpha \ge \nu \\ \|L^*_{\nu,\alpha,\beta}\|_\infty & : \beta-\alpha < \nu - 1/2 \end{cases}$$

in the Laguerre setting,

$$\gamma_n^{(\nu)}(\alpha,\beta) \sim \begin{cases} n^\nu & : \beta-\alpha \ge \nu \\ 2^{\beta-\alpha-\nu}\|L^*_{\nu,\alpha,\beta}\|_\infty n^{2\nu-\beta+\alpha} & : \beta-\alpha < \nu - 1/2 \end{cases}$$

in the Gegenbauer setting, and

$$\eta_n^{(\nu)}(\alpha,\beta) \sim n^{(|\beta-\alpha|+\nu)/2} \begin{cases} 2^{(\beta-\alpha+\nu)/2} & : \beta-\alpha \ge 0 \\ 2^{(\beta-\alpha-\nu)/2} \cdot \max\left\{\|H^{(0)}_{\nu,\alpha,\beta}\|_\infty, \|H^{(1)}_{\nu,\alpha,\beta}\|_\infty\right\} & : \beta-\alpha < -1/2 \end{cases}$$

in the Hermite setting, where $L^*_{\nu,\alpha,\beta}$, $H^{(0)}_{\nu,\alpha,\beta}$, and $H^{(1)}_{\nu,\alpha,\beta}$ are certain integral operators on $L^2(0,1)$. It is immediately noticeable that, in every case there is a small gap, always of the same size $1/2$. The conjecture raised in Section 5.4 is that this is due to the techniques used, not inherent to the problem itself, and that this limitation may be overcome by a more elaborate analysis. It is shown that, although the operator $L^*_{\nu,\alpha,\beta}$ is no longer a Hilbert-Schmidt operator for $\beta - \alpha \geq \nu - 1/2$, it still belongs to some Schatten class and is thus compact. While this does not provide an answer right away, it may help in finding the correct result in that case.

Conjecture 5.8 extends Theorem 1.1 to all possible parameter differences. The similarity to the Gegenbauer and Hermite setting imposes us to formulate related conjectures in these cases.

Conjecture 6.1. *Let $\alpha, \beta > -1$ be real numbers, ν be a positive integer, and put $\omega = \beta - \alpha - \nu$. Then,*

$$\gamma_n^{(\nu)}(\alpha, \beta) \sim \begin{cases} n^\nu & : \omega \geq 0 \\ 2^\omega \|L^*_{\nu,\alpha,\beta}\|_\infty n^{2\nu-\beta+\alpha} & : \omega < 0, \end{cases}$$

where $L^*_{\nu,\alpha,\beta}$ is the Volterra integral operator on $L^2(0,1)$ given by

$$(L^*_{\nu,\alpha,\beta} f)(x) = \frac{1}{\Gamma(-\omega)} \int_0^x x^{-\alpha/2} y^{\beta/2} (x-y)^{-\omega-1} f(y) \mathrm{d}y.$$

Conjecture 6.2. *Let $\alpha, \beta > -1/2$ be real numbers, and let ν be a positive integer. Then,*

$$\eta_n^{(\nu)}(\alpha, \beta) \sim C_\nu(\alpha, \beta) n^{(|\beta-\alpha|+\nu)/2}$$

with

$$C_\nu(\alpha, \beta) = \begin{cases} 2^{(\beta-\alpha+\nu)/2} & : \beta - \alpha \geq 0 \\ 2^{(\beta-\alpha-\nu)/2} \cdot \max\left\{ \|H^{(0)}_{\nu,\alpha,\beta}\|_\infty, \|H^{(1)}_{\nu,\alpha,\beta}\|_\infty \right\} & : \beta - \alpha < 0, \end{cases}$$

where $H^{(0)}_{\nu,\alpha,\beta}$ and $H^{(1)}_{\nu,\alpha,\beta}$ are the integral operators on $L^2(0,1)$ defined by

$$(H^{(0)}_{\nu,\alpha,\beta} f)(x) = \frac{2^\nu \Gamma(\lceil \nu/2 \rceil + 1)}{\Gamma(\alpha - \beta + \lceil \nu/2 \rceil)} \int_x^1 x^{\beta/2-1/4} y^{-\alpha/2+1/4+(\lfloor \nu/2 \rfloor - \lceil \nu/2 \rceil)/2} (y-x)^{\alpha-\beta+\lceil \nu/2 \rceil - 1} \times \sum_{\ell=0}^{\lceil \nu/2 \rceil} \binom{\beta}{\ell} \binom{\beta-\alpha-\ell}{\lceil \nu/2 \rceil - \ell} \left(\frac{x}{y-x} \right)^{\lceil \nu/2 \rceil - \ell} f(y) \mathrm{d}y,$$

and

$$(H^{(1)}_{\nu,\alpha,\beta} f)(x) = \frac{2^\nu \Gamma(\lfloor \nu/2 \rfloor + 1)}{\Gamma(\alpha - \beta + \lfloor \nu/2 \rfloor)} \int_x^1 x^{\beta/2+1/4} y^{-\alpha/2-1/4+(\lceil \nu/2 \rceil - \lfloor \nu/2 \rfloor)/2} (y-x)^{\alpha-\beta+\lfloor \nu/2 \rfloor - 1} \times \sum_{\ell=0}^{\lfloor \nu/2 \rfloor} \binom{\beta}{\ell} \binom{\beta-\alpha-\ell}{\lfloor \nu/2 \rfloor - \ell} \left(\frac{x}{y-x} \right)^{\lfloor \nu/2 \rfloor - \ell} f(y) \mathrm{d}y.$$

Now that these cases are almost completely handled, what can be done next? First and foremost, there is always room for more generalizations. For example, we generalized the Legendre case to the Gegenbauer case. Why stop there and not consider Jacobi norms? As with the Gegenbauer polynomials, the so called connection coefficients are known in the Jacobi case, and a similar result for the derivatives holds (see, e. g. [2, page 357]). More precisely,

$$\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu} J_n^{(\gamma,\delta)}(t) = \frac{(n+\gamma+\delta+1)_\nu}{2^\nu} J_{n-\nu}^{(\gamma+\nu,\delta+\nu)}(t).$$

Suppose $J_n^{(\gamma,\delta)}(t) = \sum_{k=0}^n c_{nk} J_k^{(\alpha,\beta)}(t)$. Then, the connection coefficient c_{nk} is given by

$$c_{nk} = \frac{(n+\gamma+\delta+1)_k (k+\gamma+1)_{n-k} (2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{(n-k)!\Gamma(2k+\alpha+\beta+2)} \times {}_3F_2\left(\begin{matrix} -n+k,\ n+k+\gamma+\delta+1,\ k+\alpha+1 \\ k+\gamma+1,\ 2k+\alpha+\beta+2 \end{matrix};1\right).$$

Starting from there, it is no problem to set up the matrix representation. Indeed, the proof for the matrix representation presented in Section 2.2 was just an adaption of the proof given in [2] for the Jacobi polynomials. Admittedly, this looks a lot more complex than in the Gegenbauer case, and it introduces two more parameters to care for.

Another direction would be to consider multivariate polynomials. A first study for the special case $\alpha = \beta$ was done by Böttcher and Dörfler [9]. Again, the integral operators already present in the univariate case reappear. The next step, here, is to consider different parameters.

To come back one more time to the univariate case, one could also ask the question what happens when the norms not only differ with regards to the parameters used, but also in the type of norm, say, e. g., relate the Gegenbauer norm of the derivative to the Laguerre norm of the polynomial itself.

To conclude, there are several directions one can follow from here. The complete answer to the mentioned conjectures would only be a first step.

List of Figures

The pictures where created with the help of TikZ, gnuplot, and two programs by the author – one to create the matrix plots and another one for generating the necessary data.

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Theses

on the thesis

"Best constants in Markov-type inequalities with mixed weights"

presented by Dipl.-Math. Holger Langenau

1. Markov-type inequalities give upper bounds on the norm of the νth derivative of an algebraic polynomial in terms of the norm of the polynomial itself. Such an inequality is

$$\|f^{(\nu)}\|_\beta \le C_n^{(\nu)}(\alpha,\beta)\|f\|_\alpha \quad \text{for all} f \in \mathcal{P}_n, \tag{$\star$}$$

   where $\mathcal{P}_n$ denotes the space of algebraic polynomials with complex coefficients of degree at most n. In our context, the norm $\|\cdot\|_\alpha$ is $\|f\|_\alpha^2 = \int_\Omega |f(t)|^2 u(t,\alpha)\mathrm{d}t$, with

$$\begin{aligned} \Omega &= (0,\infty), & u(t,\alpha) &= t^\alpha \mathrm{e}^{-t} & &\text{(Laguerre)},\\ \Omega &= (-1,1), & u(t,\alpha) &= (1-t^2)^\alpha & &\text{(Gegenbauer)},\\ \Omega &= (-\infty,\infty), & u(t,\alpha) &= |t|^{2\alpha}\mathrm{e}^{-t^2} & &\text{(Hermite)}. \end{aligned}$$

2. The best constant $C_n^{(\nu)}(\alpha,\beta)$ in ($\star$) is determined by the operator norm of the differential operator mapping from $(\mathcal{P}_n, \|\cdot\|_\alpha)$ to $(\mathcal{P}_n, \|\cdot\|_\beta)$. The resulting value can be expressed as the spectral norm of the matrix representation with respect to orthonormal bases associated with the chosen norms. While there are only a few special cases in which the constant can be given explicitly, asymptotically sharp bounds can be found in any of the considered cases.

3. The norm of the matrix representation heavily depends on the number $\omega = \beta - \alpha - \nu$ in the Laguerre and Gegenbauer cases and the number $\beta - \alpha$ in the Hermite case. Depending on the sign of this number, two really different settings emerge. Therefore, the methods for determining the best constants vary tremendously.

4. If $\omega \ge 0$ is an integer, the matrices are banded and allow therefore for a simple treatment. To derive an upper bound on the norm, the matrix is decomposed into a sum of diagonal matrices. Thus, the norm is bounded by the sum of all diagonal's maximal absolute values. For obtaining a lower bound, it is relatively easy to show that the matrices (scaled by some factor) converge in the norm to a well-understood infinite Toeplitz matrix.

5. If $\omega > 0$ is arbitrary, an upper bound on the norm can be given by interpolating between the constants known from the integral case. This is done with the help of a theorem by Stein, which requires the finite-dimensional operator to be extended to an infinite-dimensional operator on $L^2(\Omega)$. This is done by keeping α fixed and setting the operator to zero on the orthogonal complement of the space associated with α.

6. Although the matrix under consideration is not banded anymore for arbitrary $\omega > 0$, large valued entries still concentrate alongside the main diagonal. By choosing a vector that is close to an optimal, norm-realizing vector, a lower bound can be found in that case, too. The idea has to be balanced between accounting the norm maximizing matrix parts and retaining a small norm of the vector itself. The number of nonzero entries of the vector will be slowly increasing with the dimension. By letting the dimension go to infinity, the obtained lower bound is asymptotically the same as the upper bound given before.

7. For $\omega < 0$, the previous methods fail. However, one can construct an integral operator with piecewise constant kernel associated with the matrix under investigation. Employing a result by Widom and Shampine, the norm of the matrix is n times the norm of the received operator. If even $\omega < -1/2$, letting $n \to \infty$, the scaled versions of this operator converge to another integral operator in the Hilbert-Schmidt norm, and therefore in the operator norm. Thus, the matrix norm is completely determined by the norm of this integral operator limit. The same limit appears for the Laguerre and the Gegenbauer cases. The limit of the Gegenbauer case is, up to a constant, unitarily equivalent to the more accessible Laguerre case operator.

8. The classical Hermite problem only considers $\alpha = \beta = 0$. It can be generalized as stated below ($\star$). The methods used for the Laguerre and Gegenbauer cases continue to work for the extended Hermite case and only differ in the details. But, the distinction is now made depending on the number $\beta - \alpha$. In addition, the case $\beta = \alpha \neq 0$ here has to be treated separately. Although the matrix is not banded anymore, Geršgorin's theorem implies that the bound obtained for the integral cases $\beta - \alpha \geq \lceil \nu/2 \rceil$ is still valid. The integral operators coming into play for $\beta - \alpha < -1/2$ are a bit more complex than the ones of the Laguerre and Gegenbauer cases.

9. In the convergence proofs, the restriction $\omega < -1/2$ (resp. $\beta - \alpha < -1/2$) had to be made. This is due to the fact that the pointwise defined limit operator is no longer a Hilbert-Schmidt operator without the assumption. However, in the Laguerre case, the integral operator in question can be shown to belong to some Schatten class. More precisely, the operator belongs to the 2^nth Schatten class whenever $\omega < -1/2^n$. Therefore, it is compact for any $\omega < 0$. Similar statements for all three cases might be helpful to close the last gap.