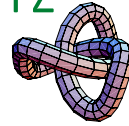


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# Duality investigations for multi-composed optimization problems with applications in location theory

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## Duality investigations for multi-composed optimization problems with applications in location theory

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### Report

The goal of this thesis is two-fold. On the one hand, it pursues to provide a contribution to the conjugate duality by proposing a new duality concept, which can be understood as an umbrella for different meaningful perturbation methods. On the other hand, this thesis aims to investigate minimax location problems by means of the duality concept introduced in the first part of this work, followed by a numerical approach using epigraphical splitting methods.

After summarizing some elements of the convex analysis as well as introducing important results needed later, we consider an optimization problem with geometric and cone constraints, whose objective function is a composition of  $n + 1$  functions. For this problem we propose a conjugate dual problem, where the functions involved in the objective function of the primal problem are decomposed. Furthermore, we formulate generalized interior point regularity conditions for strong duality and give necessary and sufficient optimality conditions. As applications of this approach we determine the formulae of the conjugate as well as the biconjugate of the objective function of the primal problem and analyze an optimization problem having as objective function the sum of reciprocals of concave functions.

In the second part of this thesis we discuss in the sense of the introduced duality concept three classes of minimax location problems. The first one consists of nonlinear and linear single minimax location problems with geometric constraints, where the maximum of nonlinear or linear functions composed with gauges between pairs of a new and existing points will be minimized. The version of the nonlinear location problem is additionally considered with set-up costs. The second class of minimax location problems deals with multifacility location problems as suggested by Drezner (1991), where for each given point the sum of weighted distances to all facilities plus set-up costs is determined and the maximal value of these sums is to be minimized. As the last and third class the classical multifacility location problem with geometrical constraints is considered in a generalized form where the maximum of gauges between pairs of new facilities and the maximum of gauges between pairs of new and existing facilities will be minimized. To each of these location problems associated dual problems will be formulated as well as corresponding duality statements and necessary and sufficient optimality conditions. To illustrate the results of the duality approach and to give a more detailed characterization of the relations between the location problems and their corresponding duals, we consider examples in the Euclidean space.

This thesis ends with a numerical approach for solving minimax location problems by epigraphical splitting methods. In this framework, we give formulae for the projections onto the epigraphs of several sums of powers of weighted norms as well as formulae for the projection onto the epigraphs of gauges. Numerical experiments document the usefulness of our approach for the discussed location problems.

### Keywords

composed functions; conjugate functions; Lagrange duality; conjugate duality; generalized interior point regularity conditions; weak and strong duality; optimality conditions; reciprocals of

concave functions; power functions; gauges; nonlinear minimax location problems; multifacility minimax location problems; epigraphical projection; projection operators

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# Chapter 1

## Introduction

Conjugate duality is a powerful instrument to analyze optimization problems and has for that reason a wide range of applications. Over the last couple of years, an important field of applications arises in areas such as vector variational inequalities [1], facility location theory [75], machine learning [13], image restoration [15], portfolio optimization [19] and monotone operator theory [48], to mention only a few of them. In many cases, the objective function of an optimization problem occurring in the mentioned research areas may be written as a composition of two or more functions. This presentation makes not only the derivation of duality assertions easier, but also the handling of optimization problems from the numerical point of view.

But until now there is no duality approach for the more general situation, namely, where the optimization problem is considered as the minimization of an objective function that is a composition of more than two functions. The advantage of this consideration is that the objective function of a certain optimization problem can be split into a certain number of functions to refine and improve some theoretical and numerical aspects.

Therefore, the goal of this thesis is to consider an optimization problem with geometric and cone constraints, whose objective function is a composition of  $n + 1$  functions and to deliver a detailed duality approach for this type of problems. For short, we call such problems multi-composed optimization problems. In fact, this study is more general than in [7, 8, 17, 20, 23, 55] and can furthermore be understood as a combination of all kinds of meaningful perturbation methods. To be more precise, we extended the already existing duality schemes to derive a more detailed characterization of the set of optimal solutions and to give a unified framework with a corresponding conjugate dual problem, regularity conditions as well as strong duality statements. As applications we present the formulae of the conjugate and the biconjugate of a multi-composed function, i.e. a function that is a composition of  $n + 1$  functions. Moreover, we discuss an optimization problem having as objective function the sum of reciprocals of concave functions.

The results presented in the first part of this thesis open a new approach to investigating facility location problems. Such kind of optimization problems are known for their numerous applications in areas like computer science, telecommunications, transportation and emergency facilities programming. In the framework of continuous optimization where the distances are measured by gauges, two kinds of location problems are particularly significant. The first one consists of the so-called minisum location problems and has the objective to determine a new point such that the sum of distances between the new and given points is minimal (see [20, 23, 36, 50, 60, 63, 66, 68]). The second type contains the so-called minimax location problems, where a new point is sought such that the maximum of distances between the new and given points will be minimized (see [38, 40, 45, 58, 62, 74]). In this work we study more general problems where the gauges may additionally be composed with a nonlinear function, i.e. we consider besides linear also nonlinear minimax location problems (see [33] and [44]).

The second class of location problems we consider was proposed in 1991 by Drezner in [35] and describes the following emergency scenario. A certain number of emergency calls arise and ask for an ambulance. To each of these demand points an ambulance is sent to load and transport the

patient to a hospital. The location of the ambulance-station and the hospital must not be necessary on the same site. This assumption may shorten the response time for the patients, especially for the farthest one, in the situation when for example a hospital is completely overcrowded or short of medical supplies. The aim is now to determine the location of the ambulance-station and the hospital such that the maximum time required before the farthest patient arrives at the hospital will be minimized. In this case the maximum time is naturally defined as the sum of the travel time of the ambulance from the ambulance-station to the patient and the travel time to the hospital plus some set-up costs. Set-up costs like the loading time at the emergency and the unloading time at the hospital of the patient are few examples to cite.

While Drezner suggested a model for the case of the Euclidean norm, Michelot and Plastria [67] work in a higher dimensional space where the distances are measured by a norm. In this thesis we generalize this location model to the situation where the distances are measured by mixed gauges defined on a Fréchet space. The goal is then to describe these type of location problems in the framework of the introduced duality concept.

Apart from these two classes of location problems, we also consider a more general and complex problem, namely, the so-called multifacility minimax location problem (see [39, 67]), which has attracted less attention in the literature compared to the multifacility minisum location problems (see [43, 56, 61, 78]). The objective of the multifacility minimax location problems is to determine several new points such that either the maximum of distances between pairs of new points or the maximum of distances between new and existing points is minimal.

The last part of this work focuses on solving methods for minimax location problems. In this context we first present formulae for projectors onto the epigraphs of several sums of powers of weighted norms as well as onto the epigraphs of gauges. These formulae allow to combine the epigraphical projection method, developed in [28] for constrained convex optimization problems, with a parallel splitting algorithm (see [2] and [29]) minimizing a finite sum of functions. To show the usefulness of the presented formulae we compare our solving method with the one presented in [30] by Cornejo and Michelot.

Next we give a description of the contents, emphasizing the most important points.

In **Chapter 2** we first collect some elements from the field of convex analysis and present important statements that are used in this thesis. While in Section 2.1 notations and preliminary results are listed on convex sets, Section 2.2 is dedicated to convex scalar and vector functions.

After introducing the basics, we consider in **Chapter 3** a multi-composed optimization problem with geometric and cone constraints. We give an equivalent formulation of this problem and use the reformulated optimization problem to construct a corresponding conjugate dual problem to the main problem, followed by a weak duality theorem. The convenience of this approach is that the functions involved in the composed objective function of the original problem can be decomposed in the formulation of the conjugate dual problem or, to formulate it more precisely, their conjugates.

Section 3.2 is devoted to generalized interior point regularity conditions guaranteeing strong duality. Moreover, by using the strong duality theorem we formulate some optimality conditions for the original problem and its corresponding conjugate dual problem. Besides of this approach, we discover in Section 3.3 the formula of the conjugate of a multi-composed function. We find also a formula of its biconjugate function and close this section with a theorem which characterizes some topological properties of this function.

In Section 3.4, as a further application of our approach, we consider a convex optimization problem having as objective function the sum of reciprocals of concave functions. For this problem we formulate a conjugate dual problem and state a strong duality theorem from which we derive necessary and sufficient optimality conditions. The approach done in this chapter is based on our paper [79].

In **Chapter 4**, which is related to our articles [80, 81] and [82], we analyze three classes of location problems starting with some properties of gauge functions in Section 4.1. Then, we consider single minimax location problems in Section 4.2. In Subsection 4.2.1, we apply the

approach done in Chapter 3 to nonlinear single minimax location problems with set-up costs in a Fréchet space and give necessary and sufficient optimality conditions. Further, in Subsection 4.2.2 we consider linear single minimax location problems without set-up costs in a Hilbert space. After presenting associated duality statements, we describe the relation between the optimal solutions of the primal problem and its dual. In Subsection 4.2.3 as well as in Subsection 4.2.4 the location problems will be studied in a Fréchet space followed by a characterization to a Hilbert space endowed with a norm. Here we formulate a second dual problem reducing the number of constraints and dual variables compared with the first formulated dual problem in the previous sections.

In Section 4.3 we study extended multifacility location problems introduced by Drezner in [35]. In Subsection 4.3.1 we construct corresponding conjugate dual problems and prove strong duality from which we derive some optimality conditions. Afterwards, we consider a special case of these location problems where the weights have a multiplicative structure like treated by Michelot and Plastria in [67] and describe the relation to their conjugate dual problems with norms as distance measures. In Subsection 4.3.2, we also deal with location problems without set-up costs. Besides the set of strong duality assertions and optimality conditions we give geometrical characterizations of the set of optimal solutions of the conjugate dual problem as well as illustrating examples.

The analysis of classic multifacility minimax location problems in Section 4.4 provides duality statements in the sense of Chapter 3. In concrete terms, this means that we formulate an associated conjugate dual problem as well as derive necessary and sufficient optimality conditions in Subsection 4.4.1. Further, we introduce another dual problem reducing the number of dual variables compared to the first formulated dual problem. Continuing in this vein, we also employ a duality approach including statements of strong duality and optimality conditions. As the most location problems are considered in Euclidean spaces, we particularize in Subsection 4.4.2 the latter case in this context and show that we have a full symmetry between the location problem, its dual problem and the Lagrange dual problem of the dual problem, which means that the Lagrange dual is identical to the location problem. Finally, we close this paper with an example showing on the one hand how an optimal solution of the location problem can be recovered from an optimal solution of the associated conjugate dual problem and on the other hand how we can geometrically interpret an optimal dual solution.

Along with a theoretical consideration, we are interested in **Chapter 5** in a numerical method for solving minimax location problems. For this purpose, we present in Section 5.2 some formulae of projections onto the epigraphs of several sums of powers of weighted norms and onto the epigraphs of gauges. In Section 5.3 we first bring the extended multifacility minimax location problem in a form of an unconstrained optimization problem where its objective function is a sum of functions. This reformulation allows us then to use the parallel splitting algorithm (see [2, 28, 29]) combined with the formulae from the Section 5.2 to solve minimax location problems. In addition, we solve the numerical examples by the method proposed by Cornejo and Michelot in [30], where the sum of powers of weighted norms is split such that the formulae of the projectors onto the epigraphs of the powers of weighted norms are relevant. This splitting scheme makes it necessary to introduce additional variables, which in turn goes at the expense of the numerical performance. It is shown that the parallel splitting algorithm combined with the presented projection formulae performs very well on these kind of location problems and outperforms the method given in [30].



## Chapter 2

# Notations and preliminary results

This chapter serves as an introduction and aims to make this thesis as self-contained as possible. We introduce here basic notions from the convex analysis and give important statements on convex sets, convex scalar and vector functions. For readers interested in convex analysis we refer to [7, 24, 37, 57, 70, 83, 84].

### 2.1 Convex sets

Let  $X$  be a Hausdorff locally convex space and  $X^*$  its topological dual space endowed with the weak\* topology  $w(X^*, X)$ . For  $x \in X$  and  $x^* \in X^*$ , let  $\langle x^*, x \rangle := x^*(x)$  be the value of the linear continuous functional  $x^*$  at  $x$ .

A set  $K \subseteq X$  is called *convex* if it holds  $\lambda x + (1 - \lambda)y \in K$  for all  $x, y \in K$  and  $\lambda \in [0, 1]$  and if  $K$  additionally satisfies the condition  $\lambda K \subseteq K$  for all  $\lambda \geq 0$ , then  $K$  is said to be a *convex cone*.

Given a set  $S \subseteq X$  and  $x \in X$ , then the *normal cone* to  $S$  at  $x$ , defined by

$$N_S(\bar{x}) := \{x^* \in X^* : \langle x^*, y - \bar{x} \rangle \leq 0 \ \forall y \in S\},$$

is a convex cone.

Consider a convex cone  $K \subseteq X$ , which induces on  $X$  a *partial ordering relation* “ $\leq_K$ ”, defined by  $\leq_K := \{(x, y) \in X \times X : y - x \in K\}$ , i.e. for  $x, y \in X$  it holds  $x \leq_K y \Leftrightarrow y - x \in K$ . Note that we assume that all cones we consider contain the origin, which we denote by  $0_X$ . Further, we attach to  $X$  a greatest element with respect to “ $\leq_K$ ”, denoted by  $+\infty_K$ , which does not belong to  $X$  and denote  $\bar{X} = X \cup \{+\infty_K\}$ . Then it holds  $x \leq_K +\infty_K$  for all  $x \in \bar{X}$ . We write  $x \leq_K y$  if and only if  $x \leq_K y$  and  $x \neq y$ . Further, we write  $\leq_{\mathbb{R}_+} =: \leq$  and  $\leq_{\mathbb{R}_+} =: <$ .

On  $\bar{X}$  we consider the following operations and conventions:  $x + (+\infty_K) = (+\infty_K) + x := +\infty_K$  for all  $x \in X \cup \{+\infty_K\}$  and  $\lambda \cdot (+\infty_K) := +\infty_K$  for all  $\lambda \in [0, +\infty]$ . Further,  $K^* := \{x^* \in X^* : \langle x^*, x \rangle \geq 0 \text{ for all } x \in K\}$  is the *dual cone* of  $K$  and we take by convention  $\langle x^*, +\infty_K \rangle := +\infty$  for all  $x^* \in K^*$ . By a slight abuse of notation we denote the extended real space  $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$  and consider on it the following operations and conventions:  $\lambda + (+\infty) = (+\infty) + \lambda := +\infty$  for all  $\lambda \in [-\infty, +\infty]$ ,  $\lambda + (-\infty) = (-\infty) + \lambda := -\infty$  for all  $\lambda \in [-\infty, +\infty]$ ,  $\lambda \cdot (+\infty) := +\infty$  for all  $\lambda \in [0, +\infty]$ ,  $\lambda \cdot (+\infty) := -\infty$  for all  $\lambda \in [-\infty, 0)$ ,  $\lambda \cdot (-\infty) := -\infty$  for all  $\lambda \in (0, +\infty]$ ,  $\lambda \cdot (-\infty) := +\infty$  for all  $\lambda \in [-\infty, 0)$  and  $0(-\infty) := 0$ .

For a set  $S \subseteq X$  the *conic hull* is defined by  $\text{cone}(S) := \{\lambda x : x \in S, \lambda \geq 0\}$ . Further, the prefix *int* we use to denote the *interior* of a set  $S \subseteq X$ , while the prefixes *cl*, *ri*, *core* and *sqli* are used to denote the *closure*, *relative interior*, *algebraic interior* and the *strong quasi relative interior*, respectively, where in the case of having a convex set  $S \subseteq X$  it holds (see [31])

$$\begin{aligned} \text{core}(S) &= \{x \in S : \text{cone}(S - x) = X\}, \\ \text{sqli}(S) &= \{x \in S : \text{cone}(S - x) \text{ is a closed linear subspace}\}. \end{aligned}$$

Note that if  $\text{cone}(S - x)$  is a linear subspace, then  $x \in S$ .

The next statement was given in [5] for the *quasi relative interior*,  $\text{qri}(S) = \{x \in S : \text{cl}(\text{cone}(S - x)) \text{ is a linear subspace}\}$ , we show the validity for the strong quasi relative interior.

**Lemma 2.1.** *Let  $A \subseteq X$  and  $B \subseteq Z$  be non-empty convex subsets. Then, it holds*

$$0_{X \times Z} \in \text{sqri}(A \times B) \Leftrightarrow 0_X \in \text{sqri}(A) \text{ and } 0_Z \in \text{sqri}(B).$$

**Proof.** First, let us recall that if  $A$  and  $B$  are convex and  $0_X \in A$  and  $0_Z \in B$ , then

$$\text{cone}(A \times B) = \text{cone}(A) \times \text{cone}(B).$$

Now, let us assume that  $0_{X \times Z} \in \text{sqri}(A \times B)$ , then  $\text{cone}(A \times B)$  is a closed linear subspace of  $X \times Z$ , which implies that  $0_{X \times Z} = (0_X, 0_Z) \in A \times B$ . But this means that  $\text{cone}(A \times B) = \text{cone}(A) \times \text{cone}(B)$  and hence,  $\text{cone}(A)$  and  $\text{cone}(B)$  are closed linear subspaces, i.e.  $0_X \in \text{sqri}(A)$  and  $0_Z \in \text{sqri}(B)$ .

On the other hand, let  $0_X \in \text{sqri}(A)$  and  $0_Z \in \text{sqri}(B)$ , then  $\text{cone}(A)$  and  $\text{cone}(B)$  are closed linear subspaces and so,  $0_X \in A$  and  $0_Z \in B$ . From here follows that  $\text{cone}(A \times B) = \text{cone}(A) \times \text{cone}(B)$  and thus,  $\text{cone}(A \times B)$  is a closed linear subspace, i.e.  $0_{X \times Z} \in \text{sqri}(A \times B)$ .  $\square$

## 2.2 Convex functions

### 2.2.1 Scalar functions

For a given function  $f : X \rightarrow \overline{\mathbb{R}}$  we consider its *effective domain*  $\text{dom } f := \{x \in X : f(x) < +\infty\}$  and call  $f$  *proper* if  $\text{dom } f \neq \emptyset$  and  $f(x) > -\infty$  for all  $x \in X$ . The *epigraph* of  $f$  is  $\text{epi } f = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$ . Recall that a function  $f : X \rightarrow \overline{\mathbb{R}}$  is called *convex* if  $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$  for all  $x, y \in X$  and all  $\lambda \in [0, 1]$ . For a subset  $A \subseteq X$ , its *indicator function*  $\delta_A : X \rightarrow \overline{\mathbb{R}}$  is

$$\delta_A(x) := \begin{cases} 0, & \text{if } x \in A, \\ +\infty, & \text{otherwise,} \end{cases}$$

and its *support function*  $\sigma_A : X^* \rightarrow \overline{\mathbb{R}}$  is  $\sigma_A(x^*) = \sup_{x \in A} \langle x^*, x \rangle$ .

The *conjugate function* of  $f$  with respect to the non-empty subset  $S \subseteq X$  is defined by

$$f_S^* : X^* \rightarrow \overline{\mathbb{R}}, \quad f_S^*(x^*) = \sup_{x \in S} \{\langle x^*, x \rangle - f(x)\}.$$

In the case  $S = X$ ,  $f_S^*$  turns into the classical *Fenchel-Moreau conjugate function* of  $f$  denoted by  $f^*$ .

A function  $f : X \rightarrow \overline{\mathbb{R}}$  is called *lower semicontinuous at  $\bar{x} \in X$*  if  $\liminf_{x \rightarrow \bar{x}} f(x) \geq f(\bar{x})$  and when this function is lower semicontinuous at all  $x \in X$ , then we call it *lower semicontinuous* (l.s.c. for short).

Let  $W \subseteq X$  be a non-empty set, then a function  $f : X \rightarrow \overline{\mathbb{R}}$  is called *K-increasing on  $W$* , if from  $x \leq_K y$  follows  $f(x) \leq f(y)$  for all  $x, y \in W$ . When  $W = X$ , then we call the function  $f$  *K-increasing*.

We also use the notion of *subdifferentiability* to formulate optimality conditions. If we take an arbitrary  $x \in X$  such that  $f(x) \in \mathbb{R}$ , then we call the set

$$\partial f(x) := \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x \rangle \forall y \in X\}$$

the (convex) *subdifferential* of  $f$  at  $x$ , where the elements are called the *subgradients* of  $f$  at  $x$ . Moreover, if  $\partial f(x) \neq \emptyset$ , then we say that  $f$  is *subdifferentiable at  $x$*  and if  $f(x) \notin \mathbb{R}$ , then we make the convention that  $\partial f(x) := \emptyset$ . Note, that the subgradients can be characterized by the conjugate function, especially this means

$$x^* \in \partial f(x) \Leftrightarrow f(x) + f^*(x^*) = \langle x^*, x \rangle \quad \forall x \in X, \quad x^* \in X^*, \quad (2.1)$$

i.e. the *Young-Fenchel inequality* is fulfilled with equality.

Furthermore, let  $\mathcal{H}$  be a real Hilbert space equipped with the scalar product  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ , where the associated norm  $\| \cdot \|_{\mathcal{H}}$  is defined by  $\|y\|_{\mathcal{H}} := \sqrt{\langle y, y \rangle_{\mathcal{H}}}$  for all  $y \in \mathcal{H}$ . Some important results in Hilbert spaces can be found in Chapter 5.

### 2.2.2 Vector functions

Let  $Z$  be another Hausdorff locally convex space partially ordered by the convex cone  $Q \subseteq Z$  and  $Z^*$  its topological dual space endowed with the weak\* topology  $w(Z^*, Z)$ . The *domain* of a vector function  $F : X \rightarrow \bar{Z} = Z \cup \{+\infty_Q\}$  is  $\text{dom } F := \{x \in X : F(x) \neq +\infty_Q\}$ .  $F$  is called *proper* if  $\text{dom } F \neq \emptyset$ .

When  $F(\lambda x + (1 - \lambda)y) \leq_Q \lambda F(x) + (1 - \lambda)F(y)$  holds for all  $x, y \in X$  and all  $\lambda \in [0, 1]$  the function  $F$  is said to be *Q-convex*.

The *Q-epigraph* of a vector function  $F$  is  $\text{epi}_Q F = \{(x, z) \in X \times Z : F(x) \leq_Q z\}$  and when  $Q$  is closed we say that  $F$  is *Q-epi closed* if  $\text{epi}_Q F$  is a closed set.

For a  $z^* \in Q^*$  we define the function  $(z^*F) : X \rightarrow \mathbb{R}$  by  $(z^*F)(x) := \langle z^*, F(x) \rangle$ . Then  $\text{dom}(z^*F) = \text{dom } F$ . Moreover, it is easy to see that if  $F$  is *Q-convex*, then  $(z^*F)$  is convex for all  $z^* \in Q^*$ . Let us point out that by the operations we defined on a Hausdorff locally convex space attached with a maximal element and on the extended real space, there holds  $0f = \delta_{\text{dom } f}$  and  $(0_{Z^*}F) = \delta_{\text{dom } F}$ .

The vector function  $F$  is called *positively Q-lower semicontinuous at*  $x \in X$  if  $(z^*F)$  is lower semicontinuous at  $x$  for all  $z^* \in Q^*$ . The function  $F$  is called *positively Q-lower semicontinuous* if it is positively *Q-lower semicontinuous* at every  $x \in X$ . Note that if  $F$  is positively *Q-lower semicontinuous*, then it is also *Q-epi closed*, while the inverse statement is not true in general (see: [7, Proposition 2.2.19]). Let us mention that in the case  $Z = \mathbb{R}$  and  $Q = \mathbb{R}_+$ , the notion of *Q-epi closedness* falls into the classical notion of lower semicontinuity.

$F : X \rightarrow \bar{Z}$  is called *(K, Q)-increasing on W*, if from  $x \leq_K y$  follows  $F(x) \leq_Q F(y)$  for all  $x, y \in W$ . When  $W = X$ , we call this function *(K, Q)-increasing*.

We give now some statements that will be useful later, beginning with one whose proof is straightforward.

**Lemma 2.2.** *Let  $V$  be a Hausdorff locally convex space partially ordered by the convex cone  $U$ ,  $F : X \rightarrow \bar{Z}$  be a proper and *Q-convex* function and  $G : Z \rightarrow \bar{V}$  be an *U-convex* and *(Q, U)-increasing* function on  $F(\text{dom } F) \subseteq \text{dom } G$  with the convention  $G(+\infty_Q) = +\infty_U$ . Then the function  $(G \circ F) : X \rightarrow \bar{V}$  is *U-convex*.*

**Lemma 2.3.** (cf. [49]) *Let  $Y$  be a Hausdorff locally convex space,  $Q$  also closed,  $h : X \times Y \rightarrow \bar{Z}$  and  $F : X \rightarrow \bar{Z}$  proper vector functions and  $G : Y \rightarrow Z$  a continuous vector functions, where  $h$  is defined by  $h(x, y) := F(x) + G(y)$ . Then  $F$  is *Q-epi closed* if and only if  $h$  is *Q-epi closed*.*

**Proof.** “ $\Rightarrow$ ”: Let  $(x_\alpha, y_\alpha, z_\alpha)_\alpha \subseteq \text{epi}_Q h$  be a net such that  $(x_\alpha, y_\alpha, z_\alpha) \rightarrow (\bar{x}, \bar{y}, \bar{z})$ . Then  $F(x_\alpha) + G(y_\alpha) \leq z_\alpha$  for any  $\alpha$ , followed by  $(x_\alpha, z_\alpha - G(y_\alpha))_\alpha \subseteq \text{epi}_Q F$  and  $(y_\alpha, G(y_\alpha))_\alpha \subseteq \text{epi}_Q G$ . Because  $G$  is continuous and  $y_\alpha \rightarrow \bar{y}$ , it follows that  $G(y_\alpha) \rightarrow G(\bar{y})$ . Then  $(x_\alpha, z_\alpha - G(y_\alpha)) \rightarrow (\bar{x}, \bar{z} - G(\bar{y})) \in \text{epi}_Q F$ , because this set is closed. One has then  $F(\bar{x}) \leq_Q \bar{z} - G(\bar{y})$ , i.e.  $(\bar{x}, \bar{y}, \bar{z}) \in \text{epi}_Q h$ . As the convergent nets  $(x_\alpha)_\alpha$ ,  $(y_\alpha)_\alpha$  and  $(z_\alpha)_\alpha$  were arbitrarily chosen, it follows that  $\text{epi}_Q h$  is closed, i.e.  $h$  is *Q-epi closed*.

“ $\Leftarrow$ ”: Let  $(x_\alpha, z_\alpha)_\alpha \subseteq \text{epi}_Q F$  such that  $(x_\alpha, z_\alpha) \rightarrow (\bar{x}, \bar{z})$ . Take also  $(y_\alpha)_\alpha \subseteq Y$  such that  $y_\alpha \rightarrow \bar{y}$ . Because  $G$  is continuous, one has  $G(y_\alpha) \rightarrow G(\bar{y})$ . Then  $(x_\alpha, y_\alpha, z_\alpha + G(y_\alpha))_\alpha \subseteq \text{epi}_Q h$ , which is closed, consequently  $(\bar{x}, \bar{y}, \bar{z} + G(\bar{y})) \in \text{epi}_Q h$ , i.e.  $F(\bar{x}) + G(\bar{y}) \leq_Q \bar{z} + G(\bar{y})$ . Therefore  $F(\bar{x}) \leq_Q \bar{z}$ , i.e.  $(\bar{x}, \bar{z}) \in \text{epi}_Q F$ . As the convergent nets  $(x_\alpha)_\alpha$  and  $(z_\alpha)_\alpha$  were arbitrarily chosen, it follows that  $\text{epi}_Q F$  is closed, i.e.  $F$  is *Q-epi closed*.  $\square$

**Remark 2.1.** *Note that a continuous proper vector function  $G : Y \rightarrow \bar{Z}$ , where  $Y$  is a Hausdorff locally convex space, has a full domain, thus one can directly take  $G : Y \rightarrow Z$  in this situation. The question whether the equivalence in Lemma 2.3 remains valid if one considers a proper vector function  $G : Y \rightarrow \bar{Z}$  that is not necessarily continuous is still open.*

**Remark 2.2.** *If we set  $Y = Z$  and  $G(y) = -y$  for all  $y \in Y$ , then Lemma 2.3 says that  $F$  is  $Q$ -epi closed if and only if the vector function defined by  $(x, y) \in X \times Y \mapsto F(x) - y$  is  $Q$ -epi closed. For this special case a similar statement can be found in [79, Lemma 2.1], but under the additional hypothesis  $\text{int } Q \neq \emptyset$ .*



## Chapter 3

# Lagrange duality for multi-composed optimization problems

The goal of this chapter is to consider an optimization problem with geometric and cone constraints, whose objective function is a composition of  $n + 1$  functions and to deliver a full duality approach for this type of problems.

By considering such a multi-composed optimization problem there are several ways to formulate a corresponding conjugate dual problem where the composed functions involved in the objective function of the primal problem, or, to be more precise, their conjugates, are separated and to give associated duality statements.

The first method is the direct applying of the perturbation theory (see [7, 37, 52, 53, 64, 70]). A second approach is presented in this chapter and starts in Section 3.1 by reformulating the primal problem as an optimization problem with set and cone constraints and continues by using the Lagrange duality concept. The question is now, which of these two methods is more suitable? It is shown in Section 3.2 that the second method asks for weaker hypotheses on the involved functions of the primal problem for guaranteeing strong duality. As applications, we present the formulae of the conjugate and the biconjugate of a multi-composed function in Section 3.3, i.e. a function that is a composition of  $n + 1$  functions. Moreover, we discuss in the Section 3.4 an optimization problem having as objective function the sum of reciprocals of concave functions.

### 3.1 The multi-composed optimization problem and its conjugate dual

As already mentioned, our aim is to formulate a conjugate dual problem to an optimization problem with geometric and cone constraints having as objective function the composition of  $n + 1$  functions. In other words, we consider the following problem

$$(P^C) \quad \inf_{x \in \mathcal{A}} (f \circ F^1 \circ \dots \circ F^n)(x), \\ \mathcal{A} = \{x \in S : g(x) \in -Q\},$$

where  $Z$  is a Hausdorff locally convex space partially ordered by the convex cone  $Q \subseteq Z$  and  $X_i$  is a Hausdorff locally convex space partially ordered by the convex cone  $K_i \subseteq X_i$ ,  $i = 0, \dots, n - 1$ . Moreover,

- $S$  is a non-empty subset of the Hausdorff locally convex space  $X_n$ ,
- $f : X_0 \rightarrow \overline{\mathbb{R}}$  is proper and  $K_0$ -increasing on  $F^1(\text{dom } F^1) + K_0 \subseteq \text{dom } f$ ,

- $F^i : X_i \rightarrow \bar{X}_{i-1} = X_{i-1} \cup \{+\infty_{K_{i-1}}\}$  is proper and  $(K_i, K_{i-1})$ -increasing on  $F^{i+1}(\text{dom } F^{i+1}) + K_i \subseteq \text{dom } F^i$  for  $i = 1, \dots, n-2$ ,
- $F^{n-1} : X_{n-1} \rightarrow \bar{X}_{n-2} = X_{n-2} \cup \{+\infty_{K_{n-2}}\}$  is proper and  $(K_{n-1}, K_{n-2})$ -increasing on  $F^n(\text{dom } F^n \cap \mathcal{A}) + K_{n-1} \subseteq \text{dom } F^{n-1}$ ,
- $F^n : X_n \rightarrow \bar{X}_{n-1} = X_{n-1} \cup \{+\infty_{K_{n-1}}\}$  is a proper function and
- $g : X_n \rightarrow \bar{Z}$  is a proper function fulfilling  $S \cap g^{-1}(-Q) \cap ((F^n)^{-1} \circ \dots \circ (F^1)^{-1})(\text{dom } f) \neq \emptyset$ .

Additionally, we make the convention that  $f(+\infty_{K_0}) = +\infty$  and  $F^i(+\infty_{K_i}) = +\infty_{K_{i-1}}$ , i.e.  $f : \bar{X}_0 \rightarrow \bar{\mathbb{R}}$  and  $F^i : \bar{X}_i \rightarrow \bar{X}_{i-1}$ ,  $i = 1, \dots, n-1$ .

**Remark 3.1.** For the rest of this paper it is preferable to make the following arrangement. In the situation when  $n = 1$  we set  $\{1, \dots, n-1\} = \{1, \dots, n-2\} = \emptyset$  and when  $n = 2$ ,  $\{1, \dots, n-2\} = \emptyset$ . In particular, this means for the case  $n = 1$  that  $F^1 : X_1 \rightarrow \bar{X}_0$  is a proper function and for the case  $n = 2$  that  $F^1 : \bar{X}_1 \rightarrow \bar{X}_0$  is a proper and  $(K_1, K_0)$ -increasing function on  $F^2(\text{dom } F^2 \cap \mathcal{A}) + K_1 \subseteq \text{dom } F^1$  and  $F^2 : X_2 \rightarrow \bar{X}_1$  a proper function.

Let us now consider the following problem

$$(\tilde{P}^C) \quad \inf_{(y^0, \dots, y^n) \in \tilde{\mathcal{A}}} \tilde{f}(y^0, \dots, y^n),$$

where

$$\begin{aligned} \tilde{\mathcal{A}} = \{ & (y^0, \dots, y^{n-1}, y^n) \in X_0 \times \dots \times X_{n-1} \times S : \\ & g(y^n) \in -Q, \ h^i(y^i, y^{i-1}) \in -K_{i-1}, \ i = 1, \dots, n\}. \end{aligned}$$

The functions  $\tilde{f} : X_0 \times \dots \times X_n \rightarrow \bar{\mathbb{R}}$  and  $h^i : X_i \times X_{i-1} \rightarrow \bar{X}_{i-1}$  are defined as

$$\tilde{f}(y^0, \dots, y^n) = f(y^0) \text{ and } h^i(y^i, y^{i-1}) = F^i(y^i) - y^{i-1} \text{ for } i = 1, \dots, n.$$

**Lemma 3.1.** Let  $(y^0, \dots, y^n)$  be feasible to  $(\tilde{P}^C)$ , then it holds  $f((F^1 \circ \dots \circ F^n)(y^n)) \leq f(y^0)$ .

**Proof.** Let  $(y^0, \dots, y^n)$  be feasible to  $(\tilde{P}^C)$ , then we have

$$F^n(y^n) \leq_{K_{n-1}} y^{n-1}, \dots, F^1(y^1) \leq_{K_0} y^0.$$

Moreover, since  $F^{n-1}$  is  $(K_{n-1}, K_{n-2})$ -increasing on  $F^n(\text{dom } F^n \cap \mathcal{A}) + K_{n-1}$  and  $F^i$  is  $(K_i, K_{i-1})$ -increasing on  $F^{i+1}(\text{dom } F^{i+1}) + K_i$  for  $i = 1, \dots, n-2$ , it follows  $(F^{n-1} \circ \dots \circ F^n)(y^n) \leq_{K_{n-2}} F^{n-1}(y^{n-1}) \leq_{K_{n-2}} y^{n-2}$  and so on  $(F^1 \circ \dots \circ F^n)(y^n) \leq_{K_0} F^1(y^1) \leq_{K_0} y^0$ . Since  $f$  is  $K_0$ -increasing on  $F^1(\text{dom } F^1) + K_0$  we get the desired inequality  $f((F^1 \circ \dots \circ F^n)(y^n)) \leq f(y^0)$ .  $\square$

**Remark 3.2.** If  $F^n$  is an affine function, then it can be useful to set  $K_{n-1} = \{0_{X_{n-1}}\}$ , because in this case  $F^{n-1}$  does not need to be monotone to ensure the inequality of the previous lemma.

If we denote by  $v(P^C)$  and  $v(\tilde{P}^C)$  the optimal objective values of the problems  $(P^C)$  and  $(\tilde{P}^C)$ , respectively, then the following relation between the optimal objective values is always true.

**Theorem 3.1.** It holds  $v(P^C) = v(\tilde{P}^C)$ .

**Proof.** Let  $x$  be a feasible element to  $(P^C)$  and set  $y^n = x$ ,  $y^{n-1} = F^n(y^n)$ ,  $y^{n-2} = F^{n-1}(y^{n-1})$ , ...,  $y^0 = F^1(y^1)$ . If there exists an  $i \in \{2, \dots, n\}$  such that  $F^i(y^i) \notin \text{dom } F^{i-1}$  or  $F^1(y^1) \notin \text{dom } f$  or there exists an  $i \in \{1, \dots, n\}$  such that  $F^i(y^i) = +\infty_{K_{i-1}}$ , then it obviously holds  $f((F^1 \circ \dots \circ F^n)(y^n)) = +\infty \geq v(\tilde{P}^C)$ . Otherwise it holds  $F^i(y^i) - y^{i-1} = 0 \in -K_{i-1}$  for  $i = 1, \dots, n$ . Moreover, by the feasibility of  $y^n$  it holds  $g(y^n) \in -Q$ , which implies the feasibility of  $(y^0, \dots, y^n)$  to the problem  $(\tilde{P}^C)$  and  $f((F^1 \circ \dots \circ F^n)(y^n)) = f(y^0) = \tilde{f}(y^0, \dots, y^n) \geq v(\tilde{P}^C)$ .

Hence it holds  $f((F^1 \circ \dots \circ F^n)(y^n)) \geq v(\tilde{P}^C)$  for all  $y^n$  feasible to  $(P^C)$ , which means that  $v(P^C) \geq v(\tilde{P}^C)$ .

Let now  $(y^0, \dots, y^n)$  be feasible to  $(\tilde{P}^C)$ . If  $y^0 \notin \text{dom } f$ , then obviously we have  $v(P^C) \leq f((F^1 \circ \dots \circ F^n)(y^n)) \leq f(y^0) = \tilde{f}(y^0, \dots, y^n) = +\infty$ . On the other hand, since  $(y^0, \dots, y^n)$  is feasible to  $(\tilde{P}^C)$  it holds  $h^i(y^i, y^{i-1}) \in -K_{i-1}$  for  $i = 1, \dots, n$  (i.e.  $F^i(y^i) - y^{i-1} \in -K_{i-1}$  for  $i = 1, \dots, n$ ) and  $g(y^n) \in -Q$ . By Lemma 3.1 we have  $v(P^C) \leq f((F^1 \circ \dots \circ F^n)(y^n)) \leq f(y^0) = \tilde{f}(y^0, \dots, y^n)$  and by taking the infimum over  $(y^0, \dots, y^n)$  on the right-hand side we get  $v(P^C) \leq v(\tilde{P}^C)$ .

Summarizing, we get the desired result  $v(P^C) = v(\tilde{P}^C)$ .  $\square$

**Remark 3.3.** The assumption that  $f$  is  $K_0$ -increasing on  $F^1(\text{dom } F^1) + K_0 \subseteq \text{dom } f$  was made to allow functions which are not necessarily monotone on their whole effective domain. But in some situations the inclusion  $F^1(\text{dom } F^1) + K_0 \subseteq \text{dom } f$  may not be fulfilled. As an example consider the convex optimization problem  $(P^G)$  in Section 3.4.

To overcome these circumstances one can alternatively assume that  $f$  is  $K_0$ -increasing on  $\text{dom } f$  and  $F^1(\text{dom } F^1) \subseteq \text{dom } f$ . For the functions  $F^1, \dots, F^{n-1}$  one can formulate in the same way alternative assumptions. To be more precise, we can alternatively ask that  $F^i$  is  $(K_i, K_{i-1})$ -increasing on  $\text{dom } F^i$  and  $F^{i+1}(\text{dom } F^{i+1}) \subseteq \text{dom } F^i$ ,  $i = 1, \dots, n-2$ , and  $F^{n-1}$  is  $(K_{n-1}, K_{n-2})$ -increasing on  $\text{dom } F^{n-1}$  and  $F^n(\text{dom } F^n \cap \mathcal{A}) \subseteq \text{dom } F^{n-1}$ . One can observe that under these alternative assumptions Lemma 3.1 and especially Theorem 3.1 still hold.

As we have seen by Theorem 3.1, the problem  $(P^C)$  can be associated to the problem  $(\tilde{P}^C)$ . In the next step we want to determine the corresponding conjugate dual problems to the problems  $(P^C)$  and  $(\tilde{P}^C)$ .

As we take a careful look at the optimization problem  $(\tilde{P}^C)$ , we can see that this problem can be rewritten in the form

$$(\tilde{P}^C) \quad \inf_{\substack{\tilde{y} \in \tilde{S}, \\ \tilde{h}(\tilde{y}) \in -\tilde{K}}} \tilde{f}(\tilde{y}), \quad (3.1)$$

where  $\tilde{y} := (y^0, \dots, y^n) \in \tilde{X} := X_0 \times \dots \times X_n$ ,  $\tilde{Z} := X_0 \times \dots \times X_{n-1} \times Z$  ordered by  $\tilde{K} := K_0 \times \dots \times K_{n-1} \times Q$ ,  $\tilde{S} := X_0 \times \dots \times X_{n-1} \times S$  and  $\tilde{h} : \tilde{X} \rightarrow \tilde{Z} = \tilde{Z} \cup \{+\infty_{\tilde{K}}\}$  is defined as

$$\tilde{h}(\tilde{y}) := \begin{cases} (h^1(y^1, y^0), \dots, h^n(y^n, y^{n-1}), g(y^n)), & \text{if } (y^i, y^{i-1}) \in \text{dom } h^i, \ i = 1, \dots, n, \\ & y^n \in \text{dom } g, \\ +\infty_{\tilde{K}} & \text{otherwise.} \end{cases}$$

Note that by the definition of  $h^i$  we have

$$\text{dom } h^i = \text{dom } F^i \times X_{i-1}, \ i = 1, \dots, n,$$

which yields

$$\text{dom } \tilde{h} = X_0 \times \text{dom } F^1 \times \dots \times (\text{dom } F^n \cap \text{dom } g). \quad (3.2)$$

At this point, let us additionally remark that the assumption from the beginning,  $S \cap g^{-1}(-Q) \cap ((F^n)^{-1} \circ \dots \circ (F^1)^{-1})(\text{dom } f) \neq \emptyset$ , implies also that  $\text{dom } \tilde{f} \cap \tilde{S} \cap \tilde{h}^{-1}(-\tilde{K}) \neq \emptyset$ , but the inverse is not true. This means

$$\begin{aligned} & S \cap g^{-1}(-Q) \cap ((F^n)^{-1} \circ \dots \circ (F^1)^{-1})(\text{dom } f) \neq \emptyset \\ \Leftrightarrow & \exists (y^0, y^1, \dots, y^{n-1}, y^n) \in \text{dom } f \times X_1 \times \dots \times X_{n-1} \times S \text{ such that} \\ & F^1(y^1) - y^0 = 0 \in -K_0, \dots, F^n(y^n) - y^{n-1} = 0 \in -K_{n-1} \text{ and } g(y^n) \in -Q \\ \Rightarrow & \exists \tilde{y} \in \tilde{S} \cap \text{dom } \tilde{f} \text{ such that } \tilde{h}(\tilde{y}) \in -\tilde{K} \\ \Leftrightarrow & \text{dom } \tilde{f} \cap \tilde{S} \cap \tilde{h}^{-1}(-\tilde{K}) \neq \emptyset. \end{aligned}$$

The corresponding Lagrange dual problem  $(\tilde{D}^C)$  with  $\tilde{z}^* := (z^{0*}, \dots, z^{(n-1)*}, z^{n*}) \in \tilde{K}^* := K_0^* \times \dots \times K_{n-1}^* \times Q^*$  as the dual variable to the problem  $(\tilde{P}^C)$  is

$$(\tilde{D}^C) \quad \sup_{\tilde{z}^* \in \tilde{K}^*} \inf_{\tilde{y} \in \tilde{S}} \{ \tilde{f}(\tilde{y}) + \langle \tilde{z}^*, \tilde{h}(\tilde{y}) \rangle \},$$

which can equivalently be written as

$$(\tilde{D}^C) \quad \sup_{\substack{z^{n*} \in Q^*, \ z^{i*} \in K_i^* \\ i=0, \dots, n-1}} \inf_{\substack{y^n \in S, \ y^i \in X_i \\ i=0, \dots, n-1}} \left\{ \tilde{f}(y^0, \dots, y^n) + \sum_{i=1}^n \langle z^{(i-1)*}, h^i(y^i, y^{i-1}) \rangle + \langle z^{n*}, g(y^n) \rangle \right\}.$$

Through the definitions we made above for  $\tilde{f}$  and  $h^i$  and since we set  $x = y^n$ , we can deduce the conjugate dual problem  $(D^C)$  to problem  $(P^C)$

$$\begin{aligned} (D^C) \quad & \sup_{\substack{z^{n*} \in Q^*, \ z^{i*} \in K_i^* \\ i=0, \dots, n-1}} \inf_{\substack{x \in S, \ y^i \in X_i \\ i=0, \dots, n-1}} \left\{ f(y^0) + \langle z^{(n-1)*}, F^n(x) - y^{n-1} \rangle + \langle z^{n*}, g(x) \rangle + \right. \\ & \left. \sum_{i=1}^{n-1} \langle z^{(i-1)*}, F^i(y^i) - y^{i-1} \rangle \right\} \\ = \quad & \sup_{\substack{z^{n*} \in Q^*, \ z^{i*} \in K_i^* \\ i=0, \dots, n-1}} \left\{ \inf_{x \in S} \{ \langle z^{(n-1)*}, F^n(x) \rangle + \langle z^{n*}, g(x) \rangle \} - \sup_{y^0 \in X_0} \{ \langle z^{0*}, y^0 \rangle - f(y^0) \} - \right. \\ & \left. \sum_{i=1}^{n-1} \sup_{y^i \in X_i} \{ \langle z^{i*}, y^i \rangle - \langle z^{(i-1)*}, F^i(y^i) \rangle \} \right\}. \end{aligned}$$

Hence, the conjugate dual problem  $(D^C)$  to problem  $(P^C)$  has the following form

$$\begin{aligned} (D^C) \quad & \sup_{\substack{z^{n*} \in Q^*, \ z^{i*} \in K_i^* \\ i=0, \dots, n-1}} \left\{ \inf_{x \in S} \{ \langle z^{(n-1)*}, F^n(x) \rangle + \langle z^{n*}, g(x) \rangle \} - f^*(z^{0*}) \right. \\ & \left. - \sum_{i=1}^{n-1} (z^{(i-1)*} F^i)^*(z^{i*}) \right\}. \end{aligned} \quad (3.3)$$

The optimal objective values of the problems  $(\tilde{D}^C)$  and  $(D^C)$  are of course equal, i.e.  $v(\tilde{D}^C) = v(D^C)$ . The next result arises from the definition of the dual problem and is always fulfilled.

**Theorem 3.2** (weak duality). *Between the primal problem  $(P^C)$  and its conjugate dual problem weak duality always holds, i.e.  $v(P^C) \geq v(D^C)$ .*

**Proof.** By Theorem 3.1.1 in [7] it holds  $v(\tilde{P}^C) \geq v(\tilde{D}^C)$ . Moreover, by Theorem 3.1 and since  $v(\tilde{D}^C) = v(D^C)$  we have  $v(P^C) = v(\tilde{P}^C) \geq v(\tilde{D}^C) = v(D^C)$ .  $\square$

**Remark 3.4.** Let  $Z_i$  be a locally convex Hausdorff space partially ordered by the non-empty convex cone  $Q_i$ ,  $i = 0, \dots, n-1$ . Then the introduced concept covers also optimization problems of the form

$$(P^{CC}) \quad \inf_{x \in \mathcal{L}} \varphi(x),$$

with

$$\mathcal{L} := \{x \in S : (G^1 \circ \dots \circ G^n)(x) \in -Q_0\},$$

where  $\varphi : X_n \rightarrow \mathbb{R}$  is proper,  $G^i : \bar{Z}_i \rightarrow \bar{Z}_{i-1}$  is proper and  $(Q_i, Q_{i-1})$ -increasing on  $G^{i+1}(\text{dom } G^{i+1}) + Q_i \subseteq \text{dom } G^i$ ,  $i = 1, \dots, n-1$ , and  $G^n : X_n \rightarrow \bar{Z}_{n-1}$  is proper. The problem  $(P^{CC})$  can equivalently be rewritten as

$$(P^{CC}) \quad \inf_{x \in X_n} \{ \varphi(x) + \delta_S(x) + (\delta_{-Q_0} \circ G^1 \circ \dots \circ G^n)(x) \}$$

and by setting  $X_0 := \mathbb{R} \times Z_0$ ,  $K_0 := \mathbb{R}_+ \times Q_0$ ,  $X_i := \mathbb{R} \times Z_i$ ,  $K_i := \mathbb{R}_+ \times Q_i$ ,  $i = 1, \dots, n-1$  and by defining the following functions

- $f : \bar{X}_0 \times \mathbb{R}, f(y^0) := y_1^0 + \delta_{-Q_0}(y_2^0)$  with  $y^0 = (y_1^0, y_2^0) \in X_0$ ,
- $F^i : \bar{X}_i \rightarrow \bar{X}_{i-1}, F^i(y_1^i, y_2^i) := (y_1^i, G^i(y_2^i))$ ,  $i = 1, \dots, n-1$  with  $y^i = (y_1^i, y_2^i) \in X_i$ ,
- $F^n : X_n \rightarrow \bar{X}_{n-1}, F^n(x) := (\varphi(x) + \delta_S(x), G^n(x))$ ,

the problem  $(P^{CC})$  turns into a special case of the problem  $(P^C)$

$$(P^{CC}) \quad \inf_{x \in X_n} (f \circ F^1 \circ \dots \circ F^n)(x)$$

with  $\mathcal{A} \equiv X_n$ .

## 3.2 Regularity conditions, strong duality and optimality conditions

In this section we want to characterize strong duality through the so-called generalized interior point regularity conditions. Besides we provide some optimality conditions for the primal problem and its corresponding conjugate dual problem. For this purpose we additionally assume for the rest of this chapter that  $S \subseteq X_n$  is a convex set,  $f$  is a convex function,  $F^i$  is a  $K_{i-1}$ -convex function for  $i = 1, \dots, n$  and  $g$  is a  $Q$ -convex function. Hence, as can be easily seen,  $(f \circ F^1 \circ \dots \circ F^n)$  is a convex function and  $(P^C)$  is a convex optimization problem. Moreover, the problem  $(\tilde{P}^C)$  is also convex.

**Remark 3.5.** Let us point out that for the convexity of  $(f \circ F^1 \circ \dots \circ F^n)$  we ask that the function  $f$  is convex and  $K_0$ -increasing on  $F^1(\text{dom } F^1) + K_0$  and the function  $F^i$  is  $K_{i-1}$ -convex and fulfills also the property of monotonicity for  $i = 1, \dots, n-1$ , while the function  $F^n$  need just be  $K_{n-1}$ -convex (see Theorem 2.2). This means that if  $F^n$  is an affine function, we do not need the monotonicity of  $F^{n-1}$ , since the composition of an affine function and a function, which fulfills the property of convexity, fulfills also the property of convexity. In this context let us pay also attention to Remark 3.2, i.e. one can choose  $K_{n-1} = \{0_{X_{n-1}}\}$ .

To derive regularity conditions which secure strong duality for the pair  $(P^C)$ -( $D^C$ ), we first consider regularity conditions for strong duality between the problems  $(\tilde{P}^C)$  and  $(\tilde{D}^C)$ , which were presented in [7]. The first one is the well-known *Slater constraint qualification*

$$(\widetilde{RC}_1^C) \quad \left| \quad \exists \tilde{y}' \in \text{dom } \tilde{f} \cap \tilde{S} \text{ such that } \tilde{h}(\tilde{y}') \in -\text{int } \tilde{K}. \right.$$

Using the definitions of  $\tilde{f}$  and  $\tilde{h}$  as well as  $\tilde{S}$  and  $\tilde{K}$  we get

$$\begin{aligned} \text{dom } \tilde{f} \cap \tilde{S} &= (\text{dom } f \times X_1 \times \dots \times X_n) \cap (X_0 \times X_1 \times \dots \times X_{n-1} \times S) \\ &= \text{dom } f \times X_1 \times \dots \times X_{n-1} \times S \end{aligned} \tag{3.4}$$

and

$$\text{int } \tilde{K} = \text{int}(K_0 \times \dots \times K_{n-1} \times Q) = \text{int } K_0 \times \dots \times \text{int } K_{n-1} \times \text{int } Q.$$

Therefore the condition  $(\widetilde{RC}_1^C)$  can in the context of the primal-dual pair  $(P^C)$ -( $D^C$ ) be rewritten as follows

$$(RC_1^C) \quad \left| \begin{array}{l} \exists (y^{0'}, y^{1'}, \dots, y^{(n-1)'}, y^{n'}) \in \text{dom } f \times X_1 \times \dots \times X_{n-1} \times S \text{ such that} \\ F^i(y^{i'}) - y^{(i-1)'} \in -\text{int } K_{i-1}, \quad i = 1, \dots, n, \text{ and } g(y^{n'}) \in -\text{int } Q. \end{array} \right.$$

The condition  $(RC_1^C)$  can also equivalently be formulated as

$$(RC_{1'}^C) \quad \left| \begin{array}{l} \exists x' \in S \text{ such that } g(x') \in -\text{int } Q \text{ and } F^n(x') \in (F^{n-1})^{-1}((F^{n-2})^{-1}(\dots \\ (F^1)^{-1}(\text{dom } f - \text{int } K_0) - \text{int } K_1 \dots) - \text{int } K_{n-2}) - \text{int } K_{n-1}. \end{array} \right.$$

This can be seen as follows: The assumption that there exists  $x' \in S$  such that

$$F^n(x') \in (F^{n-1})^{-1}((F^{n-2})^{-1}(\dots(F^1)^{-1}(\text{dom } f - \text{int } K_0) - \text{int } K_1 \dots) - \text{int } K_{n-2}) - \text{int } K_{n-1}$$

implies that there exists  $(y^{0'}, \dots, y^{(n-1)'}) \in X_0 \times \dots \times X_{n-1}$  such that

$$\begin{aligned} y^{(n-1)'} &\in (F^{n-1})^{-1}((F^{n-2})^{-1}(\dots(F^1)^{-1}(\text{dom } f - \text{int } K_0) - \text{int } K_1 \dots) - \text{int } K_{n-2}) \\ y^{(n-2)'} &\in (F^{n-2})^{-1}((F^{n-3})^{-1}(\dots(F^1)^{-1}(\text{dom } f - \text{int } K_0) - \text{int } K_1 \dots) - \text{int } K_{n-3}) \\ &\vdots \\ y^{1'} &\in (F^1)^{-1}(\text{dom } f - \text{int } K_0) \\ y^{0'} &\in \text{dom } f. \end{aligned}$$

Therefore, by setting  $x' = y^{n'}$  the elements  $(y^{0'}, \dots, y^{n'}) \in \text{dom } f \times X_1 \times \dots \times X_{n-1} \times S$  fulfill  $F^n(y^{n'}) - y^{(n-1)'} \in -\text{int } K_{n-1}$ , ...,  $F^1(y^{1'}) - y^{0'} \in -\text{int } K_0$  and from here we can now affirm that the condition  $(RC_{1'}^C)$  is fulfilled.

On the other hand, if there exists  $(y^{0'}, \dots, y^{n'}) \in \text{dom } f \times X_1 \times \dots \times X_{n-1} \times S$  such that  $g(y^{n'}) \in -\text{int } Q$  and  $F^i(y^{i'}) - y^{(i-1)'} \in -\text{int } K_{i-1}$  for  $i = 1, \dots, n$ , then we set  $y^{n'} = x'$  and get

$$F^n(x') - y^{(n-1)'} \in -\text{int } K_{n-1} \Rightarrow F^n(x') \in y^{(n-1)'} - \text{int } K_{n-1}. \quad (3. 5)$$

Further, we have

$$\begin{aligned} F^{n-1}(y^{(n-1)'}) - y^{(n-2)'} &\in -\text{int } K_{n-2} \Rightarrow F^{n-1}(y^{(n-1)'}) \in y^{(n-2)'} - \text{int } K_{n-2} \\ &\Rightarrow y^{(n-1)'} \in (F^{n-1})^{-1}(y^{(n-2)'} - \text{int } K_{n-2}). \end{aligned} \quad (3. 6)$$

From (3. 5) and (3. 6) follows

$$F^n(x') \in (F^{n-1})^{-1}(y^{(n-2)'} - \text{int } K_{n-2}) - \text{int } K_{n-1}. \quad (3. 7)$$

Since

$$\begin{aligned} F^{n-2}(y^{(n-2)'}) - y^{(n-3)'} &\in -\text{int } K_{n-3} \Rightarrow F^{n-2}(y^{(n-2)'}) \in y^{(n-3)'} - \text{int } K_{n-3} \\ &\Rightarrow y^{(n-2)'} \in (F^{n-2})^{-1}(y^{(n-3)'} - \text{int } K_{n-3}) \end{aligned}$$

we get for (3. 7)

$$F^n(x') \in (F^{n-1})^{-1}((F^{n-2})^{-1}(y^{(n-3)'} - \text{int } K_{n-3}) - \text{int } K_{n-2}) - \text{int } K_{n-1}.$$

If we continue in this manner until  $y^{0'} \in \text{dom } f$  we get finally

$$F^n(x') \in (F^{n-1})^{-1}((F^{n-2})^{-1}(\dots(F^1)^{-1}(\text{dom } f - \text{int } K_0) - \text{int } K_1 \dots) - \text{int } K_{n-2}) - \text{int } K_{n-1}.$$

This means that  $(RC_{1'}^C)$  is equivalent to  $(RC_1^C)$ . Additionally, we consider a class of regularity conditions which assume that the underlying spaces are Fréchet spaces:

$$(\widetilde{RC}_2^C) \quad \left| \begin{array}{l} \widetilde{X} \text{ and } \widetilde{Z} \text{ are Fréchet spaces, } \widetilde{S} \text{ is closed, } \widetilde{f} \text{ is lower semicontinuous,} \\ \widetilde{h} \text{ is } \widetilde{K}\text{-epi closed and } 0_{\widetilde{Z}} \in \text{sqli}(\widetilde{h}(\text{dom } \widetilde{f} \cap \widetilde{S} \cap \text{dom } \widetilde{h}) + \widetilde{K}). \end{array} \right.$$

If we exchange sqri for core or int we get stronger versions of this regularity condition:

$$\begin{aligned} (\widetilde{RC}_{2'})^C & \left| \begin{array}{l} \tilde{X} \text{ and } \tilde{Z} \text{ are Fréchet spaces, } \tilde{S} \text{ is closed, } \tilde{f} \text{ is lower semicontinuous,} \\ \tilde{h} \text{ is } \tilde{K}\text{-epi closed and } 0_{\tilde{Z}} \in \text{core}(\tilde{h}(\text{dom } \tilde{f} \cap \tilde{S} \cap \text{dom } \tilde{h}) + \tilde{K}), \end{array} \right. \\ (\widetilde{RC}_{2''})^C & \left| \begin{array}{l} \tilde{X} \text{ and } \tilde{Z} \text{ are Fréchet spaces, } \tilde{S} \text{ is closed, } \tilde{f} \text{ is lower semicontinuous,} \\ \tilde{h} \text{ is } \tilde{K}\text{-epi closed and } 0_{\tilde{Z}} \in \text{int}(\tilde{h}(\text{dom } \tilde{f} \cap \tilde{S} \cap \text{dom } \tilde{h}) + \tilde{K}), \end{array} \right. \end{aligned}$$

where the last two conditions are equivalent (see [7]). If we work in finite dimensional spaces the regularity condition  $(\widetilde{RC}_2)^C$  can be written in the following way (see [7])

$$(\widetilde{RC}_3)^C \left| \begin{array}{l} \dim(\text{lin}(\tilde{h}(\text{dom } \tilde{f} \cap \tilde{S} \cap \text{dom } \tilde{h}) + \tilde{K})) < +\infty \text{ and} \\ 0_{\tilde{Z}} \in \text{ri}(\tilde{h}(\text{dom } \tilde{f} \cap \tilde{S} \cap \text{dom } \tilde{h}) + \tilde{K}). \end{array} \right.$$

To derive corresponding regularity conditions for the primal-dual pair  $(P^C)-(D^C)$  formulated with the involved functions we first consider the formulae (3.2) and (3.4), which imply that

$$\begin{aligned} & \tilde{h}(\text{dom } \tilde{f} \cap \tilde{S} \cap \text{dom } \tilde{h}) \\ = & \tilde{h}(\text{dom } f \times \text{dom } F^1 \times \dots \times \text{dom } F^{n-1} \times (\text{dom } F^n \cap \text{dom } g \cap S)) \\ = & h^1(\text{dom } F^1 \times \text{dom } f) \times h^2(\text{dom } F^2 \times \text{dom } F^1) \times \dots \times \\ & h^{n-1}(\text{dom } F^{n-1} \times \text{dom } F^{n-2}) \times \\ & h^n((\text{dom } F^n \cap \text{dom } g \cap S) \times \text{dom } F^{n-1}) \times g(\text{dom } F^n \cap \text{dom } g \cap S) \\ = & (F^1(\text{dom } F^1) - \text{dom } f) \times (F^2(\text{dom } F^2) - \text{dom } F^1) \times \dots \times \\ & (F^{n-1}(\text{dom } F^{n-1}) - \text{dom } F^{n-2}) \times \\ & (F^n(\text{dom } F^n \cap \text{dom } g \cap S) - \text{dom } F^{n-1}) \times g(\text{dom } F^n \cap \text{dom } g \cap S) \end{aligned}$$

and from here we get by Lemma 2.1 that

$$\begin{aligned} 0_{\tilde{Z}} \in \text{sqri} & \left( (F^1(\text{dom } F^1) - \text{dom } f + K_0) \times \dots \right. \\ & \times (F^{n-1}(\text{dom } F^{n-1}) - \text{dom } F^{n-2} + K_{n-2}) \\ & \times (F^n(\text{dom } F^n \cap \text{dom } g \cap S) - \text{dom } F^{n-1} + K_{n-1}) \\ & \left. \times (g(\text{dom } F^n \cap \text{dom } g \cap S) + Q) \right) \end{aligned}$$

is equivalent to

$$\begin{aligned} 0_{X_0} & \in \text{sqri}(F^1(\text{dom } F^1) - \text{dom } f + K_0), \\ 0_{X_i} & \in \text{sqri}(F^i(\text{dom } F^i) - \text{dom } F^{i-1} + K_{i-1}), \quad i = 2, \dots, n-1, \\ 0_{X_n} & \in \text{sqri}(F^n(\text{dom } F^n \cap \text{dom } g \cap S) - \text{dom } F^{n-1} + K_{n-1}) \text{ and} \\ 0_Z & \in \text{sqri}(g(\text{dom } F^n \cap \text{dom } g \cap S) + Q). \end{aligned}$$

Now, let  $\varrho : X_0 \times \dots \times X_n \times X_0 \times \dots \times X_{n-1} \times Z \rightarrow X_0^2 \times \dots \times X_{n-1}^2 \times X_n \times Z$  be defined by  $\varrho(y^0, \dots, y^n, v^0, \dots, v^n) := (y^0, v^0, \dots, y^n, v^n)$ . Further, let us define the functions  $\varrho_{X_i}^n : X_i \times X_{i-1} \times X_{i-1} \rightarrow X_{i-1} \times X_{i-1} \times X_i$  by  $\varrho_{X_i}^n(y^i, y^{i-1}, v^{i-1}) := (y^{i-1}, v^{i-1}, y^i)$ ,  $i = 1, \dots, n$ . Obviously, the defined functions are homeomorphisms and map open sets into open sets and closed sets into closed sets. More precisely, this means that  $\varrho(\text{epi}_{\tilde{K}} \tilde{h})$  is closed if and only if  $\text{epi}_{\tilde{K}} \tilde{h}$  is a closed set and  $\varrho_{X_i}^n(\text{epi}_{K_{i-1}} h^i)$  is closed if and only if  $\text{epi}_{K_{i-1}} h^i$  is a closed set,  $i = 1, \dots, n$ . Furthermore, we have

$$\begin{aligned}
\text{epi}_{\tilde{K}} \tilde{h} &= \{(y^0, \dots, y^n, v^0, \dots, v^n) \in X_0 \times \dots \times X_n \times X_0 \times \dots \times X_{n-1} \times Z : \\
&\quad (y^1, y^0, v^0) \in \text{epi}_{K_0} h^1, \\
&\quad \vdots \\
&\quad (y^n, y^{n-1}, v^{n-1}) \in \text{epi}_{K_{n-1}} h^n, \\
&\quad (y^n, v^n) \in \text{epi}_Q g\} \\
&= \{(y^0, \dots, y^n, v^0, \dots, v^n) \in X_0 \times \dots \times X_n \times X_0 \times \dots \times X_{n-1} \times Z : \\
&\quad (y^0, v^0, y^1) \in \varrho_{X_1}^n(\text{epi}_{K_0} h^1), \\
&\quad \vdots \\
&\quad (y^{n-1}, v^{n-1}, y^n) \in \varrho_{X_n}^n(\text{epi}_{K_{n-1}} h^n), \\
&\quad (y^n, v^n) \in \text{epi}_Q g\} \\
&= \left\{ (y^0, \dots, y^n, v^0, \dots, v^n) \in X_0 \times \dots \times X_n \times X_0 \times \dots \times X_{n-1} \times Z : \right. \\
&\quad (y^0, v^0, y^1, v^1, y^2, v^2, \dots, y^{n-1}, v^{n-1}, y^n, v^n) \in \\
&\quad \left( \varrho_{X_1}^n(\text{epi}_{K_0} h^1) \right) \times X_1 \times X_2^2 \times \dots \times X_{n-1}^2 \times X_n \times Z, \\
&\quad \vdots \\
&\quad (y^0, v^0, y^1, v^1, \dots, y^{n-2}, v^{n-2}, y^{n-1}, v^{n-1}, y^n, v^n) \in \\
&\quad X_0^2 \times X_1^2 \times \dots \times X_{n-2}^2 \times \left( \varrho_{X_n}^n(\text{epi}_{K_{n-1}} h^n) \right) \times Z, \\
&\quad (y^0, v^0, \dots, y^{n-1}, v^{n-1}, y^n, v^n) \in X_0^2 \times \dots \times X_{n-1}^2 \times \text{epi}_Q g \left. \right\} \\
&= \left\{ (y^0, \dots, y^n, v^0, \dots, v^n) \in X_0 \times \dots \times X_n \times X_0 \times \dots \times X_{n-1} \times Z : \right. \\
&\quad (y^0, v^0, \dots, y^{i-2}, v^{i-2}, y^{i-1}, v^{i-1}, y^i, v^i, y^{i+1}, v^{i+1}, \dots, y^n, v^n) \in \\
&\quad X_0^2 \times \dots \times X_{i-2}^2 \times \left( \varrho_{X_i}^n(\text{epi}_{K_{i-1}} h^i) \right) \times X_i \times X_{i+1}^2 \times \dots \times X_n \times Z, \quad i = 1, \dots, n, \\
&\quad (y^0, v^0, \dots, y^{n-1}, v^{n-1}, y^n, v^n) \in X_0^2 \times \dots \times X_{n-1}^2 \times \text{epi}_Q g \left. \right\},
\end{aligned}$$

so we can write

$$\begin{aligned}
\varrho(\text{epi}_{\tilde{K}} \tilde{h}) &= \left( \bigcap_{i=1}^n \left( X_0^2 \times \dots \times X_{i-2}^2 \times \left( \varrho_{X_i}^n(\text{epi}_{K_{i-1}} h^i) \right) \times X_i \times X_{i+1}^2 \times \dots \times X_n \times Z \right) \right) \\
&\quad \bigcap \left( X_0^2 \times \dots \times X_{n-1}^2 \times \text{epi}_Q g \right)
\end{aligned}$$

and get as a consequence that  $\text{epi}_{\tilde{K}} \tilde{h}$  is closed if  $\text{epi}_{K_{i-1}} h^i$ ,  $i = 1, \dots, n$ , and  $\text{epi}_Q g$  are closed sets.

Vice versa, let  $\text{epi}_{\tilde{K}} \tilde{h}$  be closed, i.e.  $\varrho(\text{epi}_{\tilde{K}} \tilde{h})$  is closed, and

$$(y_\alpha^1, y_\alpha^0, v_\alpha^0)_\alpha \subseteq \text{epi}_{K_0} h^1, \dots, (y_\alpha^n, y_\alpha^{n-1}, v_\alpha^{n-1})_\alpha \subseteq \text{epi}_{K_{n-1}} h^n \text{ and } (y_\alpha^n, v_\alpha^n)_\alpha \subseteq \text{epi}_Q g,$$

i.e.  $(y_\alpha^0, \dots, y_\alpha^n, v_\alpha^0, \dots, v_\alpha^n)_\alpha \subseteq \text{epi}_{\tilde{K}} \tilde{h}$ . As  $\text{epi}_{\tilde{K}} \tilde{h}$  is closed, we have that  $(y_\alpha^0, \dots, y_\alpha^n, v_\alpha^0, \dots, v_\alpha^n)_\alpha \rightarrow (y^0, \dots, y^n, v^0, \dots, v^n) \in \text{epi}_{\tilde{K}} \tilde{h}$ , but this means that

$$(y^1, y^0, v^0) \in \text{epi}_{K_0} h^1, \dots, (y^n, y^{n-1}, v^{n-1}) \in \text{epi}_{K_{n-1}} h^n \text{ and } (y^n, v^n) \in \text{epi}_Q g,$$

which implies the closedness of  $\text{epi}_{K_0} h^1, \dots, \text{epi}_{K_{n-1}} h^n$  and  $\text{epi}_Q g$ .



Besides, we know by Lemma 2.3 that for a non-empty closed convex cone  $K_{i-1}$  it holds that  $\text{epi}_{K_{i-1}} h^i$  is closed if and only if  $\text{epi}_{K_{i-1}} F^i$  is closed,  $i = 1, \dots, n$ . Bringing now the last facts together implies that for non-empty closed convex cones  $K_{i-1}$ ,  $i = 1, \dots, n$ , it holds that  $\text{epi}_{\tilde{K}} \tilde{h}$  is closed if and only if  $\text{epi}_Q g$  and  $\text{epi}_{K_{i-1}} F^i$  are closed sets,  $i = 1, \dots, n$ .

Moreover, since  $\tilde{S}$  is closed if and only if  $S$  is closed and  $\tilde{f}$  is lower semicontinuous if and only if  $f$  is lower semicontinuous (follows from the fact that  $\text{epi } f$  is closed if and only if  $\text{epi } \tilde{f}$  is closed), we get the following regularity condition for the primal-dual pair  $(P^C)-(D^C)$  (note that if  $X_i$  is a Fréchet space,  $i = 0, \dots, n$ , then  $\tilde{X} = X_0 \times \dots \times X_n$  is a Fréchet space, too)

$$(RC_2^C) \quad \left| \begin{array}{l} X_0, \dots, X_n \text{ and } Z \text{ are Fréchet spaces, } f \text{ is l.s.c., } S \text{ is closed, } g \text{ is } Q\text{-epi,} \\ \text{closed, } K_{i-1} \text{ is closed, } F^i \text{ is } K_{i-1}\text{-epi closed, } i = 1, \dots, n, \\ 0_{X_0} \in \text{sqri}(F^1(\text{dom } F^1) - \text{dom } f + K_0), \\ 0_{X_{i-1}} \in \text{sqri}(F^i(\text{dom } F^i) - \text{dom } F^{i-1} + K_{i-1}), i = 2, \dots, n-1, \\ 0_{X_{n-1}} \in \text{sqri}(F^n(\text{dom } F^n \cap \text{dom } g \cap S) - \text{dom } F^{n-1} + K_{n-1}) \text{ and} \\ 0_Z \in \text{sqri}(g(\text{dom } F^n \cap \text{dom } g \cap S) + Q). \end{array} \right.$$

In the same way we get equivalent formulations of the regularity conditions  $(RC_{2'}^C)$  and  $(RC_{2''}^C)$  using core and int, respectively, instead sqri. The same holds also for the condition  $(RC_3^C)$ .

As we have seen, the condition  $(RC_i^C)$  is equivalent to  $(\widetilde{RC}_i^C)$ ,  $i \in \{1, 2, 2', 2'', 3\}$ . Moreover, since on the one hand Theorem 3.1 is always fulfilled and on the other hand the optimal objective values between  $(\tilde{D}^C)$  and  $(D^C)$  are equal, it holds the following theorem (see Theorem 3.2.9 and 3.2.10 in [7]).

**Theorem 3.3** (strong duality). *If one of the conditions  $(RC_i^C)$ ,  $i \in \{1, 1', 2, 2', 2'', 3\}$ , is fulfilled, then between  $(P^C)$  and  $(D^C)$  strong duality holds, i.e.  $v(P^C) = v(D^C)$  and the conjugate dual problem has an optimal solution.*

**Remark 3.6.** *If for some  $i \in \{1, \dots, n\}$  the function  $F^i$  is positively  $K_{i-1}$ -lower semicontinuous, then we can omit asking that  $F^i$  is  $K_{i-1}$ -epi closed in the regularity conditions  $(RC_i^C)$ ,  $i \in \{2, 2', 2''\}$ , because the positive  $K_{i-1}$ -lower semicontinuity of  $F^i$  implies the positive  $K_{i-1}$ -lower semicontinuity of  $h^i$ , which then implies the  $K_{i-1}$ -epi closedness of  $h^i$ .*

**Remark 3.7.** *Besides the used regularity conditions there are also the so-called closedness type conditions guaranteeing strong duality. Such regularity conditions were studied in different contexts, like strong duality, subdifferential calculus etc. (see [22]). These types of regularity conditions were also studied in [42] and [41], to cite only few of them.*

*We have also extensively studied closedness type conditions in the context of multi-composed optimization problems with the focus on stable strong duality and  $\epsilon$ -optimality conditions in our article [49]. As applications we considered problems from fractional programming and entropy optimization (see also [3, 4, 6, 25, 32, 46, 73]).*

We come now to the point where we can give necessary and sufficient optimality conditions for the primal-dual pair  $v(P^C)-v(D^C)$ .

**Theorem 3.4** (optimality conditions). *(a) Suppose that one of the regularity conditions  $(RC_i^C)$ ,  $i \in \{1, 1', 2, 2', 2'', 3\}$ , is fulfilled and let  $\bar{x} \in \mathcal{A}$  be an optimal solution of the problem  $(P^C)$ . Then there exists  $(\bar{z}^{0*}, \dots, \bar{z}^{(n-1)*}, \bar{z}^{n*}) \in K_0^* \times \dots \times K_{n-1}^* \times Q^*$ , an optimal solution to  $(D^C)$ , such that*

- (i)  $f((F^1 \circ \dots \circ F^n)(\bar{x})) + f^*(\bar{z}^{0*}) = \langle \bar{z}^{0*}, (F^1 \circ \dots \circ F^n)(\bar{x}) \rangle$ ,
- (ii)  $(\bar{z}^{(i-1)*} F^i)((F^{i+1} \circ \dots \circ F^n)(\bar{x})) + (\bar{z}^{(i-1)*} F^i)^*(\bar{z}^{i*}) = \langle \bar{z}^{i*}, (F^{i+1} \circ \dots \circ F^n)(\bar{x}) \rangle$ ,  $i = 1, \dots, n-1$ ,
- (iii)  $(\bar{z}^{(n-1)*} F^n)(\bar{x}) + (\bar{z}^{n*} g)(\bar{x}) + ((\bar{z}^{(n-1)*} F^n) + (\bar{z}^{n*} g))_S^*(0_{X_n^*}) = 0$ ,
- (iv)  $\langle \bar{z}^{n*}, g(\bar{x}) \rangle = 0$ .

(b) If there exists  $\bar{x} \in \mathcal{A}$  such that for some  $(\bar{z}^{0*}, \dots, \bar{z}^{(n-1)*}, \bar{z}^{n*}) \in K_0^* \times \dots \times K_{n-1}^* \times Q^*$  the conditions (i)-(iv) are fulfilled, then  $\bar{x}$  is an optimal solution of  $(P^C)$ ,  $(\bar{z}^{0*}, \dots, \bar{z}^{n*})$  is an optimal solution for  $(D^C)$  and  $v(P^C) = v(D^C)$ .

**Proof.** First, we consider part (a). By Theorem 3.3, strong duality holds for the primal-dual pair  $(P^C)$ -( $D^C$ ), which means that there exists  $(\bar{z}^{0*}, \dots, \bar{z}^{(n-1)*}, \bar{z}^{n*}) \in K_0^* \times \dots \times K_{n-1}^* \times Q^*$ , an optimal solution to  $(D^C)$ , such that the following equality holds

$$\begin{aligned} & (f \circ F^1 \circ \dots \circ F^n)(\bar{x}) \\ &= \inf_{x \in S} \{ \langle \bar{z}^{(n-1)*}, F^n(x) \rangle + \langle \bar{z}^{n*}, g(x) \rangle \} - f^*(\bar{z}^{0*}) - \sum_{i=1}^{n-1} (\bar{z}^{(i-1)*} F^i)^*(\bar{z}^{i*}). \end{aligned}$$

Furthermore, since by definition it holds

$$\begin{aligned} & \sum_{i=1}^n (\bar{z}^{(i-1)*} F^i)((F^{i+1} \circ \dots \circ F^n)(\bar{x})) \\ &= \langle \bar{z}^{0*}, (F^1 \circ \dots \circ F^n)(\bar{x}) \rangle + \sum_{i=1}^{n-1} \langle \bar{z}^{i*}, (F^{i+1} \circ \dots \circ F^n)(\bar{x}) \rangle, \end{aligned}$$

the assertions (i)-(iv) can be deduced immediately by the following consideration

$$\begin{aligned} & (f \circ F^1 \circ \dots \circ F^n)(\bar{x}) + f^*(\bar{z}^{0*}) + \sum_{i=1}^{n-1} (\bar{z}^{(i-1)*} F^i)^*(\bar{z}^{i*}) + \\ & ((\bar{z}^{(n-1)*} F^n) + (\bar{z}^{n*} g))_S^*(0_{X_n^*}) = 0 \\ \Leftrightarrow & (f \circ F^1 \circ \dots \circ F^n)(\bar{x}) + f^*(\bar{z}^{0*}) + \sum_{i=1}^{n-1} (\bar{z}^{(i-1)*} F^i)^*(\bar{z}^{i*}) \\ & + ((\bar{z}^{(n-1)*} F^n) + (\bar{z}^{n*} g))_S^*(0_{X_n^*}) + (\bar{z}^{n*} g)(\bar{x}) - \langle \bar{z}^{n*}, g(\bar{x}) \rangle \\ & + \sum_{i=1}^n (\bar{z}^{(i-1)*} F^i)((F^{i+1} \circ \dots \circ F^n)(\bar{x})) \\ & - \langle \bar{z}^{0*}, (F^1 \circ \dots \circ F^n)(\bar{x}) \rangle - \sum_{i=1}^{n-1} \langle \bar{z}^{i*}, (F^{i+1} \circ \dots \circ F^n)(\bar{x}) \rangle = 0 \\ \Leftrightarrow & [(f \circ F^1 \circ \dots \circ F^n)(\bar{x}) + f^*(\bar{z}^{0*}) - \langle \bar{z}^{0*}, (F^1 \circ \dots \circ F^n)(\bar{x}) \rangle] + \\ & \sum_{i=1}^{n-1} [(\bar{z}^{(i-1)*} F^i)((F^{i+1} \circ \dots \circ F^n)(\bar{x})) + (\bar{z}^{(i-1)*} F^i)^*(\bar{z}^{i*}) - \langle \bar{z}^{i*}, (F^{i+1} \circ \dots \circ F^n)(\bar{x}) \rangle] \\ & + [(\bar{z}^{(n-1)*} F^n)(\bar{x}) + (\bar{z}^{n*} g)(\bar{x}) + ((\bar{z}^{(n-1)*} F^n) + (\bar{z}^{n*} g))_S^*(0_{X_n^*})] \\ & + [-\langle \bar{z}^{n*}, g(\bar{x}) \rangle] = 0. \end{aligned}$$

By the Young-Fenchel inequality and the constraints of the primal and dual problem, all the terms within the brackets are non-negative and consequently must be equal to zero.

Concerning the proof of part (b) we observe that all considerations and calculations within the proof of part (a) can be done in the reverse direction.  $\square$

**Remark 3.8.** The conditions (i)-(iv) can equivalently be expressed as

- (i)  $\bar{z}^{0*} \in \partial f((F^1 \circ \dots \circ F^n)(\bar{x}))$ ,
- (ii)  $\bar{z}^{i*} \in \partial(\bar{z}^{(i-1)*} F^i)((F^{i+1} \circ \dots \circ F^n)(\bar{x}))$ ,  $i = 1, \dots, n-1$ ,
- (iii)  $0_{X_n^*} \in \partial((\bar{z}^{(n-1)*} F^n) + (\bar{z}^{n*} g) + \delta_S)(\bar{x})$ ,

$$(iv) \langle \bar{z}^{n*}, g(\bar{x}) \rangle = 0.$$

**Remark 3.9.** In fact, we can write every optimization problem in the form of a multi-composed problem by splitting the objective function into  $n + 1$  composed functions. Allowing  $n$  to be greater than one, as considered until now in the literature, is very fruitful for dealing with a big variety of optimization problems because many such problems cannot be described with an objective function composed by only two functions (case  $n = 1$ , cf. the optimization problem presented in Section 3.4). The problems and results in several papers (cf. [7, 8, 17, 20, 21, 23, 55]) turn out to be special cases of our approach for the situation when only two functions are composed in the objective function, i.e.  $n = 1$ . For this reason our analysis sums up the approaches and the results in the mentioned papers into one unified theory. Similarly to the difficulties of the usual perturbation theory, where every perturbation is accompanied by additional conditions, like monotonicity and convexity, needed for strong duality, we derive situations in our approach, where for each additional splitting function involved in the multi-composed objective function, also additional properties must be fulfilled to ensure strong duality.

But viewed from the other side, one can derive a conjugate dual problem, where the associated objective function is easier to handle due to the fact that the conjugate functions of the functions involved in the objective function of the primal problem are split. Furthermore, one can derive by the corresponding optimality conditions listed above a more detailed characterization of the set of optimal solutions of the dual and the primal problems as well as their relationship to each other. Such information can be used on the one hand to give a geometrical interpretation of the dual problem and on the other hand to improve the approach of optimization problems from the numerical point of view. In Chapter 4 we consider in this context minimax location problems as an application of the presented theory.

### 3.3 The conjugate function of a multi-composed function

Before we continue with our further approach we want to calculate the conjugate of the function  $(f \circ F^1 \circ \dots \circ F^n)$ , or, to be more precise, we determine the function

$$\gamma(x) = (f \circ F^1 \circ \dots \circ F^n)(x), \quad x \in X_n,$$

its conjugate function

$$\gamma^*(x^*) = \sup_{x \in X_n} \{ \langle x^*, x \rangle - (f \circ F^1 \circ \dots \circ F^n)(x) \}, \quad x^* \in X_n^*.$$

With this in mind, we consider for fixed  $x^* \in X_n^*$  the problem

$$(P^K) \quad \inf_{x \in X_n} \{ (f \circ F^1 \circ \dots \circ F^n)(x) - \langle x^*, x \rangle \}$$

and the equivalent primal problem

$$(\tilde{P}^K) \quad \inf_{\substack{(y^0, \dots, y^n) \in X_0 \times \dots \times X_n, \\ F^i(y^i) - y^{i-1} \in -K_{i-1}, \quad i=1, \dots, n}} \{ \tilde{f}(y^0, y^1, \dots, y^n) - \langle x^*, y^n \rangle \}.$$

In the same way like in the proof of Theorem 3.1 one can show that it holds  $v(P^K) = v(\tilde{P}^K)$  (where  $v(P^K)$  and  $v(\tilde{P}^K)$  denote the optimal objective values of the problems  $(P^K)$  and  $(\tilde{P}^K)$ , respectively). The corresponding Lagrange dual problem to problem  $(\tilde{P}^K)$  looks like

$$(\tilde{D}^K) \quad \sup_{\substack{z^{i*} \in K_i^*, \\ i=0, \dots, n-1}} \inf_{\substack{y^i \in X_i, \\ i=0, \dots, n}} \left\{ \tilde{f}(y^0, y^1, \dots, y^n) + \sum_{i=1}^n \langle z^{(i-1)*}, F^i(y^i) - y^{i-1} \rangle - \langle x^*, y^n \rangle \right\}$$

$$\begin{aligned}
&= \sup_{\substack{z^{i*} \in K_i^*, \\ i=0, \dots, n-1}} \left\{ - \sup_{y^0 \in X_0} \{ \langle z^{0*}, y^0 \rangle - f(y^0) \} - \right. \\
&\quad \left. \sup_{\substack{y^n \in S, \ y^i \in X_i, \\ i=1, \dots, n-1}} \left\{ \sum_{i=1}^{n-1} \langle z^{i*}, y^i \rangle + \langle x^*, y^n \rangle - \sum_{i=1}^n \langle z^{(i-1)*}, F^i(y^i) \rangle \right\} \right\} \\
&= \sup_{\substack{z^{i*} \in K_i^*, \\ i=0, \dots, n-1}} \left\{ - f^*(z^{0*}) - (z^{(n-1)*} F^n)^*(x^*) - \sum_{i=1}^{n-1} (z^{(i-1)*} F^i)^*(z^{i*}) \right\}.
\end{aligned}$$

Hence, we define the conjugate dual problem corresponding to the primal problem  $(P^K)$  as

$$(D^K) \quad \sup_{\substack{z^{i*} \in K_i^*, \\ i=0, \dots, n-1}} \left\{ - f^*(z^{0*}) - (z^{(n-1)*} F^n)^*(x^*) - \sum_{i=1}^{n-1} (z^{(i-1)*} F^i)^*(z^{i*}) \right\}.$$

Let us notice that for all  $x^* \in X_n^*$  one has  $\text{dom } \tilde{f} = \text{dom}(\tilde{f} + \langle x^*, \cdot \rangle)$ . To guarantee strong duality between the problem  $(P^K)$  and its conjugate dual problem  $(D^K)$ , we use the regularity conditions we introduced above. Therefore, we set  $Z = X$  ordered by the trivial cone  $Q = X$  and define the function  $g : X \rightarrow X$  by  $g(x) := x$  such that  $g$  is  $Q$ -epi closed and

$$0_X \in \text{sqli}(g(X) + Q) = \text{sqli}(X + Q) = X.$$

Hence, we get for the pair  $(P^K)$ -( $D^K$ ) the following regularity conditions. The first one looks like

$$(RC_1^K) \quad \left| \begin{array}{l} \exists (y^{0'}, y^{1'}, \dots, y^{n'}) \in \text{dom } f \times X_1 \times \dots \times X_n \text{ such that} \\ F^i(y^{i'}) - y^{(i-1)'} \in -\text{int } K_{i-1}, \ i = 1, \dots, n \end{array} \right.$$

and can also be written as

$$(RC_{1'}^K) \quad \left| \begin{array}{l} \exists x' \in X_n \text{ such that } F^n(x') \in (F^{n-1})^{-1}((F^{n-2})^{-1}(\dots \\ (F^1)^{-1}(\text{dom } f - \text{int } K_0) - \text{int } K_{n-1} \dots) - \text{int } K_{n-2}) - \text{int } K_{n-1}. \end{array} \right.$$

For the interior point regularity condition we get

$$(RC_2^K) \quad \left| \begin{array}{l} X_0, \dots, X_n \text{ are Fréchet spaces, } f \text{ is l.s.c.,} \\ K_{i-1} \text{ is closed, } F^i \text{ is } K_{i-1}\text{-epi closed, } i = 1, \dots, n, \\ 0_{X_0} \in \text{sqli}(F^1(\text{dom } F^1) - \text{dom } f + K_0) \text{ and} \\ 0_{X_{i-1}} \in \text{sqli}(F^i(\text{dom } F^i) - \text{dom } F^{i-1} + K_{i-1}), i = 2, \dots, n. \end{array} \right.$$

In the same way we get representations for  $(RC_i^K)$ ,  $i = 2', 2'', 3$ .

By Theorem 3.3 we can state the following one:

**Theorem 3.5** (strong duality). *Let  $f : \bar{X}_0 \rightarrow \bar{\mathbb{R}}$  be proper, convex and  $K_0$ -increasing on  $F^1(\text{dom } F^1) + K_0$ ,  $F^i : \bar{X}_i \rightarrow \bar{X}_{i-1}$ , be proper,  $K_{i-1}$ -convex and  $(K_i, K_{i-1})$ -increasing on  $F^{i+1}(\text{dom } F^{i+1}) + K_i$ ,  $i = 1, \dots, n-1$  and  $F^n : X_n \rightarrow \bar{X}_{n-1}$  be proper and  $K_{n-1}$ -convex. If one of the conditions  $(RC_i^K)$ ,  $i \in \{1, 1', 2, 2', 2'', 3\}$ , is fulfilled, then between  $(P^K)$  and  $(D^K)$  strong duality holds, i.e.  $v(P^K) = v(D^K)$  and the conjugate dual problem has an optimal solution.*

Furthermore, it holds the following theorem.

**Theorem 3.6.** *Let  $f : \bar{X}_0 \rightarrow \bar{\mathbb{R}}$  be proper, convex and  $K_0$ -increasing on  $F^1(\text{dom } F^1) + K_0$ ,  $F^i : \bar{X}_i \rightarrow \bar{X}_{i-1}$ , be proper,  $K_{i-1}$ -convex and  $(K_i, K_{i-1})$ -increasing on  $F^{i+1}(\text{dom } F^{i+1}) + K_i$ ,  $i = 1, \dots, n-1$  and  $F^n : X_n \rightarrow \bar{X}_{n-1}$  be proper and  $K_{n-1}$ -convex. If one of the regularity conditions  $(RC_i^K)$ ,  $i \in \{1, 1', 2, 2', 2'', 3\}$ , is fulfilled, then the conjugate function of  $\gamma$  is given by*

$$\gamma^*(x^*) = \min_{\substack{z^{i*} \in K_i^*, \\ i=0, \dots, n-1}} \left\{ f^*(z^{0*}) + (z^{(n-1)*} F^n)^*(x^*) + \sum_{i=1}^{n-1} (z^{(i-1)*} F^i)^*(z^{i*}) \right\} \quad (3.8)$$

for all  $x^* \in X_n^*$ .

**Proof.** By using Theorem 3.5 it follows that

$$\begin{aligned}\gamma^*(x^*) &= \sup_{x \in X} \{ \langle x^*, x \rangle - (f \circ F^1 \circ \dots \circ F^n)(x) \} \\ &= \min_{\substack{y^{i*} \in K_i^*, \\ i=0, \dots, n-1}} \left\{ f^*(y^{0*}) + (y^{(n-1)*} F^n)^*(x^*) + \sum_{i=1}^{n-1} (y^{(i-1)*} F^i)^*(y^{i*}) \right\} \quad \forall x^* \in X_n^*.\end{aligned}$$

□

**Remark 3.10.** The advantage of the introduced concept is that a “complicated” function  $\gamma$  can be split into  $n+1$  “simple” functions such that the calculation of the conjugate can be simplified by calculating just the conjugates of the  $n+1$  “simple” functions.

**Example 3.1.** Let us consider the following generalized signomial function  $\gamma : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  defined by

$$\gamma(x, y) = \begin{cases} \max \left\{ \frac{1}{x_1^{p_1} y_1^{q_1}}, \dots, \frac{1}{x_n^{p_n} y_n^{q_n}} \right\}, & \text{if } (x, y) \in \text{int } \mathbb{R}_+^n \times \text{int } \mathbb{R}_+^n \\ +\infty, & \text{otherwise,} \end{cases}$$

with  $p_i, q_i \geq 0$  for all  $i = 1, \dots, n$ , and  $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ ,  $y = (y_1, \dots, y_n)^T \in \mathbb{R}^n$ . Then, we split the function  $\gamma$  into the functions

- $f : \overline{\mathbb{R}^n} \rightarrow \overline{\mathbb{R}}$  defined by

$$f(y^0) := \begin{cases} \max\{y_1^0, \dots, y_n^0\}, & \text{if } y^0 = (y_1^0, \dots, y_n^0)^T \in \mathbb{R}_+^n, \\ +\infty, & \text{otherwise,} \end{cases}$$

- $F^1 : \overline{\mathbb{R}^n} \rightarrow \overline{\mathbb{R}^n}$ , defined by

$$F^1(y^1) := \begin{cases} (e^{y_1^1}, \dots, e^{y_n^1})^T, & \text{if } y^1 = (y_1^1, \dots, y_n^1)^T \in \mathbb{R}^n \\ +\infty_{\mathbb{R}_+^n}, & \text{otherwise,} \end{cases}$$

and

- $F^2 : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \overline{\mathbb{R}^n}$ , defined by

$$F^2(x, y) := \begin{cases} (-p_1 \ln x_1 - q_1 \ln y_1, \dots, -p_n \ln x_n - q_n \ln y_n)^T, & \text{if } x, y \in \text{int } \mathbb{R}_+^n, \\ +\infty_{\mathbb{R}_+^n}, & \text{otherwise,} \end{cases}$$

such that  $\gamma$  is writeable as

$$\gamma(x, y) = (f \circ F^1 \circ F^2)(x, y) \quad (3.9)$$

and set  $K_0 = K_1 = \mathbb{R}_+^n$ . Without much effort one can observe that  $f$  is proper, convex and  $\mathbb{R}_+^n$ -increasing on  $F^1(\text{dom } F^1) + \mathbb{R}_+^n = \text{int } \mathbb{R}_+^n + \mathbb{R}_+^n = \text{int } \mathbb{R}_+^n \subseteq \mathbb{R}_+^n$ ,  $F^1$  is proper,  $\mathbb{R}_+^n$ -convex and  $(\mathbb{R}_+^n, \mathbb{R}_+^n)$ -increasing on  $F^2(\text{dom } F^2) + \mathbb{R}_+^n = \mathbb{R}^n$  and  $F^2$  is proper and  $\mathbb{R}_+^n$ -convex. Moreover, it is easy to verify that the regularity condition  $(RC_1^K)$  looks in this special case like

$$(RC_1^{K^e}) \quad | \quad \exists (x', y') \in \mathbb{R}^n \times \mathbb{R}^n \text{ such that } -p_i \ln x'_i - q_i \ln y'_i \in \mathbb{R}, \quad i = 1, \dots, n,$$

which, of course, is always fulfilled. Thus, we can apply the formula (3.8) of Theorem 3.6 for the determination of the conjugate function of  $\gamma$ :

$$\gamma^*(x^*, y^*) = \min_{z^{0*}, z^{1*} \in \mathbb{R}_+^n} \{ f^*(z^{0*}) + (z^{0*} F^1)^*(z^{1*}) + (z^{1*} F^2)^*(x^*, y^*) \} \quad \forall (x^*, y^*) \in \mathbb{R}^n \times \mathbb{R}^n. \quad (3.10)$$

Now, we have to calculate the conjugate functions involved in the formula (3. 10). We have for  $z^{0*} = (z_1^{0*}, \dots, z_n^{0*})^T \in \mathbb{R}_+^n$ :

$$\begin{aligned} f^*(z^{0*}) &= \sup_{(y_1^0, \dots, y_n^0)^T \in \mathbb{R}_+^n} \left\{ \sum_{i=1}^n z_i^{0*} y_i^0 - f(y^0) \right\} = \sup_{(y_1^0, \dots, y_n^0)^T \in \mathbb{R}_+^n} \left\{ \sum_{i=1}^n z_i^{0*} y_i^0 - \max\{y_1^0, \dots, y_n^0\} \right\} \\ &= \sup_{(y_1^0, \dots, y_n^0)^T \in \mathbb{R}_+^n} \left\{ \sum_{i=1}^n z_i^{0*} y_i^0 - \inf_{\substack{t \in \mathbb{R}_+, \\ t = 1, \dots, n}} t \right\} = \sup_{\substack{y_i^0 \in \mathbb{R}_+, \\ i=1, \dots, n}} \left\{ \sum_{i=1}^n z_i^{0*} y_i^0 - t \right\}. \end{aligned}$$

As one may see,  $f^*$  can be expressed as a supremum of a linear function and thus, by elementary calculations, we have that

$$f^*(z^{0*}) = \begin{cases} 0, & \text{if } \sum_{i=1}^n z_i^{0*} \leq 1, (z_1^{0*}, \dots, z_n^{0*})^T \in \mathbb{R}_+^n, \\ +\infty, & \text{otherwise.} \end{cases} \quad (3. 11)$$

From (3. 10) and (3. 11) follows for the conjugate function of  $\gamma$

$$\gamma^*(x^*, y^*) = \min_{\substack{z_i^{0*}, z_i^{1*} \in \mathbb{R}_+, \\ \sum_{i=1}^n z_i^{0*} \leq 1, \\ i=1, \dots, n,}} \left\{ (z^{0*} F^1)^*(z^{1*}) + (z^{1*} F^2)^*(x^*, y^*) \right\}. \quad (3. 12)$$

Furthermore, we have for  $z_i^{0*} \geq 0, i = 1, \dots, n$ ,

$$\begin{aligned} (z^{0*} F^1)^*(z^{1*}) &= \sup_{y_i^1 \in \mathbb{R}, i=1, \dots, n} \left\{ \sum_{i=1}^n z_i^{1*} y_i^1 - \sum_{i=1}^n z_i^{0*} e^{y_i^1} \right\} \\ &= \sum_{i=1}^n \sup_{y_i^1 \in \mathbb{R}} \{ z_i^{1*} y_i^1 - z_i^{0*} e^{y_i^1} \} \end{aligned}$$

with (see [7] or also [24])

$$\sup_{y_i^1 \in \mathbb{R}} \{ z_i^{1*} y_i^1 - z_i^{0*} e^{y_i^1} \} = \begin{cases} z_i^{1*} \left( \ln \frac{z_i^{1*}}{z_i^{0*}} - 1 \right), & \text{if } z_i^{0*}, z_i^{1*} > 0, \\ 0, & \text{if } z_i^{1*} = 0, z_i^{0*} \geq 0, \\ +\infty, & \text{otherwise,} \end{cases} \quad (3. 13)$$

for  $i = 1, \dots, n$  and for  $z_i^{1*} \geq 0, i = 1, \dots, n$ , it holds

$$\begin{aligned} (z^{1*} F^2)^*(x^*, y^*) &= \sup_{x_i, y_i > 0, i=1, \dots, n} \left\{ \sum_{i=1}^n x_i^* x_i + \sum_{i=1}^n y_i^* y_i + \sum_{i=1}^n z_i^{1*} p_i \ln x_i + \sum_{i=1}^n z_i^{1*} q_i \ln y_i \right\} \\ &= \sum_{i=1}^n \left( \sup_{x_i > 0} \{ x_i^* x_i + z_i^{1*} p_i \ln x_i \} + \sup_{y_i > 0} \{ y_i^* y_i + z_i^{1*} q_i \ln y_i \} \right) \end{aligned}$$

for all  $x^* = (x_1^*, \dots, x_n^*)^T, y^* = (y_1^*, \dots, y_n^*)^T \in \mathbb{R}^n$ , where (see [24])

$$\sup_{x_i > 0} \{ x_i^* x_i + z_i^{1*} p_i \ln x_i \} = \begin{cases} -z_i^{1*} p_i \left( 1 + \ln \left( -\frac{x_i^*}{z_i^{1*} p_i} \right) \right), & \text{if } x_i^* < 0, z_i^{1*}, p_i > 0, \\ 0, & \text{if } x_i^* \leq 0 \text{ and } z_i^{1*} = 0 \text{ or } x_i^* \leq 0 \text{ and } p_i = 0, \\ +\infty, & \text{otherwise,} \end{cases} \quad (3. 14)$$

and likewise

$$\sup_{y_i > 0} \{ y_i^* y_i + z_i^{1*} q_i \ln y_i \} = \begin{cases} -z_i^{1*} q_i \left( 1 + \ln \left( -\frac{y_i^*}{z_i^{1*} q_i} \right) \right), & \text{if } y_i^* < 0, z_i^{1*}, q_i > 0, \\ 0, & \text{if } y_i^* \leq 0 \text{ and } z_i^{1*} = 0 \text{ or } y_i^* \leq 0 \text{ and } q_i = 0, \\ +\infty, & \text{otherwise,} \end{cases} \quad (3. 15)$$

for  $i = 1, \dots, n$ . Finally, we define the function  $\xi : \mathbb{R} \rightarrow \{0, 1\}$  by

$$\xi(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{otherwise,} \end{cases} \quad (3.16)$$

which leads, by using (3.12), (3.13), (3.14), (3.15) and (3.16), to the following formula of the conjugate function of  $\gamma$

$$\gamma^*(x^*, y^*) = \min_{\substack{\sum_{i=1}^n z_i^{0*} \leq 1, \ z_i^{0*} \geq 0, \\ z_i^{1*} \geq 0, \ i=1, \dots, n}} \left\{ \sum_{i=1}^n z_i^{1*} [(\ln z_i^{1*} - \ln z_i^{0*} - 1) \xi(z_i^{0*}) - p_i (1 + \ln x_i^* - \ln z_i^{1*} p_i) - q_i (1 + \ln y_i^* - \ln z_i^{1*} q_i)] \right\}$$

for all  $x_i^*, y_i^* \geq 0, i = 1, \dots, n$ , with the convention  $0 \ln 0 = 0$ .

Next, we give an alternative representation for  $\gamma$ . But, first pay attention to the following function

$$\beta(x^*) := \inf_{\substack{z^{i*} \in X_i^*, \\ i=0, \dots, n-1}} \left\{ f^*(z^{0*}) + (z^{(n-1)*} F^n)^*(x^*) + \sum_{i=1}^{n-1} (z^{(i-1)*} F^i)^*(z^{i*}) \right\} \forall x^* \in X_n^*.$$

If  $f : X_0 \rightarrow \overline{\mathbb{R}}$  is a  $K_0$ -increasing function on  $\{F^1(\text{dom } F^1) + K_0\} - K_0$ , it follows by [7, Proposition 2.3.11] that

$$f^*(z^{0*}) = +\infty \ \forall z^{0*} \notin K_0^*, \text{ i.e. } \text{dom } f^* \subseteq K_0^*,$$

and thus it holds

$$\beta(x^*) = \inf_{\substack{z^{0*} \in K_0^*, \ z^{i*} \in X_i^*, \\ i=1, \dots, n-1}} \left\{ f^*(z^{0*}) + (z^{(n-1)*} F^n)^*(x^*) + \sum_{i=1}^{n-1} (z^{(i-1)*} F^i)^*(z^{i*}) \right\}$$

for all  $x^* \in X_n^*$ . Moreover, if  $F^1 : X_1 \rightarrow \overline{X}_0$  is  $(K_1, K_0)$ -increasing on  $\{F^2(\text{dom } F^2) + K_1\} - K_1$ , then  $(z^{0*} F^1) : X_1 \rightarrow \overline{\mathbb{R}}$  is  $K_1$ -increasing on  $\{F^2(\text{dom } F^2) + K_1\} - K_1$  for  $z^{0*} \in K_0^*$ . By using again [7, Proposition 2.3.11] one gets for  $z^{0*} \in K_0^*$

$$(z^{0*} F^1)^*(z^{1*}) = +\infty \ \forall z^{1*} \notin K_1^*, \text{ i.e. } \text{dom}(z^{0*} F^1) \subseteq K_1^*$$

and we can write

$$\beta(x^*) = \inf_{\substack{z^{0*} \in K_0^*, \ z^{1*} \in K_1^*, \\ z^{i*} \in X_i^*, \ i=2, \dots, n-1}} \left\{ f^*(z^{0*}) + (z^{(n-1)*} F^n)^*(x^*) + \sum_{i=1}^{n-1} (z^{(i-1)*} F^i)^*(z^{i*}) \right\}$$

for all  $x^* \in X_n^*$ . If we proceed in this way, it follows that

$$(z^{(i-1)*} F^i)^*(z^{i*}) = +\infty \ \forall z^{i*} \notin K_i^*, \text{ i.e. } \text{dom}(z^{(i-1)*} F^i)^* \subseteq K_i^*, \ i = 2, \dots, n-1,$$

and therefore, it holds

$$\beta(x^*) = \inf_{\substack{z^{i*} \in K_i^*, \\ i=0, \dots, n-1}} \left\{ f^*(z^{0*}) + (z^{(n-1)*} F^n)^*(x^*) + \sum_{i=1}^{n-1} (z^{(i-1)*} F^i)^*(z^{i*}) \right\}$$

for all  $x^* \in X_n^*$ . For the conjugate function of  $\beta$  one has

$$\begin{aligned}
\beta^*(x) &= \sup_{x^* \in X_n^*} \{ \langle x^*, x \rangle - \beta(x^*) \} \\
&= \sup_{x^* \in X_n^*} \left\{ \langle x^*, x \rangle - \inf_{\substack{z^{i*} \in X_i^*, \\ i=0, \dots, n-1}} \left\{ f^*(z^{0*}) + (z^{(n-1)*} F^n)^*(x^*) + \sum_{i=1}^{n-1} (z^{(i-1)*} F^i)^*(z^{i*}) \right\} \right\} \\
&= \sup_{\substack{x^* \in X_n^*, \\ z^{i*} \in X_i^*, \\ i=0, \dots, n-1}} \left\{ \langle x^*, x \rangle - f^*(z^{0*}) - (z^{(n-1)*} F^n)^*(x^*) - \sum_{i=1}^{n-1} (z^{(i-1)*} F^i)^*(z^{i*}) \right\} \\
&= \sup_{\substack{z^{i*} \in X_i^*, \\ i=0, \dots, n-1}} \left\{ \sup_{x^* \in X_n^*} \{ \langle x^*, x \rangle - (z^{(n-1)*} F^n)^*(x^*) \} - f^*(z^{0*}) - \sum_{i=1}^{n-1} (z^{(i-1)*} F^i)^*(z^{i*}) \right\} \\
&= \sup_{\substack{z^{i*} \in X_i^*, \\ i=0, \dots, n-1}} \left\{ (z^{(n-1)*} F^n)^{**}(x) - f^*(z^{0*}) - \sum_{i=1}^{n-1} (z^{(i-1)*} F^i)^*(z^{i*}) \right\} \tag{3. 17}
\end{aligned}$$

for all  $x \in X_n$ . Since  $F^n$  is proper and  $K_{n-1}$ -convex and if we ask that  $F^n$  is also positively  $K_{n-1}$ -lower semicontinuous, (3. 17) can by using the Fenchel-Moreau Theorem be written as

$$\beta^*(x) = \sup_{\substack{z^{i*} \in X_i^*, \\ i=0, \dots, n-1}} \left\{ (z^{(n-1)*} F^n)(x) - f^*(z^{0*}) - \sum_{i=1}^{n-1} (z^{(i-1)*} F^i)^*(z^{i*}) \right\} \tag{3. 18}$$

for all  $x \in X_n$ . If we additionally ask that the function  $F^i$  is positively  $K_{i-1}$ -lower semicontinuous,  $i = 1, \dots, n-1$ , and if we assume that  $f$  is lower semicontinuous, then one gets for (3. 18) by using again the Fenchel-Moreau Theorem

$$\begin{aligned}
\beta^*(x) &= \sup_{\substack{z^{i*} \in X_i^*, \\ i=0, \dots, n-2}} \left\{ \sup_{z^{(n-1)*} \in X_{n-1}^*} \{ \langle z^{(n-1)*}, F^n(x) \rangle - (z^{(n-2)*} F^{(n-1)})^*(z^{(n-1)*}) \} - \right. \\
&\quad \left. f^*(z^{0*}) - \sum_{i=1}^{n-2} (z^{(i-1)*} F^i)^*(z^{i*}) \right\} \\
&= \sup_{\substack{z^{i*} \in X_i^*, \\ i=0, \dots, n-2}} \left\{ (z^{(n-2)*} F^{n-1})^{**}(F^n(x)) - f^*(z^{0*}) - \sum_{i=1}^{n-2} (z^{(i-1)*} F^i)^*(z^{i*}) \right\} \\
&= \sup_{\substack{z^{i*} \in X_i^*, \\ i=0, \dots, n-2}} \left\{ (z^{(n-2)*} F^{n-1})(F^n(x)) - f^*(z^{0*}) - \sum_{i=1}^{n-2} (z^{(i-1)*} F^i)^*(z^{i*}) \right\} \\
&= \sup_{\substack{z^{i*} \in X_i^*, \\ i=0, \dots, n-3}} \left\{ \sup_{z^{(n-2)*} \in X_{n-2}^*} \{ \langle z^{(n-2)*}, F^{n-1}(F^n(x)) \rangle - (z^{(n-3)*} F^{(n-2)})^*(z^{(n-2)*}) \} \right. \\
&\quad \left. - f^*(z^{0*}) - \sum_{i=1}^{n-3} (z^{(i-1)*} F^i)^*(z^{i*}) \right\} \\
&= \sup_{\substack{z^{i*} \in X_i^*, \\ i=0, \dots, n-3}} \left\{ (z^{(n-3)*} F^{n-2})(F^{n-1}(F^n(x))) - f^*(z^{0*}) - \sum_{i=1}^{n-3} (z^{(i-1)*} F^i)^*(z^{i*}) \right\} \\
&\quad \vdots \\
&= \sup_{z^{0*} \in X_0^*} \{ \langle z^{0*}, (F^1 \circ \dots \circ F^n)(x) \rangle - f^*(z^{0*}) \} = f^{**}((F^1 \circ \dots \circ F^n)(x)) \\
&= (f \circ F^1 \circ \dots \circ F^n)(x) = \gamma(x) \quad \forall x \in X_n.
\end{aligned}$$



Since the weak duality always holds, i.e.  $v(P^K) \geq v(D^K)$ , we have  $\gamma^*(x^*) \leq \beta(x^*)$  for all  $x^* \in X_n^*$ . Moreover, it holds  $\gamma(x) \geq \gamma^{**}(x)$  for all  $x \in X_n$  and from here it follows that  $\gamma(x) \geq \gamma^{**}(x) \geq \beta^*(x) = \gamma(x)$ ,  $x \in X_n$ , i.e.  $\gamma(x) = \gamma^{**}(x)$  for all  $x \in X_n$ . The latter means that  $\gamma$  is proper, convex and lower semicontinuous. Summarizing, we get the following theorem:

**Theorem 3.7.** *Let  $f : \bar{X}_0 \rightarrow \bar{\mathbb{R}}$  be a proper, convex,  $K_0$ -increasing on  $\{F^1(\text{dom } F^1) + K_0\} - K_0$  and lower semicontinuous function,  $F^i : \bar{X}_i \rightarrow \bar{X}_{i-1}$  be a proper,  $K_{i-1}$ -convex,  $(K_i, K_{i-1})$ -increasing on  $\{F^{i+1}(\text{dom } F^{i+1}) + K_i\} - K_i$  and positively  $K_{i-1}$ -lower semicontinuous function,  $i = 1, \dots, n-1$ , and  $F^n : X_n \rightarrow \bar{X}_{n-1}$  be a proper,  $K_{n-1}$ -convex and positively  $K_{n-1}$ -lower semicontinuous function. Then the function  $\gamma = f \circ F^1 \circ \dots \circ F^n : X_n \rightarrow \bar{\mathbb{R}}$  is proper, convex and lower semicontinuous and can alternatively be written as*

$$(f \circ F^1 \circ \dots \circ F^n)(x) = \sup_{\substack{z^{i*} \in X_i^*, \\ i=0, \dots, n-1}} \left\{ (z^{(n-1)*} F^n)(x) - f^*(z^{0*}) - \sum_{i=1}^{n-1} (z^{(i-1)*} F^i)^*(z^{i*}) \right\} \quad \forall x \in X_n.$$

**Remark 3.11.** *Besides the introduced duality concept there is a second way to construct a corresponding conjugate dual problem to  $(P^C)$  and to formulate associated duality statements, where the conjugates of the functions involved in the objective function of the original problem are split. This dual approach is characterized by the direct applying of the perturbation theory by defining an associated perturbation function of the following form*

$$\Phi(x, y^0, \dots, y^{n+1}) := \begin{cases} f(F^1(\dots F^{n-1}(F^n(x + y^n) + y^{n-1})\dots) + y^0), & \text{if } g(x) \in y^{n+1} - Q, \\ +\infty, & \text{otherwise,} \end{cases}$$

where  $(y^0, \dots, y^n, y^{n+1}) \in X_0 \times \dots \times X_n \times Z$  are the dual variables.

If we use this method in the context of the generalized interior point regularity conditions, then we have to impose for strong duality that the perturbation function  $\Phi$  is lower semicontinuous (see [7]). But this means, as shown in Theorem 3.7, that we have to ensure that the functions  $F^i$  are all positively  $K_{i-1}$ -lower semicontinuous, respectively. In contrast, to employ the proposed method in this chapter, we only need to secure that each of these functions is  $K_{i-1}$ -epi closed, respectively. It is well known that if a function  $F^i$  is positively  $K_{i-1}$ -lower semicontinuous, then it is also  $K_{i-1}$ -epi closed, while the inverse statement is not true in general (see Proposition 2.2.19 and Example 2.2.6. in [7]). In this sense the method introduced in this thesis asks for weaker hypothesis on the involved functions for guaranteeing strong duality.

Finally, let us turn to the question why we did not apply the Fenchel-Lagrange duality theory to the reformulated primal problem  $(\tilde{P}^C)$  with set and cone constraints. The reason is that even though that the functions  $F^n$  and  $g$  can not be split directly, one derives more complicated and stronger regularity conditions compared to the ones proposed in this work.

### 3.4 An optimization problem having as objective function the sum of reciprocals of concave functions

Let  $E_i$  be a non-empty convex subset of  $X$ ,  $i = 1, \dots, n$ , where  $X$  is a locally convex Hausdorff space partially ordered by the closed and convex cone  $K$ . Then, we consider a convex optimization problem having as objective function the sum of reciprocals of concave functions  $h_i : E_i \rightarrow \bar{\mathbb{R}}$  with strict positive values,  $i = 1, \dots, n$ , and geometric and cone constraints, i.e., the optimization problem that we discuss in this section (cf. the definitions from Section 3.1) is given by

$$(P^G) \quad \inf_{\substack{x \in S, \\ g(x) \in -Q}} \left\{ \sum_{i=1}^n \frac{1}{h_i(x)} \right\}.$$

Optimization problems of this type arise, for instance, in the study of power functions by setting  $h_i : \mathbb{R}_+ \rightarrow \bar{\mathbb{R}}$ ,  $h_i(x) = c_i x^{p_i}$  with  $c_i p_i (p_i - 1) \leq 0$ ,  $i = 1, \dots, n$ , (see [72]) and have a wide range of applications in economics, engineering and finance.

To apply the results from the previous section to  $(P^G)$ , i.e. to characterize strong duality and to derive optimality conditions, we assume that the function  $-h_i$  is  $K$ -increasing on  $E_i$ ,  $i = 1, \dots, n$ , and set  $X_0 = \mathbb{R}^n$ ,  $K_0 = \mathbb{R}_+^n$ ,  $X_1 = X^n$ ,  $K_1 = K^n$  and  $X_2 = X$ . Additionally, we define the following functions

- $f : \overline{\mathbb{R}^n} \rightarrow \overline{\mathbb{R}}$ ,

$$f(y^0) = \begin{cases} -\sum_{i=1}^n \frac{1}{y_i^0}, & \text{if } y_i^0 < 0, \ i = 1, \dots, n, \\ +\infty, & \text{otherwise,} \end{cases}$$

- $F^1 : X^n \rightarrow \overline{\mathbb{R}^n}$ ,

$$F^1(y^1) = \begin{cases} (-h_1(y_1^1), \dots, -h_n(y_n^1))^T, & \text{if } y_i^1 \in E_i, \ i = 1, \dots, n, \\ +\infty_{\mathbb{R}_+^n}, & \text{otherwise} \end{cases}$$

and

- $F^2 : X \rightarrow X^n$ ,  $F^2(x) := (x, \dots, x) \in X^n$

and we assume that  $F^2(S \cap \text{dom } g) \subseteq E_1 \times \dots \times E_n$  (cf. Remark 3.3). From here, it follows that the problem  $(P^G)$  can equivalently be written as

$$(P^G) \quad \inf_{\substack{x \in S, \\ g(x) \in -Q}} \{(f \circ F^1 \circ F^2)(x)\}$$

and by using the formula from Section 3.1 its corresponding conjugate dual problem  $(D^G)$  turns into

$$(D^G) \quad \sup_{\substack{z^{0*} \in \mathbb{R}_+^n, \ z^{1*} \in (K^*)^n, \\ z^{2*} \in Q^*}} \left\{ \inf_{x \in S} \left\{ \left\langle \sum_{i=1}^n z_i^{1*}, x \right\rangle + \langle z^{2*}, g(x) \rangle \right\} - f^*(z^{0*}) - (z^{0*} F^1)^*(z^{1*}) \right\}.$$

Furthermore, one has (see [8], [51] or [54]):

$$f^*(z^{0*}) = \sum_{i=1}^n \sup_{y_i^0 < 0} \left\{ z_i^{0*} y_i^0 + \frac{1}{y_i^0} \right\} = -2 \sum_{i=1}^n \sqrt{z_i^{0*}}$$

for all  $z_i^{0*} \geq 0$ ,  $i = 1, \dots, n$ , and since

$$(z^{0*} F^1)^*(z^{1*}) = \sum_{i=1}^n \sup_{y_i^1 \in E_i} \{ \langle z_i^{1*}, y_i^1 \rangle + z_i^{0*} h_i(y_i^1) \} = \sum_{i=1}^n (-z_i^{0*} h_i)^*_{E_i}(z_i^{1*})$$

holds, one gets for the conjugate dual problem

$$(D^G) \quad \sup_{\substack{z^{0*} \in \mathbb{R}_+^n, \ z^{1*} \in (K^*)^n, \\ z^{2*} \in -Q^*}} \left\{ -(z^{2*} g)_S^* \left( -\sum_{i=1}^n z_i^{1*} \right) + \sum_{i=1}^n \left( 2\sqrt{z_i^{0*}} - (-z_i^{0*} h_i)^*_{E_i}(z_i^{1*}) \right) \right\}.$$

It is easy to observe that  $f$  is proper,  $\mathbb{R}_+^n$ -increasing on  $\text{dom } f = -\text{int}(\mathbb{R}_+^n)$ , convex and lower semicontinuous,  $F^1$  is proper,  $(K^n, \mathbb{R}_+^n)$ -increasing on  $\text{dom } F^1 = E_1 \times \dots \times E_n$  and  $\mathbb{R}_+^n$ -convex and that  $F^1(\text{dom } F^1) \subseteq \text{int}(-\mathbb{R}_+^n) = \text{dom } f$  (in this context pay attention on Remark 3.3). For that reason we can now attach the regularity condition  $(RC_1^G)$ , specialized for the optimization problem  $(P^G)$ ,

$$(RC_1^G) \quad \left| \begin{array}{l} \exists(y^{0'}, y^{1'}, y^{2'}) \in (-\infty, 0)^n \times X^n \times S \text{ such that } h_i(y_i^{1'}) + y_i^{0'} > 0, \\ y^{2'} - y_i^{1'} \in -\text{int } K, \ i = 1, \dots, n, \text{ and } g(y^{2'}) \in -\text{int } Q. \end{array} \right.$$

As  $h_i$  is a concave function with strict positive values on  $E_i$ , there exist  $y_i^{0*} < 0$  and  $y_i^{1*} \in E_i$  such that  $h_i(y_i^{1'}) + y_i^{0'} > 0$ ,  $i = 1, \dots, n$ , and hence  $(RC_1^G)$  reduces to

$$(RC_1^G) \quad \left| \begin{array}{l} \exists(y^{1'}, y^{2'}) \in X^n \times S \text{ such that } y^{2'} - y_i^{1'} \in -\text{int } K, \ i = 1, \dots, n, \\ \text{and } g(y^{2'}) \in -\text{int } Q. \end{array} \right.$$

or, equivalently, in the light of  $(RC_1^C)$ ,

$$(RC_1^G) \quad \left| \begin{array}{l} \exists x' \in S \text{ such that } x' \in E_i - \text{int } K, \ i = 1, \dots, n, \text{ and } g(x') \in -\text{int } Q. \end{array} \right.$$

The generalized interior point regularity conditions  $(RC_2^C)$ , specialized for  $(P^G)$ , looks like

$$(RC_2^G) \quad \left| \begin{array}{l} X \text{ and } Z \text{ are Fréchet spaces, } S \text{ is closed, } g \text{ is } Q\text{-epi closed,} \\ -h_i \text{ is lower semicontinuous, } 0_X \in \text{sqli}(\text{dom } g \cap S - E_i + K), \\ i = 1, \dots, n, \text{ and } 0_Z \in \text{sqli}(g(\text{dom } g \cap S) + Q). \end{array} \right.$$

In the same way one can formulate a specialized regularity condition  $(RC_i^G)$  in respect to the condition  $(RC_i^C)$  for  $i \in \{2', 2'', 3\}$ .

**Remark 3.12.** Recall, that in respect to Remarks 3.2 and 3.5 the function  $F^1$  does not need to be monotone, because  $F^2$  is a linear function. In this case we set, like mentioned in Remark 3.2,  $K_1 = \{0_{X^n}\} = \{0_X\}^n$ . But pay attention to the circumstance that the regularity conditions  $(RC_1^G)$  and  $(RC_1^C)$  are no more applicable in this framework, as  $\text{int}\{0_X\} = \emptyset$ .

By Theorems 3.3 and 3.4 the strong duality statement and the optimality conditions follows immediately.

**Theorem 3.8** (strong duality). *If one of the conditions  $(RC_i^G)$ ,  $i \in \{1, 1', 2, 2', 2'', 3\}$ , is fulfilled, then between  $(P^G)$  and  $(D^G)$  strong duality holds, i.e.  $v(P^G) = v(D^G)$  and the conjugate dual problem has an optimal solution.*

**Theorem 3.9** (optimality conditions). *(a) Suppose that one of the regularity conditions  $(RC_i^G)$ ,  $i \in \{1, 1', 2, 2', 2'', 3\}$ , is fulfilled and let  $\bar{x} \in S$  be an optimal solution of the problem  $(P^G)$ . Then there exists  $(\bar{z}^{0*}, \bar{z}^{1*}, \bar{z}^{2*}) \in \mathbb{R}_+^n \times (K^*)^n \times Q^*$ , an optimal solution to  $(D^G)$ , such that*

$$\begin{aligned} (i) \quad & \sum_{i=1}^n \frac{1}{h_i(\bar{x})} - 2 \sum_{i=1}^n \sqrt{\bar{z}_i^{0*}} = - \sum_{i=1}^n \bar{z}_i^{0*} h_i(\bar{x}), \\ (ii) \quad & \sum_{i=1}^n (-\bar{z}_i^{0*} h_i)_{E_i}^*(\bar{z}_i^{1*}) - \sum_{i=1}^n \bar{z}_i^{0*} h_i(\bar{x}) = \left\langle \sum_{i=1}^n \bar{z}_i^{1*}, \bar{x} \right\rangle, \\ (iii) \quad & \langle \bar{z}^{2*}, g(\bar{x}) \rangle + (\bar{z}^{2*} g)_S^* \left( - \sum_{i=1}^n \bar{z}_i^{1*} \right) = \left\langle - \sum_{i=1}^n \bar{z}_i^{1*}, \bar{x} \right\rangle, \\ (iv) \quad & \langle \bar{z}^{2*}, g(\bar{x}) \rangle = 0. \end{aligned}$$

*(b) If there exists  $\bar{x} \in S$  such that for some  $(\bar{z}^{0*}, \bar{z}^{1*}, \bar{z}^{2*}) \in \mathbb{R}_+^n \times (K^*)^n \times Q^*$  the conditions (i)-(iv) are fulfilled, then  $\bar{x}$  is an optimal solution of  $(P^G)$ ,  $(\bar{z}^{0*}, \bar{z}^{1*}, \bar{z}^{2*})$  is an optimal solution for  $(D^G)$  and  $v(P^G) = v(D^G)$ .*

**Remark 3.13.** In view of the Young-Fenchel inequality, we can refine the conditions (i) and (ii) of Theorem 3.9 like follows

$$\begin{aligned} (i) \quad & \bar{z}_i^{0*} h_i(\bar{x}) = 2\sqrt{\bar{z}_i^{0*}} - \frac{1}{h_i(\bar{x})}, \ i = 1, \dots, n, \\ (ii) \quad & (-\bar{z}_i^{0*} h_i)_{E_i}^*(\bar{z}_i^{1*}) - \bar{z}_i^{0*} h_i(\bar{x}) = \langle \bar{z}_i^{1*}, \bar{x} \rangle, \ i = 1, \dots, n. \end{aligned}$$

In the end of this section we give, for completeness, alternative representations of the optimality conditions presented in Theorem 3.9 and refined in the previous remark.

**Remark 3.14.** *In accordance with Remarks 3.8 and 3.13 the optimality conditions (i)-(iv) of Theorem 3.9 can equivalently be rewritten as*

- (i)  $\bar{z}_i^{0*} \in \partial \left(-\frac{1}{\cdot}\right) (-h_i(\bar{x})), i = 1, \dots, n,$
- (ii)  $\bar{z}_i^{1*} \in \partial(-\bar{z}_i^{0*} h_i)(\bar{x}), i = 1, \dots, n,$
- (iii)  $-\sum_{i=1}^n \bar{z}_i^{1*} \in \partial((\bar{z}^{2*} g) + \delta_S)(\bar{x}),$
- (iv)  $\langle \bar{z}^{2*}, g(\bar{x}) \rangle = 0.$

**Remark 3.15.** *One may see that the function  $F^2$  has been introduced in order to split the functions  $h_i, i = 1, \dots, n,$  and  $g$  or, more precisely, to decompose their conjugate functions in the formulation of the dual problem  $(D^G)$ . As a further advantage one gets a detailed characterization of the set of optimality conditions presented in Theorem 3.9, Remark 3.13 and Remark 3.14. Other duality schemes may be employed for approaching this kind of optimization problems, too, however, the separation of the conjugates of the involved functions in the corresponding dual problems may fail to happen. This also underlines the benefit of the introduced multi-composed duality concept.*

## Chapter 4

# Duality results for minimax location problems

In the recent years, location problems attracted enormous attention in the scientific community and a large number of papers studying minisum and minimax location problems have been published (see [20, 23, 33–36, 38, 40, 44, 45, 50, 58, 60–62, 67, 68, 74, 75]). This is due to the fact that location problems cover many practical situations occurring for example in urban area models, computer science, telecommunication and also in emergency facilities location programming.

In this chapter, which is mainly based on our articles [80, 81] and [82], minimax location problems form the focal point of our approach. In particular, we are interested to give a duality approach for nonlinear and linear minimax location problems with geometric constraints, where the version of the nonlinear location problem is additionally equipped with set-up costs. For this purpose, we apply the duality theory developed in the previous chapter, which allows us to formulate more detailed dual problems as well as associated duality statements as in the mentioned papers. To be more exact, we study three classes of location problems, namely, single, extended multifacility and classical multifacility minimax location problems and to each of them, we consider different settings to specialize the associated duality results.

But first, some properties of gauges will be listed in the next section. Gauge functions are a generalization of norms and can be understood as infimal distances to sets. The use of these functions allows to consider more general location models, especially, in situations when asymmetric distance measures are of interest.

### 4.1 Some properties of the gauge function

Let us start this section by proving the following statements that we also shall use in the sequel.

**Lemma 4.1.** *Let  $a_i \in \mathbb{R}_+$  be a given point and  $h_i : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  with  $h_i(x) \in \mathbb{R}_+$ , if  $x \in \mathbb{R}_+$ , and  $h_i(x) = +\infty$ , otherwise, be a proper, lower semicontinuous and convex function,  $i = 1, \dots, n$ . Then the conjugate of the function  $g : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$  defined by*

$$g(x_1, \dots, x_n) := \begin{cases} \max\{h_1(x_1) + a_1, \dots, h_n(x_n) + a_n\}, & \text{if } x_i \in \mathbb{R}_+, i = 1, \dots, n, \\ +\infty, & \text{otherwise,} \end{cases}$$

is given by  $g^* : \mathbb{R}^n \rightarrow \mathbb{R}$ ,

$$g^*(x_1^*, \dots, x_n^*) = \min_{\substack{\sum_{i=1}^n z_i^{0*} \leq 1, \\ z_i^{0*} \geq 0, \\ i=1, \dots, n}} \left\{ \sum_{i=1}^n [(z_i^{0*} h_i)^*(x_i^*) - z_i^{0*} a_i] \right\}.$$

**Proof.** We set  $X_0 = X_1 = \mathbb{R}^n$  and  $K_0 = \mathbb{R}_+^n$ . Further, we define the function  $f : \overline{\mathbb{R}^n} \rightarrow \overline{\mathbb{R}}$  by

$$f(y_1^0, \dots, y_n^0) := \begin{cases} \max\{y_1^0 + a_1, \dots, y_n^0 + a_n\}, & \text{if } y_i^0 \in \mathbb{R}_+, i = 1, \dots, n, \\ +\infty, & \text{otherwise,} \end{cases}$$

and the function  $F^1 : \mathbb{R}^n \rightarrow \overline{\mathbb{R}^n}$  by

$$F^1(x_1, \dots, x_n) := \begin{cases} (h_1(x_1), \dots, h_n(x_n))^T, & \text{if } x_i \in \mathbb{R}_+, i = 1, \dots, n, \\ +\infty_{\mathbb{R}_+^n}, & \text{otherwise.} \end{cases}$$

Hence, the function  $g$  can be written as

$$g(x_1, \dots, x_n) = (f \circ F^1)(x_1, \dots, x_n).$$

It can easy be verified that the function  $f$  is proper, convex, lower semicontinuous and  $\mathbb{R}_+^n$ -increasing on  $F^1(\text{dom } F^1) + K_0 \subseteq \mathbb{R}_+^n$  (as  $f$  is the pointwise supremum of proper, convex and lower semicontinuous functions) and the function  $F^1$  is proper,  $\mathbb{R}_+^n$ -epi closed and  $\mathbb{R}_+^n$ -convex. Therefore, it follows by Theorem 3.6 (note also that  $0_{\mathbb{R}^n} \in \text{sqli}(F^1(\text{dom } F^1) - \text{dom } f + K_0) = \text{sqli}(F^1(\text{dom } F^1) - \mathbb{R}_+^n + \mathbb{R}_+^n) = \mathbb{R}^n$ ) that

$$g^*(x_1^*, \dots, x_n^*) = \min_{\substack{y_i^{0*} \in \mathbb{R}_+, \\ i=1, \dots, n}} \{f^*(y_1^{0*}, \dots, y_n^{0*}) + ((y_1^{0*}, \dots, y_n^{0*})^T F^1)^*(x_1^*, \dots, x_n^*)\}.$$

For the conjugate of the function  $f$  we have

$$\begin{aligned} f^*(y^{0*}) &= \sup_{y_i^0 \in \mathbb{R}, i=1, \dots, n} \left\{ \sum_{i=1}^n y_i^{0*} y_i^0 - f(y_1^0, \dots, y_n^0) \right\} \\ &= \sup_{y_i^0 \in \mathbb{R}_+, i=1, \dots, n} \left\{ \sum_{i=1}^n y_i^{0*} y_i^0 - \max_{1 \leq i \leq n} \{y_i^0 + a_i\} \right\} \\ &= \sup_{y_i^0 \in \mathbb{R}_+, i=1, \dots, n} \left\{ \sum_{i=1}^n y_i^{0*} y_i^0 - \min_{\substack{t \in \mathbb{R}_+, \\ y_i^0 + a_i \leq t, i=1, \dots, n}} t \right\} \\ &= \sup_{\substack{t \in \mathbb{R}_+, y_i^0 \in \mathbb{R}_+, \\ y_i^0 + a_i \leq t, i=1, \dots, n}} \left\{ \sum_{i=1}^n y_i^{0*} y_i^0 - t \right\}. \end{aligned} \quad (4.1)$$

Now, let us consider for any  $y^{0*} \in \mathbb{R}_+^n$  the following primal optimization problem

$$(P^{max}) \quad \inf_{\substack{t \in \mathbb{R}_+, y_i^0 \in \mathbb{R}_+, \\ y_i^0 + a_i \leq t, i=1, \dots, n}} \left\{ t - \sum_{i=1}^n y_i^{0*} y_i^0 \right\}. \quad (4.2)$$

and its corresponding Lagrange dual problem

$$\begin{aligned} (D^{max}) \quad & \sup_{\lambda_i \geq 0, i=1, \dots, n} \inf_{\substack{t \in \mathbb{R}_+, y_i^0 \in \mathbb{R}_+, \\ i=1, \dots, n}} \left\{ t - \sum_{i=1}^n y_i^{0*} y_i^0 + \sum_{i=1}^n \lambda_i (y_i^0 + a_i - t) \right\} \\ &= \sup_{\substack{\lambda_i \geq 0, \\ i=1, \dots, n}} \left\{ - \sup_{t \in \mathbb{R}_+} \left\{ \left( \sum_{i=1}^n \lambda_i - 1 \right) t \right\} - \right. \\ & \quad \left. \sup_{\substack{y_i^0 \in \mathbb{R}_+, \\ i=1, \dots, n}} \left\{ \sum_{i=1}^n (y_i^{0*} - \lambda_i) y_i^0 \right\} + \sum_{i=1}^n \lambda_i a_i \right\} = \sup_{\substack{\sum_{i=1}^n \lambda_i \leq 1, \lambda_i \geq 0, \\ y_i^{0*} \leq \lambda_i, i=1, \dots, n}} \left\{ \sum_{i=1}^n \lambda_i a_i \right\}. \end{aligned}$$

As the Slater constraint qualification is fulfilled, it holds  $v(P^{max}) = v(D^{max})$  and the dual has an optimal solution, thus one gets for the conjugate function of  $f$

$$f^*(y^{0*}) = \min_{\substack{\sum_{i=1}^n \lambda_i \leq 1, \lambda_i \geq 0, \\ y_i^{0*} \leq \lambda_i, i=1, \dots, n}} \left\{ - \sum_{i=1}^n \lambda_i a_i \right\}. \quad (4.3)$$

Furthermore, one has

$$\begin{aligned} & ((y_1^{0*}, \dots, y_n^{0*})^T F^1)^*(x_1^*, \dots, x_n^*) \\ &= \sup_{x_i \in \mathbb{R}, i=1, \dots, n} \left\{ \sum_{i=1}^n x_i^* x_i - (y_1^{0*}, \dots, y_n^{0*})^T F^1(x_1, \dots, x_n) \right\} \\ &= \sup_{x_i \in \mathbb{R}_+, i=1, \dots, n} \left\{ \sum_{i=1}^n x_i^* x_i - \sum_{i=1}^n y_i^{0*} h_i(x_i) \right\} \\ &= \sum_{i=1}^n \sup_{x_i \in \mathbb{R}_+} \{x_i^* x_i - y_i^{0*} h_i(x_i)\} = \sum_{i=1}^n (y_i^{0*} h_i)^*(x_i^*), \end{aligned} \quad (4.4)$$

and so, the conjugate function of  $g$  turns into

$$\begin{aligned} g^*(x_1^*, \dots, x_n^*) &= \min_{\substack{y_i^{0*} \geq 0, \\ i=1, \dots, n}} \left\{ \min_{\substack{\sum_{i=1}^n \lambda_i \leq 1, \lambda_i \geq 0, \\ y_i^{0*} \leq \lambda_i, i=1, \dots, n}} \left\{ - \sum_{i=1}^n \lambda_i a_i \right\} + \sum_{i=1}^n (y_i^{0*} h_i)^*(x_i^*) \right\} \\ &= \min_{\substack{\sum_{i=1}^n \lambda_i \leq 1, \lambda_i \geq 0, \\ 0 \leq y_i^{0*} \leq \lambda_i, i=1, \dots, n}} \left\{ \sum_{i=1}^n [(y_i^{0*} h_i)^*(x_i^*) - \lambda_i a_i] \right\}. \end{aligned} \quad (4.5)$$

We fix  $x_i^* \in \mathbb{R}^n$ ,  $i = 1, \dots, n$ , and emphasize that the problem

$$(P^g) \quad \min_{\substack{\sum_{i=1}^n \lambda_i \leq 1, \lambda_i \geq 0, \\ 0 \leq y_i^{0*} \leq \lambda_i, i=1, \dots, n}} \left\{ \sum_{i=1}^n [(y_i^{0*} h_i)^*(x_i^*) - \lambda_i a_i] \right\} \quad (4.6)$$

is equivalent to

$$(\tilde{P}^g) \quad \min_{\substack{\sum_{i=1}^n z_i^{0*} \leq 1, z_i^{0*} \geq 0, \\ i=1, \dots, n}} \left\{ \sum_{i=1}^n [(z_i^{0*} h_i)^*(x_i^*) - z_i^{0*} a_i] \right\} \quad (4.7)$$

in the sense that  $v(P^g) = v(\tilde{P}^g)$  (where  $v(P^g)$  and  $v(\tilde{P}^g)$  denote the optimal objective values of the problems  $(P^g)$  and  $(\tilde{P}^g)$ , respectively).

To see this, take first a feasible element  $(\lambda_1, \dots, \lambda_n, y_1^{0*}, \dots, y_n^{0*}) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$  of the problem  $(P^g)$  and set  $z_i^{0*} = \lambda_i$ ,  $i = 1, \dots, n$ , then it follows from  $\sum_{i=1}^n \lambda_i \leq 1$ ,  $\lambda_i, y_i^{0*} \geq 0$ ,  $y_i^{0*} \leq \lambda_i$ ,  $i = 1, \dots, n$ , that  $\sum_{i=1}^n z_i^{0*} \leq 1$ ,  $z_i^{0*} \geq 0$ ,  $i = 1, \dots, n$ , i.e.  $(z_1^{0*}, \dots, z_n^{0*})$  is feasible to the problem  $(\tilde{P}^g)$ . From  $y_i^{0*} \leq z_i^{0*}$ , we have that  $y_i^{0*} h_i(x_i) \leq z_i^{0*} h_i(x_i)$  and by [7, Proposition 2.3.2.(c)] follows that  $(y_i^{0*} h_i)^*(x_i^*) \geq (z_i^{0*} h_i)^*(x_i^*)$ . Hence it holds

$$\sum_{i=1}^n [(y_i^{0*} h_i)^*(x_i^*) - \lambda_i a_i] \geq \sum_{i=1}^n [(z_i^{0*} h_i)^*(x_i^*) - z_i^{0*} a_i] \geq v(\tilde{P}^g) \quad (4.8)$$

for all  $(\lambda_1, \dots, \lambda_n, y_1^{0*}, \dots, y_n^{0*})$  feasible to  $(P^g)$ , i.e.  $v(P^g) \geq v(\tilde{P}^g)$ .

Now, take a feasible element  $(z_1^{0*}, \dots, z_n^{0*}) \in \mathbb{R}_+^n$  of the problem  $(\tilde{P}^g)$  and set  $y_i^{0*} = \lambda_i = z_i^{0*}$  for all  $i = 1, \dots, n$ , then we have from  $\sum_{i=1}^n z_i^{0*} \leq 1$ ,  $z_i^{0*} \geq 0$ ,  $i = 1, \dots, n$ , that  $\sum_{i=1}^n \lambda_i \leq 1$ ,  $\lambda_i, y_i^{0*} \geq 0$ ,  $y_i^{0*} = \lambda_i$ ,  $i = 1, \dots, n$ , which means that  $(\lambda_1, \dots, \lambda_n, y_1^{0*}, \dots, y_n^{0*})$  is a feasible element of  $(P^g)$  and it holds

$$\sum_{i=1}^n [(z_i^{0*} h_i)^*(x_i^*) - z_i^{0*} a_i] = \sum_{i=1}^n [(y_i^{0*} h_i)^*(x_i^*) - \lambda_i a_i] \geq v(P^g) \quad (4.9)$$

for all  $(z_1^{0*}, \dots, z_n^{0*})$  feasible to  $v(\tilde{P}^g)$ , which implies  $v(P^g) \leq v(\tilde{P}^g)$ . Finally, it follows that  $v(P^g) = v(\tilde{P}^g)$  and thus, the conjugate function of  $g$  is given by

$$g^*(x_1^*, \dots, x_n^*) = \min_{\substack{\sum_{i=1}^n z_i^{0*} \leq 1, \ z_i^{0*} \geq 0, \\ i=1, \dots, n}} \left\{ \sum_{i=1}^n [(z_i^{0*} h_i)^*(x_i^*) - z_i^{0*} a_i] \right\} \quad (4.10)$$

and takes only finite values.  $\square$

**Lemma 4.2.** Let  $a_i \in \mathbb{R}_+$  be a given point and  $h_i : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  with  $h_i(x) \in \mathbb{R}_+$ , if  $x \in \mathbb{R}_+$ , and  $h_i(x) = +\infty$ , otherwise, be a proper, lower semicontinuous and convex function,  $i = 1, \dots, n$ . Then the function  $g : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ ,

$$g(x_1, \dots, x_n) = \begin{cases} \max\{h_1(x_1) + a_1, \dots, h_n(x_n) + a_n\}, & \text{if } x_i \in \mathbb{R}_+, \ i = 1, \dots, n, \\ +\infty, & \text{otherwise,} \end{cases}$$

can equivalently be expressed as

$$g(x_1, \dots, x_n) = \max_{\substack{\sum_{i=1}^n z_i^{0*} \leq 1, \ z_i^{0*} \geq 0, \\ i=1, \dots, n}} \left\{ \sum_{i=1}^n z_i^{0*} [h_i(x_i) + a_i] \right\} \quad \forall x_i \in \mathbb{R}, \ i = 1, \dots, n.$$

**Proof.** By Lemma 4.1 and the definition of the conjugate function we have for the biconjugate function of  $g$

$$\begin{aligned} g^{**}(x_1, \dots, x_n) &= \sup_{\substack{x_i^* \in \mathbb{R}, \\ i=1, \dots, n}} \left\{ \sum_{i=1}^n x_i^* x_i - \min_{\substack{\sum_{i=1}^n z_i^{0*} \leq 1, \\ z_i^{0*} \geq 0, \ i=1, \dots, n}} \left\{ \sum_{i=1}^n [(z_i^{0*} h_i)^*(x_i^*) - z_i^{0*} a_i] \right\} \right\} \\ &= \sup_{\substack{x_i^* \in \mathbb{R}, \ z_i^{0*} \geq 0, \\ i=1, \dots, n, \ \sum_{i=1}^n z_i^{0*} \leq 1}} \left\{ \sum_{i=1}^n x_i^* x_i - \sum_{i=1}^n [(z_i^{0*} h_i)^*(x_i^*) - z_i^{0*} a_i] \right\} \\ &= \sup_{\substack{z_i^{0*} \geq 0, \ i=1, \dots, n, \\ \sum_{i=1}^n z_i^{0*} \leq 1}} \left\{ \sum_{i=1}^n \left[ \sup_{x_i^* \in \mathbb{R}} \{x_i^* x_i - (z_i^{0*} h_i)^*(x_i^*)\} + z_i^{0*} a_i \right] \right\} \\ &= \sup_{\substack{z_i^{0*} \geq 0, \ i=1, \dots, n, \\ \sum_{i=1}^n z_i^{0*} \leq 1}} \left\{ \sum_{i=1}^n [(z_i^{0*} h_i)^{**}(x_i) + z_i^{0*} a_i] \right\} \quad \forall x_i \in \mathbb{R}, \ i = 1, \dots, n. \quad (4.11) \end{aligned}$$

As  $h_i$ ,  $i = 1, \dots, n$ , are proper, convex and lower semicontinuous functions it follows by the Fenchel-Moreau Theorem that

$$g^{**}(x_1, \dots, x_n) = \sup_{\substack{z_i^{0*} \geq 0, \ i=1, \dots, n, \\ \sum_{i=1}^n z_i^{0*} \leq 1}} \left\{ \sum_{i=1}^n [z_i^{0*} h_i(x_i) + z_i^{0*} a_i] \right\} \quad (4.12)$$



for all  $x_i \in \mathbb{R}$ ,  $i = 1, \dots, n$ , and moreover, as  $g$  is also a proper, convex and lower semicontinuous function it follows by using again the Fenchel-Moreau Theorem that  $g = g^{**}$ , i.e.

$$g(x_1, \dots, x_n) = \max_{\substack{z_i^{0*} \geq 0, \ i=1, \dots, n, \\ \sum_{i=1}^n z_i^{0*} \leq 1}} \left\{ \sum_{i=1}^n [z_i^{0*} h_i(x_i) + z_i^{0*} a_i] \right\} \quad (4.13)$$

for all  $x_i \in \mathbb{R}$ ,  $i = 1, \dots, n$ . □

**Remark 4.1.** Note that the statement in Lemma 4.2 can also be proved in a simpler way, as the maximum of finitely many real numbers is the maximum over the convex hull of finitely many real numbers.

**Remark 4.2.** If we consider the situation when the given points  $a_i$ ,  $i = 1, \dots, n$ , are arbitrary, i.e.  $a_i \in \mathbb{R}$ , then it can easily be verified that the conjugate function of  $f$  in (4.1) looks like

$$f^*(y^{0*}) = \sup_{\substack{y_i^0 \in \mathbb{R}_+, \ t \in \mathbb{R}, \\ y_i^0 + a_i \leq t, \ i=1, \dots, n}} \left\{ \sum_{i=1}^n y_i^{0*} y_i^0 - t \right\} \quad (4.14)$$

(notice that here  $t \in \mathbb{R}$  instead of  $t \in \mathbb{R}_+$ ).

If we now construct to the conjugate function in (4.14) a primal problem in the sense of  $(P^{max})$  in (4.2), then the corresponding Lagrange dual problem  $(D^{max})$  has the form

$$(D^{max}) \quad \sup_{\substack{\sum_{i=1}^n \lambda_i = 1, \ \lambda_i \geq 0, \\ y_i^{0*} \leq \lambda_i, \ i=1, \dots, n}} \left\{ \sum_{i=1}^n \lambda_i a_i \right\}.$$

Analogously to the calculations done above in (4.3) - (4.13) one derives for the conjugate function of  $g$ ,

$$g^*(x_1^*, \dots, x_n^*) = \min_{\substack{\sum_{i=1}^n z_i^{0*} = 1, \ z_i^{0*} \geq 0, \\ i=1, \dots, n}} \left\{ \sum_{i=1}^n [(z_i^{0*} h_i)^*(x_i^*) - z_i^{0*} a_i] \right\},$$

while its biconjugate is then given by

$$g^{**}(x_1, \dots, x_n) = g(x_1, \dots, x_n) = \max_{\substack{\sum_{i=1}^n z_i^{0*} = 1, \ z_i^{0*} \geq 0, \\ i=1, \dots, n}} \left\{ \sum_{i=1}^n z_i^{0*} [h_i(x_i) + a_i] \right\}$$

for all  $x_i \in \mathbb{R}$ ,  $i = 1, \dots, n$ .

In the following, let  $X$  be a Hausdorff locally convex space partially ordered by the convex cone  $K \subseteq X$  and  $X^*$  its topological dual space endowed with the weak\* topology  $w(X^*, X)$ . Further, let  $Y_i$  be another Hausdorff locally convex space partially ordered by the convex cone  $Q_i \subseteq Y_i$  and  $Y_i^*$  its topological dual space endowed with the weak\* topology  $w(Y_i^*, Y_i)$ . Now, we collect some properties of the *gauge function* (a.k.a. *Minkowski functional*) of the subset  $C \subseteq X$ ,  $\gamma_C : X \rightarrow \overline{\mathbb{R}}$  defined by

$$\gamma_C(x) := \begin{cases} \inf\{\lambda > 0 : x \in \lambda C\}, & \text{if } \{\lambda > 0 : x \in \lambda C\} \neq \emptyset, \\ +\infty, & \text{otherwise.} \end{cases}$$

When in the literature the question of continuity of the gauge function arises, then it is often assumed that  $0_X \in \text{int } C$  (see [2, 7, 24, 47, 57, 83, 84]). We start with a statement where this assumption is weakened to  $0_X \in C$ .

**Theorem 4.1.** *Let  $C \subseteq X$  be a convex and closed set with  $0_X \in C$ , then the gauge function  $\gamma_C$  is proper, convex and lower semicontinuous.*

**Proof.** Let us define the function  $g : X^* \rightarrow \overline{\mathbb{R}}$  by

$$g(x^*) := \begin{cases} 0, & \text{if } \sigma_C(x^*) \leq 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

It is obvious that  $g$  is proper, convex and lower semicontinuous. For the corresponding conjugate function of  $g$  one has

$$g^*(x) = \sup_{x^* \in X^*} \{\langle x^*, x \rangle - g(x^*)\} = \sup_{\substack{x^* \in X^*, \\ \sigma_C(x^*) \leq 1}} \langle x^*, x \rangle.$$

There is  $g^*(x) = \sup_{x^* \in X^*} \{\langle x^*, x \rangle - g(x^*)\} \geq \langle 0_{X^*}, x \rangle - g(0_{X^*}) = 0$  since  $g(0_{X^*}) = 0$  for all  $x \in X$ , and  $g^*(0_X) = \sup_{x^* \in X^*} \{-g(x^*)\} = 0$ , i.e.  $g^*$  is proper. At this point it is important to say that from  $0_X \in C$  follows that  $\gamma_C(0_X) = 0$ , i.e.  $g^*(0_X) = \gamma_C(0_X)$ .

Let us now assume that  $x \neq 0_X$  and consider for fixed  $x \in X$  the following convex optimization problem

$$(P^\gamma) \quad \inf_{\substack{x^* \in X^*, \\ \sigma_C(x^*) \leq 1}} \langle -x^*, x \rangle.$$

As  $\sigma_C(0_{X^*}) = 0 < 1$ , the Slater condition is fulfilled and hence, it holds strong duality between the problem  $(P^\gamma)$  and its corresponding Lagrange dual problem

$$(D_L^\gamma) \quad \sup_{\lambda \geq 0} \inf_{x^* \in X^*} \{\langle -x^*, x \rangle + \lambda(\sigma_C(x^*) - 1)\}.$$

Therefore, the conjugate function of  $g$  can be represented for  $x \neq 0_X$  as

$$\begin{aligned} g^*(x) &= \sup_{\substack{x^* \in X^*, \\ \sigma_C(x^*) \leq 1}} \langle x^*, x \rangle = -\max_{\lambda \geq 0} \inf_{x^* \in X^*} \{\langle -x^*, x \rangle + \lambda(\sigma_C(x^*) - 1)\} \\ &= \min_{\lambda \geq 0} \left\{ \lambda + \sup_{x^* \in X^*} \{\langle x^*, x \rangle - \lambda \sigma_C(x^*)\} \right\}. \end{aligned} \quad (4.15)$$

For  $\lambda = 0$  we verify two conceivable cases.

(a) If  $\sigma_C(x^*) < +\infty$ , then  $0 \cdot \sigma_C(x^*) = 0$  and therefore,

$$\sup_{x^* \in X^*} \{\langle x^*, x \rangle - 0 \cdot \sigma_C(x^*)\} = \sup_{x^* \in X^*} \langle x^*, x \rangle = \begin{cases} 0, & \text{if } x = 0_X, \\ +\infty, & \text{if } x \neq 0_X. \end{cases}$$

As by assumption  $x \neq 0_X$ , we have  $\sup_{x^* \in X^*} \langle x^*, x \rangle = +\infty$ , but this has no effect on the minimum in 4.15.

(b) If  $\sigma_C(x^*) = +\infty$ , then one has by convention that  $\lambda \cdot \sigma_C(x^*) = 0 \cdot (+\infty) = +\infty$  and hence,

$$\langle x^*, x \rangle - \lambda \sigma_C(x^*) = \langle x^*, x \rangle - \infty = -\infty,$$

which has no effect on  $\sup_{x^* \in X^*} \{\langle x^*, x \rangle - \lambda \sigma_C(x^*)\}$ , since  $\sigma_C$  is proper.

Hence, as the cases (a) and (b) are not relevant for  $g^*$ , we can omit the situation when  $\lambda = 0$  and can write

$$g^*(x) = \inf_{\lambda > 0} \left\{ \lambda + \lambda \sup_{x^* \in X^*} \left\{ \left\langle x^*, \frac{1}{\lambda} x \right\rangle - \sigma_C(x^*) \right\} \right\}.$$

Moreover, as  $C$  is a non-empty, closed and convex subset of  $X$ , the conjugate of the support function  $\sigma_C$  is the indicator function  $\delta_C$ , i.e.

$$g^*(x) = \inf_{\lambda > 0} \left\{ \lambda + \lambda \delta_C \left( \frac{1}{\lambda} x \right) \right\} = \inf_{\lambda > 0, \frac{1}{\lambda} x \in C} \lambda = \inf \{ \lambda > 0 : x \in \lambda C \}.$$

Taking the situations where  $x = 0_X$  and  $x \neq 0_X$  together implies that  $g^*(x) = \gamma_C(x)$  for all  $x \in X$ . Hence,  $\gamma_C$  is the conjugate function of  $g$  and by the definition of the conjugate function it follows that  $\gamma_C$  is convex and lower semicontinuous. This completes the proof.  $\square$

**Lemma 4.3.** *Let  $C \subseteq X$  be a convex and closed set with  $0_X \in C$ , then the conjugate of the gauge function  $\gamma_C$  is given by*

$$\gamma_C^*(x^*) := \begin{cases} 0, & \text{if } \sigma_C(x^*) \leq 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

**Proof.** In the proof of Theorem 4.1 we have shown that  $\gamma_C$  is the conjugate function of  $g$ , i.e.  $\gamma_C = g^*$ , and as  $g$  is proper, convex and lower semicontinuous we have  $g = g^{**}$ . As  $g^{**}$  is also the conjugate function of  $\gamma_C$ , it holds  $\gamma_C^* = g$ .  $\square$

**Remark 4.3.** (see [47, 57]) *Let  $C$  be convex and  $0_X \in \text{int } C$ , then the gauge function  $\gamma_C$  is not only convex but also sublinear and the following properties holds*

$$\begin{aligned} \gamma_C(x) &\geq 0 \quad \forall x \in X, \\ \gamma_C(0_X) &= 0, \\ \gamma_C(\mu x) &= \mu \gamma_C(x) \quad \forall \mu \geq 0, \quad \forall x \in X, \\ \gamma_C(x_1 + x_2) &\leq \gamma_C(x_1) + \gamma_C(x_2) \quad \forall x_1, x_2 \in X. \end{aligned}$$

Moreover,  $\gamma_C$  is well-defined, which means that  $\text{dom } \gamma_C = X$ , as well as continuous and

$$\text{int } C = \{x \in X : \gamma_C(x) < 1\}, \quad \text{cl}(C) = \{x \in X : \gamma_C(x) \leq 1\}$$

(see [47]).

**Remark 4.4.** *Let  $C_i \subseteq Y_i$  be a closed and convex set with  $0_{Y_i} \in \text{int } C_i$  and  $\gamma_{C_i} : Y_i \rightarrow \mathbb{R}$  be a gauge function of the set  $C_i$ ,  $i = 1, \dots, n$ . Then  $\gamma_{C_i}$  is continuous,  $i = 1, \dots, n$ , and moreover, it is an easy exercise to check that the function  $\gamma_C : Y_1 \times \dots \times Y_n \rightarrow \mathbb{R}$  defined by  $\gamma_C(x_1, \dots, x_n) := \sum_{i=1}^n \gamma_{C_i}(x_i)$ , is a gauge function fulfilling the properties listed in Remark 4.3. Especially, it holds that  $\gamma_C$  is continuous such that  $C := \{(x_1, \dots, x_n) \in Y_1 \times \dots \times Y_n : \gamma_C(x_1, \dots, x_n) \leq 1\}$ .*

**Definition 4.1.** *Let  $C \subseteq X$ . The polar set of  $C$  is defined by*

$$C^0 := \left\{ x^* \in X^* : \sup_{x \in C} \langle x^*, x \rangle \leq 1 \right\} = \{x^* \in X^* : \sigma_C(x^*) \leq 1\}$$

and by means of the polar set the dual gauge is defined by

$$\gamma_{C^0}(x^*) := \sup_{x \in C} \langle x^*, x \rangle = \sigma_C(x^*).$$

**Remark 4.5.** *Note that  $C^0$  is a convex and closed set containing the origin and by the definition of the dual gauge follows that the conjugate function of  $\gamma_C$  can equivalently be expressed by*

$$\gamma_C^*(x^*) := \begin{cases} 0, & \text{if } \gamma_{C^0}(x^*) \leq 1, \\ +\infty, & \text{otherwise.} \end{cases} = \begin{cases} 0, & \text{if } x^* \in C^0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Furthermore, if  $C$  is a convex cone, then  $C^0 = \{x^* \in X^* : \sigma_C(x^*) \leq 0\}$ , i.e.  $-C^0$  is the dual cone of  $C$ .

**Lemma 4.4.** *Let  $\gamma_{C_i} : Y_i \rightarrow \mathbb{R}$  be a gauge of the closed and convex set  $C_i \subseteq Y_i$  with  $0_{Y_i} \in \text{int } C_i$ ,  $i = 1, \dots, n$ . If the gauge  $\gamma_C : Y_1 \times \dots \times Y_n \rightarrow \mathbb{R}$  is defined by*

$$\gamma_C(x) := \sum_{i=1}^n \gamma_{C_i}(x_i), \quad x = (x_1, \dots, x_n) \in Y_1 \times \dots \times Y_n,$$

then its associated dual gauge  $\gamma_{C^0} : Y_1^* \times \dots \times Y_n^* \rightarrow \mathbb{R}$  is given by

$$\gamma_{C^0}(x^*) = \max_{1 \leq i \leq n} \left\{ \gamma_{C_i^0}(x_i^*) \right\}, \quad x^* = (x_1^*, \dots, x_n^*) \in Y_1^* \times \dots \times Y_n^*. \quad (4.16)$$

**Proof.** As  $C_i$  is closed, convex and  $0_{Y_i} \in \text{int } C_i$ , the gauge  $\gamma_{C_i}$  is continuous, convex and well-defined,  $i = 1, \dots, n$ , and thus, the gauge  $\gamma_C$  is also continuous, convex and well-defined. In the following, let  $\tilde{X}^* = Y_1^* \times \dots \times Y_n^*$  be the topological dual space of  $\tilde{X} := Y_1 \times \dots \times Y_n$  where for  $x = (x_1, \dots, x_n) \in \tilde{X}$  and  $x^* = (x_1^*, \dots, x_n^*) \in \tilde{X}^*$  we define  $\langle x^*, x \rangle := \sum_{i=1}^n \langle x_i^*, x_i \rangle$ . Hence, for the associated dual gauge of  $\gamma_C$  holds

$$\gamma_{C^0}(x^*) = \sup_{x \in C} \langle x^*, x \rangle.$$

Now, we fix  $x^* \in \tilde{X}^*$  and consider the problem

$$(P^{\gamma^0}) \quad \inf_{x \in C} \langle -x^*, x \rangle = \inf_{x \in \tilde{X}, \gamma_C(x) \leq 1} \langle -x^*, x \rangle,$$

where its associated Lagrange dual problem is

$$\begin{aligned} (D_L^{\gamma^0}) \quad & \sup_{\lambda \geq 0} \inf_{x \in \tilde{X}} \{ \langle -x^*, x \rangle + \lambda(\gamma_C(x) - 1) \} = \sup_{\lambda \geq 0} \left\{ -\lambda + \inf_{x \in \tilde{X}} \{ \langle -x^*, x \rangle + \lambda\gamma_C(x) \} \right\} \\ & = \sup_{\lambda \geq 0} \left\{ -\lambda - \sup_{x \in \tilde{X}} \{ \langle x^*, x \rangle - \lambda\gamma_C(x) \} \right\} = \sup_{\lambda \geq 0} \{ -\lambda - (\lambda\gamma_C)^*(x^*) \}. \end{aligned} \quad (4.17)$$

For  $\lambda > 0$  it holds (see Lemma 4.3 and Remark 4.5)

$$\begin{aligned} (\lambda\gamma_C)^*(x^*) &= \sup_{x \in \tilde{X}} \{ \langle x^*, x \rangle - \lambda\gamma_C(x) \} = \sup_{\substack{x_i \in Y_i, \\ i=1, \dots, n}} \left\{ \sum_{i=1}^n \langle x_i^*, x_i \rangle - \lambda \sum_{i=1}^n \gamma_{C_i}(x_i) \right\} \\ &= \sum_{i=1}^n \sup_{x_i \in Y_i} \{ \langle x_i^*, x_i \rangle - \lambda\gamma_{C_i}(x_i) \} \\ &= \sum_{i=1}^n \lambda\gamma_{C_i}^* \left( \frac{1}{\lambda} x_i^* \right) = \begin{cases} 0, & \text{if } \sigma_{C_i}(x_i^*) \leq \lambda \ \forall i = 1, \dots, n, \\ +\infty, & \text{otherwise} \end{cases} \\ &= \begin{cases} 0, & \text{if } \gamma_{C_i^0}(x_i^*) \leq \lambda \ \forall i = 1, \dots, n, \\ +\infty, & \text{otherwise} \end{cases} \end{aligned} \quad (4.18)$$

and for  $\lambda = 0$  we have

$$(0 \cdot \gamma_C)^*(x^*) = \sup_{x \in \tilde{X}} \{ \langle x^*, x \rangle \} = \begin{cases} 0, & \text{if } x_i^* = 0_{Y_i^*} \ \forall i = 1, \dots, n, \\ +\infty, & \text{otherwise.} \end{cases} \quad (4.19)$$

As  $\gamma_{C_i^0}(0_{Y_i^*}) = \sup_{x_i \in C_i} \langle 0_{Y_i^*}, x_i \rangle = 0$ , one gets by (4.18) and (4.19) for the Lagrange dual problem  $(D_L^{\gamma^0})$  that

$$(D_L^{\gamma^0}) \quad \sup_{\lambda \geq 0} \{ -\lambda - (\lambda\gamma_C)^*(x^*) \} = \sup_{\lambda \geq 0} \left\{ -\lambda : \gamma_{C_i^0}(x_i^*) \leq \lambda \ \forall i = 1, \dots, n \right\}$$

and since for the primal-dual pair  $(P^{\gamma^0})$ -( $D_L^{\gamma^0}$ ) the Slater constraint qualification is fulfilled, it holds strong duality. From the last statement we derive an alternative formula for the dual gauge  $\gamma_{C^0}$ ,

$$\gamma_{C^0}(x^*) = \sup_{x \in C} \langle x^*, x \rangle = \min_{\lambda \geq 0} \left\{ \lambda : \gamma_{C_i^0}(x_i^*) \leq \lambda \ \forall i = 1, \dots, n \right\} = \max_{1 \leq i \leq n} \left\{ \gamma_{C_i^0}(x_i^*) \right\}. \quad \square$$

Now, it is natural to ask, whether the dual gauge of  $\max_{1 \leq i \leq n} \{ \gamma_{C_i}(\cdot) \}$  is  $\sum_{i=1}^n \gamma_{C_i^0}(\cdot)$ . The next lemma gives a positive answer.

**Lemma 4.5.** Let  $\gamma_{C_i} : Y_i \rightarrow \mathbb{R}$  be a gauge of the closed and convex set  $C_i \subseteq Y_i$  with  $0_{Y_i} \in \text{int } C_i$ ,  $i = 1, \dots, n$ . If the gauge  $\gamma_C : Y_1 \times \dots \times Y_n \rightarrow \mathbb{R}$  is defined by

$$\gamma_C(x) := \max_{1 \leq i \leq n} \{\gamma_{C_i}(x_i)\}, \quad x = (x_1, \dots, x_n) \in Y_1 \times \dots \times Y_n,$$

then its associated dual gauge  $\gamma_{C^0} : Y_1^* \times \dots \times Y_n^* \rightarrow \mathbb{R}$  is given by

$$\gamma_{C^0}(x^*) = \sum_{i=1}^n \gamma_{C_i^0}(x_i^*), \quad x^* = (x_1^*, \dots, x_n^*) \in Y_1^* \times \dots \times Y_n^*. \quad (4. 20)$$

**Proof.** The main ideas here are similar to the ones in the proof of Lemma 4.4. As  $\gamma_C$  is the pointwise maximum of  $n$  continuous, convex and well-defined gauges, it is clear that  $\gamma_C$  is continuous, convex and well-defined and for the corresponding dual gauge of  $\gamma_C$  holds  $\gamma_{C^0}(x^*) = \sup_{x \in C} \{\langle x^*, x \rangle\}$ .

For fixed  $x^* := (x_1^*, \dots, x_n^*) \in \tilde{X}^* = Y_1^* \times \dots \times Y_n^*$  we consider the problem

$$(\tilde{P}^{\gamma^0}) \quad \inf_{x \in C} \langle -x^*, x \rangle = \inf_{x \in \tilde{X}, \gamma_C(x) \leq 1} \langle -x^*, x \rangle,$$

with its Lagrange dual problem (see (4. 17))

$$(\tilde{D}_L^{\gamma^0}) \quad \sup_{\lambda \geq 0} \{-\lambda - (\lambda \gamma_C)^*(x^*)\}.$$

For  $\lambda \geq 0$  one has

$$(\lambda \gamma_C)^*(x^*) = \sup_{x \in \tilde{X}} \{\langle x^*, x \rangle - \lambda \gamma_C(x)\} = \sup_{\substack{x_i \in Y_i, \\ i=1, \dots, n}} \left\{ \sum_{i=1}^n \langle x_i^*, x_i \rangle - \lambda \max_{1 \leq i \leq n} \{\gamma_{C_i}(x_i)\} \right\}.$$

Now, let  $X_0 := \mathbb{R}^n$ ,  $K_0 = \mathbb{R}_+^n$ ,  $X_1 = \tilde{X}$ , the function  $f : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$  be defined by

$$f(y_1^0, \dots, y_n^0) := \begin{cases} \max\{y_1^0, \dots, y_n^0\}, & \text{if } y_i^0 \in \mathbb{R}_+, \quad i = 1, \dots, n, \\ +\infty, & \text{otherwise,} \end{cases}$$

and the function  $F^1 : X_1 \rightarrow \mathbb{R}^n$  by

$$F^1(x_1, \dots, x_n) := (\gamma_{C_1}(x_1), \dots, \gamma_{C_n}(x_n))^T$$

Hence, the gauge  $\gamma_C$  can be written as

$$\gamma_C(x_1, \dots, x_n) = (f \circ F^1)(x_1, \dots, x_n).$$

Obviously,  $f$  is proper, convex, lower semicontinuous and  $\mathbb{R}_+^n$ -increasing on  $F^1(\text{dom } F^1) + K_0 \subseteq \mathbb{R}_+^n$ , the function  $F^1$  is proper,  $\mathbb{R}_+^n$ -epi closed and  $\mathbb{R}_+^n$ -convex as well as  $0_{\mathbb{R}^n} \in \text{ri}(F^1(\text{dom } F^1) - \text{dom } f + K_0) = \mathbb{R}^n$  and thus, it follows by Theorem 3.6 that

$$\gamma_C^*(x_1^*, \dots, x_n^*) = \min_{\substack{y_i^{0*} \in \mathbb{R}_+, \\ i=1, \dots, n}} \{f^*(y_1^{0*}, \dots, y_n^{0*}) + ((y_1^{0*}, \dots, y_n^{0*})^T F^1)^*(x_1^*, \dots, x_n^*)\}.$$

From (4. 3) we have for  $a_i = 0$ ,  $i = 1, \dots, n$ , that

$$\begin{aligned} f^*(y_1^{0*}, \dots, y_n^{0*}) &= \begin{cases} 0, & \text{if } \sum_{i=1}^n \lambda_i \leq 1, \quad \lambda_i \geq 0, \quad y_i^{0*} \leq \lambda_i, \quad i = 1, \dots, n, \\ +\infty, & \text{otherwise,} \end{cases} \\ &= \begin{cases} 0, & \text{if } \sum_{i=1}^n y_i^{0*} \leq 1, \quad y_i^{0*} \geq 0, \quad i = 1, \dots, n, \\ +\infty, & \text{otherwise.} \end{cases} \end{aligned} \quad (4. 21)$$

In addition, it holds

$$((y_1^{0*}, \dots, y_n^{0*})^T F^1)^*(x_1^*, \dots, x_n^*) = \sum_{i=1}^n \sup_{x_i \in Y_i} \{\langle x_i^*, x_i \rangle - y_i^{0*} \gamma_{C_i}(x_i)\} = \sum_{i=1}^n (y_i^{0*} \gamma_{C_i})^*(x_i^*).$$

For  $y_i^{0*} > 0$  holds

$$(y_i^{0*} \gamma_{C_i})^*(x_i^*) = \begin{cases} 0, & \text{if } \gamma_{C_i^0}(x_i^*) \leq y_i^{0*}, \\ +\infty, & \text{otherwise,} \end{cases}$$

and if  $y_i^{0*} = 0$ , then

$$(0 \cdot \gamma_{C_i})^*(x_i^*) = \sup_{x_i \in Y_i} \{\langle x_i^*, x_i \rangle\} = \begin{cases} 0, & \text{if } x_i^* = 0_{Y_i^*}, \\ +\infty, & \text{otherwise.} \end{cases}$$

This implies that

$$\begin{aligned} ((y_1^{0*}, \dots, y_n^{0*})^T F^1)^*(x_1^*, \dots, x_n^*) &= \sum_{i=1}^n (y_i^{0*} \gamma_{C_i})^*(x_i^*) \\ &= \begin{cases} 0, & \text{if } \gamma_{C_i^0}(x_i^*) \leq y_i^{0*}, \quad i = 1, \dots, n, \\ +\infty, & \text{otherwise,} \end{cases} \end{aligned} \quad (4.22)$$

and hence, one has by (4.21) and (4.22)

$$\begin{aligned} \gamma_C^*(x^*) &= \begin{cases} 0, & \text{if } \sum_{i=1}^n y_i^{0*} \leq 1, \quad y_i^{0*} \geq 0, \quad \gamma_{C_i^0}(x_i^*) \leq y_i^{0*}, \quad i = 1, \dots, n, \\ +\infty, & \text{otherwise.} \end{cases} \\ &= \begin{cases} 0, & \text{if } \sum_{i=1}^n \gamma_{C_i^0}(x_i^*) \leq 1, \\ +\infty, & \text{otherwise.} \end{cases} \end{aligned}$$

For  $\lambda > 0$  it follows

$$(\lambda \gamma_C)^*(x^*) = \lambda \gamma_C^*\left(\frac{1}{\lambda} x^*\right) = \begin{cases} 0, & \text{if } \sum_{i=1}^n \gamma_{C_i^0}(x_i^*) \leq \lambda, \\ +\infty, & \text{otherwise.} \end{cases} \quad (4.23)$$

Moreover, by (4.19) follows for  $\lambda = 0$

$$(0 \cdot \gamma_C)^*(x^*) = \sum_{i=1}^n \sup_{x_i \in Y_i} \{\langle x_i^*, x_i \rangle\} = \begin{cases} 0, & \text{if } x_i^* = 0_{Y_i^*} \quad \forall i = 1, \dots, n, \\ +\infty, & \text{otherwise.} \end{cases} \quad (4.24)$$

and as  $\sum_{i=1}^n \gamma_{C_i^0}(0_{Y_i^*}) = \sum_{i=1}^n \sup\{\langle 0_{Y_i^*}, x_i \rangle\} = 0$ , we have by (4.23) and (4.24) for the Lagrange dual problem

$$(\tilde{D}_L^{\gamma^0}) \sup_{\lambda \geq 0} \{-\lambda - (\lambda \gamma_C)^*(x^*)\} = \sup_{\lambda \geq 0} \left\{ -\lambda : \sum_{i=1}^n \gamma_{C_i^0}(x_i^*) \leq \lambda \right\}.$$

It is obvious that the Slater constraint qualification for the primal-dual problem  $(\tilde{P}^{\gamma^0}) - (\tilde{D}_L^{\gamma^0})$  is fulfilled and thus, strong duality holds, i.e.,

$$\gamma_{C^0}(x^*) = \sup_{x \in C} \{\langle x^*, x \rangle\} = \min_{\lambda \geq 0} \left\{ \lambda : \sum_{i=1}^n \gamma_{C_i^0}(x_i^*) \leq \lambda \right\} = \sum_{i=1}^n \gamma_{C_i^0}(x_i^*). \quad \square$$

## 4.2 Single minimax location problems

### 4.2.1 Constrained location problems with set-up costs in Fréchet spaces

Let us now focus our discussion on problems with given non-negative set-up costs  $a_i \in \mathbb{R}_+$  and distinct points  $p_i$ ,  $i = 1, \dots, n$  (where  $n \geq 2$ ). Consider the following geometrically constrained minimax location problem

$$(P_{h,a}^S) \quad \inf_{x \in S} \max_{1 \leq i \leq n} \{h_i(\gamma_{C_i}(x - p_i)) + a_i\},$$

where

- $S$  is a non-empty, closed and convex subset of the Fréchet space  $X$ ,
- $C_i$  is a closed and convex subset of  $X$  such that  $0_X \in \text{int } C_i$  and
- $h_i : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  with  $h_i(x) \in \mathbb{R}_+$ , if  $x \in \mathbb{R}_+$ , and  $h_i(x) = +\infty$ , otherwise, is a proper, convex, lower semicontinuous and increasing function on  $\mathbb{R}_+$ ,  $i = 1, \dots, n$ .

Hence, it is clear that the defined gauges are continuous and convex functions, which implies that the problem  $(P_{h,a}^S)$  is a convex optimization problem. The case where the set-up costs are arbitrary, i.e.  $a_i \in \mathbb{R}$ , will be discussed in Remark 4.9.

For applying the duality concept developed in Chapter 3 for multi-composed optimization problems, we set  $X_0 = \mathbb{R}^n$  ordered by  $K_0 = \mathbb{R}_+^n$ ,  $X_1 = X^n$  ordered by the trivial cone  $K_1 = \{0_{X^n}\}$  and  $X_2 = X$  and introduce the following functions:

- $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$  defined by

$$f(y^0) := \begin{cases} \max_{1 \leq i \leq n} \{h_i(y_i^0) + a_i\}, & \text{if } y^0 = (y_1^0, \dots, y_n^0)^T \in \mathbb{R}_+^n, \quad i = 1, \dots, n, \\ +\infty, & \text{otherwise,} \end{cases}$$

- $F^1 : X^n \rightarrow \mathbb{R}^n$  defined by  $F^1(y^1) := (\gamma_{C_1}(y_1^1), \dots, \gamma_{C_n}(y_n^1))^T$  with  $y^1 = (y_1^1, \dots, y_n^1) \in X^n$  and
- $F^2 : X \rightarrow X^n$  defined by  $F^2(x) := (x - p_1, \dots, x - p_n)$ .

These definitions yield the following equivalent representation for the considered problem

$$(P_{h,a}^S) \quad \inf_{x \in S} (f \circ F^1 \circ F^2)(x).$$

The function  $f$  is proper, convex,  $\mathbb{R}_+^n$ -increasing on  $F^1(\text{dom } F^1) + K_0 = \text{dom } f = \mathbb{R}_+^n$  and lower semicontinuous. Additionally, one can verify that the function  $F^1$  is proper,  $\mathbb{R}_+^n$ -convex and  $\mathbb{R}_+^n$ -epi closed. Furthermore, since the function  $F^2$  is affine, it follows that the function  $F^1$  does not need to be monotone (see Remark 3.5).

By setting  $Z = X$  ordered by the trivial cone  $Q = X$  and defining the function  $g : X \rightarrow X$  by  $g(x) := x$ , we have that  $Q^* = \{0_{X^*}\}$ , i.e.  $z^{2*} = 0_{X^*}$ , and thus, the conjugate dual problem corresponding to  $(P_{h,a}^S)$ , in accordance with the concept from the previous chapter, looks like

$$(D_{h,a}^S) \quad \sup_{\substack{z_i^{0*} \in \mathbb{R}_+, \quad z_i^{1*} \in X^*, \\ i=1, \dots, n}} \left\{ \inf_{x \in S} \left\{ \sum_{i=1}^n \langle z_i^{1*}, x - p_i \rangle \right\} - f^*(z^{0*}) - (z^{0*} F^1)^*(z^{1*}) \right\},$$

where  $z^{0*} = (z_1^{0*}, \dots, z_n^{0*})^T \in \mathbb{R}_+^n$  and  $z^{1*} = (z_1^{1*}, \dots, z_n^{1*}) \in (X^*)^n$ . It remains to determine the conjugate functions of  $f$  and  $(z^{0*} F^1)$ . For the conjugate function of  $f$  one gets by Lemma 4.1

$$f^*(z_1^{0*}, \dots, z_n^{0*}) = \min_{\substack{\sum_{i=1}^n \lambda_i \leq 1, \quad \lambda_i \geq 0, \\ i=1, \dots, n}} \left\{ \sum_{i=1}^n [(\lambda_i h_i)^*(z_i^{0*}) - \lambda_i a_i] \right\},$$

while for the conjugate function of  $(z^{0*}F^1)$  we have

$$\begin{aligned} (z^{0*}F^1)^*(z^{1*}) &= \sup_{z_i^1 \in X, i=1, \dots, n} \left\{ \sum_{i=1}^n \langle z_i^{1*}, z_i^1 \rangle - \sum_{i=1}^n z_i^{0*} \gamma_{C_i}(z_i^1) \right\} \\ &= \sum_{i=1}^n \sup_{z_i^1 \in X} \{ \langle z_i^{1*}, z_i^1 \rangle - z_i^{0*} \gamma_{C_i}(z_i^1) \} = \sum_{i=1}^n (z_i^{0*} \gamma_{C_i})^*(z_i^{1*}). \end{aligned} \quad (4.25)$$

Therefore, the conjugate dual problem  $(D_{h,a}^S)$  turns into

$$(D_{h,a}^S) \sup_{\substack{\sum_{i=1}^n \lambda_i \leq 1, \lambda_i, z_i^{0*} \geq 0, \\ z_i^{1*} \in X^*, i=1, \dots, n}} \left\{ \inf_{x \in S} \left\{ \sum_{i=1}^n \langle z_i^{1*}, x - p_i \rangle \right\} - \sum_{i=1}^n [(\lambda_i h_i)^*(z_i^{0*}) - \lambda_i a_i] - \sum_{i=1}^n (z_i^{0*} \gamma_{C_i})^*(z_i^{1*}) \right\}.$$

By separating the sum  $\sum_{i=1}^n (\lambda_i h_i)^*$  into the terms with  $\lambda_i > 0$  and the terms with  $\lambda_i = 0$  as well as  $\sum_{i=1}^n (z_i^{0*} \gamma_{C_i})^*$  into the terms with  $z_i^{0*} > 0$  and the terms with  $z_i^{0*} = 0$  in  $(D_{h,a}^S)$ , the dual problem turns into

$$\begin{aligned} (D_{h,a}^S) \sup_{\substack{\lambda_i, z_i^{0*} \geq 0, z_i^{1*} \in X^*, i=1, \dots, n, \\ R = \{r \in \{1, \dots, n\} : \lambda_r > 0\}, I = \{i \in \{1, \dots, n\} : z_i^{0*} > 0\}, \\ \sum_{r \in R} \lambda_r \leq 1}} \left\{ \inf_{x \in S} \left\{ \sum_{i=1}^n \langle z_i^{1*}, x - p_i \rangle \right\} - \sum_{r \notin R} (0 \cdot h_r)^*(z_r^{0*}) \right. \\ \left. - \sum_{r \in R} [(\lambda_r h_r)^*(z_r^{0*}) - \lambda_r a_r] - \sum_{i \notin I} (0 \cdot \gamma_{C_i})^*(z_i^{1*}) - \sum_{i \in I} (z_i^{0*} \gamma_{C_i})^*(z_i^{1*}) \right\}. \end{aligned}$$

If  $i \in I$ , then we have (see Lemma 4.3 and Remark 4.5)

$$\begin{aligned} (z_i^{0*} \gamma_{C_i})^*(z_i^{1*}) &= z_i^{0*} \gamma_{C_i}^* \left( \frac{z_i^{1*}}{z_i^{0*}} \right) = \begin{cases} 0, & \text{if } \sigma_{C_i} \left( \frac{z_i^{1*}}{z_i^{0*}} \right) \leq 1, \\ +\infty, & \text{otherwise,} \end{cases} \\ &= \begin{cases} 0, & \text{if } \sigma_{C_i}(z_i^{1*}) \leq z_i^{0*}, \\ +\infty, & \text{otherwise,} \end{cases} = \begin{cases} 0, & \text{if } \gamma_{C_i^0}(z_i^{1*}) \leq z_i^{0*}, \\ +\infty, & \text{otherwise,} \end{cases} \end{aligned} \quad (4.26)$$

and if  $i \notin I$ , then it holds

$$(0 \cdot \gamma_{C_i})^*(z_i^{1*}) = \sup_{y_i^1 \in X} \{ \langle z_i^{1*}, y_i^1 \rangle \} = \begin{cases} 0, & \text{if } z_i^{1*} = 0_{X^*}, \\ +\infty, & \text{otherwise.} \end{cases} \quad (4.27)$$

Further, let us consider the case  $r \notin R$ , i.e.  $\lambda_r = 0$ , then one has for  $z_r^{0*} \geq 0$ ,

$$(0 \cdot h_r)^*(z_r^{0*}) = \sup_{y_r^0 \geq 0} \{ z_r^{0*} y_r^0 \} = \begin{cases} 0, & \text{if } z_r^{0*} = 0, \\ +\infty, & \text{otherwise.} \end{cases} \quad (4.28)$$

For  $r \in R$ , i.e.  $\lambda_r > 0$ , follows

$$(\lambda_r h_r)^*(z_r^{0*}) = \lambda_r h_r^* \left( \frac{z_r^{0*}}{\lambda_r} \right). \quad (4.29)$$

Hence, the equation in (4.28) implies that if  $r \notin R$ , then  $z_r^{0*} = 0$ , which means that  $I \subseteq R$ . In summary, the conjugate dual problem  $(D_{h,a}^S)$  becomes to

$$\begin{aligned} (D_{h,a}^S) \sup_{\substack{\lambda_i, z_i^{0*} \geq 0, z_i^{1*} \in X^*, i=1, \dots, n, \\ I = \{i \in \{1, \dots, n\} : z_i^{0*} > 0\} \subseteq R = \{r \in \{1, \dots, n\} : \lambda_r > 0\}, \\ \gamma_{C_i^0}(z_i^{1*}) \leq z_i^{0*}, i \in I, z_j^{1*} = 0_{X^*}, j \notin I, \sum_{r \in R} \lambda_r \leq 1}} \left\{ \inf_{x \in S} \left\{ \sum_{i \in I} \langle z_i^{1*}, x - p_i \rangle \right\} - \sum_{r \in R} \lambda_r \left[ h_r^* \left( \frac{z_r^{0*}}{\lambda_r} \right) - a_r \right] \right\}. \end{aligned} \quad (4.30)$$



**Remark 4.6.** If  $h_i : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  is defined by

$$h_i(x) := \begin{cases} x, & \text{if } x \in \mathbb{R}_+, \\ +\infty, & \text{otherwise,} \end{cases}$$

then the conjugate function of  $h_i$  is

$$h_i^*(x^*) = \begin{cases} 0, & \text{if } x^* \leq 1, \\ +\infty, & \text{otherwise,} \end{cases}, \quad i = 1, \dots, n,$$

and the conjugate dual problem  $(D_{h,a}^S)$  transforms to

$$(D_{h,a}^S) \quad \sup_{\substack{\lambda_i, z_i^{0*} \geq 0, z_i^{1*} \in X^*, i=1, \dots, n, \\ I = \{i \in \{1, \dots, n\} : z_i^{0*} > 0\} \subseteq R = \{r \in \{1, \dots, n\} : \lambda_r > 0\}, \\ z_r^{0*} \leq \lambda_r, r \in R, \gamma_{C_i^0}(z_i^{1*}) \leq z_i^{0*}, i \in I, z_j^{1*} = 0_{X^*}, j \notin I, \sum_{r \in R} \lambda_r \leq 1}} \left\{ \inf_{x \in S} \left\{ \sum_{i \in I} \langle z_i^{1*}, x - p_i \rangle \right\} + \sum_{r \in R} \lambda_r a_r \right\}.$$

This dual problem can be reduced to the following equivalent problem

$$(\tilde{D}_{h,a}^S) \quad \sup_{\substack{y_i^{0*} \geq 0, y_i^{1*} \in X^*, i=1, \dots, n, \tilde{I} = \{i \in \{1, \dots, n\} : y_i^{0*} > 0\}, \\ y_j^{1*} = 0_{X^*}, j \notin \tilde{I}, \gamma_{C_i^0}(y_i^{1*}) \leq y_i^{0*}, i \in \tilde{I}, \sum_{i \in \tilde{I}} y_i^{0*} \leq 1}} \left\{ \inf_{x \in S} \left\{ \sum_{i \in \tilde{I}} \langle y_i^{1*}, x - p_i \rangle \right\} + \sum_{i \in \tilde{I}} y_i^{0*} a_i \right\}. \quad (4.31)$$

To see the equivalence between  $(D_{h,a}^S)$  and  $(\tilde{D}_{h,a}^S)$ , take first a feasible element  $(\lambda, z^{0*}, z^{1*}) = (\lambda_1, \dots, \lambda_n, z_1^{0*}, \dots, z_n^{0*}, z_1^{1*}, \dots, z_n^{1*}) \in \mathbb{R}_+^n \times \mathbb{R}_+^n \times (X^*)^n$  of the problem  $(D_{h,a}^S)$  and set  $\tilde{I} = R$ ,  $y_i^{0*} = \lambda_i$ ,  $i \in \tilde{I}$ ,  $y_j^{0*} = 0$ ,  $j \notin \tilde{I}$  and  $y_i^{1*} = z_i^{1*}$ ,  $i \in I \subseteq \tilde{I}$ ,  $y_j^{1*} = 0_{X^*}$ ,  $j \notin I$  (i.e.  $y_i^{1*} \in X^*$ ,  $i \in \tilde{I}$  and  $y_j^{1*} = 0_{X^*}$ ,  $j \notin \tilde{I}$ ), then it follows from the feasibility of  $(\lambda, z^{0*}, z^{1*})$  that  $\sum_{i \in \tilde{I}} y_i^{0*} \leq 1$ ,  $y_i^{0*} > 0$ ,  $y_i^{1*} \in X^*$ ,  $\gamma_{C_i^0}(y_i^{1*}) \leq y_i^{0*}$ ,  $i \in \tilde{I}$  and  $y_j^{0*} = 0$ ,  $y_j^{1*} = 0_{X^*}$ ,  $j \notin \tilde{I}$ , i.e.  $(y^{0*}, y^{1*}) = (y_1^{0*}, \dots, y_n^{0*}, y_1^{1*}, \dots, y_n^{1*}) \in \mathbb{R}_+^n \times (X^*)^n$  is feasible to the problem  $(\tilde{D}_{h,a}^S)$ . Hence, it holds

$$\inf_{x \in S} \left\{ \sum_{i=1}^n \langle z_i^{1*}, x - p_i \rangle \right\} + \sum_{i=1}^n \lambda_i a_i = \inf_{x \in S} \left\{ \sum_{i=1}^n \langle y_i^{1*}, x - p_i \rangle \right\} + \sum_{i=1}^n y_i^{0*} a_i \leq v(\tilde{D}_{h,a}^S)$$

for all  $(\lambda, z^{0*}, z^{1*})$  feasible to  $(D_{h,a}^S)$ , i.e.  $v(D_{h,a}^S) \leq v(\tilde{D}_{h,a}^S)$  (where  $v(D_{h,a}^S)$  and  $v(\tilde{D}_{h,a}^S)$  denote the optimal objective values of the dual problems  $(D_{h,a}^S)$  and  $(\tilde{D}_{h,a}^S)$ , respectively).

Now, take a feasible element  $(y^{0*}, y^{1*})$  of the problem  $(\tilde{D}_{h,a}^S)$  and set  $\tilde{I} = I = R$ ,  $z_i^{0*} = \lambda_i = y_i^{0*}$  and  $z_i^{1*} = y_i^{1*}$  for  $i \in I = R$  and  $z_j^{0*} = \lambda_j = 0$  for  $j \notin I = R$ , then we have from the feasibility of  $(y^{0*}, y^{1*})$  that  $\sum_{r \in R} \lambda_r \leq 1$ ,  $z_k^{0*} = \lambda_k > 0$ ,  $k \in R$ ,  $\lambda_l = 0$ ,  $l \notin R$  and  $\gamma_{C_i^0}(z_i^{1*}) \leq z_i^{0*}$ ,  $i \in I$ , which means that  $(\lambda, z^{0*}, z^{1*})$  is a feasible element of  $(D_{h,a}^S)$  and it holds

$$\inf_{x \in S} \left\{ \sum_{i=1}^n \langle y_i^{1*}, x - p_i \rangle \right\} + \sum_{i=1}^n y_i^{0*} a_i = \inf_{x \in S} \left\{ \sum_{i=1}^n \langle z_i^{1*}, x - p_i \rangle \right\} + \sum_{i=1}^n \lambda_i a_i \leq v(D_{h,a}^S)$$

for all  $(y^{0*}, y^{1*})$  feasible to  $(\tilde{D}_{h,a}^S)$ , which implies  $v(\tilde{D}_{h,a}^S) \leq v(D_{h,a}^S)$ . Finally, it follows that  $v(\tilde{D}_{h,a}^S) = v(D_{h,a}^S)$ .

**Remark 4.7.** The index sets  $I$  and  $R$  of the dual problem  $(D_{h,a}^S)$  in (4.30) give a detailed characterization of the set of feasible solutions and are very useful in the further approach. But from the numerical aspect, these index sets make the dual in (4.30) very hard to solve, as they transform it into a discrete optimization problem.

For this reason we prefer to use for theoretical approaches the dual  $(D_{h,a}^S)$  in the form of (4. 30) and for numerical studies its equivalent dual problem  $(\widehat{D}_{h,a}^S)$ :

$$(\widehat{D}_{h,a}^S) \sup_{\substack{\lambda_i, z_i^{0*} \geq 0, z_i^{1*} \in X^*, \gamma_{C_i^0}(z_i^{1*}) \leq z_i^{0*}, \\ i=1, \dots, n, \sum_{i=1}^n \lambda_i \leq 1}} \left\{ \inf_{x \in S} \left\{ \sum_{i=1}^n \langle z_i^{1*}, x - p_i \rangle \right\} - \sum_{i=1}^n [(\lambda_i h_i)^*(z_i^{0*}) - \lambda_i a_i] \right\}. \quad (4. 32)$$

The equivalence of the dual problems  $(D_{h,a}^S)$  and  $(\widehat{D}_{h,a}^S)$  can easily be proven as follows.

Let  $(\lambda_1, \dots, \lambda_n, z_1^{0*}, \dots, z_n^{0*}, z_1^{1*}, \dots, z_n^{1*})$  be a feasible solution of  $(\widehat{D}_{h,a}^S)$ , then it follows from  $r \notin R = \{r \in \{1, \dots, n\} : \lambda_r > 0\}$  by (4. 28) that  $z_r^{0*} = 0$ , i.e.  $I = \{i \in \{1, \dots, n\} : z_i^{0*} > 0\} \subseteq R$ , and for  $i \notin I$  we have (see Remark 4.5)  $0 \leq \gamma_{C_i^0}(z_i^{1*}) \leq 0 \Leftrightarrow z_i^{1*} = 0_{X^*}$ . This means that  $(\lambda_1, \dots, \lambda_n, z_1^{0*}, \dots, z_n^{0*}, z_1^{1*}, \dots, z_n^{1*})$  is also feasible to  $(D_{h,a}^S)$  and by (4. 28) and (4. 29) follows immediately that  $v(\widehat{D}_{h,a}^S) = v(D_{h,a}^S)$ .

Conversely, by the previous considerations it is clear that any feasible solution of  $(D_{h,a}^S)$  is also a feasible solution of  $(\widehat{D}_{h,a}^S)$  such that  $v(D_{h,a}^S) = v(\widehat{D}_{h,a}^S)$ .

In this context, the dual of  $(\widetilde{D}_{h,a}^S)$  in (4. 31) looks like

$$(\widetilde{D}_{h,a}^S) \sup_{\substack{z_i^{0*} \geq 0, z_i^{1*} \in X^*, \gamma_{C_i^0}(y_i^{1*}) \leq y_i^{0*}, \\ i=1, \dots, n, \sum_{i=1}^n z_i^{0*} \leq 1}} \left\{ \inf_{x \in S} \left\{ \sum_{i=1}^n \langle z_i^{1*}, x - p_i \rangle \right\} + \sum_{i=1}^n z_i^{0*} a_i \right\}.$$

The weak duality between the primal-dual pair  $(P_{h,a}^S)-(D_{h,a}^S)$  always holds, i.e.  $v(P_{h,a}^S) \geq v(D_{h,a}^S)$ .

Our aim is now to verify whether strong duality holds. For this purpose, we verify the fulfillment of the generalized interior point regularity condition  $(RC_2^C)$ , which was imposed in the Section 3.2. Let us recall that  $f$  is lower semicontinuous,  $K_0 = \mathbb{R}_+^n$  is closed,  $S$  is closed and  $F^1$  is  $\mathbb{R}_+^n$ -epi closed.

As the function  $g : X \rightarrow X$  is defined by  $g(x) := x$ , it follows that  $g$  is continuous, thus also  $Q$ -epi closed and

$$0_X \in \text{sqli}(g(X \cap S) + Q) = \text{sqli}(S + X) = X.$$

Moreover, it holds

$$0_{\mathbb{R}^n} \in \text{sqli}(F^1(\text{dom } F^1) - \text{dom } f + K_0) = \text{sqli}(F^1(\text{dom } F^1) - \mathbb{R}_+^n + \mathbb{R}_+^n) = \mathbb{R}^n$$

and

$$0_{X^n} \in \text{sqli}(F^2(\text{dom } F^2 \cap \text{dom } g \cap S) - \text{dom } F^1 + K_1) = \text{sqli}(F^2(S) - X^n + \{0_{X^n}\}) = X^n.$$

Finally, as  $F^2$  is  $\{0_{X^n}\}$ -epi closed, the regularity condition is obviously fulfilled and we can state the following theorem as a consequence of Theorem 3.3.

**Theorem 4.2.** (strong duality) Between  $(P_{h,a}^S)$  and  $(D_{h,a}^S)$  strong duality holds, i.e.  $v(P_{h,a}^S) = v(D_{h,a}^S)$  and the conjugate dual problem has an optimal solution.

The following necessary and sufficient optimality conditions are a consequence of the previous theorem.

**Theorem 4.3.** (optimality conditions) (a) Let  $\bar{x} \in S$  be an optimal solution of the problem  $(P_{h,a}^S)$ . Then there exist  $(\bar{\lambda}_1, \dots, \bar{\lambda}_n, \bar{z}_1^{0*}, \dots, \bar{z}_n^{0*}, \bar{z}_1^{1*}, \dots, \bar{z}_n^{1*}) \in \mathbb{R}_+^n \times \mathbb{R}_+^n \times (X^*)^n$  and index sets  $\bar{I} \subseteq \bar{R} \subseteq \{1, \dots, n\}$  as an optimal solution to  $(D_{h,a}^S)$  such that

- (i)  $\max_{1 \leq j \leq n} \{h_j(\gamma_{C_j}(\bar{x} - p_j)) + a_j\} = \sum_{i \in \bar{I}} \bar{z}_i^{0*} \gamma_{C_i}(\bar{x} - p_i) - \sum_{r \in \bar{R}} \bar{\lambda}_r \left[ h_r^* \left( \frac{\bar{z}_r^{0*}}{\bar{\lambda}_r} \right) - a_r \right]$   
 $= \sum_{r \in \bar{R}} \bar{\lambda}_r [h_r(\gamma_{C_r}(\bar{x} - p_r)) + a_r],$
- (ii)  $\bar{\lambda}_r h_r^* \left( \frac{\bar{z}_r^{0*}}{\bar{\lambda}_r} \right) + \bar{\lambda}_r h_r(\gamma_{C_r}(\bar{x} - p_r)) = \bar{z}_r^{0*} \gamma_{C_r}(\bar{x} - p_r) \quad \forall r \in \bar{R},$
- (iii)  $\bar{z}_i^{0*} \gamma_{C_i}(\bar{x} - p_i) = \langle \bar{z}_i^{1*}, \bar{x} - p_i \rangle \quad \forall i \in \bar{I},$
- (iv)  $\sum_{i \in \bar{I}} \langle \bar{z}_i^{1*}, \bar{x} \rangle = -\sigma_S \left( -\sum_{i \in \bar{I}} \bar{z}_i^{1*} \right),$
- (v)  $\max_{1 \leq j \leq n} \{h_j(\gamma_{C_j}(\bar{x} - p_j)) + a_j\} = h_r(\gamma_{C_r}(\bar{x} - p_r)) + a_r \quad \forall r \in \bar{R},$
- (vi)  $\sum_{r \in \bar{R}} \bar{\lambda}_r = 1, \bar{\lambda}_k > 0, k \in \bar{R}, \bar{\lambda}_l = 0, l \notin \bar{R}, \bar{z}_i^{0*} > 0, i \in \bar{I}, \text{ and } \bar{z}_j^{0*} = 0, j \notin \bar{I},$
- (vii)  $\gamma_{C_i}^0(\bar{z}_i^{1*}) = \bar{z}_i^{0*}, \bar{z}_i^{1*} \in X^* \setminus \{0_{X^*}\}, i \in \bar{I} \text{ and } \bar{z}_j^{1*} = 0_{X^*}, j \notin \bar{I}.$

(b) If there exists  $\bar{x} \in S$  such that for some  $(\bar{\lambda}_1, \dots, \bar{\lambda}_n, \bar{z}_1^{0*}, \dots, \bar{z}_n^{0*}, \bar{z}_1^{1*}, \dots, \bar{z}_n^{1*}) \in \mathbb{R}_+^n \times \mathbb{R}_+^n \times (X^*)^n$  and the index sets  $\bar{I} \subseteq \bar{R} \subseteq \{1, \dots, n\}$  the conditions (i)-(vii) are fulfilled, then  $\bar{x}$  is an optimal solution of  $(P_{h,a}^S)$ ,  $(\bar{\lambda}_1, \dots, \bar{\lambda}_n, \bar{z}_1^{0*}, \dots, \bar{z}_n^{0*}, \bar{z}_1^{1*}, \dots, \bar{z}_n^{1*}, \bar{I}, \bar{R})$  is an optimal solution for  $(D_{h,a}^S)$  and  $v(P_{h,a}^S) = v(D_{h,a}^S)$ .

**Proof.** (a) By using Theorem 3.4 we derive the following necessary and sufficient optimality conditions

- (i)  $\max_{1 \leq j \leq n} \{h_j(\gamma_{C_j}(\bar{x} - p_j)) + a_j\} + \sum_{r \in \bar{R}} \bar{\lambda}_r \left[ h_r^* \left( \frac{\bar{z}_r^{0*}}{\bar{\lambda}_r} \right) - a_r \right] = \sum_{i \in \bar{I}} \bar{z}_i^{0*} \gamma_{C_i}(\bar{x} - p_i),$
- (ii)  $\sum_{i \in \bar{I}} \bar{z}_i^{0*} \gamma_{C_i}(\bar{x} - p_i) = \sum_{i \in \bar{I}} \langle \bar{z}_i^{1*}, \bar{x} - p_i \rangle,$
- (iii)  $\sum_{i \in \bar{I}} \langle \bar{z}_i^{1*}, \bar{x} \rangle + \sigma_S \left( -\sum_{i \in \bar{I}} \bar{z}_i^{1*} \right) = 0,$
- (iv)  $\sum_{r \in \bar{R}} \bar{\lambda}_r \leq 1, \bar{\lambda}_k > 0, k \in \bar{R}, \bar{\lambda}_l = 0, l \notin \bar{R}, \bar{z}_i^{0*} > 0, i \in \bar{I}, \text{ and } \bar{z}_j^{0*} = 0, j \notin \bar{I},$
- (v)  $\gamma_{C_i}^0(\bar{z}_i^{1*}) \leq \bar{z}_i^{0*}, \bar{z}_i^{1*} \in X^*, i \in \bar{I} \text{ and } \bar{z}_j^{1*} = 0_{X^*}, j \notin \bar{I},$

where case (iii) arises from condition (iii) of Theorem 3.4 by the following observation (note that  $\bar{z}^{2*} = 0_{X^*}$ )

$$\begin{aligned} & \sum_{i \in \bar{I}} \langle \bar{z}_i^{1*}, \bar{x} - p_i \rangle + (\bar{z}^{2*} g)(\bar{x}) + \sup_{x \in S} \left\{ -\sum_{i \in \bar{I}} \langle \bar{z}_i^{1*}, x - p_i \rangle - \langle \bar{z}^{2*}, g(x) \rangle \right\} = 0 \\ \Leftrightarrow & \sum_{i \in \bar{I}} \langle \bar{z}_i^{1*}, \bar{x} \rangle - \sum_{i \in \bar{I}} \langle \bar{z}_i^{1*}, p_i \rangle + \sup_{x \in S} \left\{ -\sum_{i \in \bar{I}} \langle \bar{z}_i^{1*}, x \rangle \right\} + \sum_{i \in \bar{I}} \langle \bar{z}_i^{1*}, p_i \rangle = 0 \\ \Leftrightarrow & \sum_{i \in \bar{I}} \langle \bar{z}_i^{1*}, \bar{x} \rangle + \sup_{x \in S} \left\{ -\sum_{i \in \bar{I}} \langle \bar{z}_i^{1*}, x \rangle \right\} = 0. \end{aligned}$$

Additionally, one has by Theorem 4.2 that  $v(P_{h,a}^S) = v(D_{h,a}^S)$ , i.e.

$$\begin{aligned}
& \max_{1 \leq j \leq n} \{h_j(\gamma_{C_j}(\bar{x} - p_j)) + a_j\} = \inf_{x \in S} \left\{ \sum_{i \in \bar{I}} \langle \bar{z}_i^{1*}, x - p_i \rangle \right\} - \sum_{r \in \bar{R}} \bar{\lambda}_r \left[ h_r^* \left( \frac{\bar{z}_r^{0*}}{\bar{\lambda}_r} \right) - a_r \right] \\
& \Leftrightarrow \max_{1 \leq j \leq n} \{h_j(\gamma_{C_j}(\bar{x} - p_j)) + a_j\} + \sigma_S \left( - \sum_{i \in \bar{I}} \bar{z}_i^{1*} \right) + \sum_{i \in \bar{I}} \langle \bar{z}_i^{1*}, p_i \rangle \\
& \quad + \sum_{r \in \bar{R}} \bar{\lambda}_r \left[ h_r^* \left( \frac{\bar{z}_r^{0*}}{\bar{\lambda}_r} \right) - a_r \right] = 0 \\
& \Leftrightarrow \max_{1 \leq j \leq n} \{h_j(\gamma_{C_j}(\bar{x} - p_j)) + a_j\} + \sigma_S \left( - \sum_{i \in \bar{I}} \bar{z}_i^{1*} \right) + \sum_{i \in \bar{I}} \langle \bar{z}_i^{1*}, p_i \rangle + \sum_{r \in \bar{R}} \bar{\lambda}_r \left[ h_r^* \left( \frac{\bar{z}_r^{0*}}{\bar{\lambda}_r} \right) - a_r \right] \\
& \quad + \sum_{r \in \bar{R}} \bar{\lambda}_r h_r(\gamma_{C_r}(\bar{x} - p_r)) - \sum_{r \in \bar{R}} \bar{\lambda}_r h_r(\gamma_{C_r}(\bar{x} - p_r)) \\
& \quad + \sum_{i \in \bar{I}} \bar{z}_i^{0*} \gamma_{C_i}(\bar{x} - p_i) - \sum_{i \in \bar{I}} \bar{z}_i^{0*} \gamma_{C_i}(\bar{x} - p_i) + \sum_{i \in \bar{I}} \langle \bar{z}_i^{1*}, \bar{x} \rangle - \sum_{i \in \bar{I}} \langle \bar{z}_i^{1*}, \bar{x} \rangle = 0 \\
& \Leftrightarrow \left[ \max_{1 \leq j \leq n} \{h_j(\gamma_{C_j}(\bar{x} - p_j)) + a_j\} - \sum_{r \in \bar{R}} (\bar{\lambda}_r h_r(\gamma_{C_r}(\bar{x} - p_r)) + \bar{\lambda}_r a_r) \right] \\
& \quad + \sum_{i \in \bar{I}} [\bar{z}_i^{0*} \gamma_{C_i}(\bar{x} - p_i) - \langle \bar{z}_i^{1*}, \bar{x} - p_i \rangle] + \left[ \sigma_S \left( - \sum_{i \in \bar{I}} \bar{z}_i^{1*} \right) + \sum_{i \in \bar{I}} \langle \bar{z}_i^{1*}, \bar{x} \rangle \right] \\
& \quad + \sum_{i \in \bar{I}} \left[ \bar{\lambda}_i h_i^* \left( \frac{\bar{z}_i^{0*}}{\bar{\lambda}_i} \right) + \bar{\lambda}_i h_i(\gamma_{C_i}(\bar{x} - p_i)) - \bar{z}_i^{0*} \gamma_{C_i}(\bar{x} - p_i) \right] \\
& \quad + \sum_{r \in \bar{R} \setminus \bar{I}} [\bar{\lambda}_r h_r^*(0) + \bar{\lambda}_r h_r(\gamma_{C_r}(\bar{x} - p_r)) - 0 \cdot \gamma_{C_r}(\bar{x} - p_r)] = 0,
\end{aligned}$$

where the last two sums arise from the fact that  $\bar{I} \subseteq \bar{R}$ . By Lemma 4.2 holds that the term within the first bracket is non-negative. Moreover, by the Young-Fenchel inequality we have that the terms within the other brackets are also non-negative and hence, it follows that all the terms within the brackets must be equal to zero. Combining the last statement with the optimality conditions (i)-(v) yields

- (i)  $\max_{1 \leq j \leq n} \{h_j(\gamma_{C_j}(\bar{x} - p_j)) + a_j\} = \sum_{i \in \bar{I}} \bar{z}_i^{0*} \gamma_{C_i}(\bar{x} - p_i) - \sum_{r \in \bar{R}} \bar{\lambda}_r \left[ h_r^* \left( \frac{\bar{z}_r^{0*}}{\bar{\lambda}_r} \right) - a_r \right]$   
 $= \sum_{r \in \bar{R}} \bar{\lambda}_r [h_r(\gamma_{C_r}(\bar{x} - p_r)) + a_r],$
- (ii)  $\bar{\lambda}_r h_r^* \left( \frac{\bar{z}_r^{0*}}{\bar{\lambda}_r} \right) + \bar{\lambda}_r h_r(\gamma_{C_r}(\bar{x} - p_r)) = \bar{z}_r^{0*} \gamma_{C_r}(\bar{x} - p_r) \quad \forall r \in \bar{R},$
- (iii)  $\bar{z}_i^{0*} \gamma_{C_i}(\bar{x} - p_i) = \langle \bar{z}_i^{1*}, \bar{x} - p_i \rangle \quad \forall i \in \bar{I},$
- (iv)  $\sum_{i \in \bar{I}} \langle \bar{z}_i^{1*}, \bar{x} \rangle = -\sigma_S \left( - \sum_{i \in \bar{I}} \bar{z}_i^{1*} \right),$
- (v)  $\sum_{r \in \bar{R}} \bar{\lambda}_r \leq 1, \bar{\lambda}_k > 0, k \in \bar{R}, \bar{\lambda}_l = 0, l \notin \bar{R}, \bar{z}_i^{0*} > 0, i \in \bar{I}, \text{ and } \bar{z}_j^{0*} = 0, j \notin \bar{I},$
- (vi)  $\gamma_{C_i}^0(\bar{z}_i^{1*}) \leq \bar{z}_i^{0*}, \bar{z}_i^{1*} \in X^*, i \in \bar{I}, \text{ and } \bar{z}_j^{1*} = 0_{X^*}, j \notin \bar{I}.$

From conditions (i) and (v) we obtain that

$$\begin{aligned} \max_{1 \leq j \leq n} \{h_j(\gamma_{C_j}(\bar{x} - p_j)) + a_j\} &= \sum_{r \in \bar{R}} (\bar{\lambda}_r h_r(\gamma_{C_r}(\bar{x} - p_r)) + \bar{\lambda}_r a_r) \\ &\leq \sum_{r \in \bar{R}} \bar{\lambda}_r \max_{1 \leq j \leq n} \{h_j(\gamma_{C_j}(\bar{x} - p_j)) + a_j\} \\ &\leq \max_{1 \leq j \leq n} \{h_j(\gamma_{C_j}(\bar{x} - p_j)) + a_j\}, \end{aligned}$$

which means on the one hand that

$$\sum_{r \in \bar{R}} \bar{\lambda}_r \max_{1 \leq j \leq n} \{h_j(\gamma_{C_j}(\bar{x} - p_j)) + a_j\} = \max_{1 \leq j \leq n} \{h_j(\gamma_{C_j}(\bar{x} - p_j)) + a_j\},$$

i.e. condition (v) can be written as

$$\sum_{r \in \bar{R}} \bar{\lambda}_r = 1, \bar{\lambda}_k > 0, k \in \bar{R}, \bar{\lambda}_l = 0, l \notin \bar{R}, \bar{z}_i^{0*} > 0, i \in \bar{I}, \text{ and } \bar{z}_j^{0*} = 0, j \notin \bar{I}, \quad (4.33)$$

and on the other hand that

$$\sum_{r \in \bar{R}} (\bar{\lambda}_r h_r(\gamma_{C_r}(\bar{x} - p_r)) + \bar{\lambda}_r a_r) = \sum_{r \in \bar{R}} \bar{\lambda}_r \max_{1 \leq j \leq n} \{h_j(\gamma_{C_j}(\bar{x} - p_j)) + a_j\} \quad (4.34)$$

or, equivalently,

$$\sum_{r \in \bar{R}} \bar{\lambda}_r \left[ \max_{1 \leq j \leq n} \{h_j(\gamma_{C_j}(\bar{x} - p_j)) + a_j\} - (h_r(\gamma_{C_r}(\bar{x} - p_r)) + a_r) \right] = 0. \quad (4.35)$$

As the brackets in the sum of (4.35) are non-negative and  $\bar{\lambda}_r > 0$  for  $r \in \bar{R}$ , it follows that the terms inside the brackets must be equal to zero, more precisely,

$$\max_{1 \leq j \leq n} \{h_j(\gamma_{C_j}(\bar{x} - p_j)) + a_j\} = h_r(\gamma_{C_r}(\bar{x} - p_r)) + a_r \quad \forall r \in \bar{R}. \quad (4.36)$$

Further, if for  $i \in I$  holds  $\gamma_{C_i}(\bar{x} - p_i) = 0$ , then we have by the condition (iii) that  $\langle \bar{z}_i^{1*}, \bar{x} - p_i \rangle = 0$ , from which follows that

$$\gamma_{C_i}(\bar{x} - p_i) \gamma_{C_i^0}(\bar{z}_i^{1*}) = \langle \bar{z}_i^{1*}, \bar{x} - p_i \rangle = 0. \quad (4.37)$$

If for  $i \in I$  holds  $\gamma_{C_i}(\bar{x} - p_i) > 0$ , then we obtain by the Young-Fenchel inequality that

$$\gamma_{C_i}(\bar{x} - p_i) \gamma_{C_i^0}(\bar{z}_i^{1*}) + (\gamma_{C_i}(\bar{x} - p_i) \gamma_{C_i^0})^*(x) \geq \langle \bar{z}_i^{1*}, x \rangle \quad \forall x \in X, \quad (4.38)$$

where

$$(\gamma_{C_i}(\bar{x} - p_i) \gamma_{C_i^0})^*(x) = \gamma_{C_i}(\bar{x} - p_i) \gamma_{C_i^0}^* \left( \frac{1}{\gamma_{C_i}(\bar{x} - p_i)} x \right) = \gamma_{C_i}(\bar{x} - p_i) \delta_{(C_i^0)^0} \left( \frac{1}{\gamma_{C_i}(\bar{x} - p_i)} x \right).$$

As by [83, Theorem 1.1.9] it holds that  $C_i^{00} := (C_i^0)^0 = C_i$ ,  $i = 1, \dots, n$ , one gets that (see Remark 4.3 and 4.5)

$$(\gamma_{C_i}(\bar{x} - p_i) \gamma_{C_i^0})^*(x) = \gamma_{C_i}(\bar{x} - p_i) \delta_{C_i} \left( \frac{1}{\gamma_{C_i}(\bar{x} - p_i)} x \right) = \begin{cases} 0, & \text{if } \gamma_{C_i}(x) \leq \gamma_{C_i}(\bar{x} - p_i), \\ +\infty, & \text{otherwise,} \end{cases} \quad (4.39)$$

for all  $x \in X$ , which implies that

$$\gamma_{C_i}(\bar{x} - p_i) \gamma_{C_i^0}(\bar{z}_i^{1*}) \geq \langle \bar{z}_i^{1*}, \bar{x} - p_i \rangle. \quad (4.40)$$

Combining now (4. 37) and (4. 40) with the conditions (iii) and (vi) yields

$$\bar{z}_i^{0*} \gamma_{C_i}(\bar{x} - p_i) = \langle \bar{z}_i^{1*}, \bar{x} - p_i \rangle \leq \gamma_{C_i^0}(\bar{z}_i^{1*}) \gamma_{C_i}(\bar{x} - p_i) \leq \bar{z}_i^{0*} \gamma_{C_i}(\bar{x} - p_i),$$

which means that condition (vi) can be expressed as

$$\gamma_{C_i^0}(\bar{z}_i^{1*}) = \bar{z}_i^{0*}, \quad \bar{z}_i^{1*} \in X^* \setminus \{0_{X^*}\}, \quad i \in \bar{I}, \quad \text{and} \quad \bar{z}_j^{1*} = 0_{X^*}, \quad j \notin \bar{I}. \quad (4. 41)$$

Taking now the optimality conditions (i)-(vi), (4. 33), (4. 36) and (4. 41) together delivers the desired statement.

(b) All the calculations done in (a), can also be made in the reverse order.  $\square$

**Remark 4.8.** *The optimality conditions (i)-(iv) of the previous theorem can also be expressed by using subdifferentials. As*

$$f(y^0) = \begin{cases} \max_{1 \leq i \leq n} \{h_i(y_i^0) + a_i\}, & \text{if } y^0 = (y_1^0, \dots, y_n^0)^T \in \mathbb{R}_+^n, \quad i = 1, \dots, n, \\ +\infty_{\mathbb{R}_+^n}, & \text{otherwise,} \end{cases}$$

and

$$f^*(z_1^{0*}, \dots, z_n^{0*}) = \min_{\substack{\sum_{i=1}^n \lambda_i \leq 1, \quad \lambda_i \geq 0, \\ i=1, \dots, n}} \left\{ \sum_{i=1}^n [(\lambda_i h_i)^*(z_i^{0*}) - \lambda_i a_i] \right\},$$

we have by the optimal condition (i) of Theorem 4.3 that

$$f(\gamma_{C_1}(\bar{x} - p_1), \dots, \gamma_{C_n}(\bar{x} - p_n)) + f^*(z_1^{0*}, \dots, z_n^{0*}) = \sum_{i \in \bar{I}} \bar{z}_i^{0*} \gamma_{C_i}(\bar{x} - p_i).$$

By (2. 1) the last equality is equivalent to

$$(z_1^{0*}, \dots, z_n^{0*}) \in \partial f(\gamma_{C_1}(\bar{x} - p_1), \dots, \gamma_{C_n}(\bar{x} - p_n)).$$

Therefore, the condition (i) of Theorem 4.3 can equivalently be written as

$$(i) \quad (\bar{z}_1^{0*}, \dots, \bar{z}_n^{0*}) \in \partial \left( \max_{1 \leq j \leq n} \{h_j(\cdot) + a_j\} \right) (\gamma_{C_1}(\bar{x} - p_1), \dots, \gamma_{C_n}(\bar{x} - p_n)),$$

In the same way, we can rewrite the conditions (ii)-(iv)

$$(ii) \quad \bar{z}_r^{0*} \in \partial(\bar{\lambda}_r h_r)(\gamma_{C_r}(\bar{x} - p_r)), \quad r \in \bar{R},$$

$$(iii) \quad \bar{z}_i^{1*} \in \partial(\bar{z}_i^{0*} \gamma_{C_i})(\bar{x} - p_i), \quad i \in \bar{I},$$

$$(iv) \quad -\sum_{i \in \bar{I}} \bar{z}_i^{1*} \in \partial \delta_S(\bar{x}) = N_S(\bar{x}).$$

Bringing the optimality conditions (i) and (ii) together yields

$$\begin{aligned} (\bar{z}_1^{0*}, \dots, \bar{z}_n^{0*}) &\in \partial \left( \max_{1 \leq j \leq n} \{h_j(\cdot) + a_j\} \right) (\gamma_{C_1}(\bar{x} - p_1), \dots, \gamma_{C_n}(\bar{x} - p_n)) \\ &\cap (\partial(\bar{\lambda}_1 h_1)(\gamma_{C_1}(\bar{x} - p_1)) \times \dots \times \partial(\bar{\lambda}_n h_n)(\gamma_{C_n}(\bar{x} - p_n))). \end{aligned}$$

Moreover, summarizing the optimality conditions (iii) and (iv) reveals that

$$\sum_{i \in \bar{I}} \bar{z}_i^{1*} \in \sum_{i \in \bar{I}} \partial(\bar{z}_i^{0*} \gamma_{C_i})(\bar{x} - p_i) \cap (-N_S(\bar{x})).$$

Finally, take also note that the optimality conditions (iii) and (vii) of Theorem 4.3 give a detailed characterization of the subdifferential of  $\bar{z}_i^{0*} \gamma_{C_i}$  at  $\bar{x} - p_i$ ,  $i = 1, \dots, n$ . More precisely,

$$\partial(\bar{z}_i^{0*} \gamma_{C_i})(\bar{x} - p_i) = \left\{ \bar{z}_i^{1*} \in X^* : \bar{z}_i^{0*} \gamma_{C_i}(\bar{x} - p_i) = \langle \bar{z}_i^{1*}, \bar{x} - p_i \rangle \text{ and } \gamma_{C_i^0}(\bar{z}_i^{1*}) = \bar{z}_i^{0*} \right\}, \quad i \in \bar{I}.$$

**Remark 4.9.** If we consider the situation when the set-up costs are arbitrary, i.e.  $a_i$  can also be negative,  $i = 1, \dots, n$ , then the conjugate function of  $f$  looks like (see Remark 4.2)

$$f^*(z_1^{0*}, \dots, z_n^{0*}) = \min_{\substack{\sum_{i=1}^n \lambda_i = 1, \lambda_i \geq 0, \\ i=1, \dots, n}} \left\{ \sum_{i=1}^n [(\lambda_i h_i)^*(z_i^{0*}) - \lambda_i a_i] \right\}.$$

As a consequence, we derive the following corresponding dual problem

$$(D_{h,a}^S) \quad \sup_{\substack{\lambda_i, z_i^{0*} \geq 0, z_i^{1*} \in X^*, i=1, \dots, n, \\ I = \{i \in \{1, \dots, n\} : z_i^{0*} > 0\} \subseteq R = \{r \in \{1, \dots, n\} : \lambda_r > 0\}, \\ \gamma_{C_i^0}(z_i^{1*}) \leq z_i^{0*}, i \in I, z_j^{1*} = 0_{X^*}, j \notin I, \sum_{r \in R} \lambda_r = 1}} \left\{ \inf_{x \in S} \left\{ \sum_{i \in I} \langle z_i^{1*}, x - p_i \rangle \right\} - \sum_{r \in R} \lambda_r \left[ h_r^* \left( \frac{z_r^{0*}}{\lambda_r} \right) - a_r \right] \right\}.$$

Therefore, all the statements given in this subsection are also true in the case of arbitrary set-up costs with the difference that  $\sum_{r \in R} \lambda_r = 1$  in the constraint set.

Minimax location problems with arbitrary set-up costs were considered for example in [35] and [67]. For readers who are also interested in minimax location problems with nonlinear set-up costs, we refer to [33] and [44].

## 4.2.2 Unconstrained location problems with set-up costs in Hilbert spaces

This subsection is devoted to the case where  $S = X = \mathcal{H}$ , where  $\mathcal{H}$  is a Hilbert space,  $a_i \geq 0$  and  $\gamma_{C_i} : \mathcal{H} \rightarrow \mathbb{R}$  is defined by  $\gamma_{C_i}(x) := \|x\|_{\mathcal{H}}$ ,  $i = 1, \dots, n$ , such that the minimax location problem  $(P_{h,a}^S)$  turns into

$$(P_{h,a}^{S,N}) \quad \inf_{x \in \mathcal{H}} \max_{1 \leq i \leq n} \{h_i(\|x - p_i\|_{\mathcal{H}}) + a_i\}.$$

Its corresponding dual problem  $(D_{h,a}^{S,N})$  transforms by (4. 30) to

$$\begin{aligned} & \sup_{\substack{\lambda_i, z_i^{0*} \geq 0, z_i^{1*} \in \mathcal{H}, i=1, \dots, n, \\ I = \{i \in \{1, \dots, n\} : z_i^{0*} > 0\} \subseteq R = \{r \in \{1, \dots, n\} : \lambda_r > 0\}, \\ \|z_i^{1*}\|_{\mathcal{H}} \leq z_i^{0*}, i \in I, \sum_{r \in R} \lambda_r \leq 1}} \left\{ \inf_{x \in \mathcal{H}} \left\{ \sum_{i \in I} \langle z_i^{1*}, x - p_i \rangle_{\mathcal{H}} \right\} - \sum_{r \in R} \lambda_r \left[ h_r^* \left( \frac{z_r^{0*}}{\lambda_r} \right) - a_r \right] \right\} \\ &= \sup_{\substack{\lambda_i, z_i^{0*} \geq 0, z_i^{1*} \in \mathcal{H}, i=1, \dots, n, \\ I = \{i \in \{1, \dots, n\} : z_i^{0*} > 0\} \subseteq R = \{r \in \{1, \dots, n\} : \lambda_r > 0\}, \\ \|z_i^{1*}\|_{\mathcal{H}} \leq z_i^{0*}, i \in I, \sum_{r \in R} \lambda_r \leq 1}} \left\{ - \sup_{x \in \mathcal{H}} \left\{ \left\langle - \sum_{i \in I} z_i^{1*}, x \right\rangle_{\mathcal{H}} \right\} \right. \\ & \quad \left. - \sum_{i \in I} \langle z_i^{1*}, p_i \rangle_{\mathcal{H}} - \sum_{r \in R} \lambda_r \left[ h_r^* \left( \frac{z_r^{0*}}{\lambda_r} \right) - a_r \right] \right\} \\ &= \sup_{\substack{\lambda_i, z_i^{0*} \geq 0, z_i^{1*} \in \mathcal{H}, i=1, \dots, n, \\ I = \{i \in \{1, \dots, n\} : z_i^{0*} > 0\} \subseteq R = \{r \in \{1, \dots, n\} : \lambda_r > 0\}, \\ \|z_i^{1*}\|_{\mathcal{H}} \leq z_i^{0*}, i \in I, \sum_{i \in I} z_i^{1*} = 0_{\mathcal{H}}, \sum_{r \in R} \lambda_r \leq 1}} \left\{ - \sum_{i \in I} \langle z_i^{1*}, p_i \rangle_{\mathcal{H}} - \sum_{r \in R} \lambda_r \left[ h_r^* \left( \frac{z_r^{0*}}{\lambda_r} \right) - a_r \right] \right\}. \end{aligned}$$

The following duality statements are direct consequences of Theorem 4.2 and 4.3.

**Theorem 4.4.** (strong duality) Between  $(P_{h,a}^{S,N})$  and  $(D_{h,a}^{S,N})$  holds strong duality, i.e.  $v(P_{h,a}^{S,N}) = v(D_{h,a}^{S,N})$  and the dual problem has an optimal solution.

**Theorem 4.5.** (optimality conditions) (a) Let  $\bar{x} \in \mathcal{H}$  be an optimal solution of the problem  $(P_{h,a}^{S,N})$ . Then there exist  $(\bar{\lambda}_1, \dots, \bar{\lambda}_n, \bar{z}^{0*}, \bar{z}^{1*}) \in \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathcal{H}^n$  and index sets  $\bar{I} \subseteq \bar{R} \subseteq \{1, \dots, n\}$  as an optimal solution to  $(D_{h,a}^{S,N})$  such that

- (i)  $\max_{1 \leq j \leq n} \{h_j(\|\bar{x} - p_j\|_{\mathcal{H}}) + a_j\} = \sum_{i \in \bar{I}} \bar{z}_i^{0*} \|\bar{x} - p_i\|_{\mathcal{H}} - \sum_{r \in \bar{R}} \bar{\lambda}_r \left[ h_r^* \left( \frac{\bar{z}_r^{0*}}{\bar{\lambda}_r} \right) - a_r \right]$   
 $= \sum_{r \in \bar{R}} \bar{\lambda}_r [h_r(\|\bar{x} - p_r\|_{\mathcal{H}}) + a_r],$
- (ii)  $\bar{\lambda}_r h_r^* \left( \frac{\bar{z}_r^{0*}}{\bar{\lambda}_r} \right) + \bar{\lambda}_r h_r(\|\bar{x} - p_r\|_{\mathcal{H}}) = \bar{z}_r^{0*} \|\bar{x} - p_r\|_{\mathcal{H}} \quad \forall r \in \bar{R},$
- (iii)  $\bar{z}_i^{0*} \|\bar{x} - p_i\|_{\mathcal{H}} = \langle \bar{z}_i^{1*}, \bar{x} - p_i \rangle_{\mathcal{H}} \quad \forall i \in \bar{I},$
- (iv)  $\sum_{i \in \bar{I}} \bar{z}_i^{1*} = 0_{\mathcal{H}},$
- (v)  $\max_{1 \leq j \leq n} \{h_j(\|\bar{x} - p_j\|_{\mathcal{H}}) + a_j\} = h_r(\|\bar{x} - p_r\|_{\mathcal{H}}) + a_r \quad \forall r \in \bar{R},$
- (vi)  $\sum_{r \in \bar{R}} \bar{\lambda}_r = 1, \quad \bar{\lambda}_k > 0, \quad k \in \bar{R}, \quad \lambda_l = 0, \quad l \notin \bar{R}, \quad \bar{z}_i^{0*} > 0, \quad i \in \bar{I}, \quad \text{and } \bar{z}_j^{0*} = 0, \quad j \notin \bar{I},$
- (vii)  $\|\bar{z}_i^{1*}\|_{\mathcal{H}} = \bar{z}_i^{0*}, \quad \bar{z}_i^{1*} \in \mathcal{H} \setminus \{0_{\mathcal{H}}\}, \quad i \in \bar{I} \quad \text{and } \bar{z}_j^{1*} = 0_{\mathcal{H}}, \quad j \notin \bar{I}.$

(b) If there exists  $\bar{x} \in \mathcal{H}$  such that for some  $(\bar{\lambda}_1, \dots, \bar{\lambda}_n, \bar{z}^{0*}, \bar{z}^{1*}) \in \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathcal{H}^n$  and the index sets  $\bar{I} \subseteq \bar{R}$  the conditions (i)-(vii) are fulfilled, then  $\bar{x}$  is an optimal solution of  $(P_{h,a}^{S,N})$ ,  $(\bar{\lambda}_1, \dots, \bar{\lambda}_n, \bar{z}^{0*}, \bar{z}^{1*}, \bar{I}, \bar{R})$  is an optimal solution for  $(D_{h,a}^{S,N})$  and  $v(P_{h,a}^{S,N}) = v(D_{h,a}^{S,N})$ .

Regarding the relation between the optimal solutions of the primal and the dual problem the following corollary can be given under the additional assumption that the function  $h_i$  is continuous and strictly increasing for all  $i = 1, \dots, n$ .

**Corollary 4.1.** *Let the function*

$$h_i : \mathbb{R} \rightarrow \bar{\mathbb{R}}, \quad h_i(x) := \begin{cases} h_i(x) \in \mathbb{R}_+, & \text{if } x \in \mathbb{R}_+, \\ +\infty, & \text{otherwise,} \end{cases}$$

be convex, continuous and strictly increasing for all  $i = 1, \dots, n$ , and  $\bar{x} \in \mathcal{H}$  an optimal solution of the problem  $(P_{h,a}^{S,N})$ . If  $(\bar{\lambda}_1, \dots, \bar{\lambda}_n, \bar{z}^{0*}, \bar{z}^{1*}) \in \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathcal{H}^n$  and  $\bar{I} \subseteq \bar{R} \subseteq \{1, \dots, n\}$  are optimal solutions of the dual problem  $(D_{h,a}^{S,N})$ , then it holds

$$\bar{x} = \frac{1}{\sum_{i \in \bar{I}} \frac{\|\bar{z}_i^{1*}\|_{\mathcal{H}}}{h_i^{-1}(v(D_{h,a}^{S,N}) - a_i)}} \sum_{i \in \bar{I}} \frac{\|\bar{z}_i^{1*}\|_{\mathcal{H}}}{h_i^{-1}(v(D_{h,a}^{S,N}) - a_i)} p_i.$$

**Proof.** The optimality conditions (iii) and (vii) of Theorem 4.5 imply that

$$\|\bar{z}_i^{1*}\|_{\mathcal{H}} \|\bar{x} - p_i\|_{\mathcal{H}} = \langle \bar{z}_i^{1*}, \bar{x} - p_i \rangle_{\mathcal{H}}, \quad i \in \bar{I},$$

By [2, Fact 2.10] there exists  $\alpha_i > 0$  such that

$$\bar{z}_i^{1*} = \alpha_i (\bar{x} - p_i), \quad i \in \bar{I}, \quad (4.42)$$

and so,  $\|\bar{z}_i^{1*}\|_{\mathcal{H}} = \alpha_i \|\bar{x} - p_i\|_{\mathcal{H}}, \quad i \in \bar{I}$ . Therefore, it follows from the optimality condition (v) of Theorem 4.5 that (note that  $\bar{I} \subseteq \bar{R}$ )

$$\begin{aligned} & \max_{1 \leq j \leq n} \{h_j(\|\bar{x} - p_j\|_{\mathcal{H}}) + a_j\} = h_i \left( \frac{1}{\alpha_i} \|\bar{z}_i^{1*}\|_{\mathcal{H}} \right) + a_i \\ \Leftrightarrow & \quad h_i^{-1} \left( \max_{1 \leq j \leq n} \{h_j(\|\bar{x} - p_j\|_{\mathcal{H}}) + a_j\} - a_i \right) = \frac{1}{\alpha_i} \|\bar{z}_i^{1*}\|_{\mathcal{H}} \\ \Leftrightarrow & \quad \alpha_i = \frac{\|\bar{z}_i^{1*}\|_{\mathcal{H}}}{h_i^{-1} \left( \max_{1 \leq j \leq n} \{h_j(\|\bar{x} - p_j\|_{\mathcal{H}}) + a_j\} - a_i \right)} = \frac{\|\bar{z}_i^{1*}\|_{\mathcal{H}}}{h_i^{-1} (v(D_{h,a}^{S,N}) - a_i)}, \quad i \in \bar{I}. \end{aligned} \quad (4.43)$$



Now, we take in (4. 42) the sum over all  $i \in \bar{I}$ , which yields by condition (iv) of Theorem 4.5

$$0_{\mathcal{H}} = \sum_{i \in \bar{I}} \bar{z}_i^{1*} = \sum_{i \in \bar{I}} \alpha_i (\bar{x} - p_i) \Leftrightarrow \bar{x} = \frac{1}{\sum_{i \in \bar{I}} \alpha_i} \sum_{i \in \bar{I}} \alpha_i p_i. \quad (4. 44)$$

Finally, bringing (4. 43) and (4. 44) together implies

$$\bar{x} = \frac{1}{\sum_{i \in \bar{I}} \frac{\|\bar{z}_i^{1*}\|_{\mathcal{H}}}{h_i^{-1}(v(D_{h,a}^{S,N}) - a_i)}} \sum_{i \in \bar{I}} \frac{\|\bar{z}_i^{1*}\|_{\mathcal{H}}}{h_i^{-1}(v(D_{h,a}^{S,N}) - a_i)} p_i.$$

□

**Example 4.1.** (a) Let  $\alpha_{is}, \beta_{is} \geq 0, s = 1, \dots, v$ , and  $h_i : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  be defined by

$$h_i(x) := \begin{cases} \max_{1 \leq s \leq v} \{\alpha_{is}x + \beta_{is}\}, & \text{if } x \in \mathbb{R}_+, \\ +\infty, & \text{otherwise,} \end{cases}$$

$i = 1, \dots, n$ , then the corresponding location problem looks like

$$(P_{h,a}^{S,N}) \quad \inf_{x \in \mathcal{H}} \max_{1 \leq i \leq n} \left\{ \max_{1 \leq s \leq v} \{\alpha_{is}\|x - p_i\|_{\mathcal{H}} + \beta_{is}\} + a_i \right\} = \inf_{x \in \mathcal{H}} \max_{\substack{1 \leq i \leq n, \\ 1 \leq s \leq v}} \{\alpha_{is}\|x - p_i\|_{\mathcal{H}} + \beta_{is} + a_i\}.$$

Moreover, we define the function

$$f_s : \mathbb{R} \rightarrow \overline{\mathbb{R}}, \quad f_s(x) := \begin{cases} \alpha_{is}x + \beta_{is}, & \text{if } x \in \mathbb{R}_+, \\ +\infty, & \text{otherwise,} \end{cases}$$

then we derive by [71, Theorem 3.2]

$$h_i^*(x^*) = \left( \max_{1 \leq s \leq v} \{f_s(\cdot)\} \right)^* (x^*) = \inf_{\substack{\sum_{s=1}^v x_s^* = x^*, \\ \tau_s \geq 0, \quad s=1, \dots, v}} \left\{ \sum_{s=1}^v (\tau_s f_s)^*(x_s^*) \right\}.$$

As the conjugate of the function  $\tau_s f_s$  is

$$\begin{aligned} (\tau_s f_s)^*(x_s^*) &= \sup_{x \in \mathbb{R}} \{x_s^* x - \tau_s f_s(x)\} = \sup_{x \geq 0} \{x_s^* x - \tau_s \alpha_{is} x - \tau_s \beta_{is}\} \\ &= -\tau_s \beta_{is} + \sup_{x \geq 0} \{(x_s^* - \tau_s \alpha_{is})x\} = \begin{cases} -\tau_s \beta_{is}, & \text{if } x_s^* \leq \tau_s \alpha_{is}, \\ +\infty, & \text{otherwise,} \end{cases} \end{aligned}$$

$s = 1, \dots, v$ , we have

$$h_i^*(x^*) = \inf_{\substack{\sum_{s=1}^v x_s^* = x^*, \\ \tau_s \geq 0, \quad x_s \leq \tau_s \alpha_{is}, \quad s=1, \dots, v}} \left\{ -\sum_{s=1}^v \tau_s \alpha_{is} \right\}, \quad i = 1, \dots, n,$$

and hence, the dual problem is given by

$$\begin{aligned} (D_{h,a}^{S,N}) \quad & \sup_{\substack{\lambda_i, z_i^{0*} \geq 0, \quad z_i^{1*} \in \mathcal{H}, \quad i=1, \dots, n, \\ I = \{i \in \{1, \dots, n\} : z_i^{0*} > 0\} \subseteq R = \{r \in \{1, \dots, n\} : \lambda_r > 0\}, \\ \|z_i^{1*}\|_{\mathcal{H}} \leq z_i^{0*}, i \in I, \sum_{i \in I} z_i^{1*} = 0_{\mathcal{H}}, \sum_{r \in R} \lambda_r \leq 1 \\ \sum_{s=1}^v x_s^* = \frac{z_r^{0*}}{\lambda_r}, \quad \sum_{s=1}^v \tau_s = 1, \quad \tau_s \geq 0, \quad x_s \leq \tau_s \alpha_{rs}, \quad s=1, \dots, v}} \left\{ -\sum_{i \in I} \langle z_i^{1*}, p_i \rangle_{\mathcal{H}} + \sum_{r \in R} \lambda_r \left[ \sum_{s=1}^n \tau_s \alpha_{rs} - a_r \right] \right\}. \end{aligned}$$

Furthermore,  $h_i^{-1}(y) = \min_{1 \leq s \leq v} \left\{ \frac{1}{\alpha_{is}} (y - \beta_{is}) \right\}$  for all  $i = 1, \dots, n$ , and thus, we have by Corollary 4.1

$$\bar{x} = \frac{1}{\sum_{i \in \bar{I}} \frac{\|\bar{z}_i^{1*}\|_{\mathcal{H}}}{\min_{1 \leq s \leq v} \left\{ \frac{1}{\alpha_{is}} (v(D_{h,a}^{S,N}) - a_i - \beta_{is}) \right\}}} \sum_{i \in \bar{I}} \frac{\|\bar{z}_i^{1*}\|_{\mathcal{H}}}{\min_{1 \leq s \leq v} \left\{ \frac{1}{\alpha_{is}} (v(D_{h,a}^{S,N}) - a_i - \beta_{is}) \right\}} p_i.$$

(b) Let  $h_i : \mathbb{R} \rightarrow \bar{\mathbb{R}}$  be defined by

$$h_i(x) := \begin{cases} w_i x^{\beta_i}, & \text{if } x \in \mathbb{R}_+, \\ +\infty, & \text{otherwise,} \end{cases}$$

with  $w_i > 0$ ,  $\beta_i > 1$ ,  $i = 1, \dots, n$ , then

$$(P_{h,a}^{S,N}) \quad \inf_{x \in \mathcal{H}} \max_{1 \leq i \leq n} \left\{ w_i \|x - p_i\|_{\mathcal{H}}^{\beta_i} + a_i \right\}$$

and since the conjugate function of  $h_i$  is given by (see [2, Example 13.2 (i)])

$$h_i^*(x^*) = w_i \frac{\beta_i - 1}{\beta_i} \left( \frac{1}{w_i} x^* \right)^{\frac{\beta_i}{\beta_i - 1}} = \frac{\beta_i - 1}{\beta_i w_i^{\frac{1}{\beta_i - 1}}} (x^*)^{\frac{\beta_i}{\beta_i - 1}}, \quad i = 1, \dots, n,$$

the associated dual problem  $(D_{h,a}^{S,N})$  is

$$\sup_{\substack{\lambda_i, z_i^{0*} \geq 0, z_i^{1*} \in \mathcal{H}, i=1, \dots, n, \\ I = \{i \in \{1, \dots, n\} : z_i^{0*} > 0\} \subseteq R = \{r \in \{1, \dots, n\} : \lambda_r > 0\}, \\ \|\bar{z}_i^{1*}\|_{\mathcal{H}} \leq z_i^{0*}, i \in I, \sum_{i \in I} z_i^{1*} = 0_{\mathcal{H}}, \sum_{r \in R} \lambda_r \leq 1}} \left\{ - \sum_{i \in I} \langle z_i^{1*}, p_i \rangle_{\mathcal{H}} - \sum_{r \in R} \lambda_r \left[ \frac{\beta_r - 1}{\beta_r (\lambda_r w_r)^{\frac{1}{\beta_r - 1}}} (z_r^{0*})^{\frac{\beta_r}{\beta_r - 1}} - a_r \right] \right\}.$$

In addition, as  $h_i^{-1}(y) = (y/w_i)^{\frac{1}{\beta_i}}$  for all  $i = 1, \dots, n$ , it holds

$$\bar{x} = \frac{1}{\sum_{i \in \bar{I}} \frac{w_i^{\frac{1}{\beta_i}} \|\bar{z}_i^{1*}\|_{\mathcal{H}}}{(v(D_{h,a}^{S,N}) - a_i)^{\frac{1}{\beta_i}}}} \sum_{i \in \bar{I}} \frac{w_i^{\frac{1}{\beta_i}} \|\bar{z}_i^{1*}\|_{\mathcal{H}}}{(v(D_{h,a}^{S,N}) - a_i)^{\frac{1}{\beta_i}}} p_i.$$

(c) Let  $h_i : \mathbb{R} \rightarrow \bar{\mathbb{R}}$  be defined by

$$h_i(x) := \begin{cases} w_i x, & \text{if } x \in \mathbb{R}_+, \\ +\infty, & \text{otherwise,} \end{cases}$$

where  $w_i > 0$ , then  $h_i^{-1}(y) = \frac{1}{w_i} y$  for all  $i = 1, \dots, n$ , and hence,

$$\bar{x} = \frac{1}{\sum_{i \in \bar{I}} \frac{w_i \|\bar{z}_i^{1*}\|_{\mathcal{H}}}{v(D_{h,a}^{S,N}) - a_i}} \sum_{i \in \bar{I}} \frac{w_i \|\bar{z}_i^{1*}\|_{\mathcal{H}}}{v(D_{h,a}^{S,N}) - a_i} p_i. \quad (4.45)$$

If  $a_i = 0$ ,  $i = 1, \dots, n$ , then formula in (4.45) reduces to

$$\bar{x} = \frac{1}{\sum_{i \in \bar{I}} w_i \|\bar{z}_i^{1*}\|_{\mathcal{H}}} \sum_{i \in \bar{I}} w_i \|\bar{z}_i^{1*}\|_{\mathcal{H}} p_i. \quad (4.46)$$

**Remark 4.10.** Let us note that all the results in this section hold also for negative set-up costs. Like already mentioned in Remark 4.9, we have in this case in the constraint set of the dual problem  $\sum_{r \in R} \lambda_r = 1$ .

### 4.2.3 Constrained location problems without set-up costs in Fréchet spaces

In this section we discuss single minimax location problems without set-up costs (i.e.  $a_i = 0$ ,  $i = 1, \dots, n$ ), where  $X$  is a Fréchet space,  $S \subseteq X$  and  $h_i : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  is defined by

$$h_i(x) := \begin{cases} x, & \text{if } x \in \mathbb{R}_+, \\ +\infty, & \text{otherwise.} \end{cases}$$

Hence, the location problem  $(P_{h,a}^S)$  turns into

$$(P^S) \quad \inf_{x \in S} \max_{1 \leq i \leq n} \{\gamma_{C_i}(x - p_i)\}$$

and by (4. 31) we can write the corresponding dual  $(D^S)$  as

$$(D^S) \quad \sup_{\substack{y_i^{0*} \geq 0, y_i^{1*} \in X^*, i=1, \dots, n, \quad I = \{i \in \{1, \dots, n\} : y_i^{0*} > 0\}, \\ y_j^{1*} = 0_{X^*}, j \notin I, \gamma_{C_i^0}(y_i^{1*}) \leq y_i^{0*}, i \in I, \sum_{i \in I} y_i^{0*} \leq 1}} \left\{ \inf_{x \in S} \left\{ \sum_{i \in I} \langle z_i^{1*}, x - p_i \rangle \right\} \right\}.$$

Let us now introduce the following optimization problem

$$(\tilde{D}^S) \quad \sup_{\substack{z_i^* \in X^*, \quad i=1, \dots, n, \quad I = \{i \in \{1, \dots, n\} : \gamma_{C_i^0}(z_i^*) > 0\}, \\ z_j^* = 0_{X^*}, \quad j \notin I, \quad \sum_{i \in I} \gamma_{C_i^0}(z_i^*) \leq 1}} \left\{ \inf_{x \in S} \left\{ \sum_{i \in I} \langle z_i^*, x - p_i \rangle \right\} \right\}, \quad (4. 47)$$

then the following theorem can be formulated.

**Theorem 4.6.** *It holds  $v(D^S) = v(\tilde{D}^S)$ .*

**Proof.** Let  $z_i^*$ ,  $i = 1, \dots, n$ , be a feasible element to  $(\tilde{D}^S)$  and set  $z_i^{1*} = z_i^*$ ,  $z_i^{0*} = \gamma_{C_i^0}(z_i^*)$  for  $i \in I$  and  $z_i^{0*} = 0$ ,  $z_i^{1*} = 0_{X^*}$  for  $i \notin I$ . Then, it is obvious that  $z_i^{0*}$  and  $z_i^{1*}$ ,  $i = 1, \dots, n$ , are feasible elements to  $(D^S)$  and it holds

$$\inf_{x \in S} \left\{ \sum_{i \in I} \langle z_i^*, x - p_i \rangle \right\} = \inf_{x \in S} \left\{ \sum_{i \in I} \langle z_i^{1*}, x - p_i \rangle \right\} \leq v(D^S) \quad (4. 48)$$

for all  $z_i^*$ ,  $i = 1, \dots, n$ , feasible to  $(\tilde{D}^S)$ , which implies  $v(\tilde{D}^S) \leq v(D^S)$ .

Vice versa, let  $z_i^{0*}$  and  $z_i^{1*}$  be feasible elements to  $(D^S)$  for  $i = 1, \dots, n$ , then we have  $\gamma_{C_i^0}(z_i^{1*}) \leq z_i^{0*}$  for  $i \in I$ ,  $\sum_{i \in I} z_i^{0*} \leq 1$  and  $z_i^{0*} = 0$ ,  $z_i^{1*} = 0_{X^*}$  for  $i \notin I$ , from which follows by setting  $z_i^* = z_i^{1*}$  for  $i \in I$  and  $z_i^* = 0_{X^*}$  for  $i \notin I$  that

$$\sum_{i \in I} \gamma_{C_i^0}(z_i^*) \leq 1,$$

in other words  $z_i^*$  is a feasible solution to  $(\tilde{D}^S)$  for all  $i = 1, \dots, n$ . Furthermore, we have that

$$\inf_{x \in S} \left\{ \sum_{i \in I} \langle z_i^{1*}, x - p_i \rangle \right\} = \inf_{x \in S} \left\{ \sum_{i \in I} \langle z_i^*, x - p_i \rangle \right\} \leq v(\tilde{D}^S) \quad (4. 49)$$

for all  $z_i^{0*}$  and  $z_i^{1*}$ ,  $i = 1, \dots, n$ , feasible to  $(D^S)$ , which implies that  $v(D^S) \leq v(\tilde{D}^S)$ . Bringing the statements (4. 48) and (4. 49) together reveals that it must hold  $v(\tilde{D}^S) = v(D^S)$ .  $\square$

**Remark 4.11.** As mentioned in Remark 4.5, we have that  $\gamma_{C^0}(z_i^*) = 0 \Leftrightarrow z_i^* = 0_{X^*}$  and therefore, the index set  $I$  of the dual  $(\tilde{D}^S)$  contains all indices such that  $z_i^* \neq 0_{X^*}$ .

Motivated by Theorem 4.6 it follows immediately the following one.

**Theorem 4.7.** (strong duality) Between  $(P^S)$  and  $(\tilde{D}^S)$  holds strong duality, i.e.  $v(P^S) = v(\tilde{D}^S)$  and the dual problem  $v(\tilde{D}^S)$  has an optimal solution.

Now, it is possible to formulate the following optimality conditions for the primal-dual pair  $(P^S)$ -( $\tilde{D}^S$ ) (note that  $a_i = 0$ ,  $i = 1, \dots, n$ ).

**Theorem 4.8.** (optimality conditions) (a) Let  $\bar{x} \in S$  be an optimal solution of the problem  $(P^S)$ . Then there exist  $\bar{z}^* \in (X^*)^n$  and an index set  $\bar{I} \subseteq \{1, \dots, n\}$  as an optimal solution to  $(\tilde{D}^S)$  such that

$$(i) \max_{1 \leq j \leq n} \{\gamma_{C_j}(\bar{x} - p_j)\} = \sum_{i \in \bar{I}} \gamma_{C_i^0}(\bar{z}_i^*) \gamma_{C_i}(\bar{x} - p_i),$$

$$(ii) \sum_{i \in \bar{I}} \langle \bar{z}_i^*, \bar{x} \rangle = -\sigma_S \left( -\sum_{i \in \bar{I}} \bar{z}_i^* \right),$$

$$(iii) \gamma_{C_i^0}(\bar{z}_i^*) \gamma_{C_i}(\bar{x} - p_i) = \langle \bar{z}_i^*, \bar{x} - p_i \rangle, \quad i \in \bar{I},$$

$$(iv) \sum_{j \in \bar{I}} \gamma_{C_j^0}(\bar{z}_j^*) = 1, \quad \bar{z}_i^* \in X^* \setminus \{0_{X^*}\}, \quad i \in \bar{I}, \text{ and } \bar{z}_i^* = 0_{X^*}, \quad i \notin \bar{I},$$

$$(v) \gamma_{C_i}(\bar{x} - p_i) = \max_{1 \leq j \leq n} \{\gamma_{C_j}(\bar{x} - p_j)\}, \quad i \in \bar{I}.$$

(b) If there exists  $\bar{x} \in S$  such that for some  $\bar{z}^* \in (X^*)^n$  and an index set  $\bar{I}$  the conditions (i)-(v) are fulfilled, then  $\bar{x}$  is an optimal solution of  $(P^S)$ ,  $(\bar{z}^*, \bar{I})$  is an optimal solution for  $(\tilde{D}^S)$  and  $v(P^S) = v(\tilde{D}^S)$ .

**Proof.** Let  $\bar{x} \in S$  be an optimal solution of  $(P^S)$ , then by Theorem 4.7 there exists  $\bar{z}^* \in (X^*)^n$  and an index set  $\bar{I} \subseteq \{1, \dots, n\}$  such that  $v(P^S) = v(\tilde{D}^S)$ , i.e.

$$\begin{aligned} & \max_{1 \leq j \leq n} \{\gamma_{C_j}(\bar{x} - p_j)\} = \inf_{x \in S} \left\{ \sum_{i \in \bar{I}} \langle \bar{z}_i^*, x - p_i \rangle \right\} \\ \Leftrightarrow & \max_{1 \leq j \leq n} \{\gamma_{C_j}(\bar{x} - p_j)\} + \sigma_S \left( -\sum_{i \in \bar{I}} \bar{z}_i^* \right) + \sum_{i \in \bar{I}} \langle \bar{z}_i^*, p_i \rangle = 0 \\ \Leftrightarrow & \max_{1 \leq j \leq n} \{\gamma_{C_j}(\bar{x} - p_j)\} + \sigma_S \left( -\sum_{i \in \bar{I}} \bar{z}_i^* \right) + \sum_{i \in \bar{I}} \langle \bar{z}_i^*, p_i \rangle \\ & + \sum_{i \in \bar{I}} \gamma_{C_i^0}(\bar{z}_i^*) \gamma_{C_i}(\bar{x} - p_i) - \sum_{i \in \bar{I}} \gamma_{C_i^0}(\bar{z}_i^*) \gamma_{C_i}(\bar{x} - p_i) + \sum_{i \in \bar{I}} \langle \bar{z}_i^*, \bar{x} \rangle - \sum_{i \in \bar{I}} \langle \bar{z}_i^*, \bar{x} \rangle = 0 \\ \Leftrightarrow & \left[ \max_{1 \leq j \leq n} \{\gamma_{C_j}(\bar{x} - p_j)\} - \sum_{i \in \bar{I}} \gamma_{C_i^0}(\bar{z}_i^*) \gamma_{C_i}(\bar{x} - p_i) \right] \\ & + \left[ \delta_S(\bar{x}) + \sigma_S \left( -\sum_{i \in \bar{I}} \bar{z}_i^* \right) + \left\langle \sum_{i \in \bar{I}} \bar{z}_i^*, \bar{x} \right\rangle \right] + \sum_{i \in \bar{I}} [\gamma_{C_i^0}(\bar{z}_i^*) \gamma_{C_i}(\bar{x} - p_i) + \langle \bar{z}_i^*, p_i - \bar{x} \rangle] = 0. \end{aligned}$$

By Lemma 4.2 holds that the term within the first bracket is non-negative and by the Young-Fenchel inequality we derive that the term within the second bracket is non-negative. Further, from  $\gamma_{C_i^0}(\bar{z}_i^*) > 0$  for  $i \in \bar{I}$ , it follows by the Young-Fenchel inequality that

$$\gamma_{C_i^0}(\bar{z}_i^*)\gamma_{C_i}(\bar{x} - p_i) + (\gamma_{C_i^0}(\bar{z}_i^*)\gamma_{C_i})^*(x^*) \geq \langle x^*, \bar{x} - p_i \rangle \quad \forall x^* \in X^*, \quad (4.50)$$

and since (see Remark 4.5)

$$(\gamma_{C_i^0}(\bar{z}_i^*)\gamma_{C_i})^*(x^*) = \gamma_{C_i^0}(\bar{z}_i^*)\gamma_{C_i}^* \left( \frac{1}{\gamma_{C_i^0}(\bar{z}_i^*)} x^* \right) = \begin{cases} 0, & \text{if } \gamma_{C_i^0}(x^*) \leq \gamma_{C_i^0}(\bar{z}_i^*), \\ +\infty, & \text{otherwise,} \end{cases} \quad (4.51)$$

one has that  $\gamma_{C_i^0}(\bar{z}_i^*)\gamma_{C_i}(\bar{x} - p_i) \geq \langle \bar{z}_i^*, \bar{x} - p_i \rangle$  for all  $i \in \bar{I}$ . This means that the terms within the other brackets are also non-negative and therefore, all the terms inside the brackets must be equal to zero. This implies the cases (i)-(iii). Further, we obtain by the first bracket

$$\begin{aligned} \max_{1 \leq j \leq n} \{\gamma_{C_j}(\bar{x} - p_j)\} &= \sum_{i \in \bar{I}} \gamma_{C_i^0}(\bar{z}_i^*)\gamma_{C_i}(\bar{x} - p_i) \\ &\leq \sum_{i \in \bar{I}} \gamma_{C_i^0}(\bar{z}_i^*) \max_{1 \leq j \leq n} \{\gamma_{C_j}(\bar{x} - p_j)\} \leq \max_{1 \leq j \leq n} \{\gamma_{C_j}(\bar{x} - p_j)\} \end{aligned}$$

and from here follows that  $\sum_{i \in \bar{I}} \gamma_{C_i^0}(\bar{z}_i^*) = 1$ , which yields condition (iv), as well as

$$\begin{aligned} \sum_{i \in \bar{I}} \gamma_{C_i^0}(\bar{z}_i^*) \max_{1 \leq j \leq n} \{\gamma_{C_j}(\bar{x} - p_j)\} &= \sum_{i \in \bar{I}} \gamma_{C_i^0}(\bar{z}_i^*)\gamma_{C_i}(\bar{x} - p_i) \\ \Leftrightarrow \sum_{i \in \bar{I}} \gamma_{C_i^0}(\bar{z}_i^*) \left[ \max_{1 \leq j \leq n} \{\gamma_{C_j}(\bar{x} - p_j)\} - \gamma_{C_i}(\bar{x} - p_i) \right] &= 0. \end{aligned} \quad (4.52)$$

As the brackets in (4.52) are non-negative and  $\gamma_{C_i^0}(\bar{z}_i^*) > 0$ ,  $i \in \bar{I}$ , we get that

$$\max_{1 \leq j \leq n} \{\gamma_{C_j}(\bar{x} - p_j)\} = \gamma_{C_i}(\bar{x} - p_i), \quad i \in \bar{I}.$$

which yields the condition (v) and completes the proof.  $\square$

#### 4.2.4 Unconstrained location problems without set-up costs in the Euclidean space

Now, we turn our attention to the case where  $S = X = \mathbb{R}^d$  and  $w_i > 0$ ,  $i = 1, \dots, n$ . Furthermore, we use as the gauge functions the Euclidean norm, i.e.  $\gamma_{C_i}(\cdot) = w_i \|\cdot\|$ ,  $i = 1, \dots, n$ . By these settings, the minimax location problem  $(P^S)$  transforms into the following one

$$(P_N^S) \quad \inf_{x \in \mathbb{R}^d} \max_{1 \leq i \leq n} \{w_i \|x - p_i\|\}.$$

By using (4.47) we obtain the following dual problem corresponding to  $(P_N^S)$ ,

$$\begin{aligned} (\tilde{D}_N^S) & \sup_{\substack{z_i^* \in \mathbb{R}^d, \quad i=1, \dots, n, \quad I = \{i \in \{1, \dots, n\} : \|z_i^*\| > 0\}, \\ z_j^* = 0_{\mathbb{R}^d}, \quad j \notin I, \quad \sum_{i \in I} \frac{1}{w_i} \|z_i^*\| \leq 1}} \inf_{x \in \mathbb{R}^d} \left\{ \sum_{i \in I} \langle z_i^*, x - p_i \rangle \right\} \\ &= \sup_{\substack{z_i^* \in \mathbb{R}^d, \quad i=1, \dots, n, \quad I = \{i \in \{1, \dots, n\} : \|z_i^*\| > 0\}, \\ z_j^* = 0_{\mathbb{R}^d}, \quad j \notin I, \quad \sum_{i \in I} \frac{1}{w_i} \|z_i^*\| \leq 1}} \left\{ -\sigma_{\mathbb{R}^d} \left( -\sum_{i \in I} z_i^* \right) - \sum_{i \in I} \langle z_i^*, p_i \rangle \right\} \\ &= \sup_{\substack{z_i^* \in \mathbb{R}^d, \quad i=1, \dots, n, \quad I = \{i \in \{1, \dots, n\} : \|z_i^*\| > 0\}, \\ z_j^* = 0_{\mathbb{R}^d}, \quad j \notin I, \quad \sum_{i \in I} \frac{1}{w_i} \|z_i^*\| \leq 1, \quad \sum_{i \in I} z_i^* = 0_{\mathbb{R}^d}}} \left\{ -\sum_{i \in I} \langle z_i^*, p_i \rangle \right\}. \end{aligned} \quad (4.53)$$

**Remark 4.12.** Note that for simplicity it is also possible to substitute  $z_i^* = -z_i^*$  for all  $i = 1, \dots, n$ , whence it follows

$$(\tilde{D}_N^S) \sup_{\substack{z_i^* \in \mathbb{R}^d, \ i=1, \dots, n, \ I=\{i \in \{1, \dots, n\} : \|z_i^*\| > 0\}, \\ z_j^* = 0_{\mathbb{R}^d}, \ j \notin I, \ \sum_{i \in I} \frac{1}{w_i} \|z_i^*\| \leq 1, \ \sum_{i \in I} z_i^* = 0_{\mathbb{R}^d}}} \left\{ \sum_{i \in I} \langle z_i^*, p_i \rangle \right\}. \quad (4.54)$$

**Theorem 4.9.** (strong duality) Between  $(P_N^S)$  and  $(\tilde{D}_N^S)$  holds strong duality, i.e.  $v(P_N^S) = v(\tilde{D}_N^S)$  and the dual problem has an optimal solution.

By Theorem 4.8 and 4.9 we derive the following necessary and sufficient optimality conditions.

**Theorem 4.10.** (optimality conditions) (a) Let  $\bar{x} \in \mathbb{R}^d$  be an optimal solution of the problem  $(P_N^S)$ . Then there exist  $\bar{z}_i^* \in \mathbb{R}^d$ ,  $i = 1, \dots, n$ , and an index set  $\bar{I}$  as an optimal solution to  $(\tilde{D}_N^S)$  such that

- (i)  $\max_{1 \leq j \leq n} \{w_j \|\bar{x} - p_j\|\} = \sum_{i \in \bar{I}} \|\bar{z}_i^*\| \|\bar{x} - p_i\|$ ,
- (ii)  $\sum_{i \in \bar{I}} \bar{z}_i^* = 0_{\mathbb{R}^d}$ ,
- (iii)  $\|\bar{z}_i^*\| \|\bar{x} - p_i\| = \langle \bar{z}_i^*, \bar{x} - p_i \rangle$ ,  $i \in \bar{I}$ ,
- (iv)  $\sum_{j \in \bar{I}} \frac{1}{w_j} \|\bar{z}_j^*\| = 1$ ,  $\bar{z}_i^* \in \mathbb{R}^d \setminus \{0_{\mathbb{R}^d}\}$  for  $i \in \bar{I}$  and  $\bar{z}_i^* = 0_{\mathbb{R}^d}$  for  $i \notin \bar{I}$ ,
- (v)  $w_i \|\bar{x} - p_i\| = \max_{1 \leq j \leq n} \{w_j \|\bar{x} - p_j\|\}$ ,  $i \in \bar{I}$ .

(b) If there exists  $\bar{x} \in \mathbb{R}^d$  such that for some  $\bar{z}_i^* \in \mathbb{R}^d$ ,  $i = 1, \dots, n$ , and an index set  $\bar{I}$  the conditions (i)-(v) are fulfilled, then  $\bar{x}$  is an optimal solution of  $(P_N^S)$ ,  $(\bar{z}^*, \bar{I})$  is an optimal solution for  $(\tilde{D}_N^S)$  and  $v(P_N^S) = v(\tilde{D}_N^S)$ .

For the length of the vectors  $z_i^*$ ,  $i \in I$ , feasible to  $(\tilde{D}_N^S)$  the following estimation from above can be made.

**Corollary 4.2.** Let  $w_s := \max_{1 \leq i \leq n} \{w_i\}$  and  $z_i^* \in \mathbb{R}^d$ ,  $i = 1, \dots, n$ , and  $I \subseteq \{1, \dots, n\}$  be a feasible solution to  $(\tilde{D}_N^S)$ , then it holds

$$\|z_i^*\| \leq \frac{w_s w_i}{w_s + w_i}, \quad i \in I.$$

**Proof.** Assume that  $z_i^* \in \mathbb{R}^d$ ,  $i = 1, \dots, n$  and  $I \subseteq \{1, \dots, n\}$  are feasible elements of the dual problem  $(\tilde{D}_N^S)$ , then one has for  $j \in I$ ,

$$\sum_{i \in I} z_i^* = 0_{\mathbb{R}^d} \Leftrightarrow z_j^* = - \sum_{\substack{i \in I \\ i \neq j}} z_i^*$$

and hence,

$$\|z_j^*\| = \left\| \sum_{\substack{i \in I \\ i \neq j}} z_i^* \right\| \leq \sum_{\substack{i \in I \\ i \neq j}} \|z_i^*\|, \quad j \in I. \quad (4.55)$$

Moreover, from the feasibility of  $z_i^*$ ,  $i \in I$ , to  $(\tilde{D}_N^S)$  and by (4.55), we have

$$\begin{aligned} 1 &\geq \sum_{i \in I} \frac{1}{w_i} \|z_i^*\| = \frac{1}{w_j} \|z_j^*\| + \sum_{\substack{i \in I \\ i \neq j}} \frac{1}{w_i} \|z_i^*\| \\ &\geq \frac{1}{w_j} \|z_j^*\| + \frac{1}{w_s} \sum_{\substack{i \in I \\ i \neq j}} \|z_i^*\| \geq \frac{1}{w_j} \|z_j^*\| + \frac{1}{w_s} \|z_j^*\| = \frac{w_s + w_j}{w_s w_j} \|z_j^*\|, \quad j \in I, \end{aligned}$$

and so,

$$\|z_j^*\| \leq \frac{w_s w_j}{w_s + w_j}, \quad j \in I.$$

□

By the next remark we point out the relation between the minimax and minisum problems.

**Remark 4.13.** *The optimal solution  $\bar{x}$  of the problem  $(P_N^S)$  is also a solution of the following generalized Fermat-Torricelli problem*

$$(P_N^{FT}) \quad \min_{x \in \mathbb{R}^d} \sum_{i \in \bar{I}} \tilde{w}_i \|x - p_i\|,$$

where  $\tilde{w}_i = \|\bar{z}_i^*\|$ ,  $i \in \bar{I}$ .

*This can be seen like follows: It is well known that  $\bar{x}$  is an optimal solution of the problem  $(P_N^{FT})$  with  $\bar{x} \neq p_i$ ,  $i \in I$ , if and only if the resultant force  $R$  at  $\bar{x}$ , defined by*

$$R(\bar{x}) := \sum_{i \in \bar{I}} \tilde{w}_i \frac{\bar{x} - p_i}{\|\bar{x} - p_i\|},$$

*is zero (see [63]). As  $\bar{x}$  is an optimal solution of  $(P_N^S)$ , we have by (4.42) that*

$$\sum_{i \in \bar{I}} \tilde{w}_i \frac{\bar{x} - p_i}{\|\bar{x} - p_i\|} = \sum_{i \in \bar{I}} \|\bar{z}_i^*\| \frac{\bar{x} - p_i}{\|\bar{x} - p_i\|} = \sum_{i \in \bar{I}} \alpha_i (\bar{x} - p_i) = \sum_{i \in \bar{I}} \bar{z}_i^* = 0_{\mathbb{R}^d},$$

*which implies that  $\bar{x}$  is also an optimal solution of the problem  $(P_N^{FT})$ . In this context, pay attention also to the fact that for the optimal solution  $\bar{x}$  of the problem  $(P_N^S)$  it holds  $\bar{x} \neq p_i$ ,  $i \in \bar{I}$ . Because if there exists  $j \in \bar{I}$  such that  $\bar{x} = p_j$ , then  $\bar{x} = p_i$  for all  $i \in \bar{I}$ , which contradicts the assumption that the given points are distinct.*

#### Geometrical interpretation.

For simplicity let us suppose that  $w_1 = \dots = w_n = 1$ , then it is well-known that the problem  $(P_N^S)$  can be interpreted as the finding of a ball with center  $\bar{x}$  and minimal radius such that all given points  $p_i$ ,  $i = 1, \dots, n$  are covered by this ball. This problem is also known as the minimum covering ball problem.

Our plan is now to give a geometrical interpretation of the set of optimal solutions of the dual problem  $(\tilde{D}_N^S)$  by using Theorem 4.10. By condition (iii) we see that for  $i \in \bar{I}$  the dual problem can geometrically be understood as the finding of vectors  $\bar{z}_i^*$ , which are parallel to the vectors  $\bar{x} - p_i$  and directed to  $\bar{x}$  fulfilling  $\sum_{i \in \bar{I}} \bar{z}_i^* = 0_{\mathbb{R}^d}$  and  $\sum_{i \in \bar{I}} \|\bar{z}_i^*\| = 1$ . Especially, conditions (iv) and (v) are telling us that for  $i \in \bar{I}$ , i.e.  $\bar{z}_i^* \neq 0_{\mathbb{R}^d}$ , the corresponding point  $p_i$  is lying on the border of the minimal covering ball and for  $i \notin \bar{I}$ , i.e.  $\bar{z}_i^* = 0_{\mathbb{R}^d}$ , the corresponding point  $p_i$  is lying inside the mentioned ball. Therefore, for  $i \in \bar{I}$  the elements  $\bar{z}_i^*$  can be interpreted as force vectors, which pull the points  $p_i$  lying on the border of the minimum covering ball inside of this ball in direction to the center, the gravity point  $\bar{x}$ , where the resultant force of the sum of these force vectors is zero. For illustration see Example 4.2 and Figure 4.1.

Another well-known geometrical characterization of the location problem  $(P_N^S)$  is to find the minimum radius of balls centered at the points  $p_i$ ,  $i = 1, \dots, n$ , such that their intersection is non-empty. In this situation, the set of optimal solutions of the dual problem can be described as force vectors fulfilling the optimality conditions of Theorem 4.10 and increasing these balls until their intersection is non-empty and the radius of the largest ball is minimal. From the conditions (iv) and (v) we obtain that a force vector  $\bar{z}_i^*$  is equal to the zero vector if  $\bar{x}$  is an element of the interior of the ball centered at point  $p_i$  with radius  $v(P_N^S)$ , which is exactly the case when  $i \notin \bar{I}$ . If  $i \in \bar{I}$ , which is exactly the case when  $\bar{x}$  is lying on the border of the ball centered at point  $p_i$  with

radius  $v(P_N^S)$ , then the corresponding force vector  $\bar{z}_i^*$  is unequal to the zero vector and moreover, by the optimality condition (iii) follows that  $\bar{z}_i^*$  is parallel to the vector  $\bar{x} - p_i$  and has the same direction.

To demonstrate the statements we made above, let us discuss the following example.

**Example 4.2.** Consider the unconstrained single minimax location problem in  $\mathbb{R}^2$  defined by the given points:

$$p_1 = (-5, -2.5)^T; p_2 = (-2, 1)^T; p_3 = (2.5, 3)^T; p_4 = (3.5, -2)^T \text{ and } p_5 = (0, -3)^T.$$

The primal problem looks in this case like follows

$$(\bar{P}_N^S) \quad \inf_{x \in \mathbb{R}^2} \max_{1 \leq i \leq 5} \{\|x - p_i\|\}$$

and by using the Matlab Optimization Toolbox we get the solution  $\bar{x} = (-0.866, -0.273)^T$  with the objective function value  $\max_{1 \leq i \leq 5} \{\|\bar{x} - p_i\|\} = 4.695$ .

For the dual problem we have the formulation (see Remark 4.7)

$$(\tilde{D}_N^S) \quad \sup_{\substack{z_i^* \in \mathbb{R}^2, \quad i=1, \dots, 5, \\ \sum_{i=1}^5 \|z_i^*\| \leq 1, \quad \sum_{i=1}^5 z_i^* = 0_{\mathbb{R}^2}}} \left\{ - \sum_{i=1}^5 \langle z_i^*, p_i \rangle \right\}. \quad (4.56)$$

with the solution

$$\begin{aligned} \bar{z}_1^* &= (0.412, 0.222)^T; \bar{z}_2^* = (0, 0)^T; \bar{z}_3^* = (-0.281, -0.273)^T; \\ \bar{z}_4^* &= (-0.131, 0.052)^T; \bar{z}_5^* = (0, 0)^T. \end{aligned}$$

The dual problem was also solved by using the Matlab Optimization Toolbox. In fact, it holds  $\bar{I} = \{1, 3, 4\}$ ,  $\langle \bar{z}_1^*, p_1 \rangle + \langle \bar{z}_3^*, p_3 \rangle + \langle \bar{z}_4^*, p_4 \rangle = 4.695$ ,  $\bar{x} = \|\bar{z}_1^*\|p_1 + \|\bar{z}_3^*\|p_3 + \|\bar{z}_4^*\|p_4 = 0.468 \cdot (-5, -2.5)^T + 0.392 \cdot (2.5, 3)^T + 0.14 \cdot (3.5, -2)^T = (-0.866, -0.273)^T$  (see (4.46)) and the points  $p_1$ ,  $p_3$  and  $p_4$  are lying on the border of the minimum covering circle as Figure 4.1 verifies.

**Remark 4.14.** Let  $w_i = 1$ ,  $i=1, \dots, n$ . Then, for the case  $n = 2$  it follows immediately by condition (iv) of Theorem 4.10 and Corollary 4.2 the well-known fact that  $\bar{x} = (1/2)(p_1 + p_2)$ .

**Remark 4.15.** Let  $w_i = 1$ ,  $i=1, \dots, n$ . If we consider the case  $d = 1$ , we can write the dual problem  $(\tilde{D}_N^S)$  as

$$(\tilde{D}_N^S) \quad \sup_{\substack{z_i^* \in \mathbb{R}, \quad i=1, \dots, n, \quad I = \{i \in \{1, \dots, n\} : |z_i^*| > 0\}, \\ z_j^* = 0, \quad j \notin I, \quad \sum_{i \in I} |z_i^*| \leq 1, \quad \sum_{i \in I} z_i^* = 0}} \left\{ - \sum_{i \in I} z_i^* p_i \right\} = \sup_{\substack{z^* \in \mathbb{R}^n, \quad \langle z^*, \mathbf{1} \rangle = 0, \\ \|z^*\|_1 \leq 1}} \{-\langle z^*, p \rangle\},$$

where  $z^* = (z_1^*, \dots, z_n^*)^T \in \mathbb{R}^n$ ,  $p = (p_1, \dots, p_n)^T \in \mathbb{R}^n$ ,  $\mathbf{1} = (1, \dots, 1)^T \in \mathbb{R}^n$  and  $\|\cdot\|_1$  is the Manhattan norm. From the second formulation of the problem  $(\tilde{D}_N^S)$  it is clear that the set of the feasible elements is the intersection of a hyperplane orthogonal to the vector  $\mathbf{1}$  and a cross-polytope (or hyperoctahedron), i.e. a convex polytope. Further, it is clear that the optimal solution of this problem can get immediately by the following consideration. Let us assume that  $p_1 < \dots < p_n$ , then it holds  $p_1 < \bar{x} < p_n$  and by condition (v) of Theorem 4.10 one gets

$$\max_{1 \leq j \leq n} \{|\bar{x} - p_j|\} = |\bar{x} - p_1| = |\bar{x} - p_n|,$$

i.e.  $\bar{I} = \{1, n\}$ . By Remark 4.14 this means  $\bar{x} = (1/2)(p_1 + p_n)$ . Moreover, by Corollary 4.2 we have that  $|\bar{z}_1^*| = |\bar{z}_n^*| = 0.5$  and by condition (iii) of Theorem 4.10 finally follows that  $\bar{z}_1^* = 0.5$  and  $\bar{z}_n^* = -0.5$ . A more detailed analysis of location problems using rectilinear distances was given in [34].



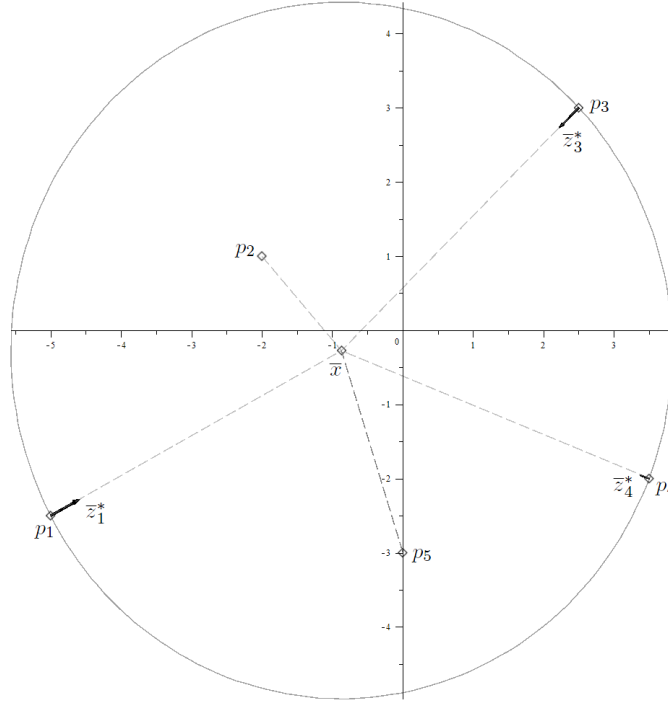


Figure 4.1: Geometrical illustration of the Example 4.2

By the next remark, we discover that the Lagrange multiplier associated with the linear equation constraint of the dual problem  $(\tilde{D}_N^S)$  is the optimal solution of the primal problem  $(P_N^S)$  and moreover, the Lagrange multiplier associated with the inequality constraint of the dual  $(\tilde{D}_N^S)$  is the optimal objective value. A similar result was shown in [61] for minisum location problems.

**Remark 4.16.** First, let us notice that the dual problem  $(\tilde{D}_N^S)$  can be written as (see Remark 4.7)

$$(\tilde{D}_N^S) \quad \sup_{\substack{z_i^* \in \mathbb{R}^d, \ i=1, \dots, n, \\ \sum_{i=1}^n \frac{1}{w_i} \|z_i^*\| \leq 1, \ \sum_{i=1}^n z_i^* = 0_{\mathbb{R}^d}}} \left\{ - \sum_{i=1}^n \langle z_i^*, p_i \rangle \right\},$$

then the Lagrange dual of the dual  $(\tilde{D}_N^S)$  looks like

$$\begin{aligned} (D\tilde{D}_N^S) & \quad \inf_{\lambda \geq 0, \ x \in \mathbb{R}^d} \sup_{z_i^* \in \mathbb{R}^d, \ i=1, \dots, n} \left\{ - \sum_{i=1}^n \langle z_i^*, p_i \rangle + \left\langle x, \sum_{i=1}^n z_i^* \right\rangle - \lambda \left( \sum_{i=1}^n \frac{1}{w_i} \|z_i^*\| - 1 \right) \right\} \\ & = \inf_{\lambda \geq 0, \ x \in \mathbb{R}^d} \left\{ \lambda + \sum_{i=1}^n \sup_{z_i^* \in \mathbb{R}^d} \left\{ \langle x - p_i, z_i^* \rangle - \frac{\lambda}{w_i} \|z_i^*\| \right\} \right\}. \end{aligned} \quad (4.57)$$

If  $\lambda = 0$ , then we get

$$\sup_{z_i^* \in \mathbb{R}^d} \langle x - p_i, z_i^* \rangle = \begin{cases} 0, & \text{if } x = p_i, \\ +\infty, & \text{otherwise,} \end{cases}$$

$i = 1, \dots, n$ , which contradicts the assumption from the beginning that the given points  $p_i, i = 1, \dots, n$

are distinct. Therefore, we can write for (4. 57)

$$\begin{aligned} (D\tilde{D}_N^S) &= \inf_{\lambda > 0, x \in \mathbb{R}^d} \left\{ \lambda + \lambda \sum_{i=1}^n \frac{1}{w_i} \sup_{z_i^* \in \mathbb{R}^d} \left\{ \left\langle \frac{w_i}{\lambda} (x - p_i), z_i^* \right\rangle - \|z_i^*\| \right\} \right\} \\ &= \inf_{\substack{\lambda > 0, x \in \mathbb{R}^d, \\ w_i \|x - p_i\| \leq \lambda, i=1, \dots, n}} \lambda = \inf_{x \in \mathbb{R}^d} \max_{1 \leq i \leq n} \{w_i \|x - p_i\|\}. \end{aligned}$$

We conclude, on the one hand, that the Lagrange dual of the dual problem  $(\tilde{D}_N^S)$  (i.e. the bidual of the primal location problem  $(P^S)$ ) is the problem  $(P^S)$ . On the other hand, we see that the Lagrange multipliers of the dual  $(D\tilde{D}_N^S)$  characterize the optimal solution and the optimal objective value of the primal problem  $(P^S)$ . Therefore, we have a complete symmetry between the primal problem  $(P_N^S)$ , the dual problem  $(\tilde{D}_N^S)$  and its Lagrange dual problem  $(D\tilde{D}_N^S)$ .

### 4.3 Extended multifacility minimax location problems

#### 4.3.1 Unconstraint location problems with set-up costs in Fréchet spaces

The location problem, which we investigate in a more general setting as suggested by Drezner in [35] and studied by Michelot and Plastria in [30, 67], is

$$(EP_a^M) \quad \inf_{(x_1, \dots, x_m) \in X^m} \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^m \gamma_{C_{ij}}(x_j - p_i) + a_i \right\},$$

where  $X$  is a Fréchet space,  $a_i \in \mathbb{R}_+$  are non-negative set-up costs,  $p_i \in X$  are distinct points and  $\gamma_{C_{ij}} : X \rightarrow \mathbb{R}$  are gauges defined by closed and convex subsets  $C_{ij}$  of  $X$  such that  $0_X \in \text{int } C_{ij}$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, m$ .

Now, set  $\tilde{X} = X^m$ ,  $x = (x_1, \dots, x_m) \in \tilde{X}$ ,  $\tilde{p}_i = (p_i, \dots, p_i) \in \tilde{X}$  and define the gauge  $\gamma_{C_i} : \tilde{X} \rightarrow \mathbb{R}$  by

$$\gamma_{C_i}(x) := \sum_{j=1}^m \gamma_{C_{ij}}(x_j), \quad x = (x_1, \dots, x_m) \in \tilde{X},$$

where  $C_i = \{x \in \tilde{X} : \gamma_{C_i}(x) \leq 1\}$ ,  $i = 1, \dots, n$ . Note that, as defined in the proof of Lemma 4.4,  $\langle x^*, x \rangle = \sum_{j=1}^m \langle x_j^*, x_j \rangle$  for  $x \in \tilde{X}$  and  $x^* \in \tilde{X}^*$ . Then, it is obvious that the location problem  $(EP_a^M)$  can also be written in a slightly different form, namely, as a single minimax location problem

$$(EP_a^M) \quad \inf_{x \in \tilde{X}} \max_{1 \leq i \leq n} \{ \gamma_{C_i}(x - \tilde{p}_i) + a_i \}.$$

We use (4. 16) of Lemma 4.4 and (4. 31) and get for the dual problem corresponding to  $(EP_a^M)$

$$(ED_a^M) \quad \sup_{\substack{z_i^{0*} \geq 0, z_{ij}^{1*} \in X^*, i=1, \dots, n, j=1, \dots, m, I = \{i \in \{1, \dots, n\} : z_i^{0*} > 0\}, \\ z_{kj}^{1*} = 0_{X^*}, k \notin I, \gamma_{C_{ij}}^0(z_{ij}^{1*}) \leq z_i^{0*}, i \in I, j=1, \dots, m, \sum_{i \in I} z_i^{0*} \leq 1}} \left\{ \inf_{x \in \tilde{X}} \left\{ \sum_{i \in I} \langle z_i^{1*}, x - \tilde{p}_i \rangle \right\} + \sum_{i \in I} z_i^{0*} a_i \right\}.$$

Because

$$\begin{aligned} \inf_{x \in \tilde{X}} \left\{ \sum_{i \in I} \langle z_i^{1*}, x - \tilde{p}_i \rangle \right\} &= \inf_{\substack{x_l \in X, \\ l=1, \dots, m}} \left\{ \sum_{i \in I} \sum_{j=1}^m \langle z_{ij}^{1*}, x_j - p_i \rangle \right\} \\ &= \sum_{j=1}^m \inf_{x_j \in X} \left\{ \sum_{i \in I} \langle z_{ij}^{1*}, x_j \rangle \right\} - \sum_{i \in I} \sum_{j=1}^m \langle z_{ij}^{1*}, p_i \rangle, \end{aligned}$$

we obtain finally for the conjugate dual problem of  $(EP_a^M)$

$$(ED_a^M) \sup_{(z_1^{0*}, \dots, z_n^{0*}, z_1^{1*}, \dots, z_n^{1*}) \in \mathcal{C}} \left\{ - \sum_{i \in I} \left[ \left\langle \sum_{j=1}^m z_{ij}^{1*}, p_i \right\rangle - z_i^{0*} a_i \right] \right\},$$

where

$$\mathcal{C} = \left\{ (z_1^{0*}, \dots, z_n^{0*}, z_1^{1*}, \dots, z_n^{1*}) \in \mathbb{R}_+^n \times (X^*)^m \times \dots \times (X^*)^m : I = \{i \in \{1, \dots, n\} : z_i^{0*} > 0\} \right. \\ \left. z_{kj}^{1*} = 0_{X^*}, k \notin I, \gamma_{C_{ij}^0}(z_{ij}^{1*}) \leq z_i^{0*}, i \in I, \sum_{i \in I} z_{ij}^{1*} = 0_{X^*}, j = 1, \dots, m, \sum_{i \in I} z_i^{0*} \leq 1 \right\}.$$

**Remark 4.17.** A similar dual problem was formulated by Michelot and Cornejo in [30] in the situation where  $X$  is the Euclidean space,  $m = 2$  and the gauges are a norm. The authors construct in their paper a Fenchel duality scheme to solve extended minimax location problems by a proximal algorithm.

**Remark 4.18.** In the sense of Remark 4.7 the dual problem  $(ED_a^M)$  is equivalent to

$$(\widehat{ED}_a^M) \sup_{(z_1^{0*}, \dots, z_n^{0*}, z_1^{1*}, \dots, z_n^{1*}) \in \widehat{\mathcal{C}}} \left\{ - \sum_{i=1}^n \left[ \left\langle \sum_{j=1}^m z_{ij}^{1*}, p_i \right\rangle - z_i^{0*} a_i \right] \right\},$$

where

$$\widehat{\mathcal{C}} = \left\{ (z_1^{0*}, \dots, z_n^{0*}, z_1^{1*}, \dots, z_n^{1*}) \in \mathbb{R}_+^n \times (X^*)^m \times \dots \times (X^*)^m : \gamma_{C_{kj}^0}(z_{kj}^{1*}) \leq z_k^{0*}, \right. \\ \left. \sum_{i=1}^n z_{ij}^{1*} = 0_{X^*}, k = 1, \dots, n, j = 1, \dots, m, \sum_{i=1}^n z_i^{0*} \leq 1 \right\}.$$

Let  $v(EP_a^M)$  be the optimal objective value of the location problem  $(EP_a^M)$  and  $v(ED_a^M)$  be the optimal objective value of the dual problem  $(ED_a^M)$ , then we obtain the following duality statement as a direct consequence of Theorem 4.2.

**Theorem 4.11.** (strong duality) Between  $(EP_a^M)$  and  $(ED_a^M)$  holds strong duality, i.e.  $v(EP_a^M) = v(ED_a^M)$  and the conjugate dual problem has an optimal solution.

The following necessary and sufficient optimality conditions are a consequence of the previous theorem.

**Theorem 4.12.** (optimality conditions) (a) Let  $(\bar{x}_1, \dots, \bar{x}_m) \in X^m$  be an optimal solution of the problem  $(EP_a^M)$ . Then there exist  $(\bar{z}_1^{0*}, \dots, \bar{z}_n^{0*}, \bar{z}_1^{1*}, \dots, \bar{z}_n^{1*}) \in \mathbb{R}_+^n \times (X^*)^m \times \dots \times (X^*)^m$  and an index set  $\bar{I} \subseteq \{1, \dots, n\}$  as an optimal solution to  $(ED_a^M)$  such that

$$(i) \max_{1 \leq u \leq n} \left\{ \sum_{j=1}^m \gamma_{C_{uj}}(\bar{x}_j - p_u) + a_u \right\} = \sum_{i \in \bar{I}} \bar{z}_i^{0*} \left( \sum_{j=1}^m \gamma_{C_{ij}}(\bar{x}_j - p_i) + a_i \right),$$

$$(ii) \bar{z}_i^{0*} \gamma_{C_{ij}}(\bar{x}_j - p_i) = \langle \bar{z}_{ij}^{1*}, \bar{x}_j - p_i \rangle, i \in \bar{I}, j = 1, \dots, m,$$

$$(iii) \sum_{i \in \bar{I}} \bar{z}_{ij}^{1*} = 0_{X^*}, j = 1, \dots, m,$$

$$(iv) \sum_{j \in \bar{I}} \bar{z}_j^{0*} = 1, \bar{z}_i^{0*} > 0, i \in \bar{I}, \text{ and } \bar{z}_k^{0*} = 0, k \notin \bar{I},$$

$$(v) \sum_{j=1}^m \gamma_{C_{ij}}(\bar{x}_j - p_i) + a_i = \max_{1 \leq u \leq n} \left\{ \sum_{j=1}^m \gamma_{C_{uj}}(\bar{x}_j - p_u) + a_u \right\}, \quad i \in \bar{I},$$

$$(vi) \max_{1 \leq l \leq m} \left\{ \gamma_{C_{il}^0}(\bar{z}_{il}^{1*}) \right\} = \bar{z}_i^{0*}, \quad (\bar{z}_{i1}^{1*}, \dots, \bar{z}_{im}^{1*}) \in X^* \times \dots \times X^* \setminus \{(0_{X^*}, \dots, 0_{X^*})\}, \quad i \in \bar{I}, \text{ and } \bar{z}_{kj}^{1*} = 0_{X^*}, \quad k \notin \bar{I}, \quad j = 1, \dots, m.$$

(b) If there exists  $(\bar{x}_1, \dots, \bar{x}_m) \in X^m$  such that for some  $(\bar{z}_1^{0*}, \dots, \bar{z}_n^{0*}, \bar{z}_1^{1*}, \dots, \bar{z}_n^{1*}) \in \mathbb{R}_+^n \times (X^*)^m \times \dots \times (X^*)^m$  and an index set  $\bar{I} \subseteq \{1, \dots, n\}$  the conditions (i)-(vi) are fulfilled, then  $(\bar{x}_1, \dots, \bar{x}_m)$  is an optimal solution of  $(EP_a^M)$ ,  $(\bar{z}_1^{0*}, \dots, \bar{z}_n^{0*}, \bar{z}_1^{1*}, \dots, \bar{z}_n^{1*}, \bar{I})$  is an optimal solution for  $(ED_a^M)$  and  $v(EP_a^M) = v(ED_a^M)$ .

**Proof.** From Theorem 4.11 we have  $v(EP_a^M) = v(ED_a^M)$ , i.e. for  $(\bar{x}_1, \dots, \bar{x}_m) \in X^m$  and  $(\bar{z}_1^{0*}, \dots, \bar{z}_n^{0*}, \bar{z}_1^{1*}, \dots, \bar{z}_n^{1*}) \in \mathbb{R}_+^n \times (X^*)^m \times \dots \times (X^*)^m$  and an index set  $\bar{I} \subseteq \{1, \dots, n\}$  it holds

$$\begin{aligned} & \max_{1 \leq u \leq n} \left\{ \sum_{j=1}^m \gamma_{C_{uj}}(\bar{x}_j - p_u) + a_u \right\} = - \sum_{i \in \bar{I}} \left[ \left\langle \sum_{j=1}^m \bar{z}_{ij}^{1*}, p_i \right\rangle - \bar{z}_i^{0*} a_i \right] \\ \Leftrightarrow & \max_{1 \leq u \leq n} \left\{ \sum_{j=1}^m \gamma_{C_{uj}}(\bar{x}_j - p_u) + a_u \right\} + \sum_{i \in \bar{I}} \left[ \left\langle \sum_{j=1}^m \bar{z}_{ij}^{1*}, p_i \right\rangle - \bar{z}_i^{0*} a_i \right] = 0 \\ \Leftrightarrow & \max_{1 \leq u \leq n} \left\{ \sum_{j=1}^m \gamma_{C_{uj}}(\bar{x}_j - p_u) + a_u \right\} + \sum_{i \in \bar{I}} \left[ \left\langle \sum_{j=1}^m \bar{z}_{ij}^{1*}, p_i \right\rangle - \bar{z}_i^{0*} a_i \right] \\ & + \sum_{i \in \bar{I}} \sum_{j=1}^m \bar{z}_i^{0*} \gamma_{C_{ij}}(\bar{x}_j - p_i) - \sum_{i \in \bar{I}} \sum_{j=1}^m \bar{z}_i^{0*} \gamma_{C_{ij}}(\bar{x}_j - p_i) \\ & + \left\langle \sum_{i \in \bar{I}} \sum_{j=1}^m \bar{z}_{ij}^{1*}, \bar{x}_j \right\rangle - \left\langle \sum_{i \in \bar{I}} \sum_{j=1}^m \bar{z}_{ij}^{1*}, \bar{x}_j \right\rangle = 0 \\ \Leftrightarrow & \left[ \max_{1 \leq u \leq n} \left\{ \sum_{j=1}^m \gamma_{C_{uj}}(\bar{x}_j - p_u) + a_u \right\} - \sum_{i \in \bar{I}} \sum_{j=1}^m \bar{z}_i^{0*} \gamma_{C_{ij}}(\bar{x}_j - p_i) - \sum_{i \in \bar{I}} \bar{z}_i^{0*} a_i \right] \\ & + \sum_{i \in \bar{I}} \sum_{j=1}^m [\bar{z}_i^{0*} \gamma_{C_{ij}}(\bar{x}_j - p_i) - \langle \bar{z}_{ij}^{1*}, \bar{x}_j - p_i \rangle] + \sum_{j=1}^m \left\langle \sum_{i \in \bar{I}} \bar{z}_{ij}^{1*}, \bar{x}_j \right\rangle = 0. \end{aligned} \quad (4.58)$$

If we define the function  $h_i : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  by

$$h_i(y) := \begin{cases} y, & \text{if } y \in \mathbb{R}_+, \\ +\infty, & \text{otherwise,} \end{cases} \quad (4.59)$$

then it follows by Lemma 4.2 that

$$\begin{aligned} g \left( \sum_{j=1}^m \gamma_{C_{1j}}(\bar{x}_j - p_1), \dots, \sum_{j=1}^m \gamma_{C_{nj}}(\bar{x}_j - p_n) \right) &= \max_{1 \leq u \leq n} \left\{ \sum_{j=1}^m \gamma_{C_{uj}}(\bar{x}_j - p_u) + a_u \right\} \\ &\geq \sum_{i \in \bar{I}} \bar{z}_i^{0*} \left[ \sum_{j=1}^m \gamma_{C_{ij}}(\bar{x}_j - p_i) + a_i \right], \end{aligned}$$

which means that the term in the first bracket of (4.58) is equal to zero. Moreover, by the Young-Fenchel inequality as well as by the fact that  $\sum_{i \in \bar{I}} \bar{z}_{ij}^{1*} = 0_{X^*}$ ,  $j = 1, \dots, m$ , we get that

the terms in the other brackets are also equal to zero. Hence, we derive the optimality conditions (i)-(iii).

By the feasibility condition,  $\sum_{i \in \bar{I}} \bar{z}_i^{0*} \leq 1$ , and the equality in the first bracket of (4. 58) it holds

$$\begin{aligned} \max_{1 \leq u \leq n} \left\{ \sum_{j=1}^m \gamma_{C_{uj}}(\bar{x}_j - p_u) + a_u \right\} &= \sum_{i \in \bar{I}} \bar{z}_i^{0*} \left( \sum_{j=1}^m \gamma_{C_{ij}}(\bar{x}_j - p_i) + a_i \right) \\ &\leq \sum_{i \in \bar{I}} \bar{z}_i^{0*} \max_{1 \leq u \leq n} \left\{ \sum_{j=1}^m \gamma_{C_{uj}}(\bar{x}_j - p_u) + a_u \right\} \\ &\leq \max_{1 \leq u \leq n} \left\{ \sum_{j=1}^m \gamma_{C_{uj}}(\bar{x}_j - p_u) + a_u \right\} \end{aligned}$$

and from here follows on the one hand that

$$\sum_{i \in \bar{I}} \bar{z}_i^{0*} = 1, \quad (4. 60)$$

and on the other hand that

$$\sum_{j=1}^m \gamma_{C_{ij}}(\bar{x}_j - p_i) + a_i = \max_{1 \leq u \leq n} \left\{ \sum_{j=1}^m \gamma_{C_{uj}}(\bar{x}_j - p_u) + a_u \right\}, \quad i \in \bar{I}. \quad (4. 61)$$

Moreover, as  $\bar{z}_i^{0*} \gamma_{C_{ij}}(\bar{x}_j - p_i) = \langle \bar{z}_{ij}^{1*}, \bar{x}_j - p_i \rangle$ ,  $i \in \bar{I}$ ,  $j = 1, \dots, m$ , one gets by the feasibility condition,

$$\gamma_{C_{ij}^0}(\bar{z}_{ij}^{1*}) \leq \bar{z}_i^{0*} \quad \forall j = 1, \dots, m, \quad i \in \bar{I}, \Leftrightarrow \max_{1 \leq l \leq m} \left\{ \gamma_{C_{ij}^0}(\bar{z}_{ij}^{1*}) \right\} \leq \bar{z}_i^{0*}, \quad i \in \bar{I}. \quad (4. 62)$$

Recall that  $\gamma_{C_i}(\bar{x} - \tilde{p}_i) = \sum_{j=1}^m \gamma_{C_{ij}}(\bar{x}_j - p_i)$  and that by Lemma 4.4 we have  $\gamma_{C_i^0}(\bar{z}_i^{1*}) = \max_{1 \leq j \leq m} \{ \gamma_{C_{ij}^0}(\bar{z}_{ij}^{1*}) \}$ , where  $\tilde{p}_i = (p_i, \dots, p_i) \in X^m$  and  $\bar{z}_i^{1*} = (\bar{z}_{i1}^{1*}, \dots, \bar{z}_{im}^{1*}) \in (X^*)^m$ ,  $i \in \bar{I}$ . Then one can show similarly to (4. 40) that

$$\gamma_{C_i^0}(\bar{z}_i^{1*}) \gamma_{C_i}(\bar{x} - \tilde{p}_i) \geq \langle \bar{z}_i^{1*}, \bar{x} - \tilde{p}_i \rangle, \quad (4. 63)$$

i.e.

$$\max_{1 \leq l \leq m} \left\{ \gamma_{C_{ij}^0}(\bar{z}_{ij}^{1*}) \right\} \sum_{j=1}^m \gamma_{C_{ij}}(\bar{x}_j - p_i) \geq \sum_{j=1}^m \langle \bar{z}_{ij}^{1*}, \bar{x}_j - p_i \rangle, \quad i \in \bar{I}. \quad (4. 64)$$

From here follows that

$$\begin{aligned} \bar{z}_i^{0*} \gamma_{C_i}(\bar{x} - \tilde{p}_i) &= \bar{z}_i^{0*} \sum_{j=1}^m \gamma_{C_{ij}}(\bar{x}_j - p_i) = \sum_{j=1}^m \langle \bar{z}_{ij}^{1*}, \bar{x}_j - p_i \rangle = \langle \bar{z}_i^{1*}, \bar{x} - \tilde{p}_i \rangle \\ &\leq \gamma_{C_i^0}(\bar{z}_i^{1*}) \gamma_{C_i}(\bar{x} - \tilde{p}_i) = \max_{1 \leq l \leq m} \left\{ \gamma_{C_{ij}^0}(\bar{z}_{ij}^{1*}) \right\} \sum_{j=1}^m \gamma_{C_{ij}}(\bar{x}_j - p_i) \\ &\leq \bar{z}_i^{0*} \sum_{j=1}^m \gamma_{C_{ij}}(\bar{x}_j - p_i), \quad i \in \bar{I}, \end{aligned}$$

and thus, the inequality in (4. 62) holds as equality. Taking now (4. 60), (4. 61) and (4. 62) as equality together yields the optimality conditions (iv)-(vi) and completes the proof.  $\square$

**Remark 4.19.** Let  $h_i : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  be defined by

$$h_i(x_i) := \begin{cases} x_i, & \text{if } x_i \in \mathbb{R}_+, \\ +\infty, & \text{otherwise,} \end{cases}$$

then the conjugate function of  $\lambda_i h_i$ ,  $\lambda_i \geq 0$ , is

$$(\lambda_i h_i)^*(x_i^*) = \begin{cases} 0, & \text{if } x_i^* \leq \lambda_i, \\ +\infty, & \text{otherwise,} \end{cases}, \quad i = 1, \dots, n.$$

In addition, we consider the function  $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ ,

$$f(y^0) = \begin{cases} \max_{1 \leq i \leq n} \{y_i^0 + a_i\}, & \text{if } y^0 = (y_1^0, \dots, y_n^0)^T \in \mathbb{R}_+^n, \quad i = 1, \dots, n, \\ +\infty, & \text{otherwise,} \end{cases}$$

and get by Lemma 4.1 that

$$f^*(z_1^{0*}, \dots, z_n^{0*}) = \min_{\substack{\sum_{i=1}^n \lambda_i \leq 1, \quad \lambda_i \geq 0, \quad z_i^{0*} \leq \lambda_i, \\ i=1, \dots, n}} \left\{ -\sum_{i=1}^n \lambda_i a_i \right\} \leq -\sum_{i=1}^n z_i^{0*} a_i$$

for all  $z_i^{0*} \leq \lambda_i$  with  $\lambda_i \geq 0$ ,  $i = 1, \dots, n$ ,  $\sum_{i=1}^n \lambda_i \leq 1$ . Hence, we have by the Young-Fenchel inequality and the optimal condition (i) of Theorem 4.12 that

$$\begin{aligned} \sum_{i \in \overline{I}} \bar{z}_i^{0*} \sum_{j=1}^m \gamma_{C_{ij}}(\bar{x}_j - p_i) &\leq f \left( \sum_{j=1}^m \gamma_{C_{1j}}(\bar{x}_j - p_1), \dots, \sum_{j=1}^m \gamma_{C_{nj}}(\bar{x}_j - p_n) \right) + f^*(\bar{z}_1^{0*}, \dots, \bar{z}_n^{0*}) \\ &\leq f \left( \sum_{j=1}^m \gamma_{C_{1j}}(\bar{x}_j - p_1), \dots, \sum_{j=1}^m \gamma_{C_{nj}}(\bar{x}_j - p_n) \right) - \sum_{i=1}^n \bar{z}_i^{0*} a_i = \sum_{i \in \overline{I}} \bar{z}_i^{0*} \sum_{j=1}^m \gamma_{C_{ij}}(\bar{x}_j - p_i), \end{aligned}$$

i.e.

$$f \left( \sum_{j=1}^m \gamma_{C_{1j}}(\bar{x}_j - p_1), \dots, \sum_{j=1}^m \gamma_{C_{nj}}(\bar{x}_j - p_n) \right) + f^*(\bar{z}_1^{0*}, \dots, \bar{z}_n^{0*}) = \sum_{i \in \overline{I}} \bar{z}_i^{0*} \sum_{j=1}^m \gamma_{C_{ij}}(\bar{x}_j - p_i)$$

and by (2. 1) this equality is equivalent to

$$(z_1^{0*}, \dots, z_n^{0*}) \in \partial f \left( \sum_{j=1}^m \gamma_{C_{1j}}(\bar{x}_j - p_1), \dots, \sum_{j=1}^m \gamma_{C_{nj}}(\bar{x}_j - p_n) \right).$$

In other words, the condition (i) of Theorem 4.12 can be written by means of the subdifferential, i.e.,

$$(i) \quad (\bar{z}_1^{0*}, \dots, \bar{z}_n^{0*}) \in \partial \left( \max_{1 \leq j \leq n} \{ \cdot + a_j \} \right) \left( \sum_{j=1}^m \gamma_{C_{1j}}(\bar{x}_j - p_1), \dots, \sum_{j=1}^m \gamma_{C_{nj}}(\bar{x}_j - p_n) \right).$$

Similarly, we can rewrite the condition (ii) of Theorem 4.12 as follows

$$(ii) \quad \bar{z}_{ij}^{1*} \in \partial(\bar{z}_i^{0*} \gamma_{C_{ij}})(\bar{x}_j - p_i), \quad i \in \overline{I}, \quad j = 1, \dots, m.$$

Moreover, combining this condition with the optimality condition (iii) of Theorem 4.12 yields that

$$0_{X^*} \in \sum_{i \in \overline{I}} \partial(\bar{z}_i^{0*} \gamma_{C_{ij}})(\bar{x}_j - p_i), \quad j = 1, \dots, m.$$

Notice also that the optimality conditions (ii) and (vi) of Theorem 4.12 give a detailed characterization of the subdifferential of  $\bar{z}_i^{0*} \gamma_{C_{ij}}$  at  $\bar{x}_j - p_i$  such that

$$\partial(\bar{z}_i^{0*} \gamma_{C_{ij}})(\bar{x}_j - p_i) = \left\{ \bar{z}_{ij}^{1*} \in X^* : \bar{z}_i^{0*} \gamma_{C_{ij}}(\bar{x}_j - p_i) = \langle \bar{z}_{ij}^{1*}, \bar{x}_j - p_i \rangle, \max_{1 \leq l \leq m} \left\{ \gamma_{C_{il}}(\bar{z}_{il}^{1*}) \right\} = \bar{z}_i^{0*} \right\}$$

for all  $i \in \bar{I}$ ,  $j = 1, \dots, m$ .

Let us now consider the extended location problem  $(EP_a^M)$  in the following framework. We set  $X = \mathcal{H}$ , where  $\mathcal{H}$  is a real Hilbert space and  $\gamma_{C_{ij}} : \mathcal{H} \rightarrow \mathbb{R}$ ,  $\gamma_{C_{ij}}(x) := w_{ij} \|x\|_{\mathcal{H}}$ , where  $w_{ij} > 0$  for  $j = 1, \dots, m$ ,  $i = 1, \dots, n$ . Hence, the location problem looks like

$$(EP_{N,a}^M) \quad \inf_{(x_1, \dots, x_m) \in \mathcal{H} \times \dots \times \mathcal{H}} \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^m w_{ij} \|x_j - p_i\|_{\mathcal{H}} + a_i \right\}.$$

For this situation, where the gauges are all identical and the distances are measured by a round norm, Michelot and Plastria examined in [67] under which conditions an optimal solution of coincidence type exists. The authors showed that if the weights have a multiplicative structure, i.e.  $w_{ij} = \lambda_i \mu_j$  with  $\lambda_i, \mu_j > 0$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, m$ , and  $\sum_{j=1}^m \mu_j = 1$ , then there exists an optimal solution of  $(EP_{N,a}^M)$  such that all new facilities coincide. Moreover, they described when the optimal solution of coincidence type is unique and presented a full characterization of the set of optimal solutions for extended multifacility location problems where the weights have a multiplicative structure.

The next statement is based on the idea of weights with a multiplicative structure and illustrates in this situation the relation between the extended location problem  $(EP_{N,a}^M)$  and its corresponding conjugate dual problem.

**Theorem 4.13.** *Let  $X = \mathcal{H}$ ,  $\gamma_{C_{ij}} : \mathcal{H} \rightarrow \mathbb{R}$  be defined by  $\gamma_{C_{ij}}(x) := w_{ij} \|x\|_{\mathcal{H}}$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, m$ , and  $w_{ij} = \lambda_i \mu_j$  with  $\lambda_i, \mu_j > 0$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, m$ , and  $\sum_{j=1}^m \mu_j = 1$ . Assume that  $\Delta_{\bar{x}} = (\bar{x}, \dots, \bar{x}) \in \underbrace{\mathcal{H} \times \dots \times \mathcal{H}}_{m\text{-times}}$  is an optimal solution of coincidence type of*

$$(EP_{N,a}^M) \quad \inf_{(x_1, \dots, x_m) \in \mathcal{H} \times \dots \times \mathcal{H}} \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^m w_{ij} \|x_j - p_i\|_{\mathcal{H}} + a_i \right\}.$$

and  $(\bar{z}_1^{0*}, \dots, \bar{z}_n^{0*}, \bar{z}_1^{1*}, \dots, \bar{z}_n^{1*}, \bar{I})$  and  $\bar{I} \subseteq \{1, \dots, n\}$  is an optimal solution of the corresponding conjugate dual problem

$$(ED_{N,a}^M) \quad \sup_{(z_1^{0*}, \dots, z_n^{0*}, z_1^{1*}, \dots, z_n^{1*}) \in \mathcal{C}} \left\{ - \sum_{i \in \bar{I}} \left[ \left\langle \sum_{j=1}^m z_{ij}^{1*}, p_i \right\rangle_{\mathcal{H}} - z_i^{0*} a_i \right] \right\},$$

where

$$\mathcal{C} = \left\{ (z_1^{0*}, \dots, z_n^{0*}, z_1^{1*}, \dots, z_n^{1*}) \in \mathbb{R}_+^n \times \underbrace{\mathcal{H}^m \times \dots \times \mathcal{H}^m}_{n\text{-times}} : I = \{i \in \{1, \dots, n\} : z_i^{0*} > 0\} \right. \\ \left. z_{kj}^{1*} = 0_{\mathcal{H}}, k \notin I, \|z_{ij}^{1*}\|_{\mathcal{H}} \leq z_i^{0*} w_{ij}, i \in I, \sum_{i \in I} z_{ij}^{1*} = 0_{\mathcal{H}}, j = 1, \dots, m, \sum_{i \in I} z_i^{0*} \leq 1 \right\}.$$

Then, it holds

$$\bar{x} = \frac{1}{\sum_{i \in \bar{I}} \frac{\lambda_i \|\bar{z}_{ij}^{1*}\|_{\mathcal{H}}}{v(ED_{N,a}^M) - a_i}} \sum_{i \in \bar{I}} \frac{\lambda_i \|\bar{z}_{ij}^{1*}\|_{\mathcal{H}}}{v(ED_{N,a}^M) - a_i} p_i \quad \forall j \in \bar{J},$$

where

$$\bar{J} := \left\{ j \in \{1, \dots, m\} : \frac{1}{w_{ij}} \|\bar{z}_{ij}^{1*}\|_{\mathcal{H}} = \max_{1 \leq l \leq m} \left\{ \frac{1}{w_{il}} \|\bar{z}_{il}^{1*}\|_{\mathcal{H}} \right\} \right\}, \quad i \in \bar{I}.$$

**Proof.** First, let us remark that the dual norm of the weighted norm  $\gamma_{C_{ij}} = w_{ij} \|\cdot\|_{\mathcal{H}}$  is given by  $\gamma_{C_{ij}^0} = (1/w_{ij}) \|\cdot\|_{\mathcal{H}}$ .

Now, let  $\Delta_{\bar{x}} = (\bar{x}, \dots, \bar{x})$  be an optimal solution of coincidence type, then the optimality conditions (ii), (iii), (v) and (vi) of Theorem 4.12 can be written as

$$(ii) \quad \bar{z}_i^{0*} w_{ij} \|\bar{x} - p_i\|_{\mathcal{H}} = \langle \bar{z}_{ij}^{1*}, \bar{x} - p_i \rangle_{\mathcal{H}}, \quad i \in \bar{I}, \quad j = 1, \dots, m,$$

$$(iii) \quad \sum_{i \in \bar{I}} \bar{z}_{ij}^{1*} = 0_{\mathcal{H}}, \quad j = 1, \dots, m,$$

$$(v) \quad \sum_{j=1}^m w_{ij} \|\bar{x} - p_i\|_{\mathcal{H}} + a_i = \max_{1 \leq u \leq n} \left\{ \sum_{j=1}^m w_{uj} \|\bar{x} - p_u\|_{\mathcal{H}} + a_u \right\}, \quad i \in \bar{I},$$

$$(vi) \quad \max_{1 \leq l \leq m} \left\{ \frac{1}{w_{il}} \|\bar{z}_{il}^{1*}\|_{\mathcal{H}} \right\} = \bar{z}_i^{0*}, \quad (\bar{z}_{i1}^{1*}, \dots, \bar{z}_{im}^{1*}) \in \mathcal{H} \times \dots \times \mathcal{H} \setminus \{(0_{\mathcal{H}}, \dots, 0_{\mathcal{H}})\}, \quad i \in \bar{I} \text{ and } \bar{z}_{kj}^{1*} = 0_{\mathcal{H}}, \quad k \notin \bar{I}, \quad j = 1, \dots, m.$$

By combining the conditions (ii) and (vi), we get

$$\|\bar{z}_{ij}^{1*}\|_{\mathcal{H}} \|\bar{x} - p_i\|_{\mathcal{H}} = \langle \bar{z}_{ij}^{1*}, \bar{x} - p_i \rangle_{\mathcal{H}}, \quad i \in \bar{I}, \quad j \in \bar{J}. \quad (4.65)$$

Moreover, by Fact 2.10 in [2] there exists  $\alpha_{ij} > 0$  such that

$$\bar{z}_{ij}^{1*} = \alpha_{ij} (\bar{x} - p_i) \quad (4.66)$$

and from here one gets that

$$\|\bar{z}_{ij}^{1*}\|_{\mathcal{H}} = \alpha_{ij} \|\bar{x} - p_i\|_{\mathcal{H}}, \quad (4.67)$$

$i \in \bar{I}, j \in \bar{J}$ . By condition (v) follows

$$\begin{aligned} \sum_{j=1}^m w_{ij} \|\bar{x} - p_i\|_{\mathcal{H}} + a_i &= \max_{1 \leq u \leq n} \left\{ \sum_{j=1}^m w_{uj} \|\bar{x} - p_u\|_{\mathcal{H}} + a_u \right\} \\ \Leftrightarrow \lambda_i \sum_{j=1}^m \mu_j \|\bar{x} - p_i\|_{\mathcal{H}} + a_i &= \max_{1 \leq u \leq n} \left\{ \lambda_u \sum_{j=1}^m \mu_j \|\bar{x} - p_u\|_{\mathcal{H}} + a_u \right\} \\ \Leftrightarrow \lambda_i \|\bar{x} - p_i\|_{\mathcal{H}} + a_i &= \max_{1 \leq u \leq n} \{ \lambda_u \|\bar{x} - p_u\|_{\mathcal{H}} + a_u \}, \quad i \in \bar{I}. \end{aligned} \quad (4.68)$$

Bringing (4.67) and (4.68) together yields

$$\begin{aligned} \frac{\lambda_i}{\alpha_{ij}} \|\bar{z}_{ij}^{1*}\|_{\mathcal{H}} + a_i &= \max_{1 \leq u \leq n} \{ \lambda_u \|\bar{x} - p_u\|_{\mathcal{H}} + a_u \} \\ \Leftrightarrow \alpha_{ij} &= \frac{\lambda_i}{\max_{1 \leq u \leq n} \{ \lambda_u \|\bar{x} - p_u\|_{\mathcal{H}} + a_u \} - a_i} \|\bar{z}_{ij}^{1*}\|_{\mathcal{H}}, \quad i \in \bar{I}, \quad j \in \bar{J}. \end{aligned} \quad (4.69)$$

Taking the sum overall  $i \in \bar{I}$  in (4.69) gives

$$\sum_{i \in \bar{I}} \alpha_{ij} = \sum_{i \in \bar{I}} \frac{\lambda_i \|\bar{z}_{ij}^{1*}\|_{\mathcal{H}}}{\max_{1 \leq u \leq n} \{ \lambda_u \|\bar{x} - p_u\|_{\mathcal{H}} + a_u \} - a_i}, \quad j \in \bar{J}. \quad (4.70)$$



Now, consider condition (iii), by (4. 66) follows

$$0_{\mathcal{H}} = \sum_{i \in \bar{I}} \bar{z}_{ij}^{1*} = \sum_{i \in \bar{I}} \alpha_{ij} (\bar{x} - p_i) \Leftrightarrow \bar{x} = \frac{1}{\sum_{i \in \bar{I}} \alpha_{ij}} \sum_{i \in \bar{I}} \alpha_{ij} p_i, \quad j \in \bar{J}. \quad (4. 71)$$

Putting (4. 69), (4. 70) and (4. 71) together reveals

$$\begin{aligned} \bar{x} &= \frac{1}{\sum_{i \in \bar{I}} \frac{\lambda_i \|\bar{z}_{ij}^{1*}\|_{\mathcal{H}}}{\max_{1 \leq u \leq n} \{\lambda_u \|\bar{x} - p_u\|_{\mathcal{H}} + a_u\} - a_i}} \sum_{i \in \bar{I}} \frac{\lambda_i \|\bar{z}_{ij}^{1*}\|_{\mathcal{H}}}{\max_{1 \leq u \leq n} \{\lambda_u \|\bar{x} - p_u\|_{\mathcal{H}} + a_u\} - a_i} p_i \\ &= \frac{1}{\sum_{i \in \bar{I}} \frac{\lambda_i \|\bar{z}_{ij}^{1*}\|_{\mathcal{H}}}{v(ED_{N,a}^M) - a_i}} \sum_{i \in \bar{I}} \frac{\lambda_i \|\bar{z}_{ij}^{1*}\|_{\mathcal{H}}}{v(ED_{N,a}^M) - a_i} p_i, \quad j \in \bar{J}, \end{aligned}$$

and the proof is finished.  $\square$

**Remark 4.20.** In the context of Theorem 4.13, it holds that  $\bar{x} - p_i$  and  $\bar{z}_{ij}^{1*}$  are parallel and so the vectors  $(1/w_{ij})\bar{z}_{ij}^{1*}$ ,  $j \in \bar{J}$ , are all parallel to each other. In other words, the vectors  $(1/w_{ij})\bar{z}_{ij}^{1*}$ ,  $j \in \bar{J}$ , are identical. In this sense, one can understand the optimal solution of the conjugate dual problem also as a solution of coincidence type.

The next statement holds for any weights, not necessary of multiplicative structure.

**Lemma 4.6.** Let  $w_{sj} := \max_{1 \leq u \leq n} \{w_{uj}\}$ ,  $X = \mathcal{H}$ ,  $\gamma_{C_{ij}} : \mathcal{H} \rightarrow \mathbb{R}$  be defined by  $\gamma_{C_{ij}}(x) := w_{ij} \|x\|_{\mathcal{H}}$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, m$ , and  $(z_1^{0*}, \dots, z_n^{0*}, z_1^{1*}, \dots, z_n^{1*})$  a feasible solution of the conjugate dual problem  $(ED_{N,a}^M)$ , then it holds

$$\|z_{ij}^{1*}\|_{\mathcal{H}} \leq \frac{w_{sj} w_{ij}}{w_{sj} + w_{ij}}, \quad i \in I, \quad j = 1, \dots, m.$$

**Proof.** Let

$$(z_1^{0*}, \dots, z_n^{0*}, z_1^{1*}, \dots, z_n^{1*}) \in \mathbb{R}^n \times \underbrace{\mathcal{H} \times \dots \times \mathcal{H}}_{m\text{-times}} \times \dots \times \underbrace{\mathcal{H} \times \dots \times \mathcal{H}}_{m\text{-times}}$$

be a feasible solution of the conjugate dual problem  $(ED_{N,a}^M)$ , then we have

- (i)  $\sum_{i \in I} z_i^{0*} \leq 1$ ,
- (ii)  $\|z_{ij}^{1*}\|_{\mathcal{H}} \leq z_i^{0*} w_{ij}$ ,  $j = 1, \dots, m$ ,  $i \in I$ ,
- (iii)  $\sum_{i \in I} z_{ij}^{1*} = 0_{\mathcal{H}}$ .

The inequalities (i) and (ii) imply the inequality

$$\sum_{i \in I} \frac{1}{w_{ij}} \|z_{ij}^{1*}\|_{\mathcal{H}} \leq 1, \quad j = 1, \dots, m. \quad (4. 72)$$

Furthermore, by (iii) we have

$$\sum_{i \in I} z_{ij}^{1*} = 0_{\mathcal{H}} \Leftrightarrow z_{kj}^{1*} = - \sum_{\substack{i \in I \\ i \neq k}} z_{ij}^{1*}, \quad k \in I, \quad j = 1, \dots, m, \quad (4. 73)$$

and hence,

$$\|z_{kj}^{1*}\|_{\mathcal{H}} = \left\| \sum_{\substack{i \in I \\ i \neq k}} z_{ij}^{1*} \right\|_{\mathcal{H}} \leq \sum_{\substack{i \in I \\ i \neq k}} \|z_{ij}^{1*}\|_{\mathcal{H}}, \quad k \in I, \quad j = 1, \dots, m. \quad (4. 74)$$

By (4. 74) we get in (4. 72)

$$\begin{aligned} 1 &\geq \frac{1}{w_{kj}} \|z_{kj}^{1*}\|_{\mathcal{H}} + \sum_{\substack{i \in I \\ i \neq k}} \frac{1}{w_{ij}} \|z_{ij}^{1*}\|_{\mathcal{H}} \geq \frac{1}{w_{kj}} \|z_{kj}^{1*}\|_{\mathcal{H}} + \frac{1}{w_{sj}} \sum_{\substack{i \in I \\ i \neq k}} \|z_{ij}^{1*}\|_{\mathcal{H}} \\ &\geq \frac{1}{w_{kj}} \|z_{kj}^{1*}\|_{\mathcal{H}} + \frac{1}{w_{sj}} \|z_{kj}^{1*}\|_{\mathcal{H}} = \frac{w_{sj} + w_{kj}}{w_{kj} w_{sj}} \|z_{kj}^{1*}\|_{\mathcal{H}}, \quad j = 1, \dots, m, \end{aligned}$$

and finally,

$$\|z_{kj}^{1*}\|_{\mathcal{H}} \leq \frac{w_{sj} w_{kj}}{w_{sj} + w_{kj}}, \quad k \in I, \quad j = 1, \dots, m. \quad \square$$

**Remark 4.21.** *If we allow also negative set-up costs, then we have in the constraint set, as stated in Remark 4.9,  $\sum_{i \in I} z_i^{0*} = 1$  instead  $\sum_{i \in I} z_i^{0*} \leq 1$ . One can easily verify that the results we presented above also holds in this case.*

### 4.3.2 Unconstrained location problems without set-up costs in Fréchet spaces

In the next, we study the case where  $X$  is a Fréchet space and  $a_i = 0$  for all  $i = 1, \dots, n$ . With this assumption the extended multifacility location problem  $(EP_a^M)$  can be stated as

$$(EP^M) \quad \inf_{(x_1, \dots, x_m) \in X^m} \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^m \gamma_{C_{ij}}(x_j - p_i) \right\}.$$

In this situation its corresponding conjugated dual problem  $(ED_a^M)$  transforms into

$$(ED^M) \quad \sup_{(z_1^{0*}, \dots, z_n^{0*}, z_1^{1*}, \dots, z_n^{1*}) \in \mathcal{C}} \left\{ - \sum_{i \in I} \left\langle \sum_{j=1}^m z_{ij}^{1*}, p_i \right\rangle \right\}.$$

Additionally, let us consider the following dual problem

$$(E\tilde{D}^M) \quad \sup_{(z_1^*, \dots, z_n^*) \in \tilde{\mathcal{C}}} \left\{ - \sum_{i \in I} \left\langle \sum_{j=1}^m z_{ij}^*, p_i \right\rangle \right\}$$

where

$$\begin{aligned} \tilde{\mathcal{C}} = & \left\{ (z_1^*, \dots, z_n^*) \in (X^*)^m \times \dots \times (X^*)^m : I = \left\{ i \in \{1, \dots, n\} : \max_{1 \leq j \leq m} \left\{ \gamma_{C_{ij}^0}(z_{ij}^*) \right\} > 0 \right\} \right. \\ & \left. z_{kj}^* = 0_{X^*}, \quad k \notin I, \quad \sum_{i \in I} z_{ij}^* = 0_{X^*}, \quad j = 1, \dots, m, \quad \sum_{i \in I} \max_{1 \leq l \leq m} \left\{ \gamma_{C_{il}^0}(z_{il}^*) \right\} \leq 1 \right\}. \end{aligned}$$

Let us denote by  $v(ED_a^M)$  and  $v(E\tilde{D}^M)$  the optimal objective values of the dual problems  $(ED_a^M)$  and  $(E\tilde{D}^M)$ , respectively, then we can state.

**Theorem 4.14.** *It holds  $v(ED^M) = v(E\tilde{D}^M)$ .*

**Proof.** The statement follows immediately by Theorem 4.6 and (4. 16).  $\square$

The next duality statements follow as direct consequences of Theorem 4.11 and Theorem 4.14.

**Theorem 4.15.** *(strong duality) Between  $(EP^M)$  and  $(E\tilde{D}^M)$  strong duality holds, i.e.  $v(EP^M) = v(E\tilde{D}^M)$  and the dual problem  $v(E\tilde{D}^M)$  has an optimal solution.*

We define

$$J_\gamma := \left\{ j \in \{1, \dots, m\} : \gamma_{C_{ij}^0}(z_{ij}^*) > 0 \right\}, \quad i \in I,$$

and obtain as a result of Theorem 4.12 (especially by using the optimality condition (vi)), Theorem 4.14 and 4.15 the following optimality conditions.

**Theorem 4.16.** (*optimality conditions*) (a) Let  $(\bar{x}_1, \dots, \bar{x}_m) \in X^m$  be an optimal solution of the problem  $(EP^M)$ . Then there exist  $(\bar{z}_1^*, \dots, \bar{z}_n^*) \in (X^*)^m \times \dots \times (X^*)^m$  and an index set  $\bar{I} \subseteq \{1, \dots, n\}$  as an optimal solution to  $(E\tilde{D}^M)$  such that

- (i)  $\max_{1 \leq u \leq n} \left\{ \sum_{j=1}^m \gamma_{C_{uj}}(\bar{x}_j - p_u) \right\} = \sum_{i \in \bar{I}} \sum_{j=1}^m \gamma_{C_{ij}^0}(\bar{z}_{ij}^*) \gamma_{C_{ij}}(\bar{x}_j - p_i),$
- (ii)  $\sum_{i \in \bar{I}} \bar{z}_{ij}^* = 0_{X^*}, \quad j = 1, \dots, m,$
- (iii)  $\gamma_{C_{ij}^0}(\bar{z}_{ij}^*) \gamma_{C_{ij}}(\bar{x}_j - p_i) = \langle \bar{z}_{ij}^*, \bar{x}_j - p_i \rangle, \quad i \in \bar{I}, \quad j = 1, \dots, m,$
- (iv)  $\sum_{i \in \bar{I}} \max_{1 \leq l \leq m} \left\{ \gamma_{C_{il}^0}(\bar{z}_{il}^*) \right\} = 1,$
- (v)  $\max_{1 \leq u \leq n} \left\{ \sum_{j=1}^m \gamma_{C_{uj}}(\bar{x}_j - p_u) \right\} = \sum_{j=1}^m \gamma_{C_{ij}}(\bar{x}_j - p_i), \quad i \in \bar{I},$
- (vi)  $\max_{1 \leq l \leq m} \left\{ \gamma_{C_{il}^0}(\bar{z}_{il}^*) \right\} = \gamma_{C_{ij}^0}(\bar{z}_{ij}^*) > 0, \quad j \in \bar{J}_\gamma, \quad (\bar{z}_{i1}^*, \dots, \bar{z}_{im}^*) \in X^* \times \dots \times X^* \setminus \{(0_{X^*}, \dots, 0_{X^*})\}, \quad i \in \bar{I},$  and  $\bar{z}_{ks}^* = 0_{X^*}, \quad k \notin \bar{I}, \quad s = 1, \dots, m.$

(b) If there exists  $(\bar{x}_1, \dots, \bar{x}_m) \in X^m$  such that for some  $(\bar{z}_1^*, \dots, \bar{z}_n^*) \in (X^*)^m \times \dots \times (X^*)^m$  and an index set  $\bar{I}$  the conditions (i)-(vi) are fulfilled, then  $(\bar{x}_1, \dots, \bar{x}_m)$  is an optimal solution of  $(EP^M)$ ,  $(\bar{z}_1^*, \dots, \bar{z}_n^*, \bar{I})$  is an optimal solution for  $(E\tilde{D}^M)$  and  $v(EP^M) = v(E\tilde{D}^M)$ .

Now, our aim is to investigate the location problem  $(EP^M)$  from the geometrical point of view. For this purpose let  $X = \mathbb{R}^d$  and the distances are measured by the Euclidean norm. Then, the problem  $(EP^M)$  turns into

$$(EP_N^M) \quad \inf_{(x_1, \dots, x_m) \in \mathbb{R}^d \times \dots \times \mathbb{R}^d} \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^m w_{ij} \|x_j - p_i\| \right\},$$

while its conjugate dual problem transforms into

$$(E\tilde{D}_N^M) \quad \sup_{(z_1^*, \dots, z_n^*) \in \tilde{C}} \left\{ - \sum_{i \in \bar{I}} \left\langle \sum_{j=1}^m z_{ij}^*, p_i \right\rangle \right\}$$

with

$$\tilde{C} = \left\{ (z_1^*, \dots, z_n^*) \in (\mathbb{R}^d)^m \times \dots \times (\mathbb{R}^d)^m : I = \left\{ i \in \{1, \dots, n\} : \max_{1 \leq j \leq m} \left\{ \frac{1}{w_{ij}} \|z_{ij}^*\| \right\} > 0 \right\} \right. \\ \left. z_{kj}^* = 0_{\mathbb{R}^d}, \quad k \notin I, \quad \sum_{i \in \bar{I}} z_{ij}^* = 0_{\mathbb{R}^d}, \quad j = 1, \dots, m, \quad \sum_{i \in \bar{I}} \max_{1 \leq l \leq m} \left\{ \frac{1}{w_{il}} \|z_{il}^*\| \right\} \leq 1 \right\}.$$

Via Theorem 4.15 and 4.16 the following statements follows immediately.

**Theorem 4.17.** (strong duality) Between  $(EP_N^M)$  and  $(ED_N^M)$  holds strong duality, i.e.  $v(EP_N^M) = v(ED_N^M)$  and the dual problem has an optimal solution.

**Theorem 4.18.** (optimality conditions) (a) Let  $(\bar{x}_1, \dots, \bar{x}_m) \in \mathbb{R}^d \times \dots \times \mathbb{R}^d$  be an optimal solution of the problem  $(EP_N^M)$ . Then there exist

$$(\bar{z}_1^*, \dots, \bar{z}_n^*) \in \underbrace{\mathbb{R}^d \times \dots \times \mathbb{R}^d}_{m\text{-times}} \times \dots \times \underbrace{\mathbb{R}^d \times \dots \times \mathbb{R}^d}_{m\text{-times}}$$

and an index set  $\bar{I}$  as an optimal solution to  $(ED_N^M)$  such that

- (i)  $\max_{1 \leq u \leq n} \left\{ \sum_{j=1}^m w_{uj} \|\bar{x}_j - p_u\| \right\} = \sum_{i \in \bar{I}} \sum_{j=1}^m \|\bar{z}_{ij}^*\| \|\bar{x}_j - p_i\|,$
- (ii)  $\sum_{i \in \bar{I}} \bar{z}_{ij}^* = 0_{\mathbb{R}^d}, j = 1, \dots, m,$
- (iii)  $\|\bar{z}_{ij}^*\| \|\bar{x}_j - p_i\| = \langle \bar{z}_{ij}^*, \bar{x}_j - p_i \rangle, i \in \bar{I}, j = 1, \dots, m,$
- (iv)  $\sum_{i \in \bar{I}} \max_{1 \leq l \leq m} \left\{ \frac{1}{w_{il}} \|\bar{z}_{il}^*\| \right\} = 1,$
- (v)  $\max_{1 \leq u \leq n} \left\{ \sum_{j=1}^m w_{uj} \|\bar{x}_j - p_u\| \right\} = \sum_{j=1}^m w_{ij} \|\bar{x}_j - p_i\|, i \in \bar{I},$
- (vi)  $\max_{1 \leq l \leq m} \left\{ \frac{1}{w_{il}} \|\bar{z}_{il}^*\| \right\} = \frac{1}{w_{ij}} \|\bar{z}_{ij}^*\|, j \in \bar{J}_\gamma = \{j \in \{1, \dots, m\} : \|\bar{z}_{ij}^*\| > 0\}, (\bar{z}_{i1}^{1*}, \dots, \bar{z}_{im}^{1*}) \in \mathbb{R}^d \times \dots \times \mathbb{R}^d \setminus \{(0_{\mathbb{R}^d}, \dots, 0_{\mathbb{R}^d})\}, i \in \bar{I}, \text{ and } \bar{z}_{kj}^* = 0_{\mathbb{R}^d}, k \notin \bar{I}, j = 1, \dots, m.$

(b) If there exists  $(\bar{x}_1, \dots, \bar{x}_m) \in \mathbb{R}^d \times \dots \times \mathbb{R}^d$  such that for some

$$(\bar{z}_1^*, \dots, \bar{z}_n^*) \in \underbrace{\mathbb{R}^d \times \dots \times \mathbb{R}^d}_{m\text{-times}} \times \dots \times \underbrace{\mathbb{R}^d \times \dots \times \mathbb{R}^d}_{m\text{-times}}$$

and an index set  $\bar{I}$  the conditions (i)-(vi) are fulfilled, then  $(\bar{x}_1, \dots, \bar{x}_m)$  is an optimal solution of  $(EP_N^M)$ ,  $(\bar{z}_1^*, \dots, \bar{z}_n^*, \bar{I})$  is an optimal solution for  $(ED_N^M)$  and  $v(EP_N^M) = v(ED_N^M)$ .

#### Geometrical interpretation.

We want now, in the concluding part of this section, to illustrate the results we presented above and describe the set of optimal solutions of the conjugate dual problem. For that end, let us first take a closer look at the optimality conditions stated in Theorem 4.18.

By the condition (iii) follows that the vectors  $\bar{z}_{ij}^*$  and  $\bar{x}_j - p_i$  are parallel and moreover, these vectors have the same direction,  $i \in \bar{I}, j = 1, \dots, m$ . From the optimality condition (vi) we additionally deduce that the vectors  $\bar{z}_{ij}^*, j = 1, \dots, m$ , are all unequal to the zero vector if  $i \in \bar{I}$ , which is the situation when the sum of the weighted distances in condition (v) is equal to the optimal objective value. In the reverse case, when  $i \notin \bar{I}$ , i.e. the sum of the weighted distances in condition (v) is less than the optimal objective value, the vectors  $\bar{z}_{ij}^*, j = 1, \dots, m$ , are all equal to the zero vector.

Therefore, it is appropriate to interpret for  $i \in \bar{I}$  the vectors  $\bar{z}_{ij}^*$  fulfilling  $\sum_{i \in \bar{I}} \bar{z}_{ij}^* = 0_{X^*}$  and  $\sum_{i \in \bar{I}} \max_{1 \leq l \leq m} \left\{ \frac{1}{w_{il}} \|\bar{z}_{il}^*\| \right\} = 1$  as force vectors pulling the given point  $p_i$  in direction to the associated gravity points  $\bar{x}_j, j = 1, \dots, m$ . As an illustration of the nature of the optimal solutions of the conjugate dual problem, let us consider the following example in the plane and especially, Figure 4.2.

**Example 4.3.** Let us consider the points  $p_1 = (0, 0)^T$ ,  $p_2 = (8, 0)^T$  and  $p_3 = (5, 6)^T$  in the plane ( $d = 2$ ). For the given weights  $w_{11} = 2$ ,  $w_{12} = 3$ ,  $w_{21} = 3$ ,  $w_{22} = 3$ ,  $w_{31} = 2$  and  $w_{32} = 2$  we want to determine  $m = 2$  new points minimizing the objective function of the location problem

$$(EP_N^M) \quad \inf_{(x_1, x_2) \in \mathbb{R}^2 \times \mathbb{R}^2} \max\{2\|x_1 - p_1\| + 3\|x_2 - p_1\|, 3\|x_1 - p_2\| + 3\|x_2 - p_2\|, \\ 2\|x_1 - p_3\| + 2\|x_2 - p_3\|\}.$$

To solve this problem, we used the Matlab Optimization Toolbox and obtained as optimal solution  $\bar{x}_1 = (6.062, 0.858)^T$ ,  $\bar{x}_2 = (2.997, 0.837)^T$  and as optimal objective value  $(EP_N^M) = 21.578$ .

The corresponding conjugate dual problem becomes to (see also Remark 4.18)

$$(ED_N^M) \quad \sup_{(z_1^*, z_2^*, z_3^*) \in \tilde{C}} \{-\langle z_{11}^* + z_{12}^*, p_1 \rangle - \langle z_{21}^* + z_{22}^*, p_2 \rangle - \langle z_{31}^* + z_{32}^*, p_3 \rangle\},$$

where

$$\tilde{C} = \left\{ (z_1^*, z_2^*, z_3^*) \in (\mathbb{R}^2 \times \mathbb{R}^2) \times (\mathbb{R}^2 \times \mathbb{R}^2) \times (\mathbb{R}^2 \times \mathbb{R}^2) : \right. \\ \left. z_{11}^* + z_{21}^* + z_{31}^* = 0_{\mathbb{R}^2}, z_{12}^* + z_{22}^* + z_{32}^* = 0_{\mathbb{R}^2}, \right. \\ \left. \max\left\{\frac{1}{2}\|z_{11}^*\|, \frac{1}{3}\|z_{12}^*\|\right\} + \max\left\{\frac{1}{3}\|z_{21}^*\|, \frac{1}{3}\|z_{22}^*\|\right\} + \max\left\{\frac{1}{2}\|z_{31}^*\|, \frac{1}{2}\|z_{32}^*\|\right\} \leq 1 \right\}.$$

The dual problem  $(ED_N^M)$  was also solved with the Matlab Optimization Toolbox. The optimal solution was

$$\begin{aligned} \bar{z}_{11}^* &= (0.803, 0.114)^T, \quad \bar{z}_{12}^* = (1.171, 0.327)^T, \\ \bar{z}_{21}^* &= (-0.909, 0.402)^T, \quad \bar{z}_{22}^* = (-0.98, 0.164)^T, \\ \bar{z}_{31}^* &= (0.106, -0.516)^T, \quad \bar{z}_{32}^* = (-0.191, -0.491)^T \end{aligned}$$

and the optimal objective function value  $v(ED_N^M) = 21.578 = v(EP_N^M)$ . See Figure 4.2 for an illustration of the relation between the optimal solutions of the primal and the conjugate dual problem.

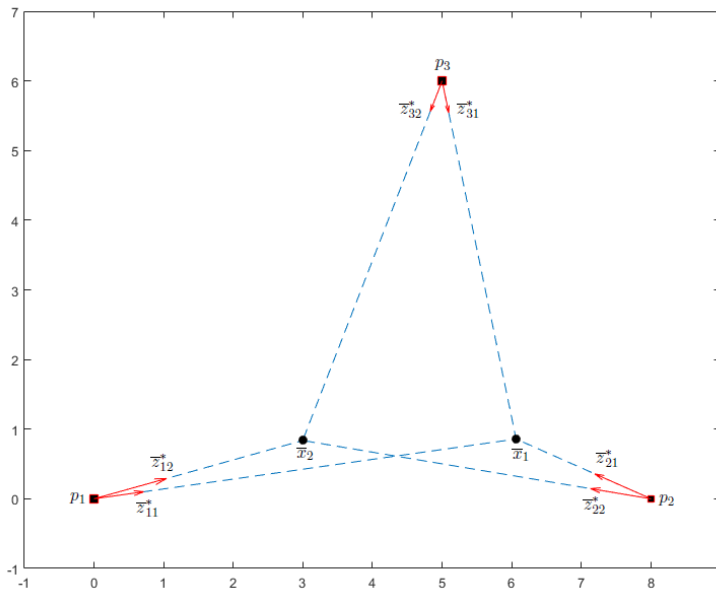


Figure 4.2: Illustration of the Example 4.3.

An alternative geometrical interpretation of the set of optimal solutions of the conjugate dual problem is based on the fact that the extended multifacility location problem ( $EP^M$ ) can be reduced to a single minimax location problem as seen in the beginning of Section 4.3.1. This means precisely that the sum of distances in the objective function of the location problem ( $EP_N^M$ ) can be understood as the finding the minimum value for  $n$  norms  $d_i$  defined by the weighted sum of Euclidean norms, i.e.  $d_i(y_1, \dots, y_m) := \sum_{j=1}^m w_{ij} \|y_j\|$  with  $y_j \in \mathbb{R}^d$ ,  $w_{ij} > 0$ ,  $j = 1, \dots, m$ , such that the associated norm balls centered at the points  $\tilde{p}_i = (p_i, \dots, p_i)$  with  $p_i \in \mathbb{R}^d$ ,  $i = 1, \dots, n$ , have a non-empty intersection. In this case, it is possible to interpret the optimal solution of the corresponding conjugate dual problem as force vectors fulfilling the conditions in point (a) of Theorem 4.18 and increasing the norm balls until their intersection is non-empty. Notice that the optimality conditions (v) and (vi) imply that the vectors  $\tilde{z}_{ij}^*$ ,  $j = 1, \dots, m$ , are equal to the zero vector if  $i \notin \bar{I}$ , which is exactly the case when  $\bar{x}$  is an element of the interior of the ball associated to the norm  $d_i$ . But this also means that the vectors  $\tilde{z}_{ij}^*$ ,  $j = 1, \dots, m$ , are all unequal to the zero vector if  $i \in \bar{I}$ , which exactly holds if  $\bar{x}$  is lying on the border of the ball associated to the norm  $d_i$ .

For a better geometrical illustration of this interpretation, let us consider an example, where  $d = 1$ . In this case the Euclidean norm reduces to the absolute value.

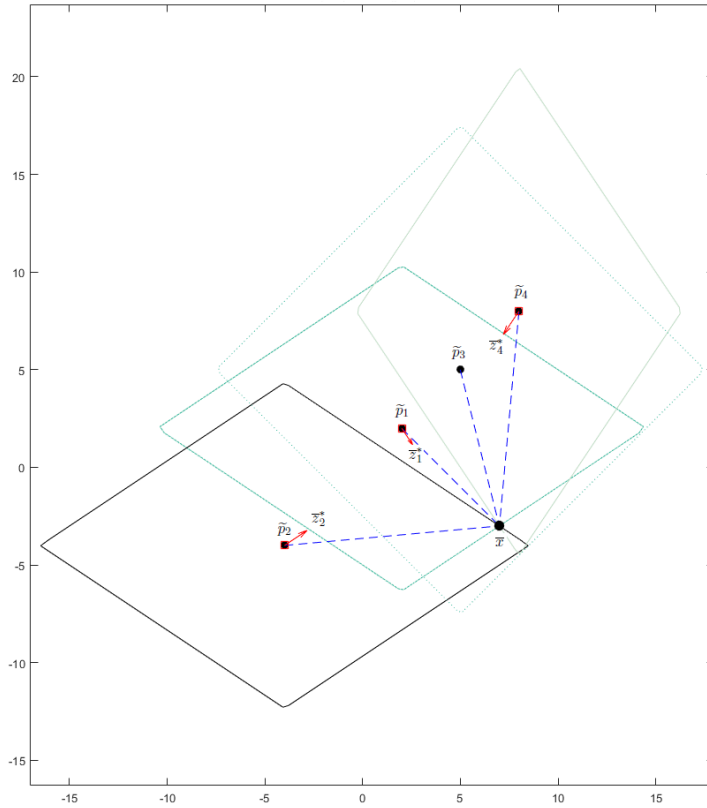


Figure 4.3: Illustration of the Example 4.4.

**Example 4.4.** For the given points  $\tilde{p}_1 = (p_1, p_1) = (2, 2)^T$ ,  $\tilde{p}_2 = (p_2, p_2) = (-4, -4)^T$ ,  $\tilde{p}_3 = (p_3, p_3) = (5, 5)^T$ ,  $\tilde{p}_4 = (p_4, p_4) = (8, 8)^T$  and the weights  $w_{11} = 2$ ,  $w_{12} = 3$ ,  $w_{21} = 2$ ,  $w_{22} = 3$ ,  $w_{31} = 2$ ,  $w_{32} = 2$ ,  $w_{41} = 3$ ,  $w_{42} = 2$  we want to locate an optimal solution  $x = (x_1, x_2)^T \in \mathbb{R}^2$

of the problem

$$(EP^M) \quad \inf_{(x_1, x_2)^T \in \mathbb{R}^2} \max\{2|x_1 - 2| + 3|x_2 - 2|, 2|x_1 + 4| + 3|x_2 + 4|, \\ 2|x_1 - 5| + 2|x_2 - 5|, 3|x_1 - 8| + 2|x_2 - 8|\}.$$

We solved the problem  $(EP^M)$  with the Matlab Optimization Toolbox and obtain as optimal solution  $\bar{x} = (\bar{x}_1, \bar{x}_2)^T = (7, -3)^T$  and as optimal objective value  $v(EP^M) = 25$ .

For the corresponding conjugate dual problem (see also Remark 4.18)

$$(E\tilde{D}^M) \quad \sup_{(z_1^*, z_2^*, z_3^*, z_4^*) \in \tilde{\mathcal{C}}} \{-2(z_{11}^* + z_{12}^*) + 4(z_{21}^* + z_{22}^*) - 5(z_{31}^* + z_{32}^*) - 8(z_{41}^* + z_{42}^*)\},$$

where

$$\begin{aligned} \tilde{\mathcal{C}} = & \left\{ (z_1^*, z_2^*, z_3^*, z_4^*) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 : \right. \\ & z_{11}^* + z_{21}^* + z_{31}^* + z_{41}^* = 0, \quad z_{12}^* + z_{22}^* + z_{32}^* + z_{42}^* = 0, \\ & \left. \max\left\{\frac{1}{2}|z_{11}^*|, \frac{1}{3}|z_{12}^*|\right\} + \max\left\{\frac{1}{2}|z_{21}^*|, \frac{1}{3}|z_{22}^*|\right\} + \max\left\{\frac{1}{2}|z_{31}^*|, \frac{1}{2}|z_{32}^*|\right\} + \max\left\{\frac{1}{3}|z_{41}^*|, \frac{1}{2}|z_{42}^*|\right\} \leq 1 \right\}, \end{aligned}$$

we obtain by using again the Matlab Optimization Toolbox the associated optimal solution

$$\begin{aligned} \bar{z}_1^* &= (\bar{z}_{11}^*, \bar{z}_{12}^*)^T = (0.333, -0.5)^T, \quad \bar{z}_2^* = (\bar{z}_{21}^*, \bar{z}_{22}^*)^T = (0.867, 1.3)^T, \\ \bar{z}_3^* &= (\bar{z}_{31}^*, \bar{z}_{32}^*)^T = (0, 0)^T, \quad \bar{z}_4^* = (\bar{z}_{41}^*, \bar{z}_{42}^*)^T = (-1.2, -0.8)^T \end{aligned}$$

and the optimal objective value  $v(E\tilde{D}^M) = 25 = v(EP^M)$ . The numerical results are illustrated in Figure 4.3. Take note that  $\bar{x}$  is lying inside the norm ball centered at the point  $\tilde{p}_3$  and that for this reason  $\bar{z}_3^*$  is equal to the zero vector.

## 4.4 Classical multifacility minimax location problems

### 4.4.1 Constrained location problems in Fréchet spaces

In this section we use the results of our previous approach to develop a conjugate dual problem of the multifacility minimax location problem with mixed gauges and geometric constraints. Furthermore, we show the validity of strong duality and derive optimality conditions for the corresponding primal-dual pair.

Let  $X$  be a Fréchet space,  $C_{jk} \subseteq X$  with  $0_X \in \text{int } C_{jk}$  for  $jk \in J := \{jk : 1 \leq j \leq m, 1 \leq k \leq m, j \neq k\}$ , and  $\tilde{C}_{ji} \subseteq X$  with  $0_X \in \text{int } \tilde{C}_{ji}$  for  $ji \in \tilde{J} := \{1 \leq j \leq m, 1 \leq i \leq t\}$ , be closed and convex as well as  $S \subseteq X^m$  non-empty, closed and convex. Moreover, let  $w_{jk} \geq 0$ ,  $jk \in J$ ,  $\tilde{w}_{ji} \geq 0$ ,  $ji \in \tilde{J}$  as well as  $\gamma_{C_{jk}} : X \rightarrow \mathbb{R}$ ,  $jk \in J$ , and  $\gamma_{\tilde{C}_{ji}} : X \rightarrow \mathbb{R}$ ,  $ji \in \tilde{J}$ , be gauges. Obviously, these gauges are convex, continuous and well-defined.

For given distinct points  $p_i \in X$ ,  $1 \leq i \leq t$ , the multifacility minimax location problem minimizes the maximum of gauges between pairs of  $m$  new facilities  $x_1, \dots, x_m$  and between pairs of  $m$  new and  $t$  existing facilities, concretely this means that

$$(P^M) \quad \inf_{x=(x_1, \dots, x_m) \in S} \max \left\{ w_{jk} \gamma_{C_{jk}}(x_j - x_k), \quad jk \in J, \quad \tilde{w}_{ji} \gamma_{\tilde{C}_{ji}}(x_j - p_i), \quad ji \in \tilde{J} \right\}.$$

We introduce the index sets  $V := \{jk \in J : w_{jk} > 0\}$  and  $\tilde{V} := \{ji \in \tilde{J} : \tilde{w}_{ji} > 0\}$ , which allows us to write the problem  $(P^M)$  as

$$(P^M) \quad \inf_{(x_1, \dots, x_m) \in S} \max \left\{ w_{jk} \gamma_{C_{jk}}(x_j - x_k), \quad jk \in V, \quad \tilde{w}_{ji} \gamma_{\tilde{C}_{ji}}(x_j - p_i), \quad ji \in \tilde{V} \right\}.$$

Take note that  $|V| \leq m(m-1)$  and  $|\tilde{V}| \leq mt$ . Now, we set  $X_0 = \mathbb{R}^{|V|} \times \mathbb{R}^{|\tilde{V}|}$  ordered by  $K_0 = \mathbb{R}_+^{|V|} \times \mathbb{R}_+^{|\tilde{V}|}$ ,  $X_1 = X^{|V|} \times X^{|\tilde{V}|}$  ordered by the trivial cone  $K_1 = \{0_{X_1}\}$  and  $X_2 = X^m$ , where the

corresponding dual spaces and dual variables are  $(z^{0*}, \tilde{z}^{0*}) = \left( (z_{jk}^{0*})_{jk \in V}, (\tilde{z}_{ji}^{0*})_{ji \in \tilde{V}} \right) \in \mathbb{R}^{|V|} \times \mathbb{R}^{|\tilde{V}|}$  and  $(z^{1*}, \tilde{z}^{1*}) = \left( (z_{jk}^{1*})_{jk \in V}, (\tilde{z}_{ji}^{1*})_{ji \in \tilde{V}} \right) \in (X^*)^{|V|} \times (X^*)^{|\tilde{V}|}$ .

We continue with the decomposition of the objective function of the problem  $(P^M)$  into the following functions:

- $f : \mathbb{R}^{|V|} \times \mathbb{R}^{|\tilde{V}|} \rightarrow \mathbb{R}$  defined by  $f(y^0, \tilde{y}^0) = \max \left\{ w_{jk} y_{jk}^0, jk \in V, \tilde{w}_{ji} \tilde{y}_{ji}^0, ji \in \tilde{V} \right\}$  if  $y^0 = (y_{jk}^0)_{jk \in V} \in \mathbb{R}_+^{|V|}$  and  $\tilde{y}^0 = (\tilde{y}_{ji}^0)_{ji \in \tilde{V}} \in \mathbb{R}_+^{|\tilde{V}|}$ , otherwise  $f(y^0, \tilde{y}^0) = +\infty$ ,
- $F^1 : X^{|V|} \times X^{|\tilde{V}|} \rightarrow \mathbb{R}^{|V|} \times \mathbb{R}^{|\tilde{V}|}$  defined by  $F^1(y^1, \tilde{y}^1) = \left( (\gamma_{C_{jk}}(y_{jk}^1))_{jk \in V}, (\gamma_{\tilde{C}_{ji}}(\tilde{y}_{ji}^1))_{ji \in \tilde{V}} \right)$ , where  $y^1 = (y_{jk}^1)_{jk \in V} \in X^{|V|}$  and  $\tilde{y}^1 = (\tilde{y}_{ji}^1)_{ji \in \tilde{V}} \in X^{|\tilde{V}|}$ ,
- $F^2 : X^m \rightarrow X^{|V|} \times X^{|\tilde{V}|}$  defined by  $F^2(x) = \left( (A_{jk}x)_{jk \in V}, (B_{ji}x - p_i)_{ji \in \tilde{V}} \right)$ , where  $A_{jk} = \begin{pmatrix} 1 & & & \\ & \dots & & \\ & & \text{Id} & \\ & & & \dots \\ & & & & -\text{Id} & \\ & & & & & \dots \\ & & & & & & \text{Id} & \\ & & & & & & & \dots \\ & & & & & & & & \text{Id} & \\ & & & & & & & & & \dots \\ & & & & & & & & & & \text{Id} \end{pmatrix}$ ,  $jk \in V$ ,  $B_{ji} = \begin{pmatrix} 1 & & & \\ & \dots & & \\ & & \text{Id} & \\ & & & \dots \\ & & & & \text{Id} & \\ & & & & & \dots \\ & & & & & & \text{Id} & \\ & & & & & & & \dots \\ & & & & & & & & \text{Id} & \\ & & & & & & & & & \dots \\ & & & & & & & & & & \text{Id} \end{pmatrix}$ ,  $ji \in \tilde{V}$ ,  $\mathbf{0}$  is the zero mapping and  $\text{Id}$  is the identity mapping, i.e.  $\mathbf{0}x_i = 0_X$  and  $\text{Id}x_i = x_i$  for all  $x_i \in X$ ,  $i = 1, \dots, m$ . In particular,  $A_{jk} : X^m \rightarrow X$  is defined as the mapping

$$x = (x_1, \dots, x_m) \mapsto$$

$$\mathbf{0}x_1 + \dots + \mathbf{0}x_{j-1} + \text{Id}x_j + \mathbf{0}x_{j+1} + \dots + \mathbf{0}x_{k-1} - \text{Id}x_k + \mathbf{0}x_{k+1} + \dots + \mathbf{0}x_m,$$

i.e.  $(x_1, \dots, x_m) \mapsto x_j - x_k$ ,  $jk \in V$ , and  $B_{ji} : X^m \rightarrow X$  is defined as the mapping

$$(x_1, \dots, x_m) \mapsto \mathbf{0}x_1 + \dots + \mathbf{0}x_{j-1} + \text{Id}x_j + \mathbf{0}x_{j+1} + \dots + \mathbf{0}x_m = x_j, \quad ji \in \tilde{V}.$$

Thus, it is easy to see that the problem  $(P^M)$  can be represented in the form

$$(P^M) \quad \inf_{x \in S} (f \circ F^1 \circ F^2)(x).$$

Like mentioned in Remark 3.5, we do not need the monotonicity assumption for the function  $F^1$ , because  $F^2$  is an affine function. Furthermore, it is clear that  $(P^M)$  is a convex optimization problem. Besides, it can easily be verified that  $f$  is proper, convex,  $\mathbb{R}_+^{|V|} \times \mathbb{R}_+^{|\tilde{V}|}$ -increasing on  $F^1(\text{dom } F^1) + K_0 = \text{dom } f = \mathbb{R}_+^{|V|} \times \mathbb{R}_+^{|\tilde{V}|}$  and lower semicontinuous and that  $F^1$  is proper and  $\mathbb{R}_+^{|V|} \times \mathbb{R}_+^{|\tilde{V}|}$ -convex as well as  $\mathbb{R}_+^{|V|} \times \mathbb{R}_+^{|\tilde{V}|}$ -epi closed.

To use the formula from Chapter 3 for the dual problem of  $(P^M)$ , we set  $Z = X^m$  ordered by the trivial cone  $Q = X^m$  and define the function  $g : X^m \rightarrow X^m$  by  $g(x_1, \dots, x_m) := (x_1, \dots, x_m)$ . As  $Q^* = \{0_{(X^*)^m}\}$ , which means that  $z^{2*} = 0_{(X^*)^m}$ , we derive for the dual problem (see 3. 3)

$$(D^M) \quad \sup_{\substack{(z^{0*}, \tilde{z}^{0*}) \in \mathbb{R}_+^{|V|} \times \mathbb{R}_+^{|\tilde{V}|}, \\ (z^{1*}, \tilde{z}^{1*}) \in (X^*)^{|V|} \times (X^*)^{|\tilde{V}|}}} \left\{ \inf_{x \in S} \left\{ \sum_{jk \in V} \langle z_{jk}^{1*}, A_{jk}x \rangle + \sum_{ji \in \tilde{V}} \langle \tilde{z}_{ji}^{1*}, B_{ji}x - p_i \rangle \right\} \right. \\ \left. - f^*(z^{0*}, \tilde{z}^{0*}) - ((z^{0*}, \tilde{z}^{0*})F^1)^*(z^{1*}, \tilde{z}^{1*}) \right\},$$

and hence, we need to calculate the conjugate functions  $f^*$  and  $((z^{0*}, \tilde{z}^{0*})F^1)^*$ . By Lemma 4.1 and Remark 4.19 we get for  $f^*$ ,

$$f^*(z^{0*}, \tilde{z}^{0*}) = \begin{cases} 0, & \text{if } z_{jk}^{0*} \leq w_{jk} \lambda_{jk}, \quad \tilde{z}_{ji}^{0*} \leq \tilde{w}_{ji} \tilde{\lambda}_{ji}, \quad \sum_{jk \in V} \lambda_{jk} + \sum_{ji \in \tilde{V}} \tilde{\lambda}_{ji} \leq 1 \\ & (\lambda_{jk})_{jk \in V} \in \mathbb{R}_+^{|V|} \text{ and } (\tilde{\lambda}_{ji})_{ji \in \tilde{V}} \in \mathbb{R}_+^{|\tilde{V}|}, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$= \begin{cases} 0, & \text{if } \sum_{jk \in V} \frac{1}{w_{jk}} z_{jk}^{0*} + \sum_{ji \in \tilde{V}} \frac{1}{\tilde{w}_{ji}} \tilde{z}_{ji}^{0*} \leq 1, \quad z^{0*} \in \mathbb{R}_+^{|V|}, \quad \tilde{z}^{0*} \in \mathbb{R}_+^{|\tilde{V}|}, \\ +\infty, & \text{otherwise,} \end{cases}$$



while for  $((z^{0*}, \tilde{z}^{0*})F^1)^*$  we obtain by using the definition of the conjugate function

$$\begin{aligned}
((z^{0*}, \tilde{z}^{0*})F^1)^*(z^{1*}, \tilde{z}^{1*}) &= \sup_{y^1 \in X^{|V|}, \tilde{y}^1 \in X^{|\tilde{V}|}} \left\{ \sum_{jk \in V} \langle z_{jk}^{1*}, y_{jk}^1 \rangle + \right. \\
&\quad \left. \sum_{ji \in \tilde{V}} \langle \tilde{z}_{ji}^{1*}, \tilde{y}_{ji}^1 \rangle - \sum_{jk \in V} z_{jk}^{0*} \gamma_{C_{jk}}(y_{jk}^1) - \sum_{ji \in \tilde{V}} \tilde{z}_{ji}^{0*} \gamma_{\tilde{C}_{ji}}(\tilde{y}_{ji}^1) \right\} \\
&= \sum_{jk \in V} \sup_{y_{jk}^1 \in X} \left\{ \langle z_{jk}^{1*}, y_{jk}^1 \rangle - z_{jk}^{0*} \gamma_{C_{jk}}(y_{jk}^1) \right\} + \sum_{ji \in \tilde{V}} \sup_{\tilde{y}_{ji}^1 \in X} \left\{ \langle \tilde{z}_{ji}^{1*}, \tilde{y}_{ji}^1 \rangle - \tilde{z}_{ji}^{0*} \gamma_{\tilde{C}_{ji}}(\tilde{y}_{ji}^1) \right\} \\
&= \sum_{jk \in V} (z_{jk}^{0*} \gamma_{C_{jk}})^*(z_{jk}^{1*}) + \sum_{ji \in \tilde{V}} (\tilde{z}_{ji}^{0*} \gamma_{\tilde{C}_{ji}})^*(\tilde{z}_{ji}^{1*})
\end{aligned}$$

for all  $(z^{0*}, \tilde{z}^{0*}) \in \mathbb{R}_+^{|V|} \times \mathbb{R}_+^{|\tilde{V}|}$  and  $z^{1*} = (z_{jk}^{1*})_{jk \in V} \in X^{|V|}$  and  $\tilde{z}^{1*} = (\tilde{z}_{ji}^{1*})_{ji \in \tilde{V}} \in X^{|\tilde{V}|}$ . Hence, the dual problem may be written as

$$(D^M) \quad \sup_{\substack{(z^{0*}, \tilde{z}^{0*}, z^{1*}, \tilde{z}^{1*}) \in \mathbb{R}_+^{|V|} \times \mathbb{R}_+^{|\tilde{V}|} \times X^{|V|} \times X^{|\tilde{V}|} \\ \sum_{jk \in V} \frac{1}{w_{jk}} z_{jk}^{0*} + \sum_{ji \in \tilde{V}} \frac{1}{\tilde{w}_{ji}} \tilde{z}_{ji}^{0*} \leq 1}} \inf_{x \in S} \Phi(z^{0*}, \tilde{z}^{0*}, z^{1*}, \tilde{z}^{1*}),$$

where

$$\begin{aligned}
\Phi(z^{0*}, \tilde{z}^{0*}, z^{1*}, \tilde{z}^{1*}) &= \inf_{x \in S} \left\{ \sum_{jk \in V} \langle z_{jk}^{1*}, A_{jk}x \rangle + \sum_{ji \in \tilde{V}} \langle \tilde{z}_{ji}^{1*}, B_{ji}x - p_i \rangle \right\} \\
&\quad - \sum_{jk \in V} (z_{jk}^{0*} \gamma_{C_{jk}})^*(z_{jk}^{1*}) - \sum_{ji \in \tilde{V}} (\tilde{z}_{ji}^{0*} \gamma_{\tilde{C}_{ji}})^*(\tilde{z}_{ji}^{1*}).
\end{aligned}$$

Let  $I := \{jk : z_{jk}^{0*} > 0\}$  and  $\tilde{I} := \{ji : \tilde{z}_{ji}^{0*} > 0\}$ , then we separate in the objective function  $\Phi$  the sum into the terms with  $z_{jk}^{0*}, \tilde{z}_{ji}^{0*} > 0$  and the terms with  $z_{jk}^{0*}, \tilde{z}_{ji}^{0*} = 0$ :

$$\begin{aligned}
\Phi(z^{0*}, \tilde{z}^{0*}, z^{1*}, \tilde{z}^{1*}) &= \inf_{x \in S} \left\{ \sum_{jk \in V} \langle z_{jk}^{1*}, A_{jk}x \rangle + \sum_{ji \in \tilde{V}} \langle \tilde{z}_{ji}^{1*}, B_{ji}x - p_i \rangle \right\} \\
&\quad - \sum_{jk \in I} (z_{jk}^{0*} \gamma_{C_{jk}})^*(z_{jk}^{1*}) - \sum_{ji \in \tilde{I}} (\tilde{z}_{ji}^{0*} \gamma_{\tilde{C}_{ji}})^*(\tilde{z}_{ji}^{1*}) \\
&\quad - \sum_{jk \notin I} (0 \cdot \gamma_{C_{jk}})^*(z_{jk}^{1*}) - \sum_{ji \notin \tilde{I}} (0 \cdot \gamma_{\tilde{C}_{ji}})^*(\tilde{z}_{ji}^{1*}).
\end{aligned}$$

Now, it holds for  $jk \in I$  that (see 4. 26)

$$(z_{jk}^{0*} \gamma_{C_{jk}})^*(z_{jk}^{1*}) = \begin{cases} 0, & \text{if } \gamma_{C_{jk}}^0(z_{jk}^{1*}) \leq z_{jk}^{0*}, \\ +\infty, & \text{otherwise,} \end{cases} \quad (4. 75)$$

and analogously, it follows for  $ji \in \tilde{I}$  that

$$(\tilde{z}_{ji}^{0*} \gamma_{\tilde{C}_{ji}})^*(\tilde{z}_{ji}^{1*}) = \begin{cases} 0, & \text{if } \gamma_{\tilde{C}_{ji}}^0(\tilde{z}_{ji}^{1*}) \leq \tilde{z}_{ji}^{0*}, \\ +\infty, & \text{otherwise.} \end{cases} \quad (4. 76)$$

For  $jk \notin I$  it holds (see 4. 27)

$$(0 \cdot \gamma_{C_{jk}})^*(z_{jk}^{1*}) = \begin{cases} 0, & \text{if } z_{jk}^{1*} = 0_{X^*}, \\ +\infty, & \text{otherwise,} \end{cases}$$

and analogously, we get for  $ji \notin I$ ,

$$(0 \cdot \gamma_{\tilde{C}_{ji}})^*(\tilde{z}_{ji}^{1*}) = \begin{cases} 0, & \text{if } \tilde{z}_{ji}^{1*} = 0_{X^*}, \\ +\infty, & \text{otherwise,} \end{cases},$$

which implies that if  $jk \notin I$ , then  $z_{jk}^{1*} = 0_{X^*}$  and if  $ji \notin I$ , then  $\tilde{z}_{ji}^{1*} = 0_{X^*}$ . Therefore, we obtain for the dual problem of the location problem  $(P^M)$ :

$$(D^M) \sup_{(z^{0*}, \tilde{z}^{0*}, z^{1*}, \tilde{z}^{1*}) \in \mathcal{B}} \inf_{x \in S} \left\{ \sum_{jk \in I} \langle z_{jk}^{1*}, A_{jk}x \rangle + \sum_{ji \in \tilde{I}} \langle \tilde{z}_{ji}^{1*}, B_{ji}x - p_i \rangle \right\},$$

where

$$\begin{aligned} \mathcal{B} &= \left\{ (z^{0*}, \tilde{z}^{0*}, z^{1*}, \tilde{z}^{1*}) \in \mathbb{R}_+^{|V|} \times \mathbb{R}_+^{|\tilde{V}|} \times (X^*)^{|V|} \times (X^*)^{|\tilde{V}|} : I = \{jk \in V : z_{jk}^{0*} > 0\}, \right. \\ &\quad \tilde{I} = \{ji \in \tilde{V} : \tilde{z}_{ji}^{0*} > 0\}, \quad z_{ef}^{1*} = 0_{X^*}, \quad ef \notin I, \quad \gamma_{C_{jk}^0}(z_{jk}^{1*}) \leq z_{jk}^{0*}, \quad jk \in I, \\ &\quad \left. \tilde{z}_{ed}^{1*} = 0_{X^*}, \quad ed \notin \tilde{I}, \quad \gamma_{\tilde{C}_{ji}^0}(\tilde{z}_{ji}^{1*}) \leq \tilde{z}_{ji}^{0*}, \quad ji \in \tilde{I}, \quad \sum_{jk \in I} \frac{1}{w_{jk}} z_{jk}^{0*} + \sum_{ji \in \tilde{I}} \frac{1}{\tilde{w}_{ji}} \tilde{z}_{ji}^{0*} \leq 1 \right\}. \end{aligned}$$

Since, the objective function of the conjugate dual problem  $(D^M)$  can also be written as

$$\begin{aligned} &\inf_{x \in S} \left\{ \sum_{jk \in I} \langle z_{jk}^{1*}, A_{jk}x \rangle + \sum_{ji \in \tilde{I}} \langle \tilde{z}_{ji}^{1*}, B_{ji}x - p_i \rangle \right\} \\ &= \inf_{x \in S} \left\{ \left\langle \sum_{jk \in I} A_{jk}^* z_{jk}^{1*} + \sum_{ji \in \tilde{I}} B_{ji}^* \tilde{z}_{ji}^{1*}, x \right\rangle \right\} - \sum_{ji \in \tilde{I}} \langle \tilde{z}_{ji}^{1*}, p_i \rangle, \end{aligned}$$

where

$$\langle A_{jk}^* z_{jk}^{1*}, x \rangle = \langle (0_{X^*}^1, \dots, 0_{X^*}, \overset{j}{z_{jk}^{1*}}, 0_{X^*}, \dots, 0_{X^*}, -\overset{k}{z_{jk}^{1*}}, 0_{X^*}, \dots, 0_{X^*}^m), (x_1, \dots, x_m) \rangle = \langle z_{jk}^{1*}, x_j - x_k \rangle$$

and

$$\langle B_{ji}^* \tilde{z}_{ji}^{1*}, x \rangle = \langle (0_{X^*}^1, \dots, 0_{X^*}, \tilde{z}_{ji}^{1*}, 0_{X^*}, \dots, 0_{X^*}^m), (x_1, \dots, x_m) \rangle = \langle \tilde{z}_{ji}^{1*}, x_j \rangle,$$

we can express  $(D^M)$  as

$$(D^M) \sup_{(z^{0*}, \tilde{z}^{0*}, z^{1*}, \tilde{z}^{1*}) \in \mathcal{B}} \left\{ -\sigma_S \left( -\sum_{jk \in I} A_{jk}^* z_{jk}^{1*} - \sum_{ji \in \tilde{I}} B_{ji}^* \tilde{z}_{ji}^{1*} \right) - \sum_{ji \in \tilde{I}} \langle \tilde{z}_{ji}^{1*}, p_i \rangle \right\}.$$

**Remark 4.22.** Take note that the problem  $(D^M)$  is equivalent to the following one

$$(\hat{D}^M) \sup_{(z^{0*}, \tilde{z}^{0*}, z^{1*}, \tilde{z}^{1*}) \in \mathcal{B}} \left\{ -\sigma_S \left( -\sum_{jk \in V} A_{jk}^* z_{jk}^{1*} - \sum_{ji \in \tilde{V}} B_{ji}^* \tilde{z}_{ji}^{1*} \right) - \sum_{ji \in \tilde{V}} \langle \tilde{z}_{ji}^{1*}, p_i \rangle \right\},$$

where

$$\begin{aligned} \hat{\mathcal{B}} &= \left\{ (z^{0*}, \tilde{z}^{0*}, z^{1*}, \tilde{z}^{1*}) \in \mathbb{R}_+^{|V|} \times \mathbb{R}_+^{|\tilde{V}|} \times (X^*)^{|V|} \times (X^*)^{|\tilde{V}|} : \gamma_{C_{jk}^0}(z_{jk}^{1*}) \leq z_{jk}^{0*}, \quad jk \in V, \right. \\ &\quad \left. \gamma_{\tilde{C}_{ji}^0}(\tilde{z}_{ji}^{1*}) \leq \tilde{z}_{ji}^{0*}, \quad ji \in \tilde{V}, \quad \sum_{jk \in V} \frac{1}{w_{jk}} z_{jk}^{0*} + \sum_{ji \in \tilde{V}} \frac{1}{\tilde{w}_{ji}} \tilde{z}_{ji}^{0*} \leq 1 \right\}, \end{aligned}$$

which can be proven as follows.

Let  $(z^{0*}, \tilde{z}^{0*}, z^{1*}, \tilde{z}^{1*}) \in \hat{\mathcal{B}}$  be a feasible solution of  $(\hat{D}^M)$ , then it holds for  $jk \notin I$  and  $ji \notin \tilde{I}$ ,

$$0 \leq \gamma_{C_{jk}^0}(z_{jk}^{1*}) = \sup_{x \in C_{jk}} \langle z_{jk}^{1*}, x \rangle \leq 0 \Leftrightarrow \langle z_{jk}^{1*}, x \rangle = 0 \quad \forall x \in C_{jk} \Leftrightarrow z_{jk}^{1*} = 0_{X^*}$$

as well as

$$0 \leq \gamma_{\tilde{C}_{ji}^0}(\tilde{z}_{ji}^{1*}) = \sup_{x \in \tilde{C}_{ji}} \langle \tilde{z}_{ji}^{1*}, x \rangle \leq 0 \Leftrightarrow \langle \tilde{z}_{ji}^{1*}, x \rangle = 0 \quad \forall x \in \tilde{C}_{ji} \Leftrightarrow \tilde{z}_{ji}^{1*} = 0_{X^*}.$$

The latter implies that from  $jk \notin I$ , i.e.  $z_{jk}^{0*} = 0$ , follows  $z_{jk}^{1*} = 0_{X^*}$  and from  $ji \notin \tilde{I}$ , i.e.  $\tilde{z}_{ji}^{0*} = 0$ ,  $\tilde{z}_{ji}^{1*} = 0_{X^*}$ . This relation means that  $\hat{\mathcal{B}} = \mathcal{B}$ , i.e. that  $(z^{0*}, \tilde{z}^{0*}, z^{1*}, \tilde{z}^{1*})$  is also a feasible solution of  $(D^M)$  and as

$$\begin{aligned} & \sigma_S \left( - \sum_{jk \in V} A_{jk}^* z_{jk}^{1*} - \sum_{ji \in \tilde{V}} B_{ji}^* \tilde{z}_{ji}^{1*} \right) + \sum_{ji \in \tilde{V}} \langle \tilde{z}_{ji}^{1*}, p_i \rangle \\ &= \sigma_S \left( - \sum_{jk \in I} A_{jk}^* z_{jk}^{1*} - \sum_{ji \in \tilde{I}} B_{ji}^* \tilde{z}_{ji}^{1*} \right) + \sum_{ji \in \tilde{I}} \langle \tilde{z}_{ji}^{1*}, p_i \rangle, \end{aligned}$$

one has immediately that  $v(D^M) = v(\hat{D}^M)$ .

Vice versa, if we take a feasible solution  $(z^{0*}, \tilde{z}^{0*}, z^{1*}, \tilde{z}^{1*})$  of the problem  $(D^M)$ , then it is obvious that we have then also a feasible solution of  $(\hat{D}^M)$ , which again implies that  $v(D^M) = v(\hat{D}^M)$ .

From the theoretical aspect a dual problem of the form  $(D^M)$  is very useful, as one has a more detailed characterization of the set of feasible solutions. But from the numerical viewpoint it is complicate to solve, as the index sets  $I$  and  $\tilde{I}$  bring an undesirable discretization in the dual problem. For this reason it is preferable to use the dual problem  $(\hat{D}^M)$  for numerical and  $(D^M)$  for theoretical studies.

We know that the weak duality between the problem  $(P^M)$  and its corresponding dual problem  $(D^M)$  always holds. Now, we are interested to know whether we also can guarantee strong duality. For this purpose we use the results from Section 3.2. As  $Z = X^m$  ordered by the trivial cone  $Q = X^m$  and  $g : X^m \rightarrow X^m$  is defined by  $g(x_1, \dots, x_m) = (x_1, \dots, x_m)$ , it is obvious that  $g$  is  $Q$ -epi closed and  $0_{X^m} \in \text{sqli}(g(x) + Q) = \text{sqli}(X^m + Q) = X^m$ . More than that, recall that  $f$  is lower semicontinuous,  $K_0 = \mathbb{R}_+^{|V|} \times \mathbb{R}_+^{|\tilde{V}|}$  is closed,  $S$  is closed and  $F^1$  is  $\mathbb{R}_+^{|V|} \times \mathbb{R}_+^{|\tilde{V}|}$ -epi closed. As

$$\begin{aligned} 0_{\mathbb{R}_+^{|V|} \times \mathbb{R}_+^{|\tilde{V}|}} &\in \text{sqli}(F^1(\text{dom } F^1) - \text{dom } f + K_0) \\ &= \text{sqli}(F^1(\text{dom } F^1) - \mathbb{R}_+^{|V|} \times \mathbb{R}_+^{|\tilde{V}|} + \mathbb{R}_+^{|V|} \times \mathbb{R}_+^{|\tilde{V}|}) \\ &= \mathbb{R}^{|V|} \times \mathbb{R}^{|\tilde{V}|}, \end{aligned}$$

$$\begin{aligned} 0_{X^{|V|} \times X^{|\tilde{V}|}} &\in \text{sqli}(F^2(\text{dom } F^2) - \text{dom } F^1 + K_1) \\ &= \text{sqli}(X^{|V|} \times X^{|\tilde{V}|} - \text{dom } F^1 + K_1) = X^{|V|} \times X^{|\tilde{V}|} \end{aligned}$$

and  $F^2$  is  $\{0_{X^{|V|} \times X^{|\tilde{V}|}}\}$ -epi closed, the generalized interior point regularity condition  $(RC_2^C)$  is fulfilled and it follows by Theorem 3.3 the following statement (note that we denote by  $v(P^M)$  and  $v(D^M)$  the optimal objective values of the problems  $(P^M)$  and  $(D^M)$ , respectively).

**Theorem 4.19.** (strong duality) *Between  $(P^M)$  and  $(D^M)$  holds strong duality, i.e.  $v(P^M) = v(D^M)$  and the conjugate dual problem has an optimal solution.*

The previous theorem implies the following necessary and sufficient optimality conditions for the primal-dual pair  $(P^M)-(D^M)$ .

**Theorem 4.20.** (optimality conditions) (a) Let  $\bar{x} \in S$  be an optimal solution of the problem  $(P^M)$ . Then there exist  $(\bar{z}^{0*}, \bar{z}^{\bar{0}*}, \bar{z}^{1*}, \bar{z}^{\bar{1}*}) \in \mathbb{R}_+^{|V|} \times \mathbb{R}_+^{|\bar{V}|} \times (X^*)^{|V|} \times (X^*)^{|\bar{V}|}$  and index sets  $\bar{I}$  and  $\bar{\bar{I}}$  as an optimal solution to  $(D^M)$  such that

- (i)  $\max \left\{ w_{ef} \gamma_{C_{ef}}(\bar{x}_e - \bar{x}_f), ef \in V, \tilde{w}_{ed} \gamma_{\tilde{C}_{ed}}(\bar{x}_e - p_d), ed \in \tilde{V} \right\}$   
 $= \sum_{jk \in \bar{I}} \bar{z}_{jk}^{0*} \gamma_{C_{jk}}(\bar{x}_j - \bar{x}_k) + \sum_{ji \in \bar{\bar{I}}} \bar{z}_{ji}^{\bar{0}*} \gamma_{\tilde{C}_{ji}}(\bar{x}_j - p_i),$
- (ii)  $\left\langle \sum_{jk \in \bar{I}} A_{jk}^* \bar{z}_{jk}^{1*} + \sum_{ji \in \bar{\bar{I}}} B_{ji}^* \bar{z}_{ji}^{\bar{1}*}, \bar{x} \right\rangle = \inf_{x \in S} \left\langle \sum_{jk \in \bar{I}} A_{jk}^* \bar{z}_{jk}^{1*} + \sum_{ji \in \bar{\bar{I}}} B_{ji}^* \bar{z}_{ji}^{\bar{1}*}, x \right\rangle,$
- (iii)  $\sum_{jk \in \bar{I}} \frac{1}{w_{jk}} \bar{z}_{jk}^{0*} + \sum_{ji \in \bar{\bar{I}}} \frac{1}{\tilde{w}_{ji}} \bar{z}_{ji}^{\bar{0}*} = 1, \bar{z}_{jk}^{0*} > 0, jk \in \bar{I}, \bar{z}_{ji}^{\bar{0}*} > 0, ji \in \bar{\bar{I}}$  and  $\bar{z}_{ef}^{0*} = 0, ef \notin \bar{I},$   
 $\bar{z}_{ed}^{\bar{0}*} = 0, ed \notin \bar{\bar{I}},$
- (iv)  $\bar{z}_{jk}^{0*} \gamma_{C_{jk}}(\bar{x}_j - \bar{x}_k) = \langle \bar{z}_{jk}^{1*}, \bar{x}_j - \bar{x}_k \rangle, jk \in \bar{I},$
- (v)  $\bar{z}_{ji}^{\bar{0}*} \gamma_{\tilde{C}_{ji}}(\bar{x}_j - p_i) = \langle \bar{z}_{ji}^{\bar{1}*}, \bar{x}_j - p_i \rangle, ji \in \bar{\bar{I}},$
- (vi)  $\max \left\{ w_{ef} \gamma_{C_{ef}}(\bar{x}_e - \bar{x}_f), ef \in V, \tilde{w}_{ed} \gamma_{\tilde{C}_{ed}}(\bar{x}_e - p_d), ed \in \tilde{V} \right\} = w_{jk} \gamma_{C_{jk}}(\bar{x}_j - \bar{x}_k), jk \in \bar{I},$
- (vii)  $\max \left\{ w_{ef} \gamma_{C_{ef}}(\bar{x}_e - \bar{x}_f), ef \in V, \tilde{w}_{ed} \gamma_{\tilde{C}_{ed}}(\bar{x}_e - p_d), ed \in \tilde{V} \right\} = \tilde{w}_{ji} \gamma_{\tilde{C}_{ji}}(\bar{x}_j - p_i), ji \in \bar{\bar{I}},$
- (viii)  $\gamma_{C_{jk}}^0(\bar{z}_{jk}^{1*}) = \bar{z}_{jk}^{0*}, \bar{z}_{jk}^{1*} \in X^* \setminus \{0_{X^*}\}, jk \in \bar{I}$  and  $\bar{z}_{ef}^{1*} = 0_{X^*}, ef \notin \bar{I},$
- (ix)  $\gamma_{\tilde{C}_{ji}}^0(\bar{z}_{ji}^{\bar{1}*}) = \bar{z}_{ji}^{\bar{0}*}, \bar{z}_{ji}^{\bar{1}*} \in X^* \setminus \{0_{X^*}\}, ji \in \bar{\bar{I}}$  and  $\bar{z}_{ed}^{\bar{1}*} = 0_{X^*}, ed \notin \bar{\bar{I}}.$

(b) If there exists  $\bar{x} \in S$  such that for some  $(\bar{z}^{0*}, \bar{z}^{\bar{0}*}, \bar{z}^{1*}, \bar{z}^{\bar{1}*}, \bar{I}, \bar{\bar{I}})$  the conditions (i)-(ix) are fulfilled, then  $\bar{x}$  is an optimal solution of  $(PC)$ ,  $(\bar{z}^{0*}, \bar{z}^{\bar{0}*}, \bar{z}^{1*}, \bar{z}^{\bar{1}*}, \bar{I}, \bar{\bar{I}})$  is an optimal solution of  $(D^M)$  and  $v(P^M) = v(D^M)$ .

**Proof.** (a) From Theorem 3.4 one gets

- (i)  $\max \left\{ w_{ef} \gamma_{C_{ef}}(\bar{x}_e - \bar{x}_f), ef \in V, \tilde{w}_{ed} \gamma_{\tilde{C}_{ed}}(\bar{x}_e - p_d), ed \in \tilde{V} \right\}$   
 $= \sum_{jk \in \bar{I}} \bar{z}_{jk}^{0*} \gamma_{C_{jk}}(\bar{x}_j - \bar{x}_k) + \sum_{ji \in \bar{\bar{I}}} \bar{z}_{ji}^{\bar{0}*} \gamma_{\tilde{C}_{ji}}(\bar{x}_j - p_i),$
- (ii)  $\sum_{jk \in \bar{I}} \bar{z}_{jk}^{0*} \gamma_{C_{jk}}(\bar{x}_j - \bar{x}_k) + \sum_{ji \in \bar{\bar{I}}} \bar{z}_{ji}^{\bar{0}*} \gamma_{\tilde{C}_{ji}}(\bar{x}_j - p_i) = \sum_{jk \in \bar{I}} \langle \bar{z}_{jk}^{1*}, \bar{x}_j - \bar{x}_k \rangle + \sum_{ji \in \bar{\bar{I}}} \langle \bar{z}_{ji}^{\bar{1}*}, \bar{x}_j - p_i \rangle,$
- (iii)  $\left\langle \sum_{jk \in \bar{I}} A_{jk}^* \bar{z}_{jk}^{1*} + \sum_{ji \in \bar{\bar{I}}} B_{ji}^* \bar{z}_{ji}^{\bar{1}*}, \bar{x} \right\rangle = -\sigma_S \left( - \sum_{jk \in \bar{I}} A_{jk}^* \bar{z}_{jk}^{1*} - \sum_{ji \in \bar{\bar{I}}} B_{ji}^* \bar{z}_{ji}^{\bar{1}*} \right),$
- (iv)  $\sum_{jk \in \bar{I}} \frac{1}{w_{jk}} \bar{z}_{jk}^{0*} + \frac{1}{\tilde{w}_{ji}} \sum_{ji \in \bar{\bar{I}}} \bar{z}_{ji}^{\bar{0}*} \leq 1, \bar{z}_{jk}^{0*} > 0, jk \in \bar{I}, \bar{z}_{ji}^{\bar{0}*} > 0, ji \in \bar{\bar{I}}$  and  $\bar{z}_{ef}^{0*} = 0, ef \notin \bar{I},$   
 $\bar{z}_{ed}^{\bar{0}*} = 0, ed \notin \bar{\bar{I}},$
- (v)  $\gamma_{C_{jk}}^0(\bar{z}_{jk}^{1*}) \leq \bar{z}_{jk}^{0*}, \bar{z}_{jk}^{1*} \in X^*, jk \in \bar{I}$  and  $\bar{z}_{ef}^{1*} = 0_{X^*}, ef \notin \bar{I},$

(vi)  $\gamma_{\tilde{C}_{ji}^0}(\tilde{z}_{ji}^{1*}) \leq \tilde{z}_{ji}^{0*}$ ,  $\tilde{z}_{ji}^{1*} \in X^*$ ,  $ji \in \tilde{I}$  and  $\tilde{z}_{ed}^{1*} = 0_{X^*}$ ,  $ed \notin \tilde{I}$ .

Condition (ii) yields

$$\sum_{jk \in \bar{I}} [\tilde{z}_{jk}^{0*} \gamma_{C_{jk}}(\bar{x}_j - \bar{x}_k) - \langle \tilde{z}_{jk}^{1*}, \bar{x}_j - \bar{x}_k \rangle] + \sum_{ji \in \tilde{I}} [\tilde{z}_{ji}^{0*} \gamma_{\tilde{C}_{ji}}(\bar{x}_j - p_i) - \langle \tilde{z}_{ji}^{1*}, \bar{x}_j - p_i \rangle] = 0 \quad (4.77)$$

and by (4.75), (4.76) and the Young-Fenchel inequality it follows that the brackets in (4.77) are non-negative and must be equal to zero, i.e.

$$\tilde{z}_{jk}^{0*} \gamma_{C_{jk}}(\bar{x}_j - \bar{x}_k) = \langle \tilde{z}_{jk}^{1*}, \bar{x}_j - \bar{x}_k \rangle, \quad jk \in \bar{I} \text{ and } \tilde{z}_{ji}^{0*} \gamma_{\tilde{C}_{ji}}(\bar{x}_j - p_i) = \langle \tilde{z}_{ji}^{1*}, \bar{x}_j - p_i \rangle, \quad ji \in \tilde{I}. \quad (4.78)$$

Similarly to the considerations done in (4.37) and (4.40) one derives that

$$\gamma_{C_{jk}^0}(\tilde{z}_{jk}^{1*}) \gamma_{C_{jk}}(\bar{x}_j - \bar{x}_k) \geq \langle \tilde{z}_{jk}^{1*}, \bar{x}_j - \bar{x}_k \rangle, \quad jk \in \bar{I}, \quad (4.79)$$

Combining the condition (v) with (4.78) and (4.79) reveals that

$$\tilde{z}_{jk}^{0*} \gamma_{C_{jk}}(\bar{x}_j - \bar{x}_k) = \langle \tilde{z}_{jk}^{1*}, \bar{x}_j - \bar{x}_k \rangle \leq \gamma_{C_{jk}^0}(\tilde{z}_{jk}^{1*}) \gamma_{C_{jk}}(\bar{x}_j - \bar{x}_k) \leq \tilde{z}_{jk}^{0*} \gamma_{C_{jk}}(\bar{x}_j - \bar{x}_k), \quad jk \in \bar{I},$$

which means that

$$\gamma_{C_{jk}^0}(\tilde{z}_{jk}^{1*}) = \tilde{z}_{jk}^{0*}, \quad jk \in \bar{I}. \quad (4.80)$$

In the same way we get

$$\gamma_{\tilde{C}_{ji}^0}(\tilde{z}_{ji}^{1*}) = \tilde{z}_{ji}^{0*}, \quad ji \in \tilde{I}. \quad (4.81)$$

Moreover, by conditions (i) and (iv) we have

$$\begin{aligned} & \max \left\{ w_{ef} \gamma_{C_{ef}}(\bar{x}_e - \bar{x}_f), \quad ef \in V, \quad \tilde{w}_{ed} \gamma_{\tilde{C}_{ed}}(\bar{x}_e - p_d), \quad ed \in \tilde{V} \right\} \\ &= \sum_{jk \in \bar{I}} \tilde{z}_{jk}^{0*} \gamma_{C_{jk}}(\bar{x}_j - \bar{x}_k) + \sum_{ji \in \tilde{I}} \tilde{z}_{ji}^{0*} \gamma_{\tilde{C}_{ji}}(\bar{x}_j - p_i) \\ &= \sum_{jk \in \bar{I}} \frac{1}{w_{jk}} \tilde{z}_{jk}^{0*} w_{jk} \gamma_{C_{jk}}(\bar{x}_j - \bar{x}_k) + \sum_{ji \in \tilde{I}} \frac{1}{\tilde{w}_{ji}} \tilde{z}_{ji}^{0*} \tilde{w}_{ji} \gamma_{\tilde{C}_{ji}}(\bar{x}_j - p_i) \\ &\leq \sum_{jk \in \bar{I}} \frac{1}{w_{jk}} \tilde{z}_{jk}^{0*} \max \left\{ w_{ef} \gamma_{C_{ef}}(\bar{x}_e - \bar{x}_f), \quad ef \in V, \quad \tilde{w}_{ed} \gamma_{\tilde{C}_{ed}}(\bar{x}_e - p_d), \quad ed \in \tilde{V} \right\} \\ &\quad + \sum_{ji \in \tilde{I}} \frac{1}{\tilde{w}_{ji}} \tilde{z}_{ji}^{0*} \max \left\{ w_{ef} \gamma_{C_{ef}}(\bar{x}_e - \bar{x}_f), \quad ef \in V, \quad \tilde{w}_{ed} \gamma_{\tilde{C}_{ed}}(\bar{x}_e - p_d), \quad ed \in \tilde{V} \right\} \\ &\leq \max \left\{ w_{ef} \gamma_{C_{ef}}(\bar{x}_e - \bar{x}_f), \quad ef \in V, \quad \tilde{w}_{ed} \gamma_{\tilde{C}_{ed}}(\bar{x}_e - p_d), \quad ed \in \tilde{V} \right\}, \end{aligned} \quad (4.82)$$

which implies that

$$\begin{aligned} & \sum_{jk \in \bar{I}} \frac{1}{w_{jk}} \tilde{z}_{jk}^{0*} \left[ \max \left\{ w_{ef} \gamma_{C_{ef}}(\bar{x}_e - \bar{x}_f), \quad ef \in V, \quad \tilde{w}_{ed} \gamma_{\tilde{C}_{ed}}(\bar{x}_e - p_d), \quad ed \in \tilde{V} \right\} - w_{jk} \gamma_{C_{jk}}(\bar{x}_j - \bar{x}_k) \right] \\ &+ \sum_{ji \in \tilde{I}} \frac{1}{\tilde{w}_{ji}} \tilde{z}_{ji}^{0*} \left[ \max \left\{ w_{ef} \gamma_{C_{ef}}(\bar{x}_e - \bar{x}_f), \quad ef \in V, \quad \tilde{w}_{ed} \gamma_{\tilde{C}_{ed}}(\bar{x}_e - p_d), \quad ed \in \tilde{V} \right\} - \tilde{w}_{ji} \gamma_{\tilde{C}_{ji}}(\bar{x}_j - p_i) \right] \\ &= 0 \end{aligned}$$

and as  $w_{jk}, \bar{z}_{jk}^{0*} > 0, jk \in \bar{I}$ , and  $\tilde{w}_{ji}, \tilde{z}_{ji}^{0*} > 0, ji \in \tilde{I}$ , it follows that

$$\max \left\{ w_{ef} \gamma_{C_{ef}}(\bar{x}_e - \bar{x}_f), ef \in V, \tilde{w}_{ed} \gamma_{\tilde{C}_{ed}}(\bar{x}_e - p_d), ed \in \tilde{V} \right\} = w_{jk} \gamma_{C_{jk}}(\bar{x}_j - \bar{x}_k), ik \in \bar{I} \quad (4.84)$$

and

$$\max \left\{ w_{ef} \gamma_{C_{ef}}(\bar{x}_e - \bar{x}_f), ef \in V, \tilde{w}_{ed} \gamma_{\tilde{C}_{ed}}(\bar{x}_e - p_d), ed \in \tilde{V} \right\} = \tilde{w}_{ji} \gamma_{\tilde{C}_{ji}}(\bar{x}_j - p_i), ji \in \tilde{I}. \quad (4.85)$$

Furthermore, we get by (4.83) that

$$\begin{aligned} & \sum_{jk \in \bar{I}} \frac{1}{w_{jk}} \bar{z}_{jk}^{0*} \max \left\{ w_{ef} \gamma_{C_{ef}}(\bar{x}_e - \bar{x}_f), ef \in V, \tilde{w}_{ed} \gamma_{\tilde{C}_{ed}}(\bar{x}_e - p_d), ed \in \tilde{V} \right\} \\ & + \sum_{ji \in \tilde{I}} \frac{1}{\tilde{w}_{ji}} \tilde{z}_{ji}^{0*} \max \left\{ w_{ef} \gamma_{C_{ef}}(\bar{x}_e - \bar{x}_f), ef \in V, \tilde{w}_{ed} \gamma_{\tilde{C}_{ed}}(\bar{x}_e - p_d), ed \in \tilde{V} \right\} \\ & = \max \left\{ w_{ef} \gamma_{C_{ef}}(\bar{x}_e - \bar{x}_f), ef \in V, \tilde{w}_{ed} \gamma_{\tilde{C}_{ed}}(\bar{x}_e - p_d), ed \in \tilde{V} \right\}, \end{aligned}$$

from which follows that

$$\sum_{jk \in \bar{I}} \frac{1}{w_{jk}} \bar{z}_{jk}^{0*} + \sum_{ji \in \tilde{I}} \frac{1}{\tilde{w}_{ji}} \tilde{z}_{ji}^{0*} = 1. \quad (4.86)$$

Combining now the conditions (i)-(vi) with (4.78), (4.80), (4.81), (4.84), (4.85) and (4.86) provides us the desired conclusion.

(b) The calculations made in (a) can also be done in the reverse direction, which completes the proof.  $\square$

**Remark 4.23.** We want to point out that the optimality condition (i) of the previous theorem can be expressed by means of the subdifferential. We have

$$f(y^0, \tilde{y}^0) = \begin{cases} \max \left\{ w_{jk} y_{jk}^0, jk \in V, \tilde{w}_{ji} \tilde{y}_{ji}^0, ji \in \tilde{V} \right\}, & \text{if } (y^0, \tilde{y}^0) \in \mathbb{R}_+^{|V|} \times \mathbb{R}_+^{|\tilde{V}|}, \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$f^*(z^{0*}, \tilde{z}^{0*}) = \begin{cases} 0, & \text{if } \sum_{jk \in V} \frac{1}{w_{jk}} z_{jk}^{0*} + \sum_{ji \in \tilde{V}} \frac{1}{\tilde{w}_{ji}} \tilde{z}_{ji}^{0*} \leq 1, z^{0*} \in \mathbb{R}_+^{|V|}, \tilde{z}^{0*} \in \mathbb{R}_+^{|\tilde{V}|}, \\ +\infty, & \text{otherwise,} \end{cases}$$

and by the optimality condition (i) of the previous theorem, it holds

$$\begin{aligned} & f \left( (\gamma_{C_{ef}}(\bar{x}_e - \bar{x}_f))_{ef \in V}, (\gamma_{\tilde{C}_{ed}}(\bar{x}_e - p_d))_{ed \in \tilde{V}} \right) + f^*(\bar{z}^{0*}, \tilde{z}^{0*}) \\ & = \sum_{jk \in \bar{I}} \bar{z}_{jk}^{0*} \gamma_{C_{jk}}(\bar{x}_j - \bar{x}_k) + \sum_{ji \in \tilde{I}} \tilde{z}_{ji}^{0*} \gamma_{\tilde{C}_{ji}}(\bar{x}_j - p_i), \end{aligned}$$

in other words, the optimality condition (i) can be rewritten as

$$(i) \quad (\bar{z}^{0*}, \tilde{z}^{0*}) \in \partial f \left( (\gamma_{C_{ef}}(\bar{x}_e - \bar{x}_f))_{ef \in V}, (\gamma_{\tilde{C}_{ed}}(\bar{x}_e - p_d))_{ed \in \tilde{V}} \right).$$

More than that, for the optimality conditions (ii), (iv) and (v) one gets by the same considerations

$$(ii) \quad - \sum_{jk \in \bar{I}} A_{jk}^* \bar{z}_{jk}^{1*} - \sum_{ji \in \tilde{I}} B_{ji}^* \tilde{z}_{ji}^{1*} \in \partial \delta_S(\bar{x}) = N_S(\bar{x}),$$

$$\begin{aligned}
(iv) \quad & \bar{z}_{jk}^{1*} \in \partial(\bar{z}_{jk}^{0*} \gamma_{C_{jk}})(\bar{x}_j - \bar{x}_k) = \partial(\bar{z}_{jk}^{0*} \gamma_{C_{jk}})(A_{jk}^* \bar{x}) \Leftrightarrow A_{jk}^* \bar{z}_{jk}^{1*} \in A_{jk}^* \partial((\bar{z}_{jk}^{0*} \gamma_{C_{jk}}) \circ A_{jk})(\bar{x}), \quad jk \in \bar{I}, \\
(v) \quad & \bar{z}_{ji}^{1*} \in \partial(\bar{z}_{ji}^{0*} \gamma_{\bar{C}_{ji}})(\bar{x}_j - p_i) = \partial(\bar{z}_{ji}^{0*} \gamma_{\bar{C}_{ji}})(B_{ji}^* \bar{x} - p_i) \\
& \Leftrightarrow B_{ji}^* \bar{z}_{ji}^{1*} \in B_{ji}^* \partial((\bar{z}_{ji}^{0*} \gamma_{\bar{C}_{ji}}) \circ B_{ji})(\cdot - p_i)(\bar{x}), \quad ji \in \bar{I}.
\end{aligned}$$

Taking (ii), (iv) and (v) together implies that

$$\begin{aligned}
& \sum_{jk \in \bar{I}} A_{jk}^* \bar{z}_{jk}^{1*} + \sum_{ji \in \bar{I}} B_{ji}^* \bar{z}_{ji}^{1*} \in \\
& \left( \sum_{jk \in \bar{I}} A_{jk}^* \partial((\bar{z}_{jk}^{0*} \gamma_{C_{jk}}) \circ A_{jk})(\bar{x}) + \sum_{ji \in \bar{I}} B_{ji}^* \partial((\bar{z}_{ji}^{0*} \gamma_{\bar{C}_{ji}}) \circ B_{ji})(\cdot - p_i)(\bar{x}) \right) \cap (-N_S(\bar{x})).
\end{aligned}$$

Finally, notice that the optimality conditions (iv), (v), (viii) and (ix) of the previous theorem give a detailed characterization of the subdifferentials of the associated gauges.

Now, we show that the dual problem  $(D^M)$  is equivalent to the problem

$$(\tilde{D}^M) \quad \sup_{(z^*, \tilde{z}^*) \in \tilde{\mathcal{B}}} \left\{ -\sigma_S \left( -\sum_{jk \in I} A_{jk}^* z_{jk}^* - \sum_{ji \in \tilde{I}} B_{ji}^* \tilde{z}_{ji}^* \right) - \sum_{ji \in \tilde{I}} \langle \tilde{z}_{ji}^*, p_i \rangle \right\}, \quad (4.87)$$

where  $(z^*, \tilde{z}^*) = ((z_{jk}^*)_{jk \in V}, (\tilde{z}_{ji}^*)_{ji \in \tilde{V}})$  and

$$\begin{aligned}
\tilde{\mathcal{B}} = & \left\{ \left( (z_{jk}^*)_{jk \in V}, (\tilde{z}_{ji}^*)_{ji \in \tilde{V}} \right) \in (X^*)^{|V|} \times (X^*)^{|\tilde{V}|} : I = \left\{ jk \in V : \gamma_{C_{jk}^0}(z_{jk}^*) > 0 \right\}, \right. \\
& \tilde{I} = \left\{ ji \in \tilde{V} : \gamma_{\bar{C}_{ji}^0}(\tilde{z}_{ji}^*) > 0 \right\}, \quad z_{ef}^* = 0_{X^*}, \quad ef \notin I, \quad \tilde{z}_{ed}^* = 0_{X^*}, \quad ed \notin \tilde{I} \\
& \left. \sum_{jk \in I} \frac{1}{w_{jk}} \gamma_{C_{jk}^0}(z_{jk}^*) + \sum_{ji \in \tilde{I}} \frac{1}{\bar{w}_{ji}} \gamma_{\bar{C}_{ji}^0}(\tilde{z}_{ji}^*) \leq 1 \right\},
\end{aligned}$$

in the sense of the next theorem, where  $v(\tilde{D}^M)$  denotes the optimal objective value of the problem  $(\tilde{D}^M)$ .

**Theorem 4.21.** *It holds  $v(D^M) = v(\tilde{D}^M)$ .*

**Proof.** Let  $(z^*, \tilde{z}^*)$  be a feasible element to  $(\tilde{D}^M)$  and set

$$z_{jk}^{1*} = z_{jk}^*, \quad z_{jk}^{0*} = \gamma_{C_{jk}^0}(z_{jk}^*) \text{ for } jk \in I, \quad z_{ef}^{1*} = 0_{X^*}, \quad z_{ef}^{0*} = 0 \text{ for } ef \notin I,$$

and

$$\tilde{z}_{ji}^{1*} = \tilde{z}_{ji}^*, \quad \tilde{z}_{ji}^{0*} = \gamma_{\bar{C}_{ji}^0}(\tilde{z}_{ji}^*) \text{ for } ji \in \tilde{I}, \quad \tilde{z}_{ed}^{1*} = 0_{X^*}, \quad \tilde{z}_{ed}^{0*} = 0 \text{ for } ed \notin \tilde{I}.$$

Then, it is clear that  $(z^{0*}, \tilde{z}^{0*}, z^{1*}, \tilde{z}^{1*})$  is a feasible element to  $(D^M)$ . Furthermore, it holds

$$\begin{aligned}
& -\sigma_S \left( -\sum_{jk \in I} A_{jk}^* z_{jk}^{1*} - \sum_{ji \in \tilde{I}} B_{ji}^* \tilde{z}_{ji}^{1*} \right) - \sum_{ji \in \tilde{I}} \langle \tilde{z}_{ji}^{1*}, p_i \rangle = \\
& -\sigma_S \left( -\sum_{jk \in I} A_{jk}^* z_{jk}^{1*} - \sum_{ji \in \tilde{I}} B_{ji}^* \tilde{z}_{ji}^{1*} \right) - \sum_{ji \in \tilde{I}} \langle \tilde{z}_{ji}^{1*}, p_i \rangle \leq v(D^M)
\end{aligned}$$

for all  $(z^*, \tilde{z}^*)$  feasible to  $(\tilde{D}^M)$ , from which follows that  $v(\tilde{D}^M) \leq v(D^M)$ .

Now, let  $(z^{0*}, \tilde{z}^{0*}, z^{1*}, \tilde{z}^{1*})$  be feasible element to  $(D^M)$ . By a careful look at the constraint set  $\mathcal{B}$  we get by setting  $z_{jk}^* = z_{jk}^{1*}$  for  $jk \in I$ ,  $\tilde{z}_{ji}^* = \tilde{z}_{ji}^{1*}$  for  $ji \in \tilde{I}$  and  $z_{ef}^* = 0_{X^*}$  for  $ef \notin I$ ,  $\tilde{z}_{ed}^* = 0_{X^*}$  for  $ed \notin \tilde{I}$  that

$$\sum_{jk \in I} \frac{1}{w_{jk}} \gamma_{C_{jk}^0}(z_{jk}^*) + \sum_{ji \in \tilde{I}} \frac{1}{\tilde{w}_{ji}} \gamma_{\tilde{C}_{ji}^0}(\tilde{z}_{ji}^*) \leq 1.$$

Therefore,  $(z^*, \tilde{z}^*)$  is feasible to  $(\tilde{D}^M)$  and we have

$$\begin{aligned} & -\sigma_S \left( -\sum_{jk \in I} A_{jk}^* z_{jk}^{1*} - \sum_{ji \in \tilde{I}} B_{ji}^* \tilde{z}_{ji}^{1*} \right) - \sum_{ji \in \tilde{I}} \langle \tilde{z}_{ji}^{1*}, p_i \rangle = \\ & -\sigma_S \left( -\sum_{jk \in I} A_{jk}^* z_{jk}^* - \sum_{ji \in \tilde{I}} B_{ji}^* \tilde{z}_{ji}^* \right) - \sum_{ji \in \tilde{I}} \langle \tilde{z}_{ji}^*, p_i \rangle \leq v(\tilde{D}^M) \end{aligned}$$

for all  $(z^{0*}, \tilde{z}^{0*}, z^{1*}, \tilde{z}^{1*})$  feasible to  $(D^M)$ , i.e.  $v(D^M) \leq v(\tilde{D}^M)$ , which completes the proof.  $\square$

The next theorem is direct consequence of Theorem 4.21.

**Theorem 4.22.** (strong duality) *Between  $(P^M)$  and  $(\tilde{D}^M)$  holds strong duality, i.e.  $v(P^M) = v(\tilde{D}^M)$  and the dual problem has an optimal solution.*

We close this subsection by the following statement, which is a result of Theorem 4.20 (especially by using the optimality conditions (viii) and (ix)), Theorem 4.21 and 4.22.

**Theorem 4.23.** (optimality conditions) (a) *Let  $\bar{x} \in S$  be an optimal solution of the problem  $(P^M)$ . Then there exist  $(\bar{z}^*, \tilde{z}^*) \in (X^*)^{|V|} \times (X^*)^{|\tilde{V}|}$  and index sets  $\bar{I}$  and  $\tilde{I}$  as an optimal solution to  $(\tilde{D}^M)$  such that*

- (i)  $\max \left\{ w_{ef} \gamma_{C_{ef}}(\bar{x}_e - \bar{x}_f), ef \in V, \tilde{w}_{ed} \gamma_{\tilde{C}_{ed}}(\bar{x}_e - p_d), ed \in \tilde{V} \right\}$   
 $= \sum_{jk \in \bar{I}} \gamma_{C_{jk}^0}(\bar{z}_{jk}^*) \gamma_{C_{jk}}(\bar{x}_j - \bar{x}_k) + \sum_{ji \in \tilde{I}} \gamma_{\tilde{C}_{ji}^0}(\tilde{z}_{ji}^*) \gamma_{\tilde{C}_{ji}}(\bar{x}_j - p_i),$
- (ii)  $\left\langle \sum_{jk \in \bar{I}} A_{jk}^* \bar{z}_{jk}^* + \sum_{ji \in \tilde{I}} B_{ji}^* \tilde{z}_{ji}^*, \bar{x} \right\rangle = -\sigma_S \left( \sum_{jk \in \bar{I}} A_{jk}^* \bar{z}_{jk}^* + \sum_{ji \in \tilde{I}} B_{ji}^* \tilde{z}_{ji}^* \right),$
- (iii)  $\gamma_{C_{jk}^0}(\bar{z}_{jk}^*) \gamma_{C_{jk}}(\bar{x}_j - \bar{x}_k) = \langle \bar{z}_{jk}^*, \bar{x}_j - \bar{x}_k \rangle, jk \in \bar{I},$
- (iv)  $\gamma_{\tilde{C}_{ji}^0}(\tilde{z}_{ji}^*) \gamma_{\tilde{C}_{ji}}(\bar{x}_j - p_i) = \langle \tilde{z}_{ji}^*, \bar{x}_j - p_i \rangle, ji \in \tilde{I},$
- (v)  $\max \left\{ w_{ef} \gamma_{C_{ef}}(\bar{x}_e - \bar{x}_f), ef \in V, \tilde{w}_{ed} \gamma_{\tilde{C}_{ed}}(\bar{x}_e - p_d), ed \in \tilde{V} \right\} = w_{jk} \gamma_{C_{jk}}(\bar{x}_j - \bar{x}_k), jk \in \bar{I},$
- (vi)  $\max \left\{ w_{ef} \gamma_{C_{ef}}(\bar{x}_e - \bar{x}_f), ef \in V, \tilde{w}_{ed} \gamma_{\tilde{C}_{ed}}(\bar{x}_e - p_d), ed \in \tilde{V} \right\} = \tilde{w}_{ji} \gamma_{\tilde{C}_{ji}}(\bar{x}_j - p_i), ji \in \tilde{I},$
- (vii)  $\sum_{jk \in \bar{I}} \frac{1}{w_{jk}} \gamma_{C_{jk}^0}(\bar{z}_{jk}^*) + \sum_{ji \in \tilde{I}} \frac{1}{\tilde{w}_{ji}} \gamma_{\tilde{C}_{ji}^0}(\tilde{z}_{ji}^*) = 1, \bar{z}_{jk}^* \in X^* \setminus \{0_{X^*}\}, jk \in \bar{I}, \tilde{z}_{ji}^* \in X^* \setminus \{0_{X^*}\}, ji \in \tilde{I},$   
and  $\bar{z}_{ef}^* = 0_{X^*}, ef \notin \bar{I}, \tilde{z}_{ed}^* = 0_{X^*}, ed \notin \tilde{I}.$

(b) *If there exists  $\bar{x} \in S$  such that for some  $(\bar{z}^*, \tilde{z}^*, \bar{I}, \tilde{I})$  the conditions (i)-(vii) are fulfilled, then  $\bar{x}$  is an optimal solution of  $(P^M)$ ,  $(\bar{z}^*, \tilde{z}^*, \bar{I}, \tilde{I})$  is an optimal solution for  $(\tilde{D}^M)$  and  $v(P^M) = v(\tilde{D}^M)$ .*



#### 4.4.2 Unconstrained multifacility minimax location problem in the Euclidean space

In this section we are interested in a detailed analysis of the situation when  $S = X^m$  and  $X = \mathbb{R}^d$  and the gauges are defined by the Euclidean norm. In addition, we set  $w_{jk} = 0$  for  $1 \leq k \leq j \leq m$  such that the index set  $V$  can be represented as  $V = \{jk : 1 \leq j < k \leq m, w_{jk} > 0\}$ , i.e.  $|V| \leq (m/2)(m-1)$ . In other words, we explore in the following the location problem

$$(P_N^M) \quad \inf_{x_i \in \mathbb{R}^d, i=1, \dots, m} \max \left\{ w_{jk} \|x_j - x_k\|, jk \in V, \tilde{w}_{ji} \|x_j - p_i\|, ji \in \tilde{V} \right\}. \quad (4.88)$$

For the dual of the location problem  $(P_N^M)$  we get by (4.87)

$$(\tilde{D}_N^M) \quad \sup_{(z^*, \tilde{z}^*) \in \tilde{\mathcal{B}}_N} \left\{ - \sum_{ji \in \tilde{I}} \langle \tilde{z}_{ji}^*, p_i \rangle \right\}, \quad (4.89)$$

where

$$\begin{aligned} \tilde{\mathcal{B}}_N = & \left\{ (z^*, \tilde{z}^*) = \left( (z_{jk}^*)_{jk \in V}, (\tilde{z}_{ji}^*)_{ji \in \tilde{V}} \right) \in (\mathbb{R}^d)^{|V|} \times (\mathbb{R}^d)^{|\tilde{V}|} : I = \{jk \in V : \|z_{jk}^*\| > 0\}, \right. \\ & \tilde{I} = \{ji \in \tilde{V} : \|\tilde{z}_{ji}^*\| > 0\}, z_{ef}^* = 0_{\mathbb{R}^d}, ef \notin I, \tilde{z}_{ed}^* = 0_{\mathbb{R}^d}, ed \notin \tilde{I}, \\ & \left. \sum_{jk \in I} \frac{1}{w_{jk}} \|z_{jk}^*\| + \sum_{ji \in \tilde{I}} \frac{1}{\tilde{w}_{ji}} \|\tilde{z}_{ji}^*\| \leq 1, \sum_{jk \in I} A_{jk}^* z_{jk}^* + \sum_{ji \in \tilde{I}} B_{ji}^* \tilde{z}_{ji}^* = \underbrace{0_{\mathbb{R}^d} \times \dots \times \mathbb{R}^d}_{m\text{-times}} \right\}. \end{aligned}$$

The next theorems are direct consequences of the results of the previous section.

**Theorem 4.24.** (strong duality) Between  $(P_N^M)$  and  $(\tilde{D}_N^M)$  strong duality holds, i.e.  $v(P_N^M) = v(\tilde{D}_N^M)$  and the dual problem has an optimal solution.

**Theorem 4.25.** (optimality conditions) (a) Let  $(\bar{x}_1, \dots, \bar{x}_m)$  be an optimal solution of the problem  $(P_N^M)$ . Then there exist  $(\bar{z}^*, \tilde{z}^*)$  and index sets  $\bar{I}$  and  $\tilde{I}$  as an optimal solution to  $(\tilde{D}_N^M)$  such that

- (i)  $\max \left\{ w_{ef} \|\bar{x}_e - \bar{x}_f\|, ef \in V, \tilde{w}_{ed} \|\bar{x}_e - p_d\|, ed \in \tilde{V} \right\}$   
 $= \sum_{jk \in \bar{I}} \|\bar{z}_{jk}^*\| \|\bar{x}_j - \bar{x}_k\| + \sum_{ji \in \tilde{I}} \|\tilde{z}_{ji}^*\| \|\bar{x}_j - p_i\|,$
- (ii)  $\sum_{jk \in \bar{I}} A_{jk}^* \bar{z}_{jk}^* + \sum_{ji \in \tilde{I}} B_{ji}^* \tilde{z}_{ji}^* = 0_{\mathbb{R}^d \times \dots \times \mathbb{R}^d},$
- (iii)  $\|\bar{z}_{jk}^*\| \|\bar{x}_j - \bar{x}_k\| = \langle \bar{z}_{jk}^*, \bar{x}_j - \bar{x}_k \rangle, jk \in \bar{I},$
- (iv)  $\|\tilde{z}_{ji}^*\| \|\bar{x}_j - p_i\| = \langle \tilde{z}_{ji}^*, \bar{x}_j - p_i \rangle, ji \in \tilde{I},$
- (v)  $\max \left\{ w_{ef} \|\bar{x}_e - \bar{x}_f\|, ef \in V, \tilde{w}_{ed} \|\bar{x}_e - p_d\|, ed \in \tilde{V} \right\} = w_{jk} \|\bar{x}_j - \bar{x}_k\|, jk \in \bar{I},$
- (vi)  $\max \left\{ w_{ef} \|\bar{x}_e - \bar{x}_f\|, ef \in V, \tilde{w}_{ed} \|\bar{x}_e - p_d\|, ed \in \tilde{V} \right\} = \tilde{w}_{ji} \|\bar{x}_j - p_i\|, ji \in \tilde{I},$
- (vii)  $\sum_{jk \in \bar{I}} \frac{1}{w_{jk}} \|\bar{z}_{jk}^*\| + \sum_{ji \in \tilde{I}} \frac{1}{\tilde{w}_{ji}} \|\tilde{z}_{ji}^*\| = 1, \bar{z}_{jk}^* \in \mathbb{R}^d \setminus \{0_{\mathbb{R}^d}\} \text{ for } jk \in \bar{I}, \tilde{z}_{ji}^* \in \mathbb{R}^d \setminus \{0_{\mathbb{R}^d}\} \text{ for } ji \in \tilde{I}$   
and  $\bar{z}_{jk}^* = 0_{\mathbb{R}^d} \text{ for } jk \notin \bar{I}, \tilde{z}_{ji}^* = 0_{\mathbb{R}^d} \text{ for } ji \notin \tilde{I}.$

(b) If there exists  $(\bar{x}_1, \dots, \bar{x}_m)$  such that for some  $(\bar{z}^*, \tilde{z}^*, \bar{I}, \tilde{I})$  the conditions (i)-(vii) are fulfilled, then  $\bar{x}$  is an optimal solution of  $(P_N^M)$ ,  $(\bar{z}^*, \tilde{z}^*, \bar{I}, \tilde{I})$  is an optimal solution for  $(\tilde{D}_N^M)$  and  $v(P_N^M) = v(\tilde{D}_N^M)$ .

**Remark 4.24.** The dual problem  $(\tilde{D}_N^M)$  can equivalently be written in the form (see Remark 4.22)

$$(\tilde{D}_N^M) \sup_{(z^*, \tilde{z}^*) \in \tilde{B}_N} \left\{ - \sum_{ji \in \tilde{V}} \langle \tilde{z}_{ji}^*, p_i \rangle \right\},$$

where

$$\tilde{B}_N = \left\{ (z^*, \tilde{z}^*) = \left( (z_{jk}^*)_{jk \in V}, (\tilde{z}_{ji}^*)_{ji \in \tilde{V}} \right) \in (\mathbb{R}^d)^{|V|} \times (\mathbb{R}^d)^{|\tilde{V}|} : \right. \\ \left. \sum_{jk \in V} \frac{1}{w_{jk}} \|z_{jk}^*\| + \sum_{ji \in \tilde{V}} \frac{1}{\tilde{w}_{ji}} \|\tilde{z}_{ji}^*\| \leq 1, \sum_{jk \in V} A_{jk}^* z_{jk}^* + \sum_{ji \in \tilde{V}} B_{ji}^* \tilde{z}_{ji}^* = 0 \underbrace{\mathbb{R}^d \times \dots \times \mathbb{R}^d}_{m \text{ times}} \right\}.$$

For its corresponding Lagrange dual problem we obtain

$$\begin{aligned} (D\tilde{D}_N^M) & \inf_{\substack{\lambda \geq 0, \\ x = (x_1, \dots, x_m) \in \mathbb{R}^d \times \dots \times \mathbb{R}^d}} \sup_{(z^*, \tilde{z}^*) \in \tilde{B}_N} \left\{ - \sum_{ji \in \tilde{V}} \langle \tilde{z}_{ji}^*, p_i \rangle + \right. \\ & \left. \left\langle x, \sum_{jk \in V} A_{jk}^T z_{jk}^* + \sum_{ji \in \tilde{V}} B_{ji}^T \tilde{z}_{ji}^* \right\rangle - \lambda \left( \sum_{jk \in V} \frac{1}{w_{jk}} \|z_{jk}^*\| + \sum_{ji \in \tilde{V}} \frac{1}{\tilde{w}_{ji}} \|\tilde{z}_{ji}^*\| - 1 \right) \right\} \\ & = \inf_{\substack{\lambda \geq 0, \\ x_i \in \mathbb{R}^d, i=1, \dots, m}} \left\{ \lambda + \sup_{(z^*, \tilde{z}^*) \in \tilde{B}_N} \left\{ - \sum_{ji \in \tilde{V}} \langle \tilde{z}_{ji}^*, p_i \rangle \right. \right. \\ & \left. \left. + \sum_{jk \in V} \langle x, A_{jk}^T z_{jk}^* \rangle + \sum_{ji \in \tilde{V}} \langle x, B_{ji}^T \tilde{z}_{ji}^* \rangle - \sum_{jk \in V} \frac{\lambda}{w_{jk}} \|z_{jk}^*\| - \sum_{ji \in \tilde{V}} \frac{\lambda}{\tilde{w}_{ji}} \|\tilde{z}_{ji}^*\| \right\} \right\} \\ & = \inf_{\substack{\lambda \geq 0, \\ x_i \in \mathbb{R}^d, i=1, \dots, m}} \left\{ \lambda + \sum_{jk \in V} \sup_{z_{jk}^* \in \mathbb{R}^d} \left\{ \langle A_{jk} x, z_{jk}^* \rangle - \frac{\lambda}{w_{jk}} \|z_{jk}^*\| \right\} \right. \\ & \left. + \sum_{ji \in \tilde{V}} \sup_{\tilde{z}_{ji}^* \in \mathbb{R}^d} \left\{ \langle B_{ji} x, \tilde{z}_{ji}^* \rangle - \langle p_i, \tilde{z}_{ji}^* \rangle - \frac{\lambda}{\tilde{w}_{ji}} \|\tilde{z}_{ji}^*\| \right\} \right\} \\ & = \inf_{\substack{\lambda \geq 0, \\ x_i \in \mathbb{R}^d, i=1, \dots, m}} \left\{ \lambda + \sum_{jk \in V} \sup_{z_{jk}^* \in \mathbb{R}^d} \left\{ \langle x_j - x_k, z_{jk}^* \rangle - \frac{\lambda}{w_{jk}} \|z_{jk}^*\| \right\} \right. \\ & \left. + \sum_{ji \in \tilde{V}} \sup_{\tilde{z}_{ji}^* \in \mathbb{R}^d} \left\{ \langle x_j - p_i, \tilde{z}_{ji}^* \rangle - \frac{\lambda}{\tilde{w}_{ji}} \|\tilde{z}_{ji}^*\| \right\} \right\}. \end{aligned}$$

The case  $\lambda = 0$  leads to  $x_j - p_i = 0$ ,  $ji \in \tilde{V}$ , and  $x_j - x_k = 0$ ,  $jk \in V$ , which contradicts our assumption that the given points  $p_i$ ,  $i = 1, \dots, n$ , are distinct, such that we can assume  $\lambda > 0$ . For this reason we can write for the Lagrange dual problem, or rather, the bidual of the location problem  $(P_N^M)$ ,

$$\begin{aligned} (D\tilde{D}_N^M) & \inf_{\substack{\lambda > 0, \\ (x_1, \dots, x_m) \in \mathbb{R}^d \times \dots \times \mathbb{R}^d}} \left\{ \lambda + \sum_{jk \in V} \frac{\lambda}{w_{jk}} \sup_{z_{jk}^* \in \mathbb{R}^d} \left\{ \left\langle \frac{w_{jk}}{\lambda} (x_j - x_k), z_{jk}^* \right\rangle - \|z_{jk}^*\| \right\} \right. \\ & \left. + \sum_{ji \in \tilde{V}} \frac{\lambda}{\tilde{w}_{ji}} \sup_{\tilde{z}_{ji}^* \in \mathbb{R}^d} \left\{ \left\langle \frac{\tilde{w}_{ji}}{\lambda} (x_j - p_i), \tilde{z}_{ji}^* \right\rangle - \|\tilde{z}_{ji}^*\| \right\} \right\} \\ & = \inf_{\substack{\lambda > 0, (x_1, \dots, x_m) \in \mathbb{R}^d \times \dots \times \mathbb{R}^d, \\ w_{jk} \|x_j - x_k\| \leq \lambda, jk \in V, \tilde{w}_{ji} \|x_j - p_i\| \leq \lambda, ji \in \tilde{V}}} \lambda \\ & = \inf_{(x_1, \dots, x_m) \in \mathbb{R}^d \times \dots \times \mathbb{R}^d} \max \left\{ w_{jk} \|x_j - x_k\|, jk \in V, \tilde{w}_{ji} \|x_j - p_i\|, ji \in \tilde{V} \right\}. \end{aligned}$$

By using the Lagrange dual concept we transformed the dual problem  $(\tilde{D}_N^M)$  back into the multifacility minimax location problem  $(P_N^M)$ , showing that one has a full symmetry between the location problem  $(P_N^M)$ , its dual problem  $(\tilde{D}_N^M)$  and the Lagrange dual problem  $(D\tilde{D}_N^M)$ . In addition, we see that the Lagrange multiplier associated to the equality constraint can be identified as the optimal solution of the multifacility minimax location problem  $(P_N^M)$  and the Lagrange multiplier associated to the inequality constraint as the optimal objective value. A similar fact was stated in [61] for the case of a multifacility minisum location problem.

The next corollary gives an estimation of the length of the vectors  $z_{jk}^*$ ,  $jk \in V$ , and  $\tilde{z}_{ji}^*$ ,  $ji \in \tilde{V}$ , feasible to the dual problem  $(\tilde{D}_N^M)$ .

**Corollary 4.3.** *Let  $\bar{w}_s := \max\{(w_{jk})_{jk \in V}, (w_{ji})_{ji \in \tilde{V}}\}$ , then for any feasible solution  $(z^*, \tilde{z}^*)$  of the problem  $(\tilde{D}_N^M)$  it holds*

$$\|z_{jk}^*\| \leq \frac{\bar{w}_s w_{jk}}{\bar{w}_s + w_{jk}} \text{ for } jk \in V \text{ and } \|\tilde{z}_{ji}^*\| \leq \frac{\bar{w}_s w_{ji}}{\bar{w}_s + w_{ji}} \text{ for } ji \in \tilde{V}.$$

**Proof.** As  $(z^*, \tilde{z}^*)$  is a feasible solution of  $(\tilde{D}_N^M)$ , it holds

$$\begin{aligned} \sum_{jk \in V} A_{jk}^* z_{jk}^* + \sum_{ji \in \tilde{V}} B_{ji}^* \tilde{z}_{ji}^* &= 0_{\mathbb{R}^d \times \dots \times \mathbb{R}^d} \Leftrightarrow -A_{uv}^* z_{uv}^* = \sum_{\substack{jk \in V, \\ jk \neq uv}} A_{jk}^* z_{jk}^* + \sum_{ji \in \tilde{V}} B_{ji}^* \tilde{z}_{ji}^* \\ \Rightarrow \|A_{uv}^* z_{uv}^*\| &= \left\| \sum_{\substack{jk \in V, \\ jk \neq uv}} A_{jk}^* z_{jk}^* + \sum_{ji \in \tilde{V}} B_{ji}^* \tilde{z}_{ji}^* \right\| \Rightarrow \|A_{uv}^* z_{uv}^*\| \leq \sum_{\substack{jk \in V, \\ jk \neq uv}} \|A_{jk}^* z_{jk}^*\| + \sum_{ji \in \tilde{V}} \|B_{ji}^* \tilde{z}_{ji}^*\| \\ \Leftrightarrow \sqrt{2} \|z_{uv}^*\| &\leq \sum_{\substack{jk \in V, \\ jk \neq uv}} \sqrt{2} \|z_{jk}^*\| + \sum_{ji \in \tilde{V}} \|\tilde{z}_{ji}^*\| \Leftrightarrow \|z_{uv}^*\| \leq \sum_{\substack{jk \in V, \\ jk \neq uv}} \|z_{jk}^*\| + \frac{1}{\sqrt{2}} \sum_{ji \in \tilde{V}} \|\tilde{z}_{ji}^*\| \\ \Rightarrow \|z_{uv}^*\| &\leq \sum_{\substack{jk \in V, \\ jk \neq uv}} \|z_{jk}^*\| + \sum_{ji \in \tilde{V}} \|\tilde{z}_{ji}^*\|, \quad uv \in V, \end{aligned}$$

and more than that, it holds

$$\begin{aligned} 1 &\geq \sum_{jk \in V} \frac{1}{w_{jk}} \|z_{jk}^*\| + \sum_{ji \in \tilde{V}} \frac{1}{\tilde{w}_{ji}} \|\tilde{z}_{ji}^*\| = \frac{1}{w_{uv}} \|z_{uv}^*\| + \sum_{\substack{jk \in V, \\ jk \neq uv}} \frac{1}{w_{jk}} \|z_{jk}^*\| + \sum_{ji \in \tilde{V}} \frac{1}{\tilde{w}_{ji}} \|\tilde{z}_{ji}^*\| \\ &\geq \frac{1}{w_{uv}} \|z_{uv}^*\| + \frac{1}{\bar{w}_s} \left( \sum_{\substack{jk \in V, \\ jk \neq uv}} \|z_{jk}^*\| + \sum_{ji \in \tilde{V}} \|\tilde{z}_{ji}^*\| \right) \geq \frac{1}{w_{uv}} \|z_{uv}^*\| + \frac{1}{\bar{w}_s} \|z_{uv}^*\| \\ &= \frac{\bar{w}_s + w_{uv}}{\bar{w}_s w_{uv}} \|z_{uv}^*\|, \end{aligned}$$

which means that

$$\|z_{jk}^*\| \leq \frac{\bar{w}_s w_{jk}}{\bar{w}_s + w_{jk}}, \quad jk \in V.$$

In the same way, we get

$$\|\tilde{z}_{ji}^*\| \leq \frac{\bar{w}_s w_{ji}}{\bar{w}_s + w_{ji}}, \quad ji \in \tilde{V}.$$

□

**Example 4.5.** For the existing facilities  $p_1 = (0, 0)^T$ ,  $p_2 = (-2, 3)^T$  and  $p_3 = (5, 8)^T$  ( $t=3$ ) we want to locate two new facilities ( $m=2$ ) in the plane ( $d=2$ ). The weights are given by

$w_{12} = \tilde{w}_{11} = \tilde{w}_{13} = \tilde{w}_{21} = \tilde{w}_{22} = 1$  and  $\tilde{w}_{12} = \tilde{w}_{23} = 0$  and define the following multifacility minimax location problem

$$(P_N^M) \quad \inf_{(x_1, x_2) \in \mathbb{R}^2 \times \mathbb{R}^2} \max \{ \|x_1 - x_2\|, \|x_1 - p_1\|, \|x_1 - p_3\|, \|x_2 - p_1\|, \|x_2 - p_2\| \},$$

i.e.  $V = \{12\}$ ,  $|V| = 1$ ,  $\tilde{V} = \{11, 13, 21, 22\}$  and  $|\tilde{V}| = 4$ . From the Matlab Optimization Toolbox we obtained the following solution  $\bar{x}_1 = (2.5, 4)$  and  $\bar{x}_2 = (0, 0)^T$ . The corresponding objective value was  $v(P_N^M) = 4.72$ .

The dual problem (see Remark 4.22)

$$(\tilde{D}_N^M) \quad \max_{(z_{12}^*, \tilde{z}_{11}^*, \tilde{z}_{13}^*, \tilde{z}_{21}^*, \tilde{z}_{22}^*) \in \tilde{\mathcal{B}}_N} \{ \langle \tilde{z}_{11}^* + \tilde{z}_{21}^*, p_1 \rangle + \langle \tilde{z}_{22}^*, p_2 \rangle + \langle \tilde{z}_{13}^*, p_3 \rangle \},$$

where

$$\begin{aligned} \tilde{\mathcal{B}}_N = \{ (z_{12}^*, \tilde{z}_{11}^*, \tilde{z}_{13}^*, \tilde{z}_{21}^*, \tilde{z}_{22}^*) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 : z_{12}^* + \tilde{z}_{11}^* + \tilde{z}_{13}^* = 0_{\mathbb{R}^2}, \\ \tilde{z}_{21}^* + \tilde{z}_{22}^* = 0_{\mathbb{R}^2}, \|z_{12}^*\| + \|\tilde{z}_{11}^*\| + \|\tilde{z}_{21}^*\| + \|\tilde{z}_{22}^*\| + \|\tilde{z}_{13}^*\| \leq 1 \}, \end{aligned}$$

was also solved by the Matlab Optimization Toolbox. The following solution was obtained

$$\bar{z}_{12}^* = \bar{z}_{11}^* = (0.13, 0.21)^T, \quad \bar{z}_{13}^* = (-0.26, -0.42)^T, \quad \bar{z}_{21}^* = \bar{z}_{22}^* = (0, 0)^T,$$

with the corresponding objective value  $v(\tilde{D}_N^M) = 4.72 = v(P_N^M)$ , i.e.  $\bar{I} = \{12\} \subseteq V$  and  $\bar{\tilde{I}} = \{11, 13\} \subseteq \tilde{V}$ .

In the situation when we have only the solution of the dual problem one can reconstruct the optimal solution of the primal problem in a recursive way by using the necessary and sufficient optimality conditions given in Theorem 4.25. By condition (iv) we know that there exists  $\tilde{\alpha}_{11} > 0$  such that

$$\bar{z}_{11}^* = \tilde{\alpha}_{11}(\bar{x}_1 - p_1), \quad \text{i.e.} \quad \|\bar{z}_{11}^*\| = \tilde{\alpha}_{11}\|\bar{x}_1 - p_1\|, \quad (4.90)$$

and as, by condition (vi) it holds

$$v(\tilde{D}_N^M) = v(P_N^M) = \|\bar{x}_1 - p_1\| = \frac{\|\bar{z}_{11}^*\|}{\tilde{\alpha}_{11}}, \quad (4.91)$$

we get by combining (4.90) and (4.91) that

$$\bar{z}_{11}^* = \frac{\|\bar{z}_{11}^*\|}{v(\tilde{D}_N^M)}(\bar{x}_1 - p_1) \Leftrightarrow \bar{x}_1 = \frac{v(\tilde{D}_N^M)}{\|\bar{z}_{11}^*\|}\bar{z}_{11}^* + p_1 = \frac{4.72}{0.25}(0.13, 0.21)^T = (2.5, 4)^T.$$

More than that, by condition (iii) there exists  $\alpha_{12} > 0$  such that

$$\bar{z}_{12}^* = \alpha_{12}(\bar{x}_1 - \bar{x}_2), \quad \text{i.e.} \quad \|\bar{z}_{12}^*\| = \alpha_{12}\|\bar{x}_1 - \bar{x}_2\|, \quad (4.92)$$

and therefore, we derive from condition (v) that

$$v(\tilde{D}_N^M) = v(P_N^M) = \|\bar{x}_1 - \bar{x}_2\| = \frac{\|\bar{z}_{12}^*\|}{\alpha_{12}}. \quad (4.93)$$

Finally, taking (4.92) and (4.93) together yields

$$\bar{z}_{12}^* = \frac{\|\bar{z}_{12}^*\|}{v(\tilde{D}_N^M)}(\bar{x}_1 - \bar{x}_2) \Leftrightarrow \bar{x}_2 = \bar{x}_1 - \frac{v(\tilde{D}_N^M)}{\|\bar{z}_{12}^*\|}\bar{z}_{12}^* = (2.5, 4)^T - \frac{4.72}{0.25}(0.13, 0.21)^T = (0, 0)^T.$$

For a geometrical illustration see Figure 4.4.

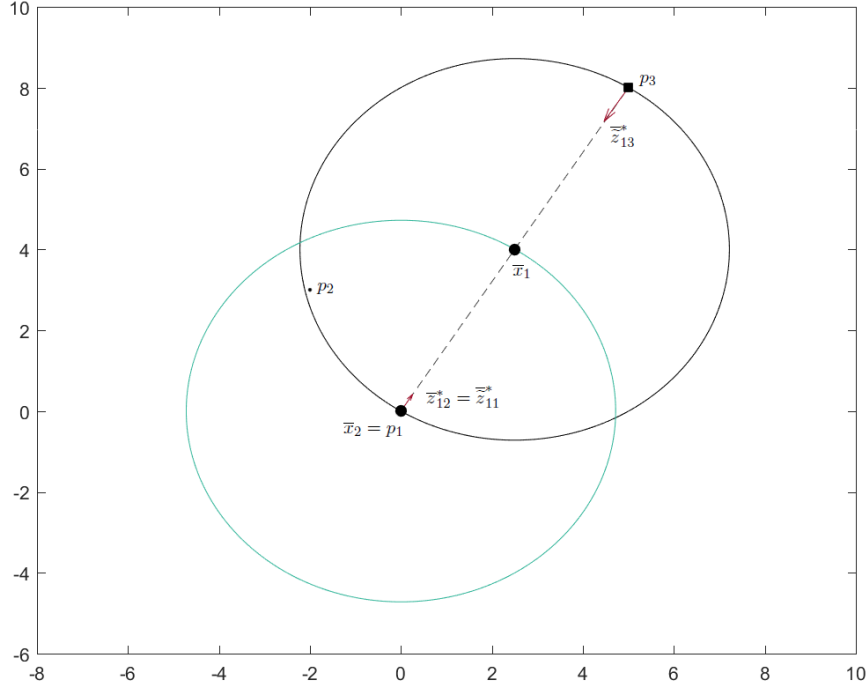


Figure 4.4: Illustration of the Example 4.5.

**Geometrical interpretation.**

In the following we provide a geometrical characterization of the set of optimal solutions of the dual problem by Theorem 4.25. By the conditions (iii) and (iv) it is clear that for  $jk \in \bar{I}$  and  $ji \in \tilde{I}$  the vectors  $\bar{z}_{jk}^*$  and  $\tilde{z}_{ji}^*$  are parallel to the vectors  $\bar{x}_j - \bar{x}_k$  and  $\bar{x}_j - p_i$  directed to  $\bar{x}_j$ , respectively. In addition, if we take into account the conditions (v), (vi) and (vii), then it is also evident that  $jk \in \bar{I}$  and  $ji \in \tilde{I}$ , i.e.  $\bar{z}_{jk}^* \neq 0_{\mathbb{R}^d}$  and  $\tilde{z}_{ji}^* \neq 0_{\mathbb{R}^d}$ , if the points  $\bar{x}_k$  and  $p_i$  are lying on the border of the minimum covering ball with radius  $v(P_N^M)$  centered in  $\bar{x}_j$ , respectively.

Vice versa, if  $jk \notin \bar{I}$  and  $ji \notin \tilde{I}$ , then  $\bar{z}_{jk}^* = 0_{\mathbb{R}^d}$  and  $\tilde{z}_{ji}^* = 0_{\mathbb{R}^d}$ , which is exactly the case when the points  $\bar{x}_k$  and  $p_i$  are lying inside the minimum covering ball centered in  $\bar{x}_j$ , respectively. Therefore, analogously to the geometrical interpretation presented in Section 4.2.4 for single minimax location problems, one can identify the vectors  $\bar{z}_{jk}^*$ ,  $jk \in \bar{I}$ , and  $\tilde{z}_{ji}^*$ ,  $ji \in \tilde{I}$ , as force vectors, which pull the points lying on the borders of the minimum covering balls inside the balls in direction to the their corresponding centers, the gravity points  $\bar{x}_j$  (see Figure 4.4).



## Chapter 5

# Solving minimax location problems via epigraphical projection

### 5.1 Motivation

As argued in a large number of papers, the proximal method is an excellent tool to solve in an efficient way optimization problems of the form

$$\min_{x \in \mathcal{H}} \left\{ \sum_{i=1}^n f_i(x) \right\}, \quad (5.1)$$

where  $\mathcal{H}$  is a real Hilbert space and  $f_i : \mathcal{H} \rightarrow \overline{\mathbb{R}}$  is a proper, lower semicontinuous and convex function,  $i = 1, \dots, n$ . This kind of problems occur for instance in areas like image processing [9, 15, 16, 28], portfolio optimization [12, 69], cluster analysis [11, 26], statistical learning theory [18], machine learning [13] and location theory [12, 14, 30, 56]. In the main step of this method it is necessary to determine the proximity operators of the functions involved in the formulation of the associated optimization problem. The *proximity operator* (a.k.a. *proximal mapping*) of a proper, lower semicontinuous and convex function  $f : \mathcal{H} \rightarrow \overline{\mathbb{R}}$  denoted by  $\text{prox}_f$  is defined by

$$\text{prox}_f x : \mathcal{H} \rightarrow \mathcal{H}, \quad \text{prox}_f x := \arg \min_{y \in \mathcal{H}} \left\{ f(y) + \frac{1}{2} \|x - y\|_{\mathcal{H}}^2 \right\} \quad \forall x \in \mathcal{H}. \quad (5.2)$$

The proximity operator can be understood as a generalization of the projection onto a convex set, as for a non-empty, closed and convex set  $A \subseteq \mathcal{H}$ , i.e.  $\delta_A$  is proper, convex and lower semicontinuous, we have

$$\text{prox}_{\delta_A} x = P_A x \quad \forall x \in \mathcal{H}, \quad (5.3)$$

where  $P_A$  is the projection operator which maps every point  $x$  in  $\mathcal{H}$  to its unique projection onto the set  $A$  (see [2]).

From (5.2) follows that the determination of the proximity operators of the functions  $f_i$ ,  $i = 1, \dots, n$ , of (5.1) requires the solving of  $n$  subproblems, where a favorable situation exists, when a closed formula of a proximity operator can be given. This in turn has a positive effect on the solving of optimization problems from the numerical point of view.

Motivated by this background, our aim is to solve numerically extended multifacility minimax location problems given by

$$(EP_N^{M,\beta}) \quad \min_{(x_1, \dots, x_m) \in \mathbb{R}^d \times \dots \times \mathbb{R}^d} \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^m w_{ij} \|x_j - p_i\|^{\beta_i} \right\}, \quad (5.4)$$

where  $w_{ij} > 0$  and  $p_i \in \mathbb{R}^d$  are distinct points,  $j = 1, \dots, m$ ,  $i = 1, \dots, n$ . In this framework we first need to rewrite this kind of location problems into the form of (5. 1) where the objective function is a sum of proper, lower semicontinuous and convex functions. For this purpose we introduce an additional variable and obtain for  $(EP_N^{M,\beta})$  the following formulation

$$\begin{aligned} (EP_N^{M,\beta}) \quad & \min_{\substack{(x_1, \dots, x_m, t) \in \mathbb{R}^d \times \dots \times \mathbb{R}^d \times \mathbb{R}, \\ \sum_{j=1}^m w_{ij} \|x_j - p_i\|^{\beta_i} \leq t, \quad i=1, \dots, n}} t = \min_{\substack{(x_1, \dots, x_m, t) \in \mathbb{R}^d \times \dots \times \mathbb{R}^d \times \mathbb{R}, \\ (x_1, \dots, x_m, t) \in \text{epi} \left( \sum_{j=1}^m w_{ij} \|\cdot - p_i\|^{\beta_i} \right), \quad i=1, \dots, n}} t \\ & = \min_{(x_1, \dots, x_m, t) \in \mathbb{R}^d \times \dots \times \mathbb{R}^d \times \mathbb{R}} \left\{ t + \sum_{i=1}^n \delta_{\text{epi} \left( \sum_{j=1}^m w_{ij} \|\cdot - p_i\|^{\beta_i} \right)}(x_1, \dots, x_m, t) \right\}. \quad (5. 5) \end{aligned}$$

Now, to apply the proximal method to  $(EP_N^{M,\beta})$  one needs to calculate the proximity operators of the functions involved in the objective function of (5. 5). For this reason and especially in the context of (5. 3), we give in Section 5.2 formulae for the projections onto the epigraphs of several sums of powers of weighted norms. As the power of norm in (5. 5) can be replaced by a gauge function, we present also formulae of projections onto the epigraphs of gauges.

To point out the benefits of the presented formulae we consider then examples of location problems in different settings and compare the numerical results with a method proposed by Cornejo and Michelot in [30]. The difference between these two methods is that the one given by Cornejo and Michelot splits the sum of powers of weighted norms by introducing  $n \cdot m$  additional variables. In this situation one gets the following presentation of the extended multifacility minimax location problem

$$(EP_N^{M,\beta}) \quad \min_{\substack{t, \quad t_{ij} \in \mathbb{R}, \quad x_j \in \mathbb{R}^d, \\ j=1, \dots, m, \quad i=1, \dots, n}} \left\{ t + \sum_{j=1}^m \sum_{i=1}^n \delta_{\text{epi}(w_{ij} \|\cdot - p_i\|^{\beta_i})}(x_j, t_{ij}) + \sum_{i=1}^n \delta_{\text{epi} \tau_i}(t_{i1}, \dots, t_{im}, t) \right\}, \quad (5. 6)$$

where  $\tau_i(t_{i1}, \dots, t_{im}) := \sum_{j=1}^m t_{ij}$ ,  $i = 1, \dots, n$ . In Section 5.3 we show that this concept makes the solving process for the considered examples of location problems very slow and the advantage of our approach more clear. The numerical tests are based on the parallel splitting algorithm, which can be found for instance in [2].

Finally, we collect some properties of Hilbert spaces, which can be found with proofs for instance in [2] and [29].

If  $f$  is Gâteaux-differentiable at  $x \in \mathcal{H}$ , then  $\partial f(x) = \{\nabla f(x)\}$ . The set of global minimizers of a function  $f : \mathcal{H} \rightarrow \mathbb{R}$  is denoted by  $\text{Argmin } f$  and if  $f$  has a unique minimizer, it is denoted by  $\arg \min_{x \in \mathcal{H}} f(x)$ . It holds

$$x \in \text{Argmin } f \Leftrightarrow 0_{\mathcal{H}} \in \partial f(x) \quad \forall x \in \mathcal{H}. \quad (5. 7)$$

It holds

$$y = \text{prox}_f x \Leftrightarrow x - y \in \partial f(y) \quad \forall x \in \mathcal{H}, \quad \forall y \in \mathcal{H}. \quad (5. 8)$$

In addition, we make for the rest of this chapter the convention that  $\frac{0}{0} = 0$  and  $\frac{1}{0} \cdot 0_{\mathcal{H}} = 0_{\mathcal{H}}$ .

In the following let  $\mathcal{H}_1 \times \dots \times \mathcal{H}_n$  be real Hilbert space endowed with inner product and norm, respectively defined by

$$\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle_{\mathcal{H}_1 \times \dots \times \mathcal{H}_n} = \sum_{i=1}^n \langle x_i, y_i \rangle_{\mathcal{H}_i} \quad \text{and} \quad \|(x_1, \dots, x_n)\|_{\mathcal{H}_1 \times \dots \times \mathcal{H}_n} = \sqrt{\sum_{i=1}^n \|x_i\|_{\mathcal{H}_i}^2},$$

where  $(x_1, \dots, x_n) \in \mathcal{H}_1 \times \dots \times \mathcal{H}_n$  and  $(y_1, \dots, y_n) \in \mathcal{H}_1 \times \dots \times \mathcal{H}_n$ .



We close this section with a lemma, which presents a formula for the projection onto a unit ball generated by the weighted sum of norms and generalizes the results given in [76] to real Hilbert spaces  $\mathcal{H}_i$ ,  $i = 1, \dots, n$ . Let  $w_i > 0$ ,  $i = 1, \dots, n$ , and  $C := \{(x_1, \dots, x_n) \in \mathcal{H}_1 \times \dots \times \mathcal{H}_n : \sum_{i=1}^n w_i \|x_i\|_{\mathcal{H}_i} \leq 1\}$ , then the following statement holds.

**Lemma 5.1.** *For all  $(x_1, \dots, x_n) \in \mathcal{H}_1 \times \dots \times \mathcal{H}_n$  it holds*

$$P_C(x_1, \dots, x_n) = \begin{cases} (x_1, \dots, x_n), & \text{if } \sum_{i=1}^n w_i \|x_i\|_{\mathcal{H}_i} \leq 1, \\ (\bar{y}_1, \dots, \bar{y}_n), & \text{otherwise,} \end{cases}$$

where

$$\bar{y}_i = \frac{\max\{\|x_i\|_{\mathcal{H}_i} - \bar{\lambda} w_i, 0\}}{\|x_i\|_{\mathcal{H}_i}} x_i, \quad i = 1, \dots, n,$$

with

$$\bar{\lambda} = \frac{\sum_{i=k+1}^n w_i^2 \tau_i - 1}{\sum_{i=k+1}^n w_i^2}$$

and  $k \in \{0, 1, \dots, n-1\}$  is the unique integer such that  $\tau_k \leq \bar{\lambda} \leq \tau_{k+1}$ , where the values  $\tau_0, \dots, \tau_n$  are defined by  $\tau_0 := 0$  and  $\tau_i := \|x_i\|_{\mathcal{H}_i}/w_i$ ,  $i = 1, \dots, n$ , and in ascending order.

**Proof.** In order to determine the projection onto the set  $C$ , we consider for fixed  $(x_1, \dots, x_n) \in \mathcal{H}_1 \times \dots \times \mathcal{H}_n$  the following optimization problem

$$\min_{\substack{(y_1, \dots, y_n) \in \mathcal{H}_1 \times \dots \times \mathcal{H}_n, \\ \sum_{i=1}^n w_i \|y_i\|_{\mathcal{H}_i} \leq 1}} \left\{ \sum_{i=1}^n \frac{1}{2} \|y_i - x_i\|_{\mathcal{H}_i}^2 \right\}. \quad (5.9)$$

Obviously, if  $\sum_{i=1}^n w_i \|x_i\|_{\mathcal{H}_i} \leq 1$ , i.e.  $(x_1, \dots, x_n) \in C$ , then the unique solution is  $\bar{y}_i = x_i$ ,  $i = 1, \dots, n$ . In the following we consider the non-trivial situation where  $\sum_{i=1}^n w_i \|x_i\|_{\mathcal{H}_i} > 1$ , i.e.  $(x_1, \dots, x_n) \notin C$  and define the function  $f : \mathcal{H}_1 \times \dots \times \mathcal{H}_n \rightarrow \mathbb{R}$  by  $f(y_1, \dots, y_n) := \sum_{i=1}^n (1/2) \|y_i - x_i\|_{\mathcal{H}_i}^2$  and the function  $g : \mathcal{H}_1 \times \dots \times \mathcal{H}_n \rightarrow \mathbb{R}$  by  $g(y_1, \dots, y_n) := \sum_{i=1}^n w_i \|y_i\|_{\mathcal{H}_i} - 1$ . Hence, by [2, Proposition 26.18] it holds for the unique solution  $(\bar{y}_1, \dots, \bar{y}_n)$  of (5.9) that

$$\nabla f(\bar{y}_1, \dots, \bar{y}_n) \in -\bar{\lambda} \partial g(\bar{y}_1, \dots, \bar{y}_n) \Leftrightarrow \bar{y}_i - x_i \in -\bar{\lambda} \partial (w_i \|\cdot\|_{\mathcal{H}_i})(\bar{y}_i), \quad i = 1, \dots, n,$$

as well as

$$\bar{\lambda} \left( \sum_{i=1}^n w_i \|\bar{y}_i\|_{\mathcal{H}_i} - 1 \right) = 0 \text{ and } \sum_{i=1}^n w_i \|\bar{y}_i\|_{\mathcal{H}_i} \leq 1,$$

where  $\bar{\lambda} \geq 0$  is the associated Lagrange multiplier of  $(\bar{y}_1, \dots, \bar{y}_n)$ . If  $\bar{\lambda} = 0$ , then  $\bar{y}_i = x_i$ ,  $i = 1, \dots, n$ , and by the feasibility condition we obtain  $\sum_{i=1}^n w_i \|x_i\|_{\mathcal{H}_i} \leq 1$ , which contradicts our assumption. Therefore,  $\bar{\lambda} > 0$  and we get by (5.8) that

$$\bar{y}_i - x_i \in -\bar{\lambda} \partial (w_i \|\cdot\|_{\mathcal{H}_i})(\bar{y}_i) \Leftrightarrow x_i - \bar{y}_i \in \partial(\bar{\lambda} w_i \|\cdot\|_{\mathcal{H}_i})(\bar{y}_i) \Leftrightarrow \bar{y}_i = \text{prox}_{\bar{\lambda} w_i \|\cdot\|_{\mathcal{H}_i}} x_i, \quad i = 1, \dots, n.$$

Using [29, Proposition 2.8] reveals that

$$\bar{y}_i = \begin{cases} x_i - \frac{\bar{\lambda} w_i}{\|x_i\|_{\mathcal{H}_i}} x_i, & \text{if } \|x_i\|_{\mathcal{H}_i} > \bar{\lambda} w_i, \\ 0_{\mathcal{H}_i}, & \text{if } \|x_i\|_{\mathcal{H}_i} \leq \bar{\lambda} w_i \end{cases} = \frac{\max\{\|x_i\|_{\mathcal{H}_i} - \bar{\lambda} w_i, 0\}}{\|x_i\|_{\mathcal{H}_i}} x_i, \quad i = 1, \dots, n,$$

and as  $\sum_{i=1}^n w_i \|\bar{y}_i\|_{\mathcal{H}_i} = 1$ , we conclude that

$$\sum_{i=1}^n w_i \max \{ \|x_i\|_{\mathcal{H}_i} - \bar{\lambda} w_i, 0 \} = 1. \quad (5.10)$$

Now, we define the function  $\kappa : \mathbb{R} \rightarrow \mathbb{R}$  by  $\kappa(\lambda) = \sum_{i=1}^n w_i^2 \max \{ \tau_i - \lambda, 0 \} - 1$ . Note, that there exists  $\tilde{\lambda} \geq \tau_i$  for all  $i = 1, \dots, n$ , such that  $\kappa(\tilde{\lambda}) = -1 < 0$ . Moreover,  $\kappa$  is a piecewise linear function with  $\kappa(0) = \sum_{i=1}^n w_i^2 \tau_i - 1$  and its slope changes at  $\lambda = \tau_i$ ,  $i = 1, \dots, n$ . To be more precise, at  $\lambda = 0$  the slope of  $\kappa$  is  $-\sum_{i=1}^n w_i^2$  and increases by  $w_1^2$  when  $\lambda = \tau_1$ . If we continue in this matter for  $i = 2, \dots, n$ , the slope keeps increasing and when  $\lambda \geq \tau_n$ ,  $\kappa(\lambda) = -1$  such that the slope is 0. In summary, to find the zero of  $\kappa$  one needs to determine the unique integer  $k \in \{0, 1, \dots, n-1\}$  such that  $\kappa(\tau_k) \geq 0$  and  $\kappa(\tau_{k+1}) \leq 0$ . In the light of the above, it holds

$$\kappa(\lambda) = \sum_{i=k+1}^n w_i^2 \tau_i - \lambda \sum_{i=k+1}^n w_i^2 - 1,$$

where  $\tau_k \leq \lambda \leq \tau_{k+1}$ , and hence, one gets for  $\bar{\lambda}$  such that  $\kappa(\bar{\lambda}) = 0$ ,

$$\bar{\lambda} = \frac{\sum_{i=k+1}^n w_i^2 \tau_i - 1}{\sum_{i=k+1}^n w_i^2}. \quad \square$$

## 5.2 Formulae of epigraphical projection

The first aim of this section is to give formulae for the projection operators onto the epigraphs of several sums of powers of weighted norms. For this purpose, we give a general formula in our central theorem, from which we deduce special cases used in our numerical tests.

The second aim is to present formulae of the projection operators onto the epigraphs of gauges. In this part of this section we use the properties of gauge functions listed in Section 4.1. Especially, by using the fact that the sum of gauges is again a gauge, we also present a formula of the projector onto the epigraph of the sum of gauges. Two examples in the cases of norms close this section.

### 5.2.1 Sum of weighted norms

Let us consider the following function  $h : \mathcal{H}_1 \times \dots \times \mathcal{H}_n \rightarrow \mathbb{R}$  defined as

$$h(x_1, \dots, x_n) := \sum_{i=1}^n w_i \|x_i\|_{\mathcal{H}_i}^{\beta_i}, \quad (5.11)$$

where  $w_i > 0$  and  $\beta_i \geq 1$ ,  $i = 1, \dots, n$ . By defining the sets

$$L := \{l \in \{1, \dots, n\} : \beta_l > 1\} \text{ and } R := \{r \in \{1, \dots, n\} : \beta_r = 1\},$$

we can state the following formula for the projection onto the epigraph of the sum of powers of weighted norms, which generalizes the results given for instance in [2, 28, 29, 69].

**Theorem 5.1.** *Assume that  $h$  is given by (5.11). Then, for every  $(x_1, \dots, x_n, \xi) \in \mathcal{H}_1 \times \dots \times \mathcal{H}_n \times \mathbb{R}$  one has*

$$P_{\text{epi } h}(x_1, \dots, x_n, \xi) = \begin{cases} (x_1, \dots, x_n, \xi), & \text{if } \sum_{i=1}^n w_i \|x_i\|_{\mathcal{H}_i}^{\beta_i} \leq \xi, \\ (\bar{y}_1, \dots, \bar{y}_n, \bar{\theta}), & \text{otherwise,} \end{cases} \quad (5.12)$$

with

$$\begin{aligned}\bar{y}_r &= \frac{\max\{\|x_r\|_{\mathcal{H}_r} - \bar{\lambda}w_r, 0\}}{\|x_r\|_{\mathcal{H}_r}}x_r, \quad r \in R, \\ \bar{y}_l &= \frac{\|x_l\|_{\mathcal{H}_l} - \eta_l(\bar{\lambda})}{\|x_l\|_{\mathcal{H}_l}}x_l, \quad l \in L, \\ \bar{\theta} &= \xi + \bar{\lambda},\end{aligned}$$

where  $\eta_l(\bar{\lambda})$  is the unique non-negative real number that solves the equation

$$\eta_l(\bar{\lambda}) + \left( \frac{\eta_l(\bar{\lambda})}{\bar{\lambda}w_l\beta_l} \right)^{\frac{1}{\beta_l-1}} = \|x_l\|_{\mathcal{H}_l}, \quad l \in L, \quad (5.13)$$

and  $\bar{\lambda} > 0$  is a solution of the equation

$$\sum_{r \in R} w_r \max\{\|x_r\|_{\mathcal{H}_r} - \lambda w_r, 0\} + \sum_{l \in L} w_l (\|x_l\|_{\mathcal{H}_l} - \eta_l(\lambda))^{\beta_l} = \lambda + \xi. \quad (5.14)$$

**Proof.** For given  $\xi \in \mathbb{R}$  and  $(x_1, \dots, x_n) \in \mathcal{H}_1 \times \dots \times \mathcal{H}_n$ , let us consider the following optimization problem

$$\min_{\substack{(y_1, \dots, y_n, \theta) \in \mathcal{H}_1 \times \dots \times \mathcal{H}_n \times \mathbb{R} \\ \sum_{i=1}^n w_i \|y_i\|_{\mathcal{H}_i}^{\beta_i} \leq \theta}} \left\{ \frac{1}{2}(\theta - \xi)^2 + \sum_{i=1}^n \frac{1}{2} \|y_i - x_i\|_{\mathcal{H}_i}^2 \right\}. \quad (5.15)$$

It is clear that in the situation when  $\sum_{i=1}^n w_i \|x_i\|_{\mathcal{H}_i}^{\beta_i} \leq \xi$ , i.e.  $(x_1, \dots, x_n, \xi) \in \text{epi } h$ , the unique solution of (5.15) is  $\bar{y}_i = x_i$ ,  $i = 1, \dots, n$ , and  $\bar{\theta} = \xi$ . Therefore, we consider in the following the non-trivial case where  $\sum_{i=1}^n w_i \|x_i\|_{\mathcal{H}_i}^{\beta_i} > \xi$ , i.e.  $(x_1, \dots, x_n, \xi) \notin \text{epi } h$ .

Let us now define the function  $f : \mathcal{H}_1 \times \dots \times \mathcal{H}_n \times \mathbb{R} \rightarrow \mathbb{R}$  by  $f(y_1, \dots, y_n, \theta) := (1/2)(\theta - \xi)^2 + \sum_{i=1}^n (1/2)\|y_i - x_i\|_{\mathcal{H}_i}^2$  and the function  $g : \mathcal{H}_1 \times \dots \times \mathcal{H}_n \times \mathbb{R} \rightarrow \mathbb{R}$  by  $g(y_1, \dots, y_n, \theta) := \sum_{i=1}^n w_i \|y_i\|_{\mathcal{H}_i}^{\beta_i} - \theta$ , then by [2, Proposition 26.18] there exists  $\bar{\lambda} \geq 0$ , such that for the unique solution  $(\bar{y}_1, \dots, \bar{y}_n, \bar{\theta})$  of (5.15) it holds

$$\nabla f(\bar{y}_1, \dots, \bar{y}_n, \bar{\theta}) \in -\bar{\lambda} \partial g(\bar{y}_1, \dots, \bar{y}_n, \bar{\theta}) \Leftrightarrow \begin{cases} \bar{y}_i - x_i \in -\bar{\lambda} \partial(w_i \|\cdot\|_{\mathcal{H}_i}^{\beta_i})(\bar{y}_i), \quad i = 1, \dots, n, \\ \bar{\theta} - \xi = \bar{\lambda}, \end{cases} \quad (5.16)$$

where  $\bar{\lambda}$  is the associated Lagrange multiplier of  $(\bar{y}_1, \dots, \bar{y}_n, \bar{\theta})$ . If  $\bar{\lambda} = 0$ , then one gets by (5.16) that  $\bar{y}_i = x_i$ ,  $i = 1, \dots, n$ , and  $\bar{\theta} = \xi$  and by the feasibility of the solution it follows that  $\sum_{i=1}^n w_i \|x_i\|_{\mathcal{H}_i}^{\beta_i} \leq \xi$ , which contradicts our assumption. Hence, it holds  $\bar{\lambda} > 0$  and by (5.8) and (5.16) we have

$$\begin{cases} x_i - \bar{y}_i \in \partial(\bar{\lambda} w_i \|\cdot\|_{\mathcal{H}_i}^{\beta_i})(\bar{y}_i), \quad i = 1, \dots, n, \\ \bar{\theta} = \bar{\lambda} + \xi, \end{cases} \Leftrightarrow \begin{cases} \bar{y}_i = \text{prox}_{\bar{\lambda} w_i \|\cdot\|_{\mathcal{H}_i}^{\beta_i}} x_i, \quad i = 1, \dots, n, \\ \bar{\theta} = \bar{\lambda} + \xi. \end{cases}$$

Further, from [29, Proposition 2.8] it follows for the case  $r \in R$ , i.e.  $\beta_r = 1$ , that

$$\bar{y}_r = \begin{cases} x_r - \frac{\bar{\lambda} w_r}{\|x_r\|_{\mathcal{H}_r}} x_r, & \text{if } \|x_r\|_{\mathcal{H}_r} > \bar{\lambda} w_r, \\ 0_{\mathcal{H}_r}, & \text{if } \|x_r\|_{\mathcal{H}_r} \leq \bar{\lambda} w_r \end{cases} = \frac{\max\{\|x_r\|_{\mathcal{H}_r} - \bar{\lambda} w_r, 0\}}{\|x_r\|_{\mathcal{H}_r}} x_r, \quad (5.17)$$

and for the case  $l \in L$ , i.e.  $\beta_l > 0$ , that

$$\bar{y}_l = x_l - \frac{\eta_l(\bar{\lambda})}{\|x_l\|_{\mathcal{H}_l}} x_l = \frac{\|x_l\|_{\mathcal{H}_l} - \eta_l(\bar{\lambda})}{\|x_l\|_{\mathcal{H}_l}} x_l, \quad (5.18)$$

where  $\eta_l(\bar{\lambda})$  is the unique non-negative real number that solves the following equation

$$\eta_l(\bar{\lambda}) + \left( \frac{\eta_l(\bar{\lambda})}{\bar{\lambda} w_l \beta_l} \right)^{\frac{1}{\beta_l - 1}} = \|x_l\|_{\mathcal{H}_l} \quad (5.19)$$

(notice that by (5.19) follows that  $\|x_l\|_{\mathcal{H}_l} - \eta_l(\bar{\lambda}) \geq 0$ ). Furthermore, the complementary slackness condition

$$\bar{\lambda} \left( \sum_{i=1}^n w_i \|\bar{y}_i\|_{\mathcal{H}_i}^{\beta_i} - \bar{\theta} \right) = 0 \quad (5.20)$$

implies that

$$\sum_{i=1}^n w_i \|\bar{y}_i\|_{\mathcal{H}_i}^{\beta_i} = \bar{\theta}, \quad (5.21)$$

and from here follows by (5.17) and (5.18) that

$$\sum_{i=1}^n w_i \|\bar{y}_i\|_{\mathcal{H}_i}^{\beta_i} = \sum_{r \in R} w_r \max\{\|x_r\|_{\mathcal{H}_r} - \bar{\lambda} w_r, 0\} + \sum_{l \in L} w_l (\|x_l\|_{\mathcal{H}_l} - \eta_l(\bar{\lambda}))^{\beta_l} = \bar{\lambda} + \xi. \quad \square \quad (5.22)$$

**Remark 5.1.** In the situation when  $\beta_i > 1$  for all  $i=1, \dots, n$ , we get by summarizing the formulae (5.13) and (5.14)

$$\begin{aligned} \eta_i(\lambda) + \left( \frac{\eta_i(\lambda)}{w_i \beta_i \left( \sum_{j=1}^n w_j (\|x_j\|_{\mathcal{H}_j} - \eta_j(\lambda))^{\beta_j} \right) - w_i \beta_i \xi} \right)^{\frac{1}{\beta_i - 1}} &= \|x_i\|_{\mathcal{H}_i} \\ \Leftrightarrow \frac{\eta_i(\lambda)}{w_i \beta_i \left( \sum_{j=1}^n w_j (\|x_j\|_{\mathcal{H}_j} - \eta_j(\lambda))^{\beta_j} \right) - w_i \beta_i \xi} &= (\|x_i\|_{\mathcal{H}_i} - \eta_i(\lambda))^{\beta_i - 1}, \quad i = 1, \dots, n. \end{aligned} \quad (5.23)$$

By setting  $\chi_i = \|x_i\|_{\mathcal{H}_i} - \eta_i(\lambda) \geq 0$ ,  $i = 1, \dots, n$ , formula (5.23) can be expressed by

$$\begin{aligned} \frac{\|x_i\|_{\mathcal{H}_i} - \chi_i}{w_i \beta_i \left( \sum_{j=1}^n w_j \chi_j^{\beta_j} \right) - w_i \beta_i \xi} &= \chi_i^{\beta_i - 1} \\ \Leftrightarrow w_i \beta_i \chi_i^{\beta_i - 1} \sum_{j=1}^n w_j \chi_j^{\beta_j} - \xi w_i \beta_i \chi_i^{\beta_i - 1} + \chi_i &= \|x_i\|_{\mathcal{H}_i} \\ \Leftrightarrow w_i^2 \beta_i \chi_i^{2\beta_i - 1} + w_i \beta_i \sum_{\substack{j=1 \\ j \neq i}}^n w_j \chi_j^{\beta_j} - \xi w_i \beta_i \chi_i^{\beta_i - 1} + \chi_i &= \|x_i\|_{\mathcal{H}_i}, \quad i = 1, \dots, n. \end{aligned}$$

Hence, it holds for every  $(x_1, \dots, x_n, \xi) \in \mathcal{H}_1 \times \dots \times \mathcal{H}_n \times \mathbb{R}$

$$P_{\text{epi } h}(x_1, \dots, x_n, \xi) = \begin{cases} (x_1, \dots, x_n, \xi), & \text{if } \sum_{i=1}^n w_i \|x_i\|_{\mathcal{H}_i}^{\beta_i} \leq \xi, \\ (\bar{y}_1, \dots, \bar{y}_n, \bar{\theta}), & \text{otherwise,} \end{cases}$$

with

$$\bar{y}_i = \frac{\bar{\chi}_i}{\|x_i\|_{\mathcal{H}_i}} x_i, \quad i = 1, \dots, n, \quad \text{and } \bar{\theta} = \sum_{i=1}^n w_i (\bar{\chi}_i)^{\beta_i},$$

where  $\bar{\chi}_i \geq 0$ ,  $i = 1, \dots, n$ , are the unique real numbers that solve a polynomial equation system of the form

$$w_i^2 \beta_i \bar{\chi}_i^{2\beta_i - 1} + w_i \beta_i \sum_{\substack{j=1 \\ j \neq i}}^n w_j \bar{\chi}_j^{\beta_j} - \xi w_i \beta_i \bar{\chi}_i^{\beta_i - 1} + \bar{\chi}_i = \|x_i\|_{\mathcal{H}_i}, \quad i = 1, \dots, n.$$

Let us additionally mention that the case where  $n = 1$  was considered for instance in [28].

An important consequence of Theorem 5.1 where  $\beta_i = 1$  for all  $i = 1, \dots, n$ , follows.

**Corollary 5.1.** *Let  $h$  be given by (5. 11) where  $\beta_i = 1$  for all  $i = 1, \dots, n$ . Then for all  $(x_1, \dots, x_n, \xi) \in \mathcal{H}_1 \times \dots \times \mathcal{H}_n \times \mathbb{R}$  it holds*

$$P_{\text{epi } h}(x_1, \dots, x_n, \xi) = \begin{cases} (x_1, \dots, x_n, \xi), & \text{if } \sum_{i=1}^n w_i \|x_i\|_{\mathcal{H}_i} \leq \xi, \\ (0_{\mathcal{H}_1}, \dots, 0_{\mathcal{H}_n}, 0), & \text{if } \xi < 0 \text{ and } \|x_i\|_{\mathcal{H}_i} \leq -\xi w_i, \ i = 1, \dots, n, \\ (\bar{y}_1, \dots, \bar{y}_n, \bar{\theta}), & \text{otherwise,} \end{cases} \quad (5. 24)$$

where

$$\bar{y}_i = \frac{\max\{\|x_i\|_{\mathcal{H}_i} - \bar{\lambda} w_i, 0\}}{\|x_i\|_{\mathcal{H}_i}} x_i, \ i = 1, \dots, n, \text{ and } \bar{\theta} = \xi + \bar{\lambda},$$

with

$$\bar{\lambda} = \frac{\sum_{i=k+1}^n w_i^2 \tau_i - \xi}{\sum_{i=k+1}^n w_i^2 + 1} \quad (5. 25)$$

and  $k \in \{0, 1, \dots, n-1\}$  is the unique integer such that  $\tau_k \leq \bar{\lambda} \leq \tau_{k+1}$ , where the values  $\tau_0, \dots, \tau_n$  are defined by  $\tau_0 := 0$  and  $\tau_i := \|x_i\|_{\mathcal{H}_i}/w_i$ ,  $i = 1, \dots, n$  and in ascending order.

**Proof.** As  $\beta_i = 1$  for all  $i = 1, \dots, n$ , Theorem 5.1 yields

$$P_{\text{epi } h}(x_1, \dots, x_n, \xi) = \begin{cases} (x_1, \dots, x_n, \xi), & \text{if } \sum_{i=1}^n w_i \|x_i\|_{\mathcal{H}_i} \leq \xi, \\ (\bar{y}_1, \dots, \bar{y}_n, \bar{\theta}), & \text{otherwise,} \end{cases}$$

with

$$\bar{y}_i = \frac{\max\{\|x_i\|_{\mathcal{H}_i} - \bar{\lambda} w_i, 0\}}{\|x_i\|_{\mathcal{H}_i}} x_i, \ i = 1, \dots, n, \text{ and } \bar{\theta} = \xi + \bar{\lambda},$$

where  $\bar{\lambda} > 0$  is a solution of the equation

$$\sum_{i=1}^n w_i \max\{\|x_i\|_{\mathcal{H}_i} - \lambda w_i, 0\} = \lambda + \xi.$$

Now, we consider the case where  $\sum_{i=1}^n w_i \|x_i\|_{\mathcal{H}_i} > \xi$  and distinguish two cases.

(a) Let  $\xi < 0$ . If  $\|x_i\|_{\mathcal{H}_i} + \xi w_i \leq 0$  for all  $i = 1, \dots, n$ , we have by  $0 \leq \bar{\theta} = \xi + \bar{\lambda}$ , i.e.  $\xi \geq -\bar{\lambda}$ , that

$$0 \geq \|x_i\|_{\mathcal{H}_i} + \xi w_i \geq \|x_i\|_{\mathcal{H}_i} - \bar{\lambda} w_i \ \forall i = 1, \dots, n, \quad (5. 26)$$

and from here follows that

$$\bar{\lambda} + \xi = \sum_{i=1}^n w_i \max\{\|x_i\|_{\mathcal{H}_i} - \bar{\lambda} w_i, 0\} = 0, \text{ i.e. } \bar{\lambda} = -\xi. \quad (5. 27)$$

But this means that  $(\bar{y}_1, \dots, \bar{y}_n, \bar{\theta}) = (0_{\mathcal{H}_1}, \dots, 0_{\mathcal{H}_n}, 0)$ , which verifies the second case of (5. 24).

If we now assume that there exists  $j \in \{1, \dots, n\}$  such that  $\|x_j\|_{\mathcal{H}_j} + \xi w_j > 0$ , then we define the function  $g : \mathbb{R} \rightarrow \mathbb{R}$  by

$$g(\lambda) := \sum_{i=1}^n w_i^2 \max\{\tau_i - \lambda, 0\} - \lambda - \xi. \quad (5. 28)$$

Moreover, this assumption yields

$$g(\lambda) = \sum_{i=1}^n w_i^2 \max\{\tau_i - \lambda, 0\} - \lambda - \xi < \sum_{i=1}^n w_i^2 \max\{\tau_i - \lambda, 0\} - \lambda + \frac{\|x_j\|_{\mathcal{H}_j}}{w_j}.$$

Now, we choose  $\tilde{\lambda} > 0$  such that  $\|x_i\|_{\mathcal{H}_i} - w_i \tilde{\lambda} < 0$  for all  $i = 1, \dots, n$ , and get

$$g(\tilde{\lambda}) < -\tilde{\lambda} + \frac{\|x_j\|_{\mathcal{H}_j}}{w_j} < 0.$$

(b) Let  $\xi \geq 0$ . If there exists  $j \in \{1, \dots, n\}$  such that  $\|x_j\|_{\mathcal{H}_j} + \xi w_j < 0$ , we derive a contradiction. Therefore, it holds  $\|x_i\|_{\mathcal{H}_i} + \xi w_i \geq 0$  for all  $i = 1, \dots, n$ , and for the function  $g$  we have

$$g(\lambda) = \sum_{i=1}^n w_i^2 \max\{\tau_i - \lambda, 0\} - \lambda - \xi \leq \sum_{i=1}^n w_i^2 \max\{\tau_i - \lambda, 0\} - \lambda.$$

Now, we can take  $\tilde{\lambda} > 0$  such that  $\|x_i\|_{\mathcal{H}_i} - w_i \tilde{\lambda} < 0$  for all  $i = 1, \dots, n$ , and derive that  $g(\tilde{\lambda}) \leq -\tilde{\lambda} < 0$ .

In summary, we can secure the existence of  $\tilde{\lambda} > 0$  such that  $g(\tilde{\lambda}) < 0$ . Additionally, take note that, if  $\lambda = 0$ , then  $g(0) = \sum_{i=1}^n w_i \|x_i\|_{\mathcal{H}_i} - \xi > 0$ . The rest of the proof is oriented on the Algorithm I given in [76] to determine the projection onto an  $l_1$ -norm ball.

Since, the values  $\tau_0, \dots, \tau_n$  are in ascending order,  $g$  is a piecewise linear function in  $\lambda$ , where the slope of  $g$  changes at  $\lambda = \tau_i$ ,  $i = 0, \dots, n$ . More precisely, at  $\lambda = 0$  the slope of  $g$  is  $-(\sum_{i=1}^n w_i^2 + 1)$  and increases by  $w_1^2$  when  $\lambda = \tau_1$ . If we proceed in this way, one may see that the slope keeps increasing when  $\lambda$  takes the values  $\tau_k$ ,  $k = 2, \dots, n$ . In the case when  $\lambda \geq \tau_n$  the slope of  $g$  is  $-1$ . Hence, to determine  $\lambda$  such that  $g(\lambda) = 0$ , we have to locate the interval where  $g$  changes its sign from a positive to a negative value. In other words, we have to find the unique integer  $k \in \{0, \dots, n-1\}$  such that  $g(\tau_k) \geq 0$  and  $g(\tau_{k+1}) \leq 0$ . Hence, we have

$$g(\lambda) = -\left(\sum_{i=k+1}^n w_i^2 + 1\right)\lambda + \sum_{i=k+1}^n w_i^2 \tau_i - \xi,$$

where  $\tau_k \leq \lambda \leq \tau_{k+1}$ . Finally, we can determine  $\bar{\lambda}$  such that  $g(\bar{\lambda}) = 0$ :

$$\bar{\lambda} = \frac{\sum_{i=k+1}^n w_i^2 \tau_i - \xi}{\sum_{i=k+1}^n w_i^2 + 1}.$$

□

**Remark 5.2.** From the ideas of the previous proof, we can now construct an algorithm to determine  $\bar{\lambda}$  of Corollary 5.1.

**Algorithm:**

- (i) If  $\sum_{i=1}^n w_i \|x_i\|_{\mathcal{H}_i} \leq \xi$ , then  $\bar{\lambda} = 0$ .
- (ii) If  $\xi < 0$  and  $\|x_i\|_{\mathcal{H}_i} \leq -\xi w_i$  for all  $i = 1, \dots, n$ , then  $\bar{\lambda} = -\xi$ .
- (iii) Otherwise, define  $\tau_0 := 0$ ,  $\tau_i := \|x_i\|_{\mathcal{H}_i} / w_i$ ,  $i = 1, \dots, n$ , and sort  $\tau_0, \dots, \tau_n$  in ascending order.
- (iv) Determine the values of  $g$  defined in (5. 28) at  $\lambda = \tau_i$ ,  $i = 0, \dots, n$ .
- (v) Find the unique  $k \in \{0, \dots, n-1\}$  such that  $g(\tau_k) \geq 0$  and  $g(\tau_{k+1}) \leq 0$ .
- (vi) Calculate  $\bar{\lambda}$  by (5. 25).

**Corollary 5.2.** *Let  $h$  be given by (5. 11) where  $\beta_i = 2$  and  $w_i = 1$  for all  $i = 1, \dots, n$ , then it holds*

$$P_{\text{epi } h}(x_1, \dots, x_n, \xi) = \begin{cases} (x_1, \dots, x_n, \xi), & \text{if } \sum_{i=1}^n \|x_i\|_{\mathcal{H}_i}^2 \leq \xi, \\ (\bar{y}_1, \dots, \bar{y}_n, \bar{\theta}), & \text{otherwise,} \end{cases}$$

where

$$\bar{y}_i = \frac{1}{2\bar{\lambda} + 1} x_i, \quad i = 1, \dots, n, \quad \text{and } \bar{\theta} = \xi + \bar{\lambda},$$

and  $\bar{\lambda} > 0$  is a solution of a cubic equation of the form

$$\lambda^3 + (1 + \xi)\lambda^2 + \frac{1}{4}(1 + 4\xi)\lambda + \frac{1}{4} \left( \xi - \sum_{i=1}^n \|x_i\|_{\mathcal{H}_i}^2 \right) = 0. \quad (5. 29)$$

**Proof.** By Theorem 5.1 we get that

$$P_{\text{epi } h}(x_1, \dots, x_n, \xi) = \begin{cases} (x_1, \dots, x_n, \xi), & \text{if } \sum_{i=1}^n \|x_i\|_{\mathcal{H}_i}^2 \leq \xi, \\ (\bar{y}_1, \dots, \bar{y}_n, \bar{\theta}), & \text{otherwise,} \end{cases}$$

with

$$\bar{y}_i = \frac{\|x_i\|_{\mathcal{H}_i} - \eta_i(\bar{\lambda})}{\|x_i\|_{\mathcal{H}_i}} x_i, \quad i = 1, \dots, n, \quad \text{and } \bar{\theta} = \xi + \bar{\lambda}, \quad (5. 30)$$

where  $\eta_i(\bar{\lambda})$  is the unique non-negative real number that solves the equation

$$\eta_i(\bar{\lambda}) + \frac{\eta_i(\bar{\lambda})}{2\bar{\lambda}} = \|x_i\|_{\mathcal{H}_i}, \quad i = 1, \dots, n, \quad (5. 31)$$

and  $\bar{\lambda} > 0$  is a solution of the equation

$$\sum_{i=1}^n (\|x_i\|_{\mathcal{H}_i} - \eta_i(\bar{\lambda}))^2 = \bar{\lambda} + \xi. \quad (5. 32)$$

From (5. 31) we get immediately

$$\eta_i(\bar{\lambda}) \left( 1 + \frac{1}{2\bar{\lambda}} \right) = \|x_i\|_{\mathcal{H}_i} \Leftrightarrow \eta_i(\bar{\lambda}) = \frac{2\bar{\lambda}}{2\bar{\lambda} + 1} \|x_i\|_{\mathcal{H}_i}, \quad i = 1, \dots, n, \quad (5. 33)$$

and in combination with (5. 32) we derive

$$\begin{aligned} \sum_{i=1}^n \left( \|x_i\|_{\mathcal{H}_i} - \frac{2\bar{\lambda}}{2\bar{\lambda} + 1} \|x_i\|_{\mathcal{H}_i} \right)^2 &= \bar{\lambda} + \xi \Leftrightarrow \frac{1}{(2\bar{\lambda} + 1)^2} \sum_{i=1}^n \|x_i\|_{\mathcal{H}_i}^2 = \bar{\lambda} + \xi \\ \Leftrightarrow (2\bar{\lambda} + 1)^2 (\bar{\lambda} + \xi) - \sum_{i=1}^n \|x_i\|_{\mathcal{H}_i}^2 &= 0 \Leftrightarrow 4\bar{\lambda}^3 + 4(1 + \xi)\bar{\lambda}^2 + (1 + 4\xi)\bar{\lambda} + \xi - \sum_{i=1}^n \|x_i\|_{\mathcal{H}_i}^2 = 0. \end{aligned}$$

In the end, formula (5. 33) implies that

$$\bar{y}_i = \frac{\|x_i\|_{\mathcal{H}_i} - \frac{2\bar{\lambda}}{2\bar{\lambda} + 1} \|x_i\|_{\mathcal{H}_i}}{\|x_i\|_{\mathcal{H}_i}} x_i = \frac{1}{2\bar{\lambda} + 1} x_i, \quad i = 1, \dots, n, \quad (5. 34)$$

which completes the proof.  $\square$

The next remark discusses the question whether the solution  $\bar{\lambda} > 0$  of Corollary 5.2 is unique.

**Remark 5.3.** Let  $(x_1, \dots, x_n, \xi) \in \mathcal{H}_1 \times \dots \times \mathcal{H}_n \times \mathbb{R}$  be such that  $\sum_{i=1}^n \|x_i\|_{\mathcal{H}_i}^2 > \xi$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $g(\lambda) := \lambda^3 + (1 + \xi)\lambda^2 + (1/4)(1 + 4\xi)\lambda + (1/4)(\xi - \sum_{i=1}^n \|x_i\|_{\mathcal{H}_i}^2)$ , then  $g'(\lambda) = 3\lambda^2 + 2(1 + \xi)\lambda + (1/4)(1 + 4\xi)$  as well as  $g''(\lambda) = 6\lambda + 2(1 + \xi)$ . From the zeros of  $g'$  we derive the local extrema of  $g$  as follows

$$\begin{aligned} \lambda_{1/2} &= -\frac{1}{3}(1 + \xi) \pm \sqrt{\frac{(1 + \xi)^2}{9} - \frac{1 + 4\xi}{12}} = -\frac{1}{3}(1 + \xi) \pm \sqrt{\frac{4(1 + 2\xi + \xi^2) - 3(1 + 4\xi)}{36}} \\ &= -\frac{1}{3}(1 + \xi) \pm \sqrt{\frac{1 - 4\xi + 4\xi^2}{36}} = -\frac{1}{3}(1 + \xi) \pm \frac{1}{6}(1 - 2\xi) \end{aligned}$$

and hence,  $\lambda_1 = -(1/6)(1 + 4\xi)$  and  $\lambda_2 = -(1/2)$ .

Further, if  $\xi > 1/2 \Leftrightarrow -1 + 2\xi > 0$ , then  $g$  is strongly monotone increasing on  $\mathbb{R}_+$ ,  $g''(\lambda_1) = 1 - 2\xi < 0$  and  $g''(\lambda_2) = -1 + 2\xi > 0$ , which means that  $g$  has in  $\lambda_1$  a local maximum and in  $\lambda_2$  a local minimum. As  $\lambda_1 < \lambda_2 < 0$  and  $g(0) = (1/4)(\xi - \sum_{i=1}^n \|x_i\|_{\mathcal{H}_i}^2) < 0$ , the function  $g$  has exactly one positive zero in this situation.

If  $\xi < 1/2 \Leftrightarrow 1 - 2\xi > 0$ , then  $g''(\lambda_1) = 1 - 2\xi > 0$  and  $g''(\lambda_2) = -1 + 2\xi < 0$  and we derive a local minimum in  $\lambda_1$  and a local maximum in  $\lambda_2$ . From  $g(0) < 0$  and  $\lambda_2 < \lambda_1$  we conclude that  $g$  has also in this situation exactly one positive zero.

Finally, let us consider the case where  $\xi = 1/2$ , then  $g$  is strongly monotone increasing on  $\mathbb{R}_+$ ,  $\lambda_1 = \lambda_2 = -1/2$  and  $g''(\lambda_1) = 0$ , i.e.  $g$  has at the point  $-(1/2)$  a saddle point. From the fact that  $g''(\lambda) \leq 0$  for all  $\lambda \in (-\infty, -(1/2)]$  and  $g''(\lambda) > 0$  for all  $\lambda \in (-(1/2), +\infty)$ , it is clear that  $g$  has again exactly one positive zero.

In conclusion, the function  $g$  has in all situations exactly one positive zero, i.e.  $\bar{\lambda} > 0$  is unique.

**Remark 5.4.** In the framework of Corollary 5.2, let us consider the case where  $n = 1$ . Then, by Remark 5.1 we have to find a real number  $\bar{\chi} \geq 0$  that solves the equation

$$2\chi^3 + (1 - 2\xi)\chi - \|x\|_{\mathcal{H}} = 0, \quad (5.35)$$

to get a formula of the projection onto the epigraph of  $h$ .

As one may see by (5.29), the arithmetic effort for the case  $n > 1$  is not much higher compared to the case  $n = 1$ . In both situations we have to solve a cubic equation to derive a formula for the projection onto the epigraph of  $h$ .

As a direct consequence of Corollary 5.1 one gets the following well-known statement (see for instance [2] or [28]).

**Corollary 5.3.** Let  $h$  be given by (5.11) where  $n = 1$ ,  $w_1 = w \geq 1$  and  $\beta_1 = 1$ , i.e.  $h(x) = w\|x\|_{\mathcal{H}}$ . Then, for every  $(x, \xi) \in \mathcal{H} \times \mathbb{R}$

$$\text{P}_{\text{epi } w\|\cdot\|_{\mathcal{H}}}(x, \xi) = \begin{cases} (x, \xi), & \text{if } w\|x\|_{\mathcal{H}} \leq \xi, \\ (0, 0), & \text{if } \|x\|_{\mathcal{H}} \leq -w\xi, \\ \left( \frac{\|x\|_{\mathcal{H}} + w\xi}{\|x\|_{\mathcal{H}}(w^2 + 1)}y, \frac{w\|x\|_{\mathcal{H}} + w^2\xi}{w^2 + 1} \right), & \text{otherwise.} \end{cases}$$

For our numerical tests we need two lemmas more.

**Lemma 5.2.** For  $p_i \in \mathcal{H}$ ,  $i = 1, \dots, n$ , it holds

$$\text{P}_{\text{epi} \left( \sum_{i=1}^n w_i \|\cdot - p_i\|_{\mathcal{H}_i}^{\beta_i} \right)}(x_1, \dots, x_n, \xi) = \text{P}_{\text{epi} \left( \sum_{i=1}^n w_i \|\cdot\|_{\mathcal{H}_i}^{\beta_i} \right)}(x_1 - p_1, \dots, x_n - p_n, \xi) + (p_1, \dots, p_n, 0).$$



**Proof.** For  $p_i \in \mathcal{H}_i$ ,  $i = 1, \dots, n$  one has

$$\begin{aligned} (x_1, \dots, x_n, \xi) &\in \text{epi} \left( \sum_{i=1}^n w_i \|\cdot - p_i\|_{\mathcal{H}_i}^{\beta_i} \right) \Leftrightarrow \sum_{i=1}^n w_i \|x_i - p_i\|_{\mathcal{H}_i}^{\beta_i} \leq \xi \\ &\Leftrightarrow (x_1 - p_1, \dots, x_n - p_n, \xi) \in \text{epi} \left( \sum_{i=1}^n w_i \|\cdot\|_{\mathcal{H}_i}^{\beta_i} \right) \\ &\Leftrightarrow (x_1, \dots, x_n, \xi) \in \text{epi} \left( \sum_{i=1}^n w_i \|\cdot\|_{\mathcal{H}_i}^{\beta_i} \right) + (p_1, \dots, p_n, 0). \end{aligned}$$

Thus, by [2, Proposition 3.17] follows

$$\begin{aligned} &P_{\text{epi} \left( \sum_{i=1}^n w_i \|\cdot - p_i\|_{\mathcal{H}_i}^{\beta_i} \right)}(x_1, \dots, x_n, \xi) = P_{\text{epi} \left( \sum_{i=1}^n w_i \|\cdot\|_{\mathcal{H}_i}^{\beta_i} \right) + (p_1, \dots, p_n, 0)}(x_1, \dots, x_n, \xi) \\ &= P_{\text{epi} \left( \sum_{i=1}^n w_i \|\cdot\|_{\mathcal{H}_i}^{\beta_i} \right)}(x_1 - p_1, \dots, x_n - p_n, \xi) + (p_1, \dots, p_n, 0). \end{aligned}$$

□

**Lemma 5.3.** Let  $w > 0$  and  $A : \mathcal{H} \rightarrow \mathcal{K}$  be a linear operator with  $AA^* = \mu \text{Id}$ ,  $\mu > 0$ , where  $\mathcal{K}$  is a real Hilbert space. Then,

$$P_{\text{epi } w\|A\cdot\|_{\mathcal{H}}}(x, \xi) = (x, \xi) + \frac{1}{\mu}(A^* \times \text{Id}) \left( P_{\text{epi } w\|\cdot\|_{\mathcal{H}}}(Ax, \xi) - (Ax, \xi) \right).$$

**Proof.** We have

$$\delta_{\text{epi}(w\|A\cdot\|_{\mathcal{H}})}(x, \xi) = \delta_{\text{epi}(w\|\cdot\|_{\mathcal{H}})}(Ax, \xi) = (\delta_{\text{epi}(w\|\cdot\|_{\mathcal{H}})} \circ (A \times \text{Id}))(x, \xi).$$

By [2, Proposition 23.32] it follows that

$$\begin{aligned} &\text{prox}_{\delta_{\text{epi } w\|A\cdot\|_{\mathcal{H}}}}(x, \xi) = \text{prox}_{\delta_{\text{epi } w\|\cdot\|_{\mathcal{H}}} \circ (A \times \text{Id})}(x, \xi) \\ &= (x, \xi) + \frac{1}{\mu}(A \times \text{Id})^T \left( \text{prox}_{\delta_{\text{epi } w\|\cdot\|_{\mathcal{H}}}}(Ax, \xi) - (Ax, \xi) \right) \\ &\Leftrightarrow P_{\text{epi } w\|A\cdot\|_{\mathcal{H}}}(x, \xi) = (x, \xi) + \frac{1}{\mu}(A^* \times \text{Id}) \left( P_{\text{epi } w\|\cdot\|_{\mathcal{H}}}(Ax, \xi) - (Ax, \xi) \right). \quad \square \end{aligned}$$

## 5.2.2 Gauges

The next considerations are devoted to gauge functions of closed convex sets defined on Hilbert spaces.

**Theorem 5.2.** Let  $C$  be a closed convex subset of  $\mathcal{H}$  such that  $0_{\mathcal{H}} \in C$ , then it holds for every  $(x, \xi) \in \mathcal{H} \times \mathbb{R}$

$$P_{\text{epi } \gamma_C}(x, \xi) = \begin{cases} (x, \xi), & \text{if } \gamma_C(x) \leq \xi, \\ (P_{\text{cl}(\text{dom } \gamma_C)}(x), \xi), & \text{if } x \notin \text{dom } \gamma_C \text{ and } \gamma_C(P_{\text{cl}(\text{dom } \gamma_C)}(x)) \leq \xi < \gamma_C(x), \\ (\bar{y}, \bar{\theta}), & \text{otherwise,} \end{cases}$$

where

$$\bar{y} = x - \bar{\lambda} P_{C^0} \left( \frac{1}{\bar{\lambda}} x \right) \text{ and } \bar{\theta} = \bar{\lambda} + \xi$$

and  $\bar{\lambda} > 0$  is a solution of an equation of the form

$$\lambda + \xi = \left\langle x, P_{C^0} \left( \frac{1}{\lambda} x \right) \right\rangle_{\mathcal{H}} - \lambda \left\| P_{C^0} \left( \frac{1}{\lambda} x \right) \right\|_{\mathcal{H}}^2.$$

**Proof.** Let us consider for fixed  $(x, \xi) \in \mathcal{H} \times \mathbb{R}$  the following optimization problem

$$\min_{\substack{(y, \theta) \in \mathcal{H} \times \mathbb{R}, \\ \gamma_C(y) \leq \theta}} \left\{ \frac{1}{2}(\theta - \xi)^2 + \frac{1}{2}\|y - x\|_{\mathcal{H}}^2 \right\}. \quad (5.36)$$

If  $\gamma_C(x) \leq \xi$ , i.e.  $(x, \xi) \in \text{epi } \gamma_C$ , then it is obvious that  $(\bar{y}, \bar{\theta}) = (x, \xi)$ . In the following we consider the non-trivial situation where  $\gamma_C(x) > \xi$ .

We define the function  $f : \mathcal{H} \times \mathbb{R} \rightarrow \mathbb{R}$  by  $f(y, \theta) := (1/2)(\theta - \xi)^2 + (1/2)\|y - x\|_{\mathcal{H}}^2$  and the function  $g : \mathcal{H} \times \mathbb{R} \rightarrow \overline{\mathbb{R}}$  by  $g(y, \theta) = \gamma_C(y) - \theta$ , then it is clear that  $f$  is continuous and strongly convex and  $g$  is proper, lower semicontinuous and convex by Theorem 4.1. As  $\gamma_C(0) < 1$ , it follows by [7, Theorem 3.3.16] (see also [7, Remark 3.3.8]) that

$$0 \in \partial(f + (\bar{\lambda}g))(\bar{y}, \bar{\theta}) \quad (5.37)$$

and

$$\begin{cases} (\bar{\lambda}g)(\bar{y}, \bar{\theta}) = 0, \\ g(\bar{y}, \bar{\theta}) \leq 0, \end{cases} \Leftrightarrow \begin{cases} \bar{\lambda}(\gamma_C(\bar{y}) - \bar{\theta}) = 0, \\ \gamma_C(\bar{y}) \leq \bar{\theta}, \end{cases} \quad (5.38)$$

where  $(\bar{y}, \bar{\theta})$  is the unique solution of (5.36) and  $\bar{\lambda} \geq 0$  the associated Lagrange multiplier. Furthermore, from [7, Theorem 3.5.13] one gets that

$$0 \in \partial(f + (\bar{\lambda}g))(\bar{y}, \bar{\theta}) \Leftrightarrow 0 \in \partial f(\bar{y}, \bar{\theta}) + \partial(\bar{\lambda}g)(\bar{y}, \bar{\theta}). \quad (5.39)$$

If  $\bar{\lambda} = 0$ , then it follows by (5.8) and (5.3)

$$\begin{aligned} 0 \in \partial f(\bar{y}, \bar{\theta}) + \partial \delta_{\text{dom } g}(\bar{y}, \bar{\theta}) &\Leftrightarrow 0 \in (\bar{y} - x, \bar{\theta} - \xi) + \partial \delta_{\text{dom } \gamma_C \times \mathbb{R}}(\bar{y}, \bar{\theta}) \\ \Leftrightarrow 0 \in (\bar{y} - x, \bar{\theta} - \xi) + \partial \delta_{\text{cl}(\text{dom } \gamma_C) \times \mathbb{R}}(\bar{y}, \bar{\theta}) &\Leftrightarrow (x - \bar{y}, \xi - \bar{\theta}) \in \partial \delta_{\text{cl}(\text{dom } \gamma_C) \times \mathbb{R}}(\bar{y}, \bar{\theta}) \\ \Leftrightarrow (\bar{y}, \bar{\theta}) = P_{\text{cl}(\text{dom } \gamma_C) \times \mathbb{R}}(x, \xi) &\Leftrightarrow \begin{cases} \bar{y} = P_{\text{cl}(\text{dom } \gamma_C)}(x), \\ \bar{\theta} = \xi, \end{cases} \end{aligned}$$

and thus, it holds by the feasibility condition (5.38) that  $\gamma_C(P_{\text{cl}(\text{dom } \gamma_C)}(x)) \leq \xi$ , from which follows that  $P_{\text{cl}(\text{dom } \gamma_C)}(x) \in \text{dom } \gamma_C$ . If  $x \in \text{dom } \gamma_C$ , this means that  $P_{\text{cl}(\text{dom } \gamma_C)}(x) = x$  and again by the feasibility condition (5.38) that  $\gamma_C(x) \leq \xi$ , which contradicts our assumption. Therefore, if  $x \notin \text{dom } \gamma_C$  and the inequalities  $\gamma_C(P_{\text{cl}(\text{dom } \gamma_C)}(x)) \leq \xi < \gamma_C(x)$  hold, then  $(\bar{y}, \bar{\theta}) = (P_{\text{cl}(\text{dom } \gamma_C)}(x), \xi)$ .

Now, let  $\bar{\lambda} > 0$ , then it follows from (5.39) and (5.8)

$$\begin{aligned} 0 \in \partial(f + (\bar{\lambda}g))(\bar{y}, \bar{\theta}) &\Leftrightarrow 0 \in \partial f(\bar{y}, \bar{\theta}) + \bar{\lambda} \partial g(\bar{y}, \bar{\theta}) \\ \Leftrightarrow \nabla f(\bar{y}, \bar{\theta}) \in -\bar{\lambda} \partial g(\bar{y}, \bar{\theta}) &\Leftrightarrow \begin{cases} \bar{y} - x \in -\bar{\lambda} \partial \gamma_C(\bar{y}), \\ \bar{\theta} - \xi = \bar{\lambda}, \end{cases} \Leftrightarrow \begin{cases} \bar{y} = \text{prox}_{\bar{\lambda} \gamma_C} x, \\ \bar{\theta} = \xi - \bar{\lambda}, \end{cases} \end{aligned} \quad (5.40)$$

by combining (5.40) and (5.38) we derive that  $\gamma_C(\bar{y}) = \xi + \bar{\lambda}$ . Finally, as by Lemma 4.3 and Remark 4.5 it holds that  $\gamma_C^* = \delta_{C^0}$ , one gets by [2, Theorem 14.3(iii)] the following equivalences

$$\begin{aligned} \gamma_C(\bar{y}) &= \xi + \bar{\lambda} \quad (5.41) \\ \Leftrightarrow \xi + \bar{\lambda} &= \gamma_C \left( \text{prox}_{\bar{\lambda} \gamma_C} x \right) + \delta_{C^0} \left( P_{C^0} \left( \frac{1}{\bar{\lambda}} x \right) \right) = \left\langle \text{prox}_{\bar{\lambda} \gamma_C} x, P_{C^0} \left( \frac{1}{\bar{\lambda}} x \right) \right\rangle_{\mathcal{H}} \\ \Leftrightarrow \xi + \bar{\lambda} &= \left\langle x - \bar{\lambda} P_{C^0} \left( \frac{1}{\bar{\lambda}} x \right), P_{C^0} \left( \frac{1}{\bar{\lambda}} x \right) \right\rangle_{\mathcal{H}}. \quad \square \end{aligned}$$

**Corollary 5.4.** Let  $C \subseteq \mathcal{H}$  be a closed convex cone, then

$$P_{\text{epi } \gamma_C}(x, \xi) = \begin{cases} (x, \xi), & \text{if } \gamma_C(x) \leq \xi, \\ (P_{\text{cl}(\text{dom } \gamma_C)}(x), \xi), & \text{if } x \notin \text{dom } \gamma_C \text{ and } \gamma_C(P_{\text{cl}(\text{dom } \gamma_C)}(x)) \leq \xi < \gamma_C(x), \\ (P_C x, \gamma_C(P_C x)), & \text{otherwise.} \end{cases}$$

**Proof.** We use Theorem 5.2. Let  $x \in \text{dom } \gamma_C$  such that  $\gamma_C(x) > \xi$ , then one has from [2, Proposition 28.22] and [2, Theorem 6.29] that

$$\bar{y} = x - \bar{\lambda} P_{C^0} \left( \frac{1}{\bar{\lambda}} x \right) = x - P_{C^0} x = P_C x. \quad (5.42)$$

Moreover, by (5.38) we have  $\gamma_C(\bar{y}) = \bar{\theta}$ , which finally yields  $P_{\text{epi}_{\gamma_C}}(x, \xi) = (P_C x, \gamma_C(P_C x))$ .  $\square$

**Corollary 5.5.** *Let  $C_i$  be a closed convex subset of  $\mathcal{H}_i$  such that  $0_{\mathcal{H}_i} \in \text{int } C_i$ ,  $i = 1, \dots, n$ , and the gauge  $\gamma_C : \mathcal{H}_1 \times \dots \times \mathcal{H}_n \rightarrow \bar{\mathbb{R}}$  be defined by  $\gamma_C(x_1, \dots, x_n) = \sum_{i=1}^n \gamma_{C_i}(x_i)$ . Then it holds for every  $(x_1, \dots, x_n, \xi) \in \mathcal{H}_1 \times \dots \times \mathcal{H}_n \times \mathbb{R}$*

$$P_{\text{epi}_{\gamma_C}}(x_1, \dots, x_n, \xi) = \begin{cases} (x_1, \dots, x_n, \xi), & \text{if } \sum_{i=1}^n \gamma_{C_i}(x_i) \leq \xi, \\ (\bar{y}_1, \dots, \bar{y}_n, \bar{\theta}), & \text{otherwise,} \end{cases}$$

where

$$\bar{y}_i = x_i - \bar{\lambda} P_{C_i^0} \left( \frac{1}{\bar{\lambda}} x_i \right), \quad i = 1, \dots, n, \quad \text{and } \bar{\theta} = \bar{\lambda} + \xi \quad (5.43)$$

and  $\bar{\lambda} > 0$  is a solution of an equation of the form

$$\lambda + \xi = \sum_{i=1}^n \left[ \left\langle x_i, P_{C_i^0} \left( \frac{1}{\lambda} x_i \right) \right\rangle_{\mathcal{H}_i} - \lambda \left\| P_{C_i^0} \left( \frac{1}{\lambda} x_i \right) \right\|_{\mathcal{H}_i}^2 \right]. \quad (5.44)$$

**Proof.** As  $0_{\mathcal{H}_i} \in \text{int } C_i$ ,  $i = 1, \dots, n$ , it is clear that the gauges are well-defined, i.e.  $\text{dom } \gamma_{C_i} = \mathcal{H}_i$ ,  $i = 1, \dots, n$ , and so,  $\text{dom } \gamma_C = \mathcal{H}_1 \times \dots \times \mathcal{H}_n$ . Further, let us recall that the polar set  $C^0$  of the set  $C$  can be characterized by the dual gauge  $\gamma_{C^0}$  as

$$C^0 = \{x = (x_1, \dots, x_n) \in \mathcal{H}_1 \times \dots \times \mathcal{H}_n : \gamma_{C^0}(x) = \gamma_{C^0}(x_1, \dots, x_n) \leq 1\}. \quad (5.45)$$

This relation holds also for the polar set  $C_i^0$  and its associated dual gauge  $\gamma_{C_i^0}$ ,  $i = 1, \dots, n$ . Moreover, in Lemma 4.4 it was shown that  $\gamma_{C^0}(x) = \max_{1 \leq i \leq n} \{\gamma_{C_i^0}(x_i)\}$  and hence, the polar set in (5.45) can be written as

$$\begin{aligned} C^0 &= \left\{ (x_1, \dots, x_n) \in \mathcal{H}_1 \times \dots \times \mathcal{H}_n : \max_{1 \leq i \leq n} \{\gamma_{C_i^0}(x_i)\} \leq 1 \right\} \\ &= \left\{ (x_1, \dots, x_n) \in \mathcal{H}_1 \times \dots \times \mathcal{H}_n : \gamma_{C_i^0}(x_i) \leq 1, \quad i = 1, \dots, n \right\} \\ &= \{x_1 \in \mathcal{H}_1 : \gamma_{C_1^0}(x_1) \leq 1\} \times \dots \times \{x_n \in \mathcal{H}_n : \gamma_{C_n^0}(x_n) \leq 1\} = C_1^0 \times \dots \times C_n^0. \end{aligned}$$

From here follows that

$$P_{C^0}(x) = P_{C_1^0 \times \dots \times C_n^0}(x_1, \dots, x_n) = P_{C_1^0}(x_1) \times \dots \times P_{C_n^0}(x_n),$$

which by using Theorem 5.2 directly implies (5.43) and (5.44).  $\square$

As one may see, the equation (5.44) of the previous corollary can be very hard to solve and hence, it can be very complicated to find a projection formula. The next two corollaries are examples, which demonstrate how one can determine the formula of the projector by using Corollary 5.5.

**Corollary 5.6.** *Let  $\gamma_C : \mathcal{H}_1 \times \dots \times \mathcal{H}_n \rightarrow \mathbb{R}$  be defined by*

$$\gamma_C(x_1, \dots, x_n) := \max_{1 \leq i \leq n} \{\|x_i\|_{\mathcal{H}_i}\} + \|x_{n+1}\|_{\mathcal{H}_{n+1}},$$

then for every  $(x_1, \dots, x_n, \xi) \in \mathcal{H}_1 \times \dots \times \mathcal{H}_n \times \mathbb{R}$

$$P_{\text{epi } \gamma_C}(x_1, \dots, x_{n+1}, \xi) = \begin{cases} (x_1, \dots, x_{n+1}, \xi), & \text{if } \max_{1 \leq i \leq n} \{\|x_i\|_{\mathcal{H}_i}\} + \|x_{n+1}\|_{\mathcal{H}_{n+1}} \leq \xi, \\ (\bar{y}_1, \dots, \bar{y}_{n+1}, \bar{\theta}), & \text{otherwise,} \end{cases}$$

where exactly one of the following four cases holds:

- (i)  $\sum_{i=1}^n \max \{\|x_i\|_{\mathcal{H}_i} - \|x_{n+1}\|_{\mathcal{H}_{n+1}} - \xi, 0\} < \|x_{n+1}\|_{\mathcal{H}_{n+1}},$   
 $\sum_{i=1}^n \max \{\|x_i\|_{\mathcal{H}_i} - \|x_{n+1}\|_{\mathcal{H}_{n+1}} - \xi, 0\} < \sum_{i=1}^n \|x_i\|_{\mathcal{H}_i},$   
 $\sum_{i=1}^n \|x_i\|_{\mathcal{H}_i} > (\|x_{n+1}\|_{\mathcal{H}_{n+1}} - \xi)/2 > -\xi$  and  $\|x_{n+1}\|_{\mathcal{H}_{n+1}} > -\xi$ , then

$$\bar{y}_i = x_i - \frac{\max \{\|x_i\|_{\mathcal{H}_i} - 2\bar{\lambda} - \xi + \|x_{n+1}\|_{\mathcal{H}_{n+1}}, 0\}}{\|x_i\|_{\mathcal{H}_i}} x_i, \quad i = 1, \dots, n,$$

$$\bar{y}_{n+1} = \frac{\|x_{n+1}\|_{\mathcal{H}_{n+1}} - \bar{\lambda}}{\|x_{n+1}\|_{\mathcal{H}_{n+1}}} x_{n+1} \quad \text{and} \quad \bar{\theta} = \bar{\lambda} + \xi,$$

where  $\bar{\lambda} > 0$  is a solution of the equation

$$\sum_{i=1}^n \max \{\|x_i\|_{\mathcal{H}_i} - 2\bar{\lambda} - \xi + \|x_{n+1}\|_{\mathcal{H}_{n+1}}, 0\} = \bar{\lambda}.$$

- (ii)  $\|x_{n+1}\|_{\mathcal{H}_{n+1}} \leq \sum_{i=1}^n \max \{\|x_i\|_{\mathcal{H}_i} - \|x_{n+1}\|_{\mathcal{H}_{n+1}} - \xi, 0\} < \sum_{i=1}^n \|x_i\|_{\mathcal{H}_i}$  and  
 $\sum_{i=1}^n \|x_i\|_{\mathcal{H}_i} > -\xi$ , then

$$\bar{y}_i = x_i - \frac{\max \{\|x_i\|_{\mathcal{H}_i} - \bar{\lambda} - \xi, 0\}}{\|x_i\|_{\mathcal{H}_i}} x_i, \quad i = 1, \dots, n, \quad \bar{y}_{n+1} = 0_{\mathcal{H}_{n+1}} \quad \text{and} \quad \bar{\theta} = \bar{\lambda} + \xi,$$

where  $\bar{\lambda} > 0$  is a solution of the equation

$$\sum_{i=1}^n \max \{\|x_i\|_{\mathcal{H}_i} - \bar{\lambda} - \xi, 0\} = \bar{\lambda}.$$

- (iii)  $\sum_{i=1}^n \|x_i\|_{\mathcal{H}_i} \leq (\|x_{n+1}\|_{\mathcal{H}_{n+1}} - \xi)/2 < \|x_{n+1}\|_{\mathcal{H}_{n+1}}$  and  $\|x_{n+1}\|_{\mathcal{H}_{n+1}} > -\xi$ , then

$$\bar{y}_i = 0_{\mathcal{H}_i}, \quad i = 1, \dots, n, \quad \bar{y}_{n+1} = \frac{\|x_{n+1}\|_{\mathcal{H}_{n+1}} + \xi}{2\|x_{n+1}\|_{\mathcal{H}_{n+1}}} x_{n+1} \quad \text{and} \quad \bar{\theta} = \frac{\|x_{n+1}\|_{\mathcal{H}_{n+1}} + \xi}{2}.$$

- (iv)  $\sum_{i=1}^n \|x_i\|_{\mathcal{H}_i} \leq -\xi$  and  $\|x_{n+1}\|_{\mathcal{H}_{n+1}} \leq -\xi$ , then  $\bar{y}_i = 0_{\mathcal{H}_i}$ ,  $i = 1, \dots, n+1$ , and  $\bar{\theta} = 0$ .

**Proof.** By Corollary 5.5 we have

$$P_{\text{epi } \gamma_C}(x_1, \dots, x_{n+1}, \xi) = \begin{cases} (x_1, \dots, x_{n+1}, \xi), & \text{if } \max_{1 \leq i \leq n} \{\|x_i\|_{\mathcal{H}_i}\} + \|x_{n+1}\|_{\mathcal{H}_{n+1}} \leq \xi, \\ (\bar{y}_1, \dots, \bar{y}_{n+1}, \bar{\theta}), & \text{otherwise,} \end{cases}$$

where

$$(\bar{y}_1, \dots, \bar{y}_n) = (x_1, \dots, x_n) - \bar{\lambda} P_{C_1^0} \left( \frac{1}{\bar{\lambda}} (x_1, \dots, x_n) \right), \quad \bar{y}_{n+1} = x_{n+1} - \bar{\lambda} P_{C_2^0} \left( \frac{1}{\bar{\lambda}} x_{n+1} \right),$$

$$\bar{\theta} = \bar{\lambda} + \xi \quad \text{and} \quad \bar{\lambda} > 0.$$

From Lemma 4.5 follows that dual gauge of  $\gamma_{C_1}(x_1, \dots, x_n) = \max_{1 \leq i \leq n} \{\|x_i\|_{\mathcal{H}_i}\}$  is given by  $\gamma_{C_1^0}(x_1, \dots, x_n) = \sum_{i=1}^n \|x_i\|_{\mathcal{H}_i}$  and hence, the polar set of  $C_1 = \{(x_1, \dots, x_n) \in \mathcal{H}_1 \times \dots \times \mathcal{H}_n :$

$\max_{1 \leq i \leq n} \{\|x_i\|_{\mathcal{H}_i}\} \leq 1\}$  is  $C_1^0 = \{(x_1, \dots, x_n) \in \mathcal{H}_1 \times \dots \times \mathcal{H}_n : \sum_{i=1}^n \|x_i\|_{\mathcal{H}_i} \leq 1\}$ . Thus, we derive for the case  $(1/\bar{\lambda})(x_1, \dots, x_n) \notin C_1^0$ , i.e.  $\sum_{i=1}^n \|x_i\|_{\mathcal{H}_i} > \bar{\lambda}$ , from Lemma 5.1 that

$$P_{C_1^0} \left( \frac{1}{\bar{\lambda}}(x_1, \dots, x_n) \right) = (\bar{z}_1, \dots, \bar{z}_n) \in \mathcal{H} \times \dots \times \mathcal{H} \text{ where } \bar{z}_i = \frac{\max\{\|x_i\|_{\mathcal{H}_i} - \bar{\lambda}\bar{\mu}, 0\}}{\bar{\lambda}\|x_i\|_{\mathcal{H}_i}} x_i, \quad (5.46)$$

$i = 1, \dots, n$ , and  $\bar{\mu} > 0$  is a solution of the equation (see (5.10) of the proof of Lemma 5.1)

$$\sum_{i=1}^n \max \left\{ \frac{1}{\bar{\lambda}} \|x_i\|_{\mathcal{H}_i} - \bar{\mu}, 0 \right\} = 1 \Leftrightarrow \sum_{i=1}^n \max \{ \|x_i\|_{\mathcal{H}_i} - \bar{\lambda}\bar{\mu}, 0 \} = \bar{\lambda}. \quad (5.47)$$

Furthermore, as  $C_2 = C_2^0 = \{x_{n+1} \in \mathcal{H}_{n+1} : \|x_{n+1}\|_{\mathcal{H}_{n+1}} \leq 1\}$ , it holds by [2, Example 3.16] (or also by Lemma 5.1 for  $n = 1$ )

$$P_{C_2^0} \left( \frac{1}{\bar{\lambda}} x_{n+1} \right) = \frac{1}{\bar{\lambda} \max \left\{ \frac{1}{\bar{\lambda}} \|x_{n+1}\|_{\mathcal{H}_{n+1}}, 1 \right\}} x_{n+1}. \quad (5.48)$$

Now, we need to consider the following four conceivable cases.

(a)  $(1/\bar{\lambda})(x_1, \dots, x_n) \notin C_1^0$ , i.e.  $\sum_{i=1}^n \|x_i\|_{\mathcal{H}_i} > \bar{\lambda}$ , and  $(1/\bar{\lambda})x_{n+1} \notin C_2^0$ , i.e.  $\|x_{n+1}\|_{\mathcal{H}_{n+1}} > \bar{\lambda}$ : Then one has  $P_{C_2^0}((1/\bar{\lambda})x_{n+1}) = (1/\|x_{n+1}\|_{\mathcal{H}_{n+1}})x_{n+1}$  and therefore, it follows together with (5.46), (5.47) and (5.48)

$$\bar{y}_i = x_i - \bar{\lambda}\bar{z}_i, \quad i = 1, \dots, n, \quad \bar{y}_{n+1} = x_{n+1} - \frac{\bar{\lambda}}{\|x_{n+1}\|_{\mathcal{H}_{n+1}}} x_{n+1} = \frac{\|x_{n+1}\|_{\mathcal{H}_{n+1}} - \bar{\lambda}}{\|x_{n+1}\|_{\mathcal{H}_{n+1}}} x_{n+1}$$

and  $\bar{\theta} = \gamma_C(\bar{y}_1, \dots, \bar{y}_{n+1}) = \bar{\lambda} + \xi$ .

As for  $\|x_i\|_{\mathcal{H}_i} - \bar{\lambda}\bar{\mu} > 0$  we have that

$$\bar{y}_i = x_i - \bar{\lambda}\bar{z}_i = x_i - \frac{\|x_i\|_{\mathcal{H}_i} - \bar{\lambda}\bar{\mu}}{\|x_i\|_{\mathcal{H}_i}} x_i = \frac{\bar{\lambda}\bar{\mu}}{\|x_i\|_{\mathcal{H}_i}} x_i$$

and for  $\|x_i\|_{\mathcal{H}_i} - \bar{\lambda}\bar{\mu} \leq 0$  that  $\bar{y}_i = x_i$ ,  $i = 1, \dots, n$ , it follows that  $\max_{1 \leq i \leq n} \{\|\bar{y}_i\|_{\mathcal{H}_i}\} = \bar{\lambda}\bar{\mu}$  and so,  $\bar{\lambda} + \xi = \gamma_C(\bar{y}_1, \dots, \bar{y}_{n+1}) = \max_{1 \leq i \leq n} \{\|\bar{y}_i\|_{\mathcal{H}_i}\} + \|\bar{y}_{n+1}\|_{\mathcal{H}_{n+1}} = \bar{\lambda}\bar{\mu} + \|x_{n+1}\|_{\mathcal{H}_{n+1}} - \bar{\lambda}$ , which means that  $\bar{\lambda}\bar{\mu} = 2\bar{\lambda} + \xi - \|x_{n+1}\|_{\mathcal{H}_{n+1}} \geq 0$ . For this reason, we can write for (5.47)

$$\sum_{i=1}^n \max \{ \|x_i\|_{\mathcal{H}_i} - 2\bar{\lambda} - \xi + \|x_{n+1}\|_{\mathcal{H}_{n+1}}, 0 \} = \bar{\lambda}.$$

Bringing the inequalities  $\|x_{n+1}\|_{\mathcal{H}_{n+1}} > \bar{\lambda}$  and  $\sum_{i=1}^n \|x_i\|_{\mathcal{H}_i} > \bar{\lambda}$  together with the last equality implies

$$\sum_{i=1}^n \max \{ \|x_i\|_{\mathcal{H}_i} - \|x_{n+1}\|_{\mathcal{H}_{n+1}} - \xi, 0 \} < \|x_{n+1}\|_{\mathcal{H}_{n+1}}$$

and

$$\sum_{i=1}^n \max \{ \|x_i\|_{\mathcal{H}_i} - \|x_{n+1}\|_{\mathcal{H}_{n+1}} - \xi, 0 \} < \sum_{i=1}^n \|x_i\|_{\mathcal{H}_i}.$$

Moreover, as  $\bar{\lambda}\bar{\mu} > 0$ , we have  $\bar{\lambda} + \xi > \|x_{n+1}\|_{\mathcal{H}_{n+1}} - \bar{\lambda}$ , which means that  $\bar{\lambda} > (\|x_{n+1}\|_{\mathcal{H}_{n+1}} - \xi)/2$ . From the assumption  $\|x_{n+1}\|_{\mathcal{H}_{n+1}} > \bar{\lambda}$  and  $\sum_{i=1}^n \|x_i\|_{\mathcal{H}_i} > \bar{\lambda}$  follows that  $\|x_{n+1}\|_{\mathcal{H}_{n+1}} > -\xi$  and  $\sum_{i=1}^n \|x_i\|_{\mathcal{H}_i} > (\|x_{n+1}\|_{\mathcal{H}_{n+1}} - \xi)/2$ . This yields (i).

(b)  $(1/\bar{\lambda})(x_1, \dots, x_n) \notin C_1^0$ , i.e.  $\sum_{i=1}^n \|x_i\|_{\mathcal{H}_i} > \bar{\lambda}$ , and  $(1/\bar{\lambda})x_{n+1} \in C_2^0$ , i.e.  $\|x_{n+1}\|_{\mathcal{H}_{n+1}} \leq \bar{\lambda}$ : Then one has that  $P_{C_2^0}((1/\bar{\lambda})x_{n+1}) = (1/\bar{\lambda})x_{n+1}$ , which means that  $\bar{y}_{n+1} = 0_{\mathcal{H}_{n+1}}$  and as

$(1/\bar{\lambda})(y_1, \dots, y_n) \notin C_1^0$  it follows, as shown in the previous case, that  $\max_{1 \leq i \leq n} \{\|\bar{y}_i\|_{\mathcal{H}_i}\} = \bar{\lambda}\bar{\mu}$ . This means that  $\bar{\theta} = \gamma_C(\bar{y}_1, \dots, \bar{y}_{n+1}) = \max_{1 \leq i \leq n} \{\|\bar{y}_i\|_{\mathcal{H}_i}\} + \|\bar{y}_{n+1}\|_{\mathcal{H}_{n+1}} = \bar{\lambda}\bar{\mu} = \bar{\lambda} + \xi$  and for (5. 47) we can write

$$\sum_{i=1}^n \max \{\|x_i\|_{\mathcal{H}_i} - \bar{\lambda} - \xi, 0\} = \bar{\lambda}.$$

As  $\|x_{n+1}\|_{\mathcal{H}_{n+1}} \leq \bar{\lambda}$ , it holds that

$$\|x_{n+1}\|_{\mathcal{H}_{n+1}} \leq \sum_{i=1}^n \max \{\|x_i\|_{\mathcal{H}_i} - \|x_{n+1}\|_{\mathcal{H}_{n+1}} - \xi, 0\} < \sum_{i=1}^n \|x_i\|_{\mathcal{H}_i}.$$

This verifies the case (ii).

(c)  $(1/\bar{\lambda})(x_1, \dots, x_n) \in C_1^0$ , i.e.  $\sum_{i=1}^n \|x_i\|_{\mathcal{H}_i} \leq \bar{\lambda}$ , and  $(1/\bar{\lambda})x_{n+1} \notin C_2^0$ , i.e.  $\|x_{n+1}\|_{\mathcal{H}_{n+1}} > \bar{\lambda}$ : Then  $P_{C_1^0}((1/\bar{\lambda})(x_1, \dots, x_n)) = (1/\bar{\lambda})(x_1, \dots, x_n)$  and  $P_{C_2^0}((1/\bar{\lambda})x_{n+1}) = (1/\|x_{n+1}\|_{\mathcal{H}_{n+1}})x_{n+1}$  implies

$$\begin{aligned} \bar{y}_i &= 0_{\mathcal{H}_i}, \quad i = 1, \dots, n, \quad \bar{y}_{n+1} = \frac{\|x_{n+1}\|_{\mathcal{H}_{n+1}} - \bar{\lambda}}{\|x_{n+1}\|_{\mathcal{H}_{n+1}}} x_{n+1} \text{ and} \\ \bar{\lambda} + \xi &= \bar{\theta} = \gamma_C(\bar{y}_1, \dots, \bar{y}_{n+1}) = \max_{1 \leq i \leq n} \{\|\bar{y}_i\|_{\mathcal{H}_i}\} + \|\bar{y}_{n+1}\|_{\mathcal{H}_{n+1}} = \|x_{n+1}\|_{\mathcal{H}_{n+1}} - \bar{\lambda}, \end{aligned}$$

where from the last equality one gets  $\bar{\lambda} = (\|x_{n+1}\|_{\mathcal{H}_{n+1}} - \xi)/2$ . But this yields

$$\bar{y}_{n+1} = \frac{\|x_{n+1}\|_{\mathcal{H}_{n+1}} - \frac{\|x_{n+1}\|_{\mathcal{H}_{n+1}} - \xi}{2}}{\|x_{n+1}\|_{\mathcal{H}_{n+1}}} x_{n+1} = \frac{\|x_{n+1}\|_{\mathcal{H}_{n+1}} + \xi}{2\|x_{n+1}\|_{\mathcal{H}_{n+1}}} x_{n+1}$$

and

$$\bar{\theta} = \|x_{n+1}\|_{\mathcal{H}_{n+1}} - \frac{\|x_{n+1}\|_{\mathcal{H}_{n+1}} - \xi}{2} = \frac{\|x_{n+1}\|_{\mathcal{H}_{n+1}} + \xi}{2}.$$

Further, one has

$$\sum_{i=1}^n \|x_i\|_{\mathcal{H}_i} \leq \frac{\|x_{n+1}\|_{\mathcal{H}_{n+1}} - \xi}{2} < \|x_{n+1}\|_{\mathcal{H}_{n+1}} \text{ as well as } \|x_{n+1}\|_{\mathcal{H}_{n+1}} > -\xi$$

and this yields (iii).

(d)  $(1/\bar{\lambda})(x_1, \dots, x_n) \in C_1^0$ , i.e.  $\sum_{i=1}^n \|x_i\|_{\mathcal{H}_i} \leq \bar{\lambda}$ , and  $(1/\bar{\lambda})x_{n+1} \in C_2^0$ , i.e.  $\|x_{n+1}\|_{\mathcal{H}_{n+1}} \leq \bar{\lambda}$ : Then  $P_{C_1^0}((1/\bar{\lambda})(x_1, \dots, x_n)) = (1/\bar{\lambda})(x_1, \dots, x_n)$  and  $P_{C_2^0}((1/\bar{\lambda})x_{n+1}) = (1/\bar{\lambda})x_{n+1}$  implies

$$\bar{y}_i = 0, \quad i = 1, \dots, n+1, \text{ and } \bar{\lambda} + \xi = \bar{\theta} = \gamma_C(\bar{y}_1, \dots, \bar{y}_{n+1}) = 0,$$

which means that  $\bar{\lambda} = -\xi$ . Hence, one gets that  $\xi < 0$  and  $\sum_{i=1}^n \|x_i\|_{\mathcal{H}_i} \leq -\xi$  as well as  $\|x_{n+1}\|_{\mathcal{H}_{n+1}} \leq -\xi$ . This verifies (iv).

As only these four cases are possible and exclude each other, we derive the statement of the corollary.  $\square$

**Corollary 5.7.** *Let  $\gamma_C : \mathcal{H}_1 \times \dots \times \mathcal{H}_n \rightarrow \mathbb{R}$  be defined by  $\gamma_C(x_1, \dots, x_n) := \max_{1 \leq i \leq n} \{(\|x_i\|_{\mathcal{H}_i})/w_i\}$ , then it holds*

$$P_{\text{epi } \gamma_C}(x_1, \dots, x_n, \xi) = \begin{cases} (x_1, \dots, x_n), & \text{if } \max_{1 \leq i \leq n} \left\{ \frac{1}{w_i} \|x_i\|_{\mathcal{H}_i} \right\} \leq \xi, \\ (\bar{y}_1, \dots, \bar{y}_n, \bar{\theta}), & \text{otherwise,} \end{cases}$$

where

$$\bar{y}_i = x_i - \frac{\max\{\|x_i\|_{\mathcal{H}_i} - (\bar{\lambda} + \xi)w_i, 0\}}{\|x_i\|_{\mathcal{H}_i}} x_i, \quad i = 1, \dots, n, \quad \text{and } \bar{\theta} = \frac{\sum_{i=k+1}^n w_i^2 \tau_i + \xi}{\sum_{i=k+1}^n w_i^2 + 1}$$

with

$$\bar{\lambda} = \frac{\sum_{i=k+1}^n w_i^2 \tau_i - \xi}{\sum_{i=k+1}^n w_i^2 + 1}$$

and  $k \in \{0, 1, \dots, n-1\}$  is the unique integer such that  $\tau_k \leq \bar{\lambda} \leq \tau_{k+1}$ , where the values  $\tau_0, \dots, \tau_n$  are defined by  $\tau_0 := 0$  and  $\tau_i := (\|x_i\|_{\mathcal{H}_i})/w_i$ ,  $i = 1, \dots, n$ , and in ascending order.

**Proof.** As  $C = \{(x_1, \dots, x_n) : \mathcal{H}_1 \times \dots \times \mathcal{H}_n : \max_{1 \leq i \leq n} \{(1/w_i)\|x_i\|_{\mathcal{H}_i}\} \leq 1\}$  (see Remark 4.3), Corollary 5.5 reveals that

$$P_{\text{epi } \gamma_C}(x_1, \dots, x_n, \xi) = \begin{cases} (x_1, \dots, x_n, \xi), & \text{if } \max_{1 \leq i \leq n} \{(1/w_i)\|x_i\|_{\mathcal{H}_i}\} \leq \xi, \\ (\bar{y}_1, \dots, \bar{y}_n, \bar{\theta}), & \text{otherwise,} \end{cases}$$

where

$$(\bar{y}_1, \dots, \bar{y}_n) = (x_1, \dots, x_n) - \bar{\lambda} P_{C^0} \left( \frac{1}{\bar{\lambda}} (x_1, \dots, x_n) \right), \quad \bar{\theta} = \bar{\lambda} + \xi \text{ and } \bar{\lambda} > 0.$$

By Lemma 4.5 the polar set of  $C$  looks like  $C^0 = \{(x_1, \dots, x_n) \in \mathcal{H}_1 \times \dots \times \mathcal{H}_n : \sum_{i=1}^n w_i \|x_i\|_{\mathcal{H}_i} \leq 1\}$  and from Lemma 5.1 we derive

$$P_{C^0} \left( \frac{1}{\bar{\lambda}} (x_1, \dots, x_n) \right) = (\bar{z}_1, \dots, \bar{z}_n) \in \mathcal{H}_1 \times \dots \times \mathcal{H}_n,$$

where

$$\bar{z}_i = \frac{\max\{\|x_i\|_{\mathcal{H}_i} - \bar{\lambda} \bar{\mu} w_i, 0\}}{\bar{\lambda} \|x_i\|_{\mathcal{H}_i}} x_i, \quad i = 1, \dots, n,$$

and  $\bar{\mu} > 0$  is a solution of the equation (see (5. 10) of the proof of Lemma 5.1)

$$\sum_{i=1}^n w_i \max\{\|x_i\|_{\mathcal{H}_i} - \bar{\lambda} \bar{\mu} w_i, 0\} = \bar{\lambda}. \quad (5. 49)$$

Therefore, it follows

$$\bar{y}_i = x_i - \frac{\max\{\|x_i\|_{\mathcal{H}_i} - \bar{\lambda} \bar{\mu} w_i, 0\}}{\|x_i\|_{\mathcal{H}_i}} x_i = \frac{\|x_i\|_{\mathcal{H}_i} - \max\{\|x_i\|_{\mathcal{H}_i} - \bar{\lambda} \bar{\mu} w_i, 0\}}{\|x_i\|_{\mathcal{H}_i}} x_i, \quad i = 1, \dots, n,$$

and as for  $\|x_i\|_{\mathcal{H}_i} - \bar{\lambda} \bar{\mu} w_i \leq 0$  one gets  $\bar{y}_i = x_i$ , i.e.  $\|\bar{y}_i\|_{\mathcal{H}_i} = \|x_i\|_{\mathcal{H}_i}$  and for  $\|x_i\|_{\mathcal{H}_i} - \bar{\lambda} \bar{\mu} w_i > 0$ ,  $\bar{y}_i = (\bar{\lambda} \bar{\mu} w_i / \|x_i\|_{\mathcal{H}_i}) x_i$ , i.e.  $\|\bar{y}_i\|_{\mathcal{H}_i} = \bar{\lambda} \bar{\mu} w_i$ ,  $i = 1, \dots, n$ , we obtain

$$\gamma_C(\bar{y}_1, \dots, \bar{y}_n) = \max_{1 \leq i \leq n} \left\{ \frac{1}{w_i} \|\bar{y}_i\|_{\mathcal{H}_i} \right\} = \bar{\lambda} \bar{\mu} = \bar{\lambda} + \xi. \quad (5. 50)$$

Bringing (5. 49) and (5. 50) together yields

$$\sum_{i=1}^n w_i \max\{\|x_i\|_{\mathcal{H}_i} - (\bar{\lambda} + \xi)w_i, 0\} = \bar{\lambda}. \quad (5. 51)$$

Clearly, if  $\|x_i\|_{\mathcal{H}_i} - \xi w_i \leq 0$  for all  $i = 1, \dots, n$ , i.e.  $\max_{1 \leq i \leq n} \{\|x_i\|_{\mathcal{H}_i}/w_i\} \leq \xi$ , then  $\|x_i\|_{\mathcal{H}_i} - \xi w_i - \lambda w_i \leq 0$  for all  $i = 1, \dots, n$ , and one gets by (5. 51) that

$$\bar{\lambda} = \sum_{i=1}^n w_i \max \{ \|x_i\|_{\mathcal{H}_i} - (\bar{\lambda} + \xi)w_i, 0 \} = 0,$$

which means that  $\bar{y}_i = x_i$  for all  $i = 1, \dots, n$ , and  $\bar{\theta} = \xi$ .

Now, let us assume that  $J := \{i \in \{1, \dots, n\} : \|x_i\|_{\mathcal{H}_i} - \xi w_i > 0\} \neq \emptyset$  and define the function  $g : \mathbb{R} \rightarrow \mathbb{R}$  by  $g(\lambda) = \sum_{i=1}^n w_i^2 \max \{ \tau_i - (\lambda + \xi), 0 \} - \lambda$ , then it follows from  $\lambda + \xi \geq 0$  that

$$g(\lambda) = \sum_{i=1}^n w_i^2 \max \{ \tau_i - (\lambda + \xi), 0 \} - \lambda \leq \sum_{i=1}^n w_i^2 \max \{ \tau_i, 0 \} - \lambda = \sum_{i \in J} w_i^2 \tau_i - \lambda.$$

If we choose  $\tilde{\lambda} > 0$  such that  $\tilde{\lambda} > \sum_{i \in J} w_i^2 \tau_i$ , then we derive that  $g(\tilde{\lambda}) < 0$ . Thus, we can secure the existence of a  $\tilde{\lambda} > 0$  such that  $g(\tilde{\lambda}) < 0$ .

As  $g$  is a piecewise linear function, one has, similarly to Corollary 5.1, to find the unique integer  $k \in \{0, 1, \dots, n-1\}$  such that  $g(\tau_k) \geq 0$  and  $g(\tau_{k+1}) \leq 0$ . This leads to

$$\sum_{i=k+1}^n w_i^2 \tau_i - \xi \sum_{i=k+1}^n w_i^2 - \bar{\lambda} \sum_{i=k+1}^n (w_i^2 + 1) = 0 \Leftrightarrow \bar{\lambda} = \frac{\sum_{i=k+1}^n w_i^2 \tau_i - \xi \sum_{i=k+1}^n w_i^2}{\sum_{i=k+1}^n w_i^2 + 1}$$

and hence,  $\bar{\theta} = \bar{\lambda} + \xi = (\sum_{i=k+1}^n w_i^2 \tau_i + \xi) / (\sum_{i=k+1}^n w_i^2 + 1)$ .  $\square$

**Remark 5.5.** In [28] the formula in the previous corollary was given for the case where  $\mathcal{H}_i = \mathbb{R}$ ,  $i = 1, \dots, n$ , in other words, where  $\gamma_C$  is the weighted  $l_\infty$ -norm.

**Remark 5.6.** Like in Lemma 5.3, one can give a formula for the projection onto the epigraph of a gauge composed with a linear operator  $A : \mathcal{H} \rightarrow \mathcal{K}$  with  $AA^* = \mu Id$ ,  $\mu > 0$ ,

$$P_{\text{epi } \gamma_C(A \cdot)}(x, \xi) = (x, \xi) + \frac{1}{\mu} (A^* \times Id) (P_{\text{epi } \gamma_C(\cdot)}(Ax, \xi) - (Ax, \xi)).$$

Moreover, it can easily be observed that for  $p \in \mathcal{H}$  holds (similar to the proof of Lemma 5.2)

$$P_{\text{epi } \gamma_C(\cdot - p)}(x, \xi) = P_{\text{epi } \gamma_C}(x - p, \xi) + (p, 0).$$

We close this section with a characterization of the subdifferential of a gauge function by the projection operator.

**Remark 5.7.** Let  $C \subseteq \mathcal{H}$  be closed and convex such that  $0_{\mathcal{H}} \in C$ , then it holds by (5. 3), (5. 8), [81, Lemma 2.1], [81, Remark 2.2] and [2, Theorem 14.3(ii)] for all  $x, y \in \mathcal{H}$  that

$$\begin{aligned} x \in \partial \gamma_C(y) &\Leftrightarrow x + y - y \in \partial \gamma_C(y) \Leftrightarrow y = \text{prox}_{\gamma_C}(x + y) \\ &\Leftrightarrow y = x + y - \text{prox}_{\gamma_C^*}(x + y) \Leftrightarrow y = x + y - \text{prox}_{\delta_{C^0}}(x + y) \\ &\Leftrightarrow x = P_{C^0}(x + y). \end{aligned}$$

From which follows that

$$\partial \gamma_C(y) = \{x \in \mathcal{H} : x = P_{C^0}(x + y)\}.$$

In addition, if  $C$  is a closed convex cone, then it follows from [2, Theorem 6.29] that

$$\partial \gamma_C(y) = \{x \in \mathcal{H} : x = x + y - P_C(x + y)\} = \{x \in \mathcal{H} : y = P_C(x + y)\}.$$



### 5.3 Numerical experiments

Our numerical tests are implemented on a PC with an Intel Core i7-6700HQ CPU with 2.6GHz and 12 GB RAM. While the numerical tests in [30] were based on the partial inverse algorithm introduced by Spingarn in [77], we use here the *parallel splitting algorithm* from [2, Proposition 27.8].

**Theorem 5.3.** (*parallel splitting algorithm*) Let  $n$  be an integer such that  $n \geq 2$  and  $f_i : \mathbb{R}^s \rightarrow \overline{\mathbb{R}}$  be a proper, lower semicontinuous and convex function for  $i = 1, \dots, n$ . Suppose that the problem

$$(P^{DR}) \quad \min_{x \in \mathbb{R}^s} \left\{ \sum_{i=1}^n f_i(x) \right\}$$

has at least one solution and that  $\text{dom } f_1 \cap \bigcap_{i=2}^n \text{int dom } f_i \neq \emptyset$ . Let  $(\mu_k)_{k \in \mathbb{N}}$  be a sequence in  $[0, 2]$  such that  $\sum_{k \in \mathbb{N}} \mu_k(2 - \mu_k) = +\infty$ , let  $\nu > 0$ , and let  $(x_{i,0})_{i=1}^n \in \mathbb{R}^s \times \dots \times \mathbb{R}^s$ . Set

$$(\forall k \in \mathbb{N}) \quad \left\{ \begin{array}{l} r_k = \frac{1}{n} \sum_{i=1}^n x_{i,k}, \\ y_{i,k} = \text{prox}_{\nu f_i} x_{i,k}, \quad i = 1, \dots, n, \\ q_k = \frac{1}{n} \sum_{i=1}^n y_{i,k}, \\ x_{i,k+1} = x_{i,k} + \mu_k(2q_k - r_k - y_{i,k}), \quad i = 1, \dots, n. \end{array} \right.$$

Then  $(r_k)_{k \in \mathbb{N}}$  converges to a solution of problem  $(P^{DR})$ .

In order to use the parallel splitting algorithm given in the previous theorem, we need to rewrite the *extended multifacility location problem*  $(EP_N^{M,\beta})$  in (5. 4) into an optimization problem with an objective function, which is a sum of proper, convex and lower semicontinuous functions.

The first way to reformulate this location problem is based on the introduction of an additional variable as presented in (5. 5):

$$(EP_N^{M,\beta}) \quad \min_{(x_1, \dots, x_m, t) \in \mathbb{R}^d \times \dots \times \mathbb{R}^d \times \mathbb{R}} \left\{ t + \sum_{i=1}^n \delta_{\text{epi} \left( \sum_{j=1}^m w_{ij} \|\cdot - p_i\|^{\beta_i} \right)}(x_1, \dots, x_m, t) \right\}. \quad (5. 52)$$

We define the functions

$$\begin{aligned} f_1 : \mathbb{R}^d \times \dots \times \mathbb{R}^d \times \mathbb{R} &\rightarrow \mathbb{R}, \quad f_1(x_1, \dots, x_m, t) = t \text{ and} \\ f_i : \mathbb{R}^d \times \dots \times \mathbb{R}^d \times \mathbb{R} &\rightarrow \overline{\mathbb{R}}, \quad f_i(x_1, \dots, x_m, t) = \delta_{\text{epi} \left( \sum_{j=1}^m w_{ij} \|\cdot - p_i\|^{\beta_i} \right)}(x_1, \dots, x_m, t), \end{aligned}$$

$i = 2, \dots, n+1$ , then  $\text{dom } f_1 = \mathbb{R}^d \times \dots \times \mathbb{R}^d \times \mathbb{R}$  and

$$\left( 0_{\mathbb{R}^d}, \dots, 0_{\mathbb{R}^d}, \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^m w_{ij} \|p_i\|^{\beta_i} \right\} + 1 \right) \in \text{int dom } f_i = \text{int epi} \left( \sum_{j=1}^m w_{ij} \|\cdot - p_i\|^{\beta_i} \right)$$

for all  $i = 2, \dots, n+1$ , i.e., it holds that  $\text{dom } f_1 \cap \bigcap_{i=2}^{n+1} \text{int dom } f_i \neq \emptyset$ . Therefore, the sequences generated by the algorithm from Theorem 5.3 converges to a solution of the location problem  $(EP_N^{M,\beta})$  and the following formulae for the proximal points associated to the functions  $f_1, \dots, f_{n+1}$  can be formulated by using (5. 8) and Lemma 5.2

$$\begin{aligned} (\bar{y}_1, \dots, \bar{y}_m, \bar{\theta}) &= \text{prox}_{\nu f_1}(x_1, \dots, x_m, t) \\ \Leftrightarrow (x_1, \dots, x_m, t) - (\bar{y}_1, \dots, \bar{y}_m, \bar{\theta}) &\in \partial(\nu f_1)(\bar{y}_1, \dots, \bar{y}_m, \bar{\theta}) = (0_{\mathbb{R}^d}, \dots, 0_{\mathbb{R}^d}, \nu) \\ \Leftrightarrow x_i &= \bar{y}_i, \quad i = 1, \dots, m, \text{ and } \bar{\theta} = t - \nu \Leftrightarrow (\bar{y}_1, \dots, \bar{y}_m, \bar{\theta}) = (x_1, \dots, x_m, t - \nu) \end{aligned}$$

and

$$\begin{aligned}
(\bar{y}_1, \dots, \bar{y}_m, \bar{\theta}) &= \text{prox}_{\nu f_i}(x_1, \dots, x_m, t) = \text{prox}_{\nu \delta_{\text{epi}\left(\sum_{j=1}^m w_{ij} \|\cdot - p_i\|^{\beta_i}\right)}}(x_1, \dots, x_m, t) \\
&= \text{P}_{\text{epi}\left(\sum_{j=1}^m w_{ij} \|\cdot - p_i\|^{\beta_i}\right)}(x_1, \dots, x_m, t) \\
&= \text{P}_{\text{epi}\left(\sum_{j=1}^m w_{ij} \|\cdot\|^{\beta_i}\right)}(x_1 - p_i, \dots, x_m - p_i, t) + (p_i, \dots, p_i, 0). \quad (5.53)
\end{aligned}$$

The second way to rewrite the extended multifacility location problem ( $EP_N^{M,\beta}$ ) into an optimization problem of the form of ( $P^{DR}$ ) makes use of the ideas of Cornejo and Michelot given in [30] and splits the sums of weighted norms by  $n \cdot m$  additional variables (see also (5.6)):

$$(EP_N^{M,\beta}) \min_{\substack{t, t_{ij} \in \mathbb{R}, x_j \in \mathbb{R}^d, \\ j=1, \dots, m, i=1, \dots, n}} \left\{ t + \sum_{j=1}^m \sum_{i=1}^n \delta_{\text{epi}(w_{ij} \|\cdot - p_i\|^{\beta_i})}(x_j, t_{ij}) + \sum_{i=1}^n \delta_{\text{epi} \tau_i}(t_{i1}, \dots, t_{im}, t) \right\}, \quad (5.54)$$

where  $\tau_i(t_{i1}, \dots, t_{im}) := \sum_{j=1}^m t_{ij}$ ,  $i = 1, \dots, n$ . Now, let  $\tilde{x} := (x_1, \dots, x_m) \in \mathbb{R}^d \times \dots \times \mathbb{R}^d$ ,  $\tilde{t} := (t_{ij})_{i=1, \dots, n, j=1, \dots, m}$ ,

$$\begin{aligned}
f_1 : \underbrace{\mathbb{R}^d \times \dots \times \mathbb{R}^d}_{m\text{-times}} \times \mathbb{R}^{mn} \times \mathbb{R} &\rightarrow \mathbb{R}, \quad f_1(\tilde{x}, \tilde{t}, t) := t, \\
f_{ij} : \mathbb{R}^d \times \dots \times \mathbb{R}^d \times \mathbb{R}^{mn} \times \mathbb{R} &\rightarrow \overline{\mathbb{R}}, \quad f_{ij}(\tilde{x}, \tilde{t}, t) := \delta_{\text{epi}(w_{ij} \|\cdot - p_i\|^{\beta_i})}(x_j, t_{ij}),
\end{aligned}$$

$j = 1, \dots, m$ ,  $i = 1, \dots, n$ , and

$$\tilde{f}_i : \mathbb{R}^d \times \dots \times \mathbb{R}^d \times \mathbb{R}^{mn} \times \mathbb{R} \rightarrow \overline{\mathbb{R}}, \quad \tilde{f}_i(\tilde{x}, \tilde{t}, t) := \delta_{\text{epi} \tau_i}(t_{i1}, \dots, t_{im}, t), \quad i = 1, \dots, n.$$

As

$$\begin{aligned}
\text{dom } f_1 &= \mathbb{R}^d \times \dots \times \mathbb{R}^d \times \mathbb{R}^{mn} \times \mathbb{R}, \\
\text{dom } f_{ij} &= \{(\tilde{x}, \tilde{t}, t) \in \mathbb{R}^d \times \dots \times \mathbb{R}^d \times \mathbb{R}^{mn} \times \mathbb{R} : (x_j, t_{ij}) \in \text{epi}(w_{ij} \|\cdot - p_i\|^{\beta_i})\}, \\
i &= 1, \dots, n, \quad j = 1, \dots, m, \\
\text{dom } \tilde{f}_i &= \{(\tilde{x}, \tilde{t}, t) \in \mathbb{R}^d \times \dots \times \mathbb{R}^d \times \mathbb{R}^{mn} \times \mathbb{R} : (t_{i1}, \dots, t_{im}, t) \in \text{epi} \tau_i\}, \\
i &= 1, \dots, n
\end{aligned}$$

and

$$\begin{aligned}
&\left(0_{\mathbb{R}^d}, \dots, 0_{\mathbb{R}^d}, \max_{\substack{1 \leq i \leq n, \\ 1 \leq j \leq m}} \{w_{ij} \|p_i\|^{\beta_i}\} + 1, \dots, \max_{\substack{1 \leq i \leq n, \\ 1 \leq j \leq m}} \{w_{ij} \|p_i\|^{\beta_i}\} + 1, m \max_{\substack{1 \leq i \leq n, \\ 1 \leq j \leq m}} \{w_{ij} \|p_i\|^{\beta_i}\} + m + 1\right) \\
&\in \text{dom } f_1 \cap \left(\bigcap_{\substack{1 \leq i \leq n, \\ 1 \leq j \leq m}} \text{int dom } f_{ij}\right) \cap \left(\bigcap_{1 \leq i \leq n} \text{int dom } \tilde{f}_i\right),
\end{aligned}$$

convergence in the sense of Theorem 5.3 can be guaranteed. Now, let  $\tilde{\bar{y}} := (\bar{y}_1, \dots, \bar{y}_m)$  and  $\tilde{\bar{\theta}} := (\bar{\theta}_{ij})_{1 \leq i \leq n, 1 \leq j \leq m}$ , then one has by (5.8) for the corresponding proximal points of the functions  $f_1, f_{ij}$ ,  $j = 1, \dots, m$ ,  $i = 1, \dots, n$ , and  $\tilde{f}_i$ ,  $i = 1, \dots, n$ ,

$$(\tilde{\bar{y}}, \tilde{\bar{\theta}}, \bar{\theta}) = \text{prox}_{\nu f_1}(\tilde{x}, \tilde{t}, t) = \underbrace{(0_{\mathbb{R}^d}, \dots, 0_{\mathbb{R}^d})}_{m\text{-times}}, \underbrace{(0, \dots, 0)}_{mn\text{-times}}, t - \nu$$

and by (5. 8) and Lemma 5.2

$$\begin{aligned}
& (\bar{y}, \bar{\theta}, \bar{\theta}) = \text{prox}_{\nu f_{ij}}(\tilde{x}, \tilde{t}, t) \Leftrightarrow (\tilde{x}, \tilde{t}, t) - (\bar{y}, \bar{\theta}, \bar{\theta}) \in \partial(\nu f_{ij})(\bar{y}, \bar{\theta}, \bar{\theta}) \\
& \Leftrightarrow (x_j, t_{ij}) - (\bar{y}_j, \bar{\theta}_{ij}) \in \partial(\nu \delta_{\text{epi}(w_{ij} \|\cdot - p_i\|^{\beta_i})})(\bar{y}_j, \bar{\theta}_{ij}) \text{ and} \\
& \bar{y}_l = x_l, \bar{\theta}_{sl} = t_{sl}, \bar{\theta} = t, \quad s = 1, \dots, n, \quad l = 1, \dots, m, \quad sl \neq ij, \\
& \Leftrightarrow (\bar{y}_j, \bar{\theta}_{ij}) = \text{prox}_{\nu \delta_{\text{epi}(w_{ij} \|\cdot - p_i\|^{\beta_i})}}(x_j, t_{ij}) = P_{\text{epi}(w_{ij} \|\cdot - p_i\|^{\beta_i})}(x_j, t_{ij}) \\
& = P_{\text{epi}(w_{ij} \|\cdot\|^{\beta_i})}(x_j - p_i, t_{ij}) + (p_i, 0) \text{ and} \\
& \bar{y}_l = x_l, \bar{\theta}_{sl} = t_{sl}, \bar{\theta} = t, \quad s = 1, \dots, n, \quad l = 1, \dots, m, \quad sl \neq ij,
\end{aligned} \tag{5. 55}$$

$j = 1, \dots, m, \quad i = 1, \dots, n$ . Moreover, by (5. 8) and [2, Example 28.17] follows

$$\begin{aligned}
& (\bar{y}, \bar{\theta}, \bar{\theta}) = \text{prox}_{\nu \tilde{f}_i}(\tilde{x}, \tilde{t}, t) \Leftrightarrow (\tilde{x}, \tilde{t}, t) - (\bar{y}, \bar{\theta}, \bar{\theta}) \in \partial(\nu \tilde{f}_i)(\bar{y}, \bar{\theta}, \bar{\theta}) \\
& \Leftrightarrow (t_{i1}, \dots, t_{im}, t) - (\bar{\theta}_{i1}, \dots, \bar{\theta}_{im}, \bar{\theta}) \in \partial(\nu \delta_{\text{epi } \tau_i})(\bar{\theta}_{i1}, \dots, \bar{\theta}_{im}, \bar{\theta}) \text{ and} \\
& (t_{l1}, \dots, t_{lm}, t) = (\bar{\theta}_{l1}, \dots, \bar{\theta}_{lm}, \bar{\theta}), \quad l = 1, \dots, n, \quad l \neq i, \quad (x_1, \dots, x_m) = (\bar{y}_1, \dots, \bar{y}_m) \\
& \Leftrightarrow (\bar{\theta}_{i1}, \dots, \bar{\theta}_{im}, \bar{\theta}) = \text{prox}_{\nu \delta_{\text{epi } \tau_i}}(t_{i1}, \dots, t_{im}, t) = P_{\text{epi } \tau_i}(t_{i1}, \dots, t_{im}, t) \\
& = \begin{cases} (\bar{\theta}_{i1}, \dots, \bar{\theta}_{im}, \bar{\theta})^T, & \text{if } \sum_{j=1}^m t_{ij} - t \leq 0, \\ (\bar{\theta}_{i1}, \dots, \bar{\theta}_{im}, \bar{\theta})^T - \frac{\sum_{j=1}^m t_{ij} - t}{m+1} (1, \dots, 1, -1)^T, & \text{if } \sum_{j=1}^m t_{ij} - t > 0, \end{cases} \\
& \text{and } (t_{l1}, \dots, t_{lm}, t) = (\bar{\theta}_{l1}, \dots, \bar{\theta}_{lm}, \bar{\theta}), \quad l = 1, \dots, n, \quad l \neq i, \quad (x_1, \dots, x_m) = (\bar{y}_1, \dots, \bar{y}_m), \\
& i = 1, \dots, n.
\end{aligned}$$

The tables below illustrate the performance of our method using the formulae from Corollary 5.1 and 5.2 for the projection onto the epigraph of the sum of powers of weighted norms (EpiSum-Norms) compared with the concept proposed by Cornejo and Michelot in [30], where only the projection onto the epigraph of a weighted norm (EpiNorm) is needed (see Corollary 5.3). We solved the problem  $(EP_N^{M, \beta})$  in  $\mathbb{R}^2$  and  $\mathbb{R}^3$  for different choices of given and new facilities. The performance results are visualized by the associated figures, where we use the following notations:

NumGivFac: Number of given facilities  
 NumNewFac: Number of new facilities  
 NumIt: Number of Iterations of the algorithm  
 CPUtime: CPU time in seconds.

We used the following parameters for initialization:  $\mu_n = 1$  for all  $n \in \mathbb{N}$ . Moreover, let us point out that we tested the algorithm of Theorem 5.3 for different values of the parameter  $\nu$ , where the most remarkable results are printed in the tables and the best of them concerning the CPU time and number of iterations are visualized in the corresponding figures.

First, we consider the situation where  $\beta_i = 1$  for all  $i = 1, \dots, n$ .

Table 5.1: Performance evaluation for NumGiFac 25 and NumNewFac 5 in  $\mathbb{R}^2$

	$\nu = 5$		$\nu = 30$		$\nu = 50$	
	NumIt	CPUtime	NumIt	CPUtime	NumIt	CPUtime
EpiSumNorms	989	1.92	185	0.45	306	0.73
EpiNorm	21171	193.57	2179	17.89	2543	19.69

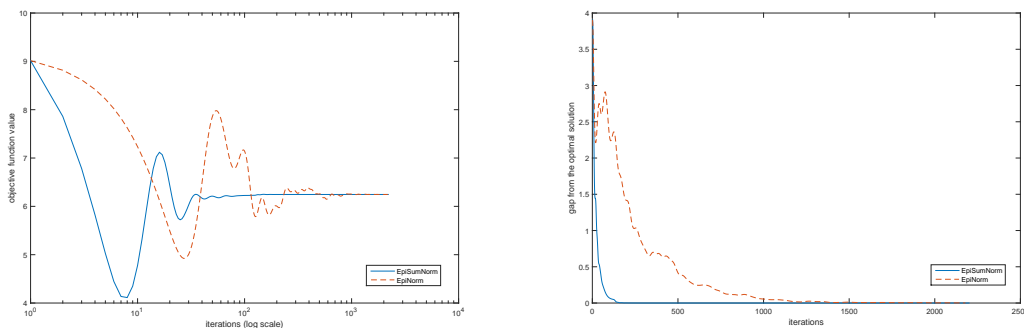


Figure 5.1: Comparison of the methods EpiSumNorm (blue solid line) and EpiNorm (red dashed line) in  $\mathbb{R}^2$  for  $\nu = 30$

Table 5.2: Performance evaluation for NumGiFac 30 and NumNewFac 10 in  $\mathbb{R}^2$

	$\nu = 18$		$\nu = 50$		$\nu = 82$	
	NumIt	CPUtime	NumIt	CPUtime	NumIt	CPUtime
EpiSumNorms	269	0.87	535	1.53	909	2.51
EpiNorm	14341	335.45	3478	71.69	4312	92.93

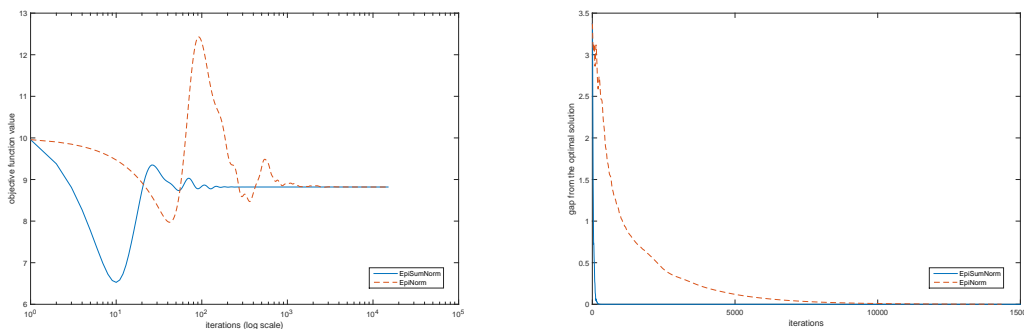


Figure 5.2: Comparison of the methods EpiSumNorm (blue solid line) and EpiNorm (red dashed line) in  $\mathbb{R}^2$  for  $\nu = 18$

Table 5.3: Performance evaluation for NumGiFac 60 and NumNewFac 20 in  $\mathbb{R}^3$

	$\nu = 98$		$\nu = 205$		$\nu = 275$	
	NumIt	CPUtime	NumIt	CPUtime	NumIt	CPUtime
EpiSumNorms	592	4.2	1129	7.53	1496	10.38
EpiNorm	28920	5653.66	15697	2951.28	15987	2983.45

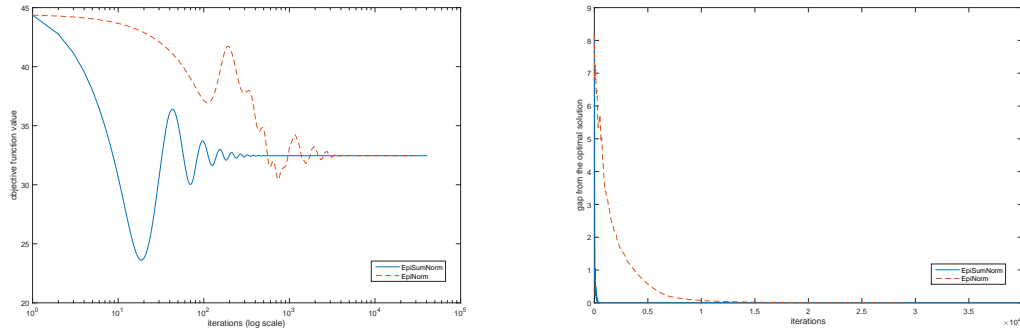


Figure 5.3: Comparison of the methods EpiSumNorm (blue solid line) and EpiNorm (red dashed line) in  $\mathbb{R}^3$  for  $\nu = 98$

In Table 5.1 it is shown that the parallel splitting algorithm converges very slow when employed in connection with the method proposed in [30], while our method performs much better. To be more precise, we used here the value 0.001 as the maximum bound from the optimal solution. The corresponding figure shows that our method EpiSumNorms regenerates after 185 iterations a solution which is within the maximum bound from the optimal solution, while the method EpiNorm needs 2179 iterations. Take also note that in this example the location problem has in the form of EpiNorm 125 additional variables, while the examples in the Table 5.2 and 5.3 have 300 and 1200 additional variables, respectively. For this reason our method by far outperforms the concept EpiNorm on such optimization problems regarding the accuracy as well as the CPU speed and number of iterations.

Finally, we consider the situation where  $w_i = 1$  and  $\beta_i = 2$  for all  $i = 1, \dots, n$ .

Table 5.4: Performance evaluation for NumGiFac 25 and NumNewFac 5 in  $\mathbb{R}^2$

	$\nu = 5$		$\nu = 39$		$\nu = 72$	
	NumIt	CUtime	NumIt	CUtime	NumIt	CUtime
EpiSumNorms	398	0.47	2664	2.58	4877	4.89
EpiNorm	10377	90.34	2782	23.51	5035	42.72

Table 5.5: Performance evaluation for NumGiFac 60 and NumNewFac 10 in  $\mathbb{R}^3$

	$\nu = 110$		$\nu = 445$		$\nu = 495$	
	NumIt	CUtime	NumIt	CUtime	NumIt	CUtime
EpiSumNorms	1684	3.78	6468	13.68	7433	17.01
EpiNorm	15131	970.24	5154	326.78	5713	356.06

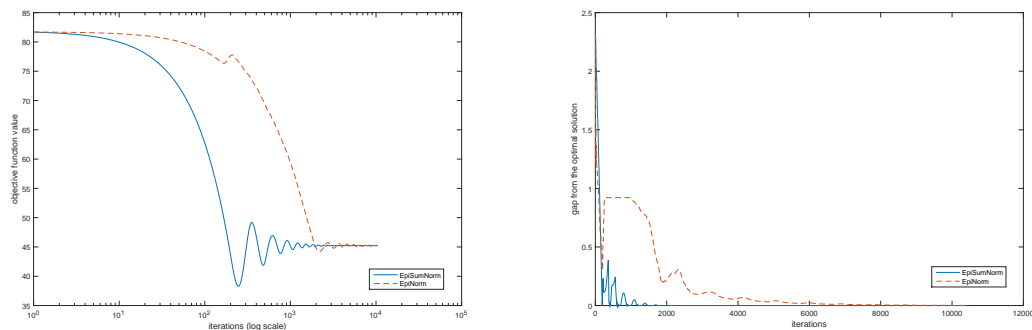


Figure 5.4: Comparison of the methods EpiSumNorm (blue solid line) and EpiNorm (red dashed line) in  $\mathbb{R}^2$  for  $\nu = 5$

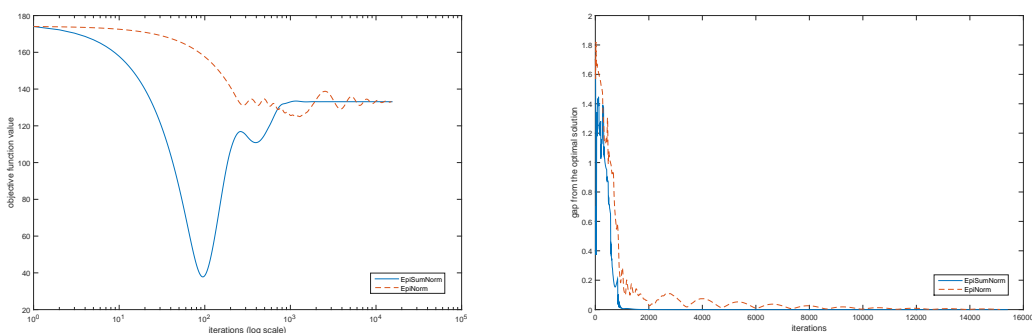


Figure 5.5: Comparison of the methods EpiSumNorm (blue solid line) and EpiNorm (red dashed line) in  $\mathbb{R}^3$  for  $\nu = 110$

The examples in the last two tables draw a similar picture as the examples in the previous ones. While the method EpiSumNorms generates a solution within the maximum bound from the optimal solution after few seconds, the method EpiNorm needs several minutes. This also points up the usefulness of our approach made in Section 5.2.

In the Appendix the corresponding source codes for the Matlab implementation are provided.

# Index of notation

## Spaces and sets

$X^*$	the topological dual space $X^*$ of $X$
$\langle x^*, x \rangle$	the value of $x^*$ at $x$
$w(X^*, X)$	weak* topology on $X^*$ induced by $X$
$\leq_K$	the partial ordering induced by the convex cone $K$
$x \leq_K$	$x \leq_K y$ and $x \neq y$
$0_X$	the zero element of $X$
$+\infty_K$	the greatest element regarding the ordering cone $K$
$\overline{X}$	the space $X$ to which the element $+\infty_K$ is added
$K^*$	the dual cone of the cone $K$
$N_S$	the normal cone of the set $S$
$\text{int}(S)$	the interior of the set $S$
$\text{ri}(S)$	the relative interior of the set $S$
$\text{cl}(S)$	the closure of the set $S$
$\text{cone}(S)$	the conic hull of the set $S$
$\text{core}(S)$	the algebraic interior of the set $S$
$\text{sqri}(S)$	the strong quasi interior of the set $S$
$A \times B$	the Cartesian product of two sets
$A + B$	the Minkowski sum of two sets
$ V $	the cardinality of the index set $V$
$C^0$	the polar set of the set $C$
$\forall$	for all
$\in$	in
$\exists$	there exists (at least one)
$\mathcal{H}$	the Hilbert space $\mathcal{H}$
$\langle \cdot, \cdot \rangle_{\mathcal{H}}$	the scalar product in Hilbert space $\mathcal{H}$
$\  \cdot \ _{\mathcal{H}}$	the norm defined by the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$

$\text{Argmin } f$	the set of global minimizers of the function $f$
$\arg \min_{x \in \mathcal{H}} f(x)$	the unique minimizer of $f$
$P_C$	the projection onto the non-empty, closed and convex set $C$
$\pm\infty$	plus and minus infinite, respectively
$\mathbb{R}$	the set of real numbers
$\overline{\mathbb{R}}$	the extended set of real numbers, $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$
$\mathbb{R}_+^n$	the non-negative orthant of $\mathbb{R}^n$
$\langle \cdot, \cdot \rangle$	the scalar product in $\mathbb{R}^n$
$\ \cdot\ $	the Euclidean norm in $\mathbb{R}^n$

### Scalar and vector functions

$\text{dom } f$	the domain of the function $f$
$\text{epi } f$	the epigraph of the function $f$
$f^*$	the conjugate of the function $f$
$f_S^*$	the conjugate of the function $f$ regarding the set $S$
$f^{**}$	the biconjugate of the function $f$
$\partial f$	the subdifferential of the function $f$
$\delta_A$	the indicator function of the set $A$
$\sigma_A$	the support function of the set $A$
$\partial f(x)$	the subdifferential of the function $f$ at $x \in X$
$\nabla f(x)$	the gradient of the function $f$ at $x \in X$
$\gamma_C$	the gauge function (a.k.a. Minkowski functional) of the set $C$
$\text{prox}_f$	the proximity operator of a function $f$
$(z^*F)$	the function $\langle z^*, F \rangle$ , where $F$ is a vector function and $z^* \in K^*$
$\text{epi}_Q F$	the $Q$ -epigraph of the vector function $F$
$F \circ G$	the composition of two functions
$\text{Id}$	the identity mapping
$\mathbf{0}$	the zero mapping
$A_{jk}, B_{ji}$	linear mappings
$v(P^C)$	the optimal objective value of the optimization problem $(P^C)$



# Appendix A

## Appendix

Here we present the Matlab source codes of the m-files for our numerical tests, which can be found on the compact disk attached to this thesis.

The file **projection\_weighted\_sum.m** calculates the projection onto the epigraph of the sum of weighted norms.

```
1 % calculate the projection onto the epigraph of the weighted sum of norms
2 function [proj_y,proj_xi] = projection_weighted_sum(w,y,xi)
3
4 k1 = size(y,1);
5 k2 = size(y,2);
6 proj_y = zeros(k1,k2);
7 nrm_y = zeros(k1,1);
8 tau_old = zeros(k1,1);
9 s = zeros(k1+1,1);
10 s_tilde = zeros(k1,1);
11
12 for i = 1:k1
13     nrm_y(i) = norm(y(i,:));
14     tau_old(i) = nrm_y(i)/w(i);
15 end
16
17 tau_old_tilde = [0;tau_old];
18
19 if dot(w,nrm_y) <= xi % check whether (y,xi) is an element of the epigraph
20     proj_xi = xi;
21     for i = 1:k1
22         proj_y(i,:) = y(i,:);
23     end
24
25 elseif xi < 0 && max(tau_old_tilde) <= -xi
26     proj_xi = 0;
27     for i = 1:k1
28         proj_y(i,:) = zeros(1,k2);
29     end
30
31 else
32     [tau_new,I] = sort(tau_old); % sort the vector tau_old in ascending order
33     w_tilde = w(I);
34     tau_new_tilde = sort(tau_old_tilde); % sort the vector tau_old_tilde in ...
35         ascending order
36
37     % determine the value of the function g (see (5.28)) at tau_new(i)
38     for i = 1:k1+1
39         for j = 1:k1
40             s_tilde(j) = w_tilde(j)^2*max(tau_new(j)-tau_new_tilde(i),0);
41         end
42         s(i) = sum(s_tilde)-tau_new_tilde(i)-xi;
```

```

42     end
43
44     % find the unique i such that g(tau_new(i))>=0 and g(tau_new(i+1))<=0
45     for i = 1:k1
46         if (s(i) >= 0) && (s(i+1) <= 0)
47             l = w_tilde(i:end);
48             r = l.^2;
49             u = tau_new(i:end);
50             lambda = (1/(sum(r)+1))*(dot(r,u)-xi);
51         end
52     end
53
54     % calculate the projection (proj_y,proj_xi)
55     for i = 1:k1
56         if (norm_y(i) > lambda*w(i))
57             proj_y(i,:) = ((norm_y(i)-lambda*w(i))/norm_y(i)).*y(i,:);
58         else
59             proj_y(i,:) = zeros(1,k2);
60         end
61     end
62
63     proj_xi = xi+lambda;
64 end
65
66 end

```

The file **EpiSumNorms.m** is one of the main files and solves the extended location problem (5. 5). The given points are generated by the command *randn* and the given weights by the command *rand*, both data sets are saved in mat-files, respectively. The optimal solution is also saved in a mat-file. In the step where the projection onto the epigraph of the sum of weighted norms is calculated the file *projection\_weighted\_sum.m* is used.

```

1  % parallel splitting algorithm
2  clear all
3  clc
4  close all
5
6  nIterations = 25000; % define the number of iterations
7
8  maxBoundFromOpt = 1e-3; % maximum gap from the optimal solution
9
10 load 'optSol_g25_n5.mat' optimalSolution;
11
12 % p is a matrix of given points
13 load 'points_g25_dim2.mat' p;
14
15 % w is a matrix of given weights
16 load 'weights_g25_n5.mat' w;
17
18 k = size(w,2); % define number of new points
19
20
21 m = size(p,1); % number of given points
22 d = size(p,2); % dimension of the underlying space
23
24 z = zeros(k,d);
25 t = zeros(k,d);
26 p_tilde = zeros(k,d);
27 x = zeros(m+1,k*d+1);
28 l = zeros(k,d);
29 xnew = zeros(m+1,k*d+1);
30 y = zeros(m+1,k*d+1);
31
32 xVerlaufDR = [];
33 xnormDR = [];

```

```

34
35 % determine a feasible solution as startpoint
36 v = zeros(1,m);
37 for i = 1:m
38     v(i) = norm(p(i,:));
39 end
40
41 vtilde = zeros(m,k);
42 for i = 1:m
43     vtilde(i,:) = v(i)*ones(1,k);
44 end
45
46 vstar = zeros(1,m);
47 for i = 1:m
48     vstar(i) = dot(w(i,:),vtilde(i,:));
49 end
50 vstarmax = max(vstar(:));
51 startPoint = [zeros(1,k*d), vstarmax]; % vector of dimension k*d+1
52
53 nu = 33; % specify the parameter nu of the algorithm
54
55 for i = 1:m+1
56     x(i,:) = startPoint; % startpoint x_0 is a feasible solution
57 end
58
59 tic
60 for nIter = 1:nIterations
61
62     rk = (1/(m+1)).*sum(x); % current solution
63
64     xVerlaufDR(end+1,:) = rk; % save the current solution
65
66     y(1,:)=x(1,:)-[zeros(1,k*d) nu]; % calculate the proximal point of f_1
67
68     % calculate the proximal points of f_2,...f_{m+1}
69     for i=2:m+1
70         t(:,:)=reshape(x(i,1:k*d),[d,k])'; % write the row vector as a matrix
71
72         % create a matrix, where the row vectors are copies of the associated ...
73         % given point
74         for j=1:k
75             p_tilde(j,:)=p(i-1,:);
76         end
77
78         % calculate the projection onto the epigraph of the weighted sum of norms
79         [z(:,:),y(i,d*k+1)]=projection_weighted_sum(w(i-1,:)', ...
80             t(:,:)-p_tilde(:,:),x(i,k*d+1));
81         l(:,:)=p_tilde(:,:)+z(:,:);
82         y(i,1:d*k)=reshape(l(:,:)',[d*k,1])';
83     end
84
85     q = (1/(m+1)).*sum(y);
86
87     for i=1:m+1
88         xnew(i,:) = x(i,:) + 2.*q - rk - y(i,:);
89     end
90
91     x = xnew;
92
93     % if the gap between the current solution and the optimal solution is small ...
94     % enough, stop and display current solution
95     xnormDR(end+1,:) = norm(rk(1:d*k)-optimalSolution);
96     if ( norm(rk(1:d*k)-optimalSolution) <= maxBoundFromOpt )
97         disp(rk);
98         semilogx(xVerlaufDR(:,end));
99         figure
100         plot(xVerlaufDR(:,end));

```

```

98         figure
99         plot(xnormDR(:,end));
100         disp(toc);
101         break;
102     end
103 end
104 toc
105
106 xVerlaufDR(end+1,:) = rk;
107 % if the defined number of iterations is reached, display current solution
108 if ( nIter == nIterations )
109     disp(rk);
110     plot(xVerlaufDR(:,end));
111     disp(toc);
112 end

```

The file **EpiNorm.m** solves the extended location problem (5. 6). The projection onto the epigraph of the weighted norm is determined by the file `projection_weighted_sum.m`.

```

1  % parallel splitting algorithm
2  clearvars -except xnormDR xVerlaufDR
3  clc
4  close all
5  pause(1.0)
6
7  nIterations = 25000; % define the number of iterations
8
9  maxBoundFromOpt = 1e-3; % maximum gap from the optimal solution
10
11 load 'optSol.g25.n5.mat' optimalSolution;
12
13 % p is a matrix of given points
14 load 'points.g25.dim2.mat' p;
15
16 % w is a matrix of given weights
17 load 'weights.g25.n5.mat' w;
18
19 k = size(w,2); % define number of new points
20
21 m = size(p,1); % number of given points
22 d = size(p,2); % dimension of the underlying space
23
24 z = zeros(k,d,m);
25 r = zeros(m,k);
26 y2 = zeros(m,d*k+m*k+1);
27 y3 = zeros(m,k*d+m*k+1);
28 x2new = zeros(k,d*k+m*k+1,m);
29 x3new = zeros(m,k*d+m*k+1);
30 x2tilde = zeros(m,d*k+m*k+1);
31 y2tilde = zeros(m,d*k+m*k+1);
32 % determine a feasible solution as startpoint
33 nrm_w = zeros(m,k);
34 for j = 1:k
35     for i = 1:m
36         nrm_w(i,j) = w(i,j)*norm(p(i,:));
37     end
38 end
39
40 nrm_w_vec = reshape(nrm_w,[m*k,1])';
41
42 sum_nrm_w = zeros(m,1);
43
44 for i = 1:m
45     sum_nrm_w(i) = sum(nrm_w(i,:));
46 end
47

```

```

48 max_sum_nrm_w = max(sum_nrm_w);
49
50 startPoint = [zeros(1,k*d) nrm_w_vec max_sum_nrm_w]; % vector of dimension ...
               k*d+m*k+1
51
52 % startpoint is a feasible solution
53 x1 = startPoint;
54
55 x2 = zeros(k,d*k+m*k+1,m);
56 for j = 1:k
57     for i = 1:m
58         x2(j,:,i) = startPoint;
59     end
60 end
61
62 x3 = zeros(m,d*k+m*k+1);
63 for i = 1:m
64     x3(i,:) = startPoint;
65 end
66
67 nu = 33; % specify the parameter nu of the algorithm
68
69 xnormM = [];
70 xVerlaufM = [];
71 tic
72 for nIter = 1:nIterations
73
74     for i = 1:m
75         x2tilde(i,:) = sum(x2(:, :, i));
76     end
77
78     % sum up x_1, x_2 and x_3 into x
79     x = [x1;x2tilde;x3];
80
81     rk = (1/(m*k+m+1)).*sum(x); % current solution
82
83     xVerlaufM(end+1,:) = rk; % save the current solution
84
85     % calculate the proximal point of f_1
86     y1 = x1-[zeros(1,k*d+m*k) nu];
87
88     % calculate the proximal points of f_2,...f_k*d
89     for j = 1:k
90         for i = 1:m
91             C = zeros(d,k*d);
92             for l = 1:k
93                 if (l == j)
94                     C(:,d*(l-1)+1:d*(l-1)+d) = eye(d);
95                 end
96             end
97
98             X_2 = reshape(x2(j,k*d+1:k*d+m*k,i), [k,m])';
99
100             % calculate the projection onto the epigraph of the weighted norm
101             [z(j,:,i), r(i,j)] = ...
                projection_weighted_sum(w(i,j), (C*x2(j,1:k*d,i))'-p(i,:), X_2(i,j));
102             z(j,:,i) = z(j,:,i)+p(i,:);
103             teta = reshape(x2(j,1:k*d,i), [d,k])';
104             zeta = reshape(x2(j,k*d+1:k*d+k*m,i), [k,m])';
105             teta(j,:) = z(j,:,i);
106             zeta(i,j) = r(i,j);
107             y2(j,:,i) = [reshape(teta', [1,k*d]) reshape(zeta', [1,m*k]) ...
                x2(j,k*d+k*m+1,i)];
108         end
109     end
110
111     % calculate the proximal points of f_k*d+1,...f_k*d+k*m+1

```

```

112     for j = 1:m
113         a = [zeros(1,m*k), -1];
114         for i = 1:m
115             if i == j
116                 a(1,k*(i-1)+1:k*(i-1)+k) = ones(1,k);
117             end
118         end
119
120         if (dot(a,x3(j,k*d+1:k*d+m*k+1)) <= 0)
121             y3(j,:) = x3(j,:);
122         elseif (dot(a,x3(j,k*d+1:k*d+m*k+1)) > 0)
123             u = norm(a)^2;
124             u.tilde = dot(a,x3(j,k*d+1:k*d+m*k+1));
125             y3(j,:)=[x3(j,1:k*d) (x3(j,k*d+1:k*d+m*k+1)-(u.tilde/u).*a)];
126         end
127     end
128
129
130
131     for i = 1:m
132         y2tilde(i,:) = sum(y2(:, :, i));
133     end
134
135
136     y = [y1;y2tilde;y3];
137
138     q = (1/(m*k+m+1)).*sum(y);
139
140     x1new = x1 + 2.*q - rk - y1;
141     x1 = x1new;
142
143     for j = 1:k
144         for i = 1:m
145             x2new(j, :, i) = x2(j, :, i) + 2.*q - rk - y2(j, :, i);
146             x2(j, :, i) = x2new(j, :, i);
147         end
148     end
149
150     for i = 1:m
151         x3new(i, :) = x3(i, :) + 2.*q - rk - y3(i, :);
152         x3(i, :) = x3new(i, :);
153     end
154
155     % if the gap between the current solution and the optimal solution is small ...
156     enough, stop and display current solution
157     xnormM(end+1,:) = norm(rk(1:d*k)-optimalSolution);
158     if (norm(rk(1:d*k)-optimalSolution) <= maxBoundFromOpt)
159         disp(rk);
160         semilogx(xVerlaufDR(:,end));
161         hold on;
162         semilogx(xVerlaufM(:,end));
163         figure
164         plot(xnormDR(:,end));
165         hold on;
166         plot(xnormM(:,end));
167         disp(toc);
168         break;
169     end
170 end
171 toc
172
173 xVerlaufM(end+1,:) = rk;
174 xnormM(end+1,:) = norm(rk(1:d*k)-optimalSolution);
175 % if the defined number of iterations is reached, display current solution
176 if ( nIter == nIterations )
177     disp(rk);

```

```

178     plot(xVerlaufM(:,end));
179     disp(toc);
180 end

```

In the next we document which files were used for solving the location problem (5. 5) in the case where  $w_i = 1$  and  $\beta_i = 2$  for all  $i = 1, \dots, n$ .

The file **projection\_squared\_sum.m** calculates the projection onto the epigraph of the sum of squared norms.

```

1  % calculate the projection onto the epigraph of the sum of squared norms
2  function [x,t] = projection_squared_sum(y,xi)
3  h = zeros(1,size(y,1));
4  for j = 1:size(y,1)
5      h(j) = norm(y(j,:))^2;
6  end
7  x = zeros(size(y,1),size(y,2));
8  if ( sum(h) <= xi ) % check whether (y,xi) is an element of the epigraph
9      for j = size(y,1)
10         x(j,:) = y(j,:);
11     end
12     t = xi;
13 else % if (y,xi) is not an element of the epigraph, define a, b, c, d
14     a = 1;
15     b = (1+xi);
16     c = (1/4)*(1+4*xi);
17     d = (1/4)*(xi-sum(h));
18     p = [a b c d];
19     r = roots(p); % determine the roots of the cubic equation with the ...
                coefficients a, b, c, d
20     for i=1:3
21         if (r(i) > 0) % find the unique positive root
22             g = r(i);
23         end
24     end
25     % calculate the projection (x,t)
26     for j = size(y,1)
27         x(j,:) = (1/((2*g)+1))*y(j,:);
28     end
29     t = xi+g;
30 end
31 end

```

The file **squaredEpiSumNorm.m** solves the location problem (5. 5). In the step where the projection onto the epigraph of the sum of squared norms is calculated the file **projection\_squared\_sum.m** is used.

```

1  % parallel splitting algorithm
2  clear all
3  clc
4  close all
5  pause(1.0)
6
7  nIterations = 25000; % define the maximum number of iterations
8
9  maxBoundFromOpt = 1e-3; % maximum gap from the optimal solution
10
11 load 'squared_optSol_g25_n5.mat' optimalSolution;
12
13 % p is a matrix of given points
14 load 'points_g25_dim2.mat' p;
15
16 k = 5; % set number of new points
17

```

```

18 m = size(p,1); % number of given points
19 d = size(p,2); % dimension of the underlying space
20
21 a = zeros(1,m+1);
22 x = zeros(m+1,k*d+1);
23
24 % determine a feasible solution as startpoint
25 v = zeros(1,m);
26 for i = 1:m
27     v(i) = k*norm(p(i,:))^2;
28 end
29 vtilde = max(v(:));
30 startPoint = [zeros(1,k*d), vtilde]; % vector of dimension k*d+1
31
32 nu = 7; % specify the parameter nu of the algorithm
33
34 for i=1:m+1
35     x(i,:) = startPoint; % startpoint x_0 is a feasible solution
36 end
37
38 z = zeros(1,k*d);
39 xVerlaufDR = [];
40 xnormDR = [];
41 ptilde = zeros(k,d);
42 tic
43 for nIter = 1:nIterations
44
45     rk = (1/(m+1)).*sum(x); % current solution
46     xVerlaufDR(end+1,:) = rk; % save the current solution
47
48     y = zeros(m+1,k*d+1);
49     y(1,:) = x(1,:)-[zeros(1,k*d) nu]; % save the current solution
50
51     % calculate the proximal points of f_2,...f_{m+1}
52     for i = 2:m+1
53         % create a matrix, where the row vectors are copies of the associated ...
54         % given point
55         for j = 1:k
56             ptilde(j,:) = p(i-1,:);
57         end
58         ptildeStar = reshape(ptilde',[d*k,1])'; % rewrite the matrix ptilde as a ...
59         % row vector
60         % calculate the projection onto the epigraph of the sum of squared norms
61         [z(:,:),y(i,d*k+1)] = ...
62         projection.squared.sum(x(i,1:k*d)-ptildeStar,x(i,k*d+1));
63         y(i,1:d*k) = ptildeStar + z(:,:);
64     end
65
66     q = (1/(m+1)).*sum(y);
67
68     xnew = zeros(m+1,k*d+1);
69
70     for i=1:m+1
71         xnew(i,:) = x(i,:) + 2.*q - rk - y(i,:);
72     end
73
74     x = xnew;
75     % if the gap between the current solution and the optimal solution is small ...
76     % enough, stop and display current solution
77     xnormDR(end+1,:) = norm(rk(1:d*k)-optimalSolution);
78     if (norm(rk(1:d*k)-optimalSolution) <= maxBoundFromOpt)
79         disp(rk);
80         semilogx(xVerlaufDR(:,end));
81         figure
82         plot(xnormDR(:,end));

```



```

81         disp(toc);
82         break;
83     end
84 end
85 end
86 toc
87 xVerlaufDR(end+1,:) = rk;
88
89 % if the defined number of iterations is reached, display current solution
90 if (nIter == nIterations)
91     disp(rk);
92     %disp(['x1 = ' num2str(rk(1)) ' x2 = ' num2str(rk(2)) ' x3 = ' ...
93           num2str(rk(3)) ' x4 = ' num2str(rk(4)) ' t = ' num2str(rk(5)) ' ...
94           iterations: ' num2str(nIterations) ' gamma = ' num2str(g)]);
95     disp(toc);
96     plot(xVerlaufDR(:,end));
97     hold on;
98 end

```

The file **squaredEpiNorm.m** solves the location problem (5. 6), where the projection onto the epigraph of the squared norm is calculated by the m-file **projection\_squared\_sum.m**.

```

1  % parallel splitting algorithm
2  clearvars -except xVerlaufDR xnormDR
3  clc
4  close all
5  pause(1.0)
6
7  nIterations = 25000; % define the number of iterations
8
9  maxBoundFromOpt = 1e-3; % maximum gap from the optimal solution
10
11 load 'squaredOptSol_g25_n5.mat' optimalSolution;
12
13 % p is a matrix of given points
14 load 'points_g25_dim2.mat' p;
15
16 k = 5; % set number of new points
17
18 m = size(p,1); % number of given points
19 d = size(p,2); % dimension of the underlying space
20
21 % determine a feasible solution as startpoint
22
23 nrm2_p = zeros(m,k);
24 for j = 1:k
25     for i = 1:m
26         nrm2_p(i,j) = norm(p(i,:))^2;
27     end
28 end
29
30 nrm2_vec = reshape(nrm2_p,[m*k,1])';
31
32 sum_nrm2 = zeros(1,m);
33 for i = 1:m
34     sum_nrm2(i) = sum(nrm2_p(i,:));
35 end
36
37 max_sum_nrm2 = max(sum_nrm2);
38
39 startPoint = [zeros(1,k*d) nrm2_vec max_sum_nrm2]; % vector of dimension k*d+m*k+1
40
41 % startpoint is a feasible solution
42 x1 = startPoint;
43
44 x2 = zeros(k,k*d+m*k+1,m);

```

```

45 for j=1:k
46     for i=1:m
47         x2(j,:,i)=startPoint;
48     end
49 end
50
51 x3 = zeros(m,k*d+m*k+1);
52 for i=1:m
53     x3(i,:) = startPoint;
54 end
55
56 nu = 7; % specify the parameter nu of the algorithm
57
58 z = zeros(k,d,m);
59 r = zeros(m,k);
60
61 y2 = zeros(m,d*k+m*k+1);
62 y3 = zeros(m,k*d+m*k+1);
63 x2new = zeros(k,d*k+m*k+1,m);
64 x3new = zeros(m,k*d+m*k+1);
65 x2tilde = zeros(m,d*k+m*k+1);
66 y2tilde = zeros(m,d*k+m*k+1);
67
68 xnormM = [];
69 xVerlaufM = [];
70
71 tic
72 for nIter = 1:nIterations
73     % sum up x_1, x_2 and x_3 into x
74     for i = 1:m
75         x2tilde(i,:) = sum(x2(:, :, i));
76     end
77
78     % sum up x_1, x_2 and x_3 into x
79     x = [x1;x2tilde;x3];
80
81     rk = (1/(m*k+m+1)).*sum(x); % current solution
82
83     xVerlaufM(end+1,:) = rk; % save the current solution
84
85     % calculate the proximal point of f_1
86     y1 = x1-[zeros(1,k*d+m*k) nu];
87
88     % calculate the proximal points of f_2,...f_k*d
89     for j = 1:k
90         for i = 1:m
91             C = zeros(d,k*d);
92             for l = 1:k
93                 if (l == j)
94                     C(:,d*(l-1)+1:d*(l-1)+d) = eye(d);
95                 end
96             end
97
98             X_2 = reshape(x2(j,k*d+1:k*d+m*k,i),[k,m])';
99
100             % calculate the projection onto the epigraph of the squared norm
101             [z(j,:,i), r(i,j)] = ...
                projection_squared_sum((C*x2(j,1:k*d,i))'-p(i,:),X_2(i,j));
102
103             z(j,:,i) = z(j,:,i)+p(i,:);
104             teta = reshape(x2(j,1:k*d,i),[d,k])';
105             zeta = reshape(x2(j,k*d+1:k*d+k*m,i),[k,m])';
106             teta(j,:) = z(j,:,i);
107             zeta(i,j) = r(i,j);
108             teta_tilde = reshape(teta',[1,k*d]);
109             zeta_tilde = reshape(zeta',[1,m*k]);
110             u_tilde = [teta_tilde zeta_tilde x2(j,k*d+k*m+1,i)];

```

```

111         y2(j,:,i) = u_tilde;
112     end
113 end
114
115 % calculate the proximal points of f_k*d+1,...f_k*d+k*m+1
116 for j = 1:m
117     a = [zeros(1,m*k), -1];
118     for i = 1:m
119         if i == j
120             a(1,k*(i-1)+1:k*(i-1)+k) = ones(1,k);
121         end
122     end
123
124     if (dot(a,x3(j,k*d+1:k*d+m*k+1)) <= 0)
125         y3(j,:)=x3(j,:);
126     elseif (dot(a,x3(j,k*d+1:k*d+m*k+1)) > 0)
127         u = norm(a)^2;
128         u_tilde = dot(a,x3(j,k*d+1:k*d+m*k+1));
129         y3(j,:) = [x3(j,1:k*d) (x3(j,k*d+1:k*d+m*k+1)-(u_tilde/u).*a)];
130     end
131
132 end
133
134 for i = 1:m
135     y2tilde(i,:) = sum(y2(:, :, i));
136 end
137
138 y = [y1;y2tilde;y3];
139
140 q = (1/(m*k+m+1)).*sum(y);
141
142 x1new = x1 + 2.*q - rk - y1;
143 x1 = x1new;
144
145 for j = 1:k
146     for i = 1:m
147         x2new(j,:,i) = x2(j,:,i) + 2.*q- rk - y2(j,:,i);
148         x2(j,:,i) = x2new(j,:,i);
149     end
150 end
151
152 for i = 1:m
153     x3new(i,:) = x3(i,:) + 2.*q - rk - y3(i,:);
154     x3(i,:) = x3new(i,:);
155 end
156 % if the gap between the current solution and the optimal solution is small ...
157 % enough, stop and display current solution
158 xnormM(end+1,:) = norm(rk(1:d*k)-optimalSolution);
159 if ( norm(rk(1:d*k)-optimalSolution) <= maxBoundFromOpt )
160     disp(rk);
161     semilogx(xVerlaufDR(:,end));
162     hold on;
163     semilogx(xVerlaufM(:,end));
164     figure
165     plot(xnormDR(:,end));
166     hold on;
167     plot(xnormM(:,end));
168     disp(toc);
169     break;
170 end
171 end
172 toc
173
174 xVerlaufM(end+1,:) = rk;
175
176 % if the defined number of iterations is reached, display current solution

```

```
177 if (nIter == nIterations)
178     disp(rk);
179     plot(xVerlaufM(:,end));
180     disp(toc);
181 end
```

# Theses

## 1. The multi-composed optimization problem

$$(P^C) \quad \inf_{x \in \mathcal{A}} (f \circ F^1 \circ \dots \circ F^n)(x),$$

$$\mathcal{A} = \{x \in S : g(x) \in -Q\}$$

is introduced, where

- $Z$  is a Hausdorff locally convex space partially ordered by the convex cone  $Q \subseteq Z$  and  $X_i$  is a Hausdorff locally convex space partially ordered by the convex cone  $K_i \subseteq X_i$  for  $i = 0, \dots, n-1$ ,
- $S$  is a non-empty subset of the Hausdorff locally convex space  $X_n$ ,
- $f : \overline{X}_0 \rightarrow \overline{\mathbb{R}}$  is proper and  $K_0$ -increasing on  $F^1(\text{dom } F^1) + K_0 \subseteq \text{dom } f$ ,
- $F^i : \overline{X}_i \rightarrow \overline{X}_{i-1}$  is proper and  $(K_i, K_{i-1})$ -increasing on  $F^{i+1}(\text{dom } F^{i+1}) + K_i \subseteq \text{dom } F^i$  for  $i = 1, \dots, n-2$ ,
- $F^{n-1} : \overline{X}_{n-1} \rightarrow \overline{X}_{n-2} = X_{n-2}$  is proper and  $(K_{n-1}, K_{n-2})$ -increasing on  $F^n(\text{dom } F^n \cap \mathcal{A}) + K_{n-1} \subseteq \text{dom } F^{n-1}$ ,
- $F^n : X_n \rightarrow \overline{X}_{n-1} = X_{n-1}$  is a proper function and
- $g : X_n \rightarrow \overline{Z}$  is a proper function fulfilling  $S \cap g^{-1}(-Q) \cap ((F^n)^{-1} \circ \dots \circ (F^1)^{-1})(\text{dom } f) \neq \emptyset$ .

To  $(P^C)$  a corresponding conjugate dual problem  $(D^C)$  is constructed, where the conjugates of the functions involved in the objective function of  $(P^C)$  are split in the formulation of the dual  $(D^C)$

$$\sup_{\substack{z^{n*} \in Q^*, \quad z^{i*} \in K_i^*, \\ i=0, \dots, n-1}} \left\{ \inf_{x \in S} \{ \langle z^{(n-1)*}, F^n(x) \rangle + \langle z^{n*}, g(x) \rangle \} - f^*(z^{0*}) - \sum_{i=1}^{n-1} \langle z^{(i-1)*}, F^i \rangle^*(z^{i*}) \right\}.$$

For the primal-dual pair  $(P^C)$ – $(D^C)$  we prove weak duality and formulate associated regularity conditions of interiority type guaranteeing strong duality, the situation when the optimal objective values of the two problems are equal and the dual has an optimal solution. In this context we give necessary and sufficient optimality conditions by using conjugate functions and subdifferentials. This approach generalizes the results from the literature and opens a new way to investigate optimization problems.

As an application an optimization problem having as objective function the sum of reciprocals of concave functions is presented (see also [79]).

2. As a further application of the previous approach, we consider the unconstrained version of  $(P^C)$  and add a linear continuous functional to the objective function to derive a formula of the conjugate of the function  $\gamma = f \circ F^1 \circ \dots \circ F^n : X_n \rightarrow \overline{\mathbb{R}}$ , where the conjugates of the involved functions in  $\gamma$  are decomposed. We use the conjugate of  $\gamma$  to calculate also a formula for its biconjugate function, which reveals an alternative representation for  $\gamma$ .

3. We consider nonlinear single minimax location problems with geometric constraints of the form

$$(P_{h,a}^S) \quad \inf_{x \in S} \max_{1 \leq i \leq n} \{h_i(\gamma_{C_i}(x - p_i)) + a_i\},$$

where  $S$  is a non-empty, closed and convex subset of the Fréchet space  $X$  and for  $i = 1, \dots, n$ ,  $a_i \in \mathbb{R}_+$  are non-negative set-up costs,  $p_i \in X$  are distinct points,  $C_i$  are closed and convex subsets of  $X$  such that  $0_X \in \text{int } C_i$ ,  $\gamma_{C_i} : X \rightarrow \mathbb{R}$  are gauge functions of  $C_i$  and  $h_i : \mathbb{R} \rightarrow \mathbb{R}$ , defined by

$$h_i(x) := \begin{cases} h_i(x) \in \mathbb{R}_+, & \text{if } x \in \mathbb{R}_+, \\ +\infty, & \text{otherwise,} \end{cases}$$

are proper, convex, lower semicontinuous and increasing functions on  $\mathbb{R}_+$ . By using the results from the first part of this thesis, we attach to  $(P_{h,a}^S)$  a conjugate dual problem  $(D_{h,a}^S)$

$$\sup_{\substack{\lambda_i, z_i^{0*} \geq 0, z_i^{1*} \in X^*, i=1, \dots, n, \\ I = \{i \in \{1, \dots, n\} : z_i^{0*} > 0\} \subseteq R = \{r \in \{1, \dots, n\} : \lambda_r > 0\}, \\ \gamma_{C_i^0}(z_i^{1*}) \leq z_i^{0*}, i \in I, z_j^{1*} = 0_{X^*}, j \notin I, \sum_{r \in R} \lambda_r \leq 1}} \left\{ \inf_{x \in S} \left\{ \sum_{i \in I} \langle z_i^{1*}, x - p_i \rangle \right\} - \sum_{r \in R} \lambda_r \left[ h_r^* \left( \frac{z_r^{0*}}{\lambda_r} \right) - a_r \right] \right\}$$

and prove strong duality in this framework. This approach allows us to formulate more detailed necessary and sufficient optimality conditions expressed via conjugate functions, dual gauges, subdifferentials and normal cones.

Moreover, we consider the primal-dual pair  $(P_{h,a}^S)-(D_{h,a}^S)$  in different settings and show in this way further connections between these two problems. For the situation when the underlying space is a Hilbert space, the subset  $S$  is the whole space and the distances are measured by the norm defined by the scalar product of the Hilbert space we give a formula which provides the optimal solution of the primal problem from the optimal solution of the dual.

In addition, we present for the linear single minimax location problem a second dual problem reducing the number of dual variables compared with the first formulated one. Then, we give in the framework of the Euclidean space without constraints a geometrical interpretation of the set of optimal solutions of this dual and show that its Lagrange dual problem coincides with the original location problem (see also [81]).

4. We consider the extended multifacility location problem in a more general setting as introduced by Drezner in [35] (see also [30, 67]):

$$(EP_a^M) \quad \inf_{(x_1, \dots, x_m) \in X^m} \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^m \gamma_{C_{ij}}(x_j - p_i) + a_i \right\},$$

where  $X$  is a Fréchet space,  $a_i \in \mathbb{R}_+$  are non-negative set-up costs,  $p_i \in X$  are distinct points and  $\gamma_{C_{ij}} : X \rightarrow \mathbb{R}$  are gauges defined by closed and convex subsets  $C_{ij}$  of  $X$  such that  $0_X \in \text{int } C_{ij}$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, m$ . We show that  $(EP_a^M)$  can be rewritten as a single minimax location problem and apply the previous results to formulate a corresponding conjugate dual problem

$$(ED_a^M) \quad \sup_{(z_1^{0*}, \dots, z_n^{0*}, z_1^{1*}, \dots, z_n^{1*}) \in \mathcal{C}} \left\{ - \sum_{i \in I} \left[ \left\langle \sum_{j=1}^m z_{ij}^{1*}, p_i \right\rangle - z_i^{0*} a_i \right] \right\},$$

where

$$\mathcal{C} = \left\{ (z_1^{0*}, \dots, z_n^{0*}, z_1^{1*}, \dots, z_n^{1*}) \in \mathbb{R}_+^n \times (X^*)^m \times \dots \times (X^*)^m : I = \{i \in \{1, \dots, n\} : z_i^{0*} > 0\} \right. \\ \left. z_{kj}^{1*} = 0_{X^*}, k \notin I, \gamma_{C_{ij}^0}(z_{ij}^{1*}) \leq z_i^{0*}, i \in I, \sum_{i \in I} z_{ij}^{1*} = 0_{X^*}, j = 1, \dots, m, \sum_{i \in I} z_i^{0*} \leq 1 \right\}$$

as well as associated necessary and sufficient optimality conditions.

Further, we study the scenario in the Hilbert space  $\mathcal{H}$  where the weights have a multiplicative structure (see [30]) and present a second dual problem for which we give a geometrical interpretation of the set of optimal solutions when  $\mathcal{H} = \mathbb{R}^d$  (see [80]).

5. Via our approach for multi-composed optimization problems we assign a conjugate dual problem  $(D^M)$  to the following multifacility minimax location problem

$$(P^M) \quad \inf_{(x_1, \dots, x_m) \in S} \max \left\{ w_{jk} \gamma_{C_{jk}}(x_j - x_k), jk \in V, \tilde{w}_{ji} \gamma_{\tilde{C}_{ji}}(x_j - p_i), ji \in \tilde{V} \right\},$$

where  $X$  is a Fréchet space,  $p_i \in X$ ,  $i = 1, \dots, t$ , are distinct points,  $C_{jk} \subseteq X$  with  $0_X \in \text{int } C_{jk}$  for  $jk \in V := \{jk : 1 \leq j \leq m, 1 \leq k \leq m, j \neq k, w_{jk} > 0\}$ , and  $\tilde{C}_{ji} \subseteq X$  with  $0_X \in \text{int } \tilde{C}_{ji}$  for  $ji \in \tilde{V} := \{1 \leq j \leq m, 1 \leq i \leq t, \tilde{w}_{ji} > 0\}$ , be closed and convex,  $S \subseteq X^m$  non-empty, closed and convex as wells as  $\gamma_{C_{jk}} : X \rightarrow \mathbb{R}$ ,  $jk \in V$ , and  $\gamma_{\tilde{C}_{ji}} : X \rightarrow \mathbb{R}$ ,  $ji \in \tilde{V}$ , be gauges. We show that strong duality holds between  $(P^M)$  and its dual

$$(D^M) \quad \sup_{(z^{0*}, \tilde{z}^{0*}, z^{1*}, \tilde{z}^{1*}) \in \mathcal{B}} \inf_{x \in S} \left\{ \sum_{jk \in I} \langle z_{jk}^{1*}, A_{jk}x \rangle + \sum_{ji \in \tilde{I}} \langle \tilde{z}_{ji}^{1*}, B_{ji}x - p_i \rangle \right\},$$

with

$$\mathcal{B} = \left\{ (z^{0*}, \tilde{z}^{0*}, z^{1*}, \tilde{z}^{1*}) \in \mathbb{R}_+^{|V|} \times \mathbb{R}_+^{|\tilde{V}|} \times (X^*)^{|V|} \times (X^*)^{|\tilde{V}|} : I = \{jk \in V : z_{jk}^{0*} > 0\}, \right. \\ \tilde{I} = \{ji \in \tilde{V} : \tilde{z}_{ji}^{0*} > 0\}, z_{ef}^{1*} = 0_{X^*}, ef \notin I, \gamma_{C_{jk}^0}(z_{jk}^{1*}) \leq z_{jk}^{0*}, jk \in I, \\ \left. \tilde{z}_{ed}^{1*} \in 0_{X^*}, ed \notin \tilde{I}, \gamma_{\tilde{C}_{ji}^0}(\tilde{z}_{ji}^{1*}) \leq \tilde{z}_{ji}^{0*}, ji \in \tilde{I}, \sum_{jk \in I} \frac{1}{w_{jk}} z_{jk}^{0*} + \sum_{ji \in \tilde{I}} \frac{1}{\tilde{w}_{ji}} \tilde{z}_{ji}^{0*} \leq 1 \right\},$$

where  $A_{jk}$ ,  $jk \in I$ , and  $B_{ji}$ ,  $ji \in \tilde{I}$ , are linear mappings and present necessary and sufficient optimality conditions using conjugate functions, dual gauges, subdifferentials and normal cones.

Apart from this approach we introduce a second dual problem reducing the number of constraints and dual variables compared with  $(D^M)$  and give a geometrical interpretation for the set of optimal solutions of this dual for  $S = X = \mathbb{R}^d$ . In the context of this dual we also demonstrate that the bidual of  $(P^M)$  is identical to  $(P^M)$  (see also [82]).

6. For solving extended multifacility location problems in Hilbert spaces  $\mathcal{H}_i$ ,  $i = 1, \dots, n$ , numerically by proximal methods we present first a general formula of the projection onto the epigraph of the function  $h : \mathcal{H}_1 \times \dots \times \mathcal{H}_n \rightarrow \mathbb{R}$ , defined by  $h(x_1, \dots, x_n) := \sum_{i=1}^n w_i \|x_i\|_{\mathcal{H}_i}^{\beta_i}$ . We consider the situations when  $\beta_i = 1$ ,  $i = 1, \dots, n$ , and  $w_i = 1$ ,  $\beta_i = 2$ ,  $i = 1, \dots, n$ , where the formulae given for instance in [2, 28, 29] turn out to be special cases for  $n = 1$  of our considerations.

Moreover, we develop a formula for the projection onto the epigraph of a gauge function  $\gamma_C : X \rightarrow \mathbb{R}$  of a closed and convex set  $C \subseteq \mathcal{H}$  with  $0_{\mathcal{H}} \in C$ . As a consequence, we derive a

formula for the projection onto the epigraph of the gauge of a closed and convex cone as well as the sum of gauges. Finally, two examples are considered to demonstrate how the latter formula can be used to determine the projector.

7. We apply the formula for the projection onto the epigraph of the weighted sum of powers of norms for solving extended multifacility location problems numerically by the parallel splitting algorithm and compare our method with the one presented in [30], where the formula of the projection onto the epigraph of the weighted power of norm is required. The numerical tests show that our method clearly outperforms the one proposed in [30] from the viewpoints of accuracy, CPU speed and number of iterations.



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# Lebenslauf

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### Publikationen

- L. Altangerel, G. Wanka, **O. Wilfer**: *An Oriented Distance Function Application to Gap Functions for Vector Variational Inequalities*. Optimization, Simulation, and Control 76, 17-34, 2013.
- G. Wanka, **O. Wilfer**: *A Lagrange Duality Approach for Multi-Composed Optimization Problems*. TOP, 2016. doi: 10.1007/s11750-016-0431-2.
- S.-M. Grad, G. Wanka, **O. Wilfer**: *Duality and  $\varepsilon$ -Optimality Conditions for Multi-composed Optimization Problems with Applications to Fractional and Entropy Optimization*. Pure and Applied Functional Analysis 2(1), 43-63, 2017.
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- G. Wanka, **O. Wilfer**: *Multifacility Minimax Location Problems via Multi-Composed Optimization*. Preprint 2016-6, ISSN 1614-8835, 2016.

### Vorträge

- *Duality for Convex Continuous Location Problems via Multi-Composed Optimization*, International Symposium on Location Decision (ISOLDE), Neapel/Capri, Italien, Juni 2014.
- *Dualitätsaussagen für Erweiterte Minimax Standortprobleme*, Workshop Optimierung, Ehrenfriedersdorf, Deutschland, Februar 2015.
- *Duality Results for Extended Multifacility Minimax Location Problems*, Seminar Optimization, Universität Wien, Österreich, März 2015.



## Erklärung gemäß §6 der Promotionsordnung

Hiermit erkläre ich an Eides Statt, dass ich die von mir eingereichte Arbeit “*Duality investigations for multi-composed optimization problems with applications in location theory*” selbstständig und nur unter Benutzung der in der Arbeit angegebenen Quellen und Hilfsmittel angefertigt habe.

Chemnitz, den 26.10.2016

Oleg Wilfer