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Asymptotic bounds and values for the norm of the Laplace operator and other partial differential operators on spaces of polynomials

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1 Introduction and results

This work is about Markov-type inequalities for the Laplace and some other partial differential operators. A Markov-type inequality is an inequality giving upper bounds for norms of linear combinations of ordinary or partial derivatives of polynomials by the norm of the polynomials, that is, an inequality of the form

$$\|\mathcal{D}f\| \leq C\|f\|, \quad \forall f \in \mathcal{P}, \tag{1.1}$$

where $\mathcal{P}$ is a normed space of polynomials, $\mathcal{D} : \mathcal{P} \to \mathcal{P}$ is a linear combination of (partial) derivatives and C is a constant not depending on f. If $\mathcal{D}$ is bounded, the best constant in (1.1) is of course the operator norm of the operator.

We here consider the following situation. For positive integers $n, N \in \mathbb{N}$ we set $E := [0,1]^N$ or $E := \Omega_N := \left\{(x_1, \ldots, x_N) \in [0,1]^N : \ x_1 + \ldots + x_N \leq 1\right\}$ and denote by $\mathcal{P}_n(E)$ the finite-dimensional linear space of all polynomials of the form

$$f(t_1, \ldots, t_N) := \sum_{i_1/n, \ldots, i_N/n \in E} c_{t_1,\ldots,i_N} t_1^{i_1} \ldots t_N^{i_N}$$

with complex coefficients. So $\mathcal{P}_n^N := \mathcal{P}_n\left([0,1]^N\right)$ consists of all complex polynomials with N variables where each variable occurs with power of at most n, whereas $\mathcal{P}_n\left(\Omega_N\right)$ includes all complex polynomials in which in every summand the sum of all powers is bounded by n. We equip this space with the Laguerre norm

$$\|f\|^2 := \int_{(0,\infty)^N} |f(t_1, \ldots, t_N)|^2 e^{-t_1} \ldots e^{-t_N} \ dt_1 \ldots dt_N, \tag{1.2}$$

the Hermite norm

$$\|f\|^2 := \int_{\mathbb{R}^N} |f(t_1, \ldots, t_N)|^2 e^{-t_1^2} \ldots e^{-t_N^2} \ dt_1 \ldots dt_N \tag{1.3}$$

or the Legendre norm

$$\|f\|^2 := \int_{[-1,1]^N} |f(t_1, \ldots, t_N)|^2 \ dt_1 \ldots dt_N. \tag{1.4}$$

The Laplace operator

$$\Delta := \frac{\partial^2}{\partial t_1^2} + \ldots + \frac{\partial^2}{\partial t_N^2} : \mathcal{P}_n(E) \to \mathcal{P}_n(E)$$

is in all three cases a linear and bounded operator, thus there exists a finite constant C depending on n, N, E and the norm on $\mathcal{P}_n(E)$ such that the inequality

$$\|\Delta f\| \leq C\|f\| \tag{1.5}$$

holds for all $f \in \mathcal{P}_n(E)$. The same holds for the operators

$$\Delta_\nu := \frac{\partial^\nu}{\partial t_1^\nu} + \ldots + \frac{\partial^\nu}{\partial t_N^\nu} \quad \text{and} \quad \Delta^2,$$

$\nu \in \mathbb{N}$. We are interested in the smallest constant such that inequality (1.5) holds for all $f \in \mathcal{P}_n(E)$, that is, we ask for the operator norm of Δ on $\mathcal{P}_n(E)$ with respect to the different weighted L^2-norms on the spaces of polynomials.

We denote this norm by $\lambda\left(\Delta \,|\mathcal{P}_n(E)\right)$ if $\mathcal{P}_n(E)$ is equipped with norm (1.2), by $\eta\left(\Delta \,|\mathcal{P}_n(E)\right)$ in the case of the norm (1.3) and by $\gamma\left(\Delta \,|\mathcal{P}_n(E)\right)$ in the case of norm (1.4) (of course we use the same notation of the norms when we replace Δ by an other linear operator on $\mathcal{P}_n(E)$). Furthermore we use the notation $a_n \sim b_n$ if $\frac{a_n}{b_n}$ goes to one as n goes to infinity.

We have the following asymptotic bounds and values for the norms.

Theorem 1.1. *For all n and N in $\mathbb{N}$ we have*

$$\frac{N}{6}(n^2-n) \leq \lambda\left(\Delta \,\middle|\mathcal{P}_n^N\right) \leq \frac{N}{2}(n^2-n).$$

Thus, asymptotically we get

$$0.1666\ldots \leq \liminf_{n\to\infty} \frac{\lambda\left(\Delta \,\middle|\mathcal{P}_n^N\right)}{Nn^2} \leq \limsup_{n\to\infty} \frac{\lambda\left(\Delta \,\middle|\mathcal{P}_n^N\right)}{Nn^2} \leq 0.5000$$

for each N. The following asymptotic result narrows the gap.

Theorem 1.2. *For each $N \in \mathbb{N}$,*

$$0.173 \leq \liminf_{n\to\infty} \frac{\lambda\left(\Delta \,\middle|\mathcal{P}_n^N\right)}{Nn^2} \leq \limsup_{n\to\infty} \frac{\lambda\left(\Delta \,\middle|\mathcal{P}_n^N\right)}{Nn^2} \leq 0.285.$$

In the case of the Hermite norm we get even an asymptotic value for the norm of the operator.

Theorem 1.3. *For all $N \in \mathbb{N}$ we have for the norm of the operator $\Delta : \mathcal{P}_n^N \to \mathcal{P}_n^N$ the asymptotic value*

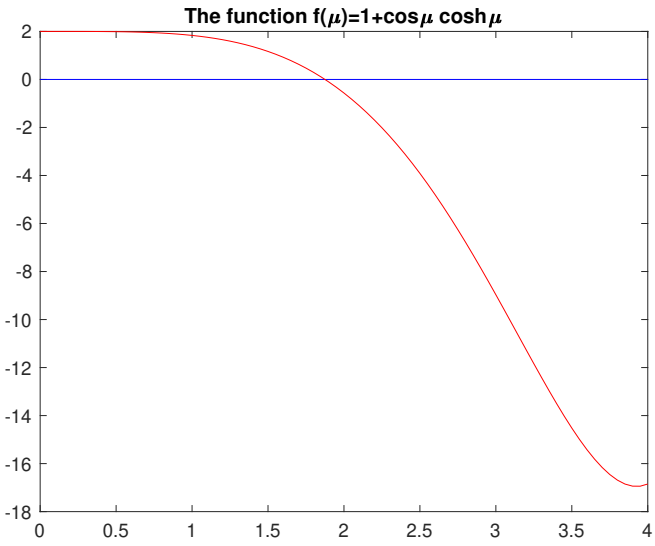
$$\lim_{n\to\infty} \frac{\eta\left(\Delta \,\middle|\mathcal{P}_n^N\right)}{Nn} = 2.$$

Finally, for the norm in the Legendre case we can establish the following asymptotic bounds.

Theorem 1.4. *In the Legendre case, for each $N \in \mathbb{N}$ the norm of the operator $\Delta : \mathcal{P}_n^N \to \mathcal{P}_n^N$ satisfies*

$$0.041667 \approx \frac{1}{24} \leq \liminf_{n\to\infty} \frac{\gamma\left(\Delta \,\middle|\mathcal{P}_n^N\right)}{Nn^4} \leq \limsup_{n\to\infty} \frac{\gamma\left(\Delta \,\middle|\mathcal{P}_n^N\right)}{Nn^4} \leq \frac{1}{4\mu_0^2} \approx 0.07110326,$$

where $\mu_0 = 1.8746\ldots$ is the smallest positive solution of the equation $1 + \cosh\mu\cos\mu = 0$.



Things are much more involved for Ω_n in place of $[0,1]^N$. In this context we are at least able to prove the following results.

Theorem 1.5. *The norm of the operator $\Delta : \mathcal{P}_n(\Omega_N) \to \mathcal{P}_n(\Omega_N)$, where we equip $\mathcal{P}_n(\Omega_N)$ with the Laguerre norm, satisfies for arbitrary $N \in \mathbb{N}$*

$$0.173 \cdot \frac{1}{N^2} \leq \liminf_{n\to\infty} \frac{\lambda(\Delta|\mathcal{P}_n(\Omega_N))}{Nn^2} \leq \limsup_{n\to\infty} \frac{\lambda(\Delta|\mathcal{P}_n(\Omega_N))}{Nn^2} \leq 0.285.$$

Theorem 1.6. *The norm of the operator $\Delta : \mathcal{P}_n(\Omega_N) \to \mathcal{P}_n(\Omega_N)$, where we equip $\mathcal{P}_n(\Omega_N)$ with the Hermite norm, satisfies for $N \in \{2,3\}$ the inequality*

$$\eta(\Delta|\mathcal{P}_n(\Omega_N)) \geq 2\sqrt{n^2-n+2}.$$

For arbitrary $N \in \mathbb{N}$ the operator norm satisfies

$$\frac{2}{N} \leq \liminf_{n\to\infty} \frac{\eta(\Delta|\mathcal{P}_n(\Omega_N))}{Nn} \leq \limsup_{n\to\infty} \frac{\eta(\Delta|\mathcal{P}_n(\Omega_N))}{Nn} \leq 2.$$

Theorem 1.7. *The norm of the operator $\Delta : \mathcal{P}_n(\Omega_N) \to \mathcal{P}_n(\Omega_N)$, where we equip $\mathcal{P}_n(\Omega_N)$ with the Legendre norm, satisfies for arbitrary $N \in \mathbb{N}$*

$$\frac{1}{24N^4} \leq \liminf_{n\to\infty} \frac{\gamma(\Delta|\mathcal{P}_n(\Omega_N))}{Nn^4} \leq \limsup_{n\to\infty} \frac{\gamma(\Delta|\mathcal{P}_n(\Omega_N))}{Nn^4} \leq \frac{1}{4\mu_0^2}.$$

The norm of Δ is in all cases the largest singular value of its matrix representation $[\Delta]$ with respect to an orthonormal basis of $\mathcal{P}_n(E)$. This matrix representation is an $(n+1)^N \times (n+1)^N$ matrix in the case $E = [0,1]^N$, which allows us to compute the largest singular value numerically for moderately sized values of n provided

$N \leq 3$, but things go to the limit of the capacity of a usual computer for, say, $n = 100$ and $N = 5$. For $N = 1$, in which case Δ is simply d^2/dt^2, Shampine [23] proved that actually $\lambda\left(D^2 \left| \mathcal{P}_n \right.\right)/n^2$ converges to $1/\mu_0^2 = 0.28441\ldots$, where μ_0 is the smallest positive solution of the equation $1 + \cos\mu \cosh\mu = 0$. Thus, for $N = 1$, the upper bound in Theorem 1.2 is well-known so far. Here are some results obtained numerically for $1 \leq N \leq 7$ in the case of the Laguerre norm.

$N=1$	n	10	50	100	500	1000
	$\lambda\left(\Delta \left\| \mathcal{P}_n^N \right.\right)/n^2$	0.2829	0.2844	0.2844	0.2844	0.2844

$N=2$	n	10	50	100	500	1000
	$\lambda\left(\Delta \left\| \mathcal{P}_n^N \right.\right)/(2n^2)$	0.2107	0.2175	0.2183	0.2189	0.2190

$N=3$	n	10	20	30	50	100
	$\lambda\left(\Delta \left\| \mathcal{P}_n^N \right.\right)/(3n^2)$	0.1950	0.1993	0.2008	0.2020	0.2029

$N=4$	n	10	20	30	50	100
	$\lambda\left(\Delta \left\| \mathcal{P}_n^N \right.\right)/(4n^2)$	0.1855	0.1906	0.1923	0.1937	0.1947

$N=5$	n	10	20	30	50
	$\lambda\left(\Delta \left\| \mathcal{P}_n^N \right.\right)/(5n^2)$	0.1804	0.1857	0.1875	0.1889

$N=6$	n	10	20	30
	$\lambda\left(\Delta \left\| \mathcal{P}_n^N \right.\right)/(6n^2)$	0.1770	0.1824	0.1843

$N=7$	n	10
	$\lambda\left(\Delta \left\| \mathcal{P}_n^N \right.\right)/(7n^2)$	0.1745

These results indicate that the lower bound in Theorem 1.2 is perhaps not too far from the truth. The following table shows numerical values for the Hermite norm.

$N=1$	n	10	50	100	500	1000
	$\eta\left(\Delta \left\| \mathcal{P}_n^N \right.\right)/n$	1.8974	1.9799	1.9900	1.9980	1.9990

$N=2$	n	10	50	100	500	1000
	$\eta\left(\Delta \left\| \mathcal{P}_n^N \right.\right)/(2n)$	1.5362	1.7863	1.8539	1.9438	1.9634

$N=3$	n	10	20	30	50	100
	$\eta\left(\Delta \left\| \mathcal{P}_n^N \right.\right)/(3n)$	1.4329	1.5870	1.6647	1.7444	1.8271

$N=4$	n	10	20	30	50	100
	$\eta\left(\Delta \left\| \mathcal{P}_n^N \right.\right)/(4n)$	1.3868	1.5581	1.6396	1.7260	1.8144

$N=5$	n	10	20	30
	$\eta\left(\Delta\left\vert\mathcal{P}_n^N\right.\right)/(5n)$	1.3603	1.5395	1.6251

$N=6$	n	10
	$\eta\left(\Delta\left\vert\mathcal{P}_n^N\right.\right)/(6n)$	1.3445

Finally, here are results for the Legendre norm.

$N=1$	n	10	50	100	500	1000
	$\gamma\left(\Delta\left\vert\mathcal{P}_n^N\right.\right)/n^4$	0.1201	0.0799	0.0754	0.0720	0.0715

$N=2$	n	10	50	100	500
	$\gamma\left(\Delta\left\vert\mathcal{P}_n^N\right.\right)/(2n^4)$	0.0910	0.0615	0.0581	0.0554

$N=3$	n	10	20	30	50	100
	$\gamma\left(\Delta\left\vert\mathcal{P}_n^N\right.\right)/(3n^4)$	0.0843	0.0670	0.0615	0.0572	0.0540

$N=4$	n	10	20	30	50
	$\gamma\left(\Delta\left\vert\mathcal{P}_n^N\right.\right)/(4n^4)$	0.0807	0.0642	0.0590	0.0549

$N=5$	n	10	20	30
	$\gamma\left(\Delta\left\vert\mathcal{P}_n^N\right.\right)/(5n^4)$	0.0786	0.0627	0.0576

Remark 1.8. The numerical examples presented in the tables in this section were computed with a program written by Holger Langenau. The program calculates the norm of an arbitrary linear partial differential operator with constant coefficients on $\mathcal{P}_n^N$ with respect to the (weighted) Laguerre, Hermite, Gegenbauer or Jacobi norms. To compute the norm of the differential operator, the program calculates the largest singular value of the corresponding matrix representation by the power method. In the Laguerre case the program benefits from the Toeplitz structure of the matrix representation to perform the matrix vector multiplications quickly.

The text has the following structure. We start with a historic overview and some motivations of the problem. Then, for each of the three norms, we give the matrix representation $[\Delta]$ of the Laplace operator with respect to an orthonormal basis in $\mathcal{P}_n^N$ for arbitrary $N \in \mathbb{N}$ and a basis in $\mathcal{P}_n(\Omega_N)$ in the Hermite case for $N = 2, 3$. The operator norm of Δ is then equal to the spectral norm of the matrix representation, that is, to the largest singular value of $[\Delta]$.

Section 4 is devoted to the proofs of the main results. The proofs of Theorems 1.1 and 1.2 can be found in [7], Theorems 1.3 and 1.4 were published in [20], Theorems 1.5, 1.6 and 1.7 are presented here for the first time.

In the first part of Section 4 we present a general estimate for the largest singular value of a matrix. We will apply this result to estimate the operator norm of Δ for the Laguerre, the Hermite and the Legendre norm on $\mathcal{P}_n^N$.

The second part of this section is devoted to the proof of Theorem 1.1. For this, we will use a method presented in [6]: given a sequence $b = (b_j)_{j=-\infty}^{\infty}$ of complex numbers, one constructs Laurent polynomials $S_n b$ and $\sigma_n b$ such that

$$\|\sigma_n b\|_\infty \leq \|T_n(b)\|_2 \leq \|S_n b\|_\infty$$

for the $n \times n$ Toeplitz matrix $T_n(b) := (b_{j-k})_{j,k=0}^{n-1}$ (Theorem 5.10 in [6]). We establish a generalization of this result to multilevel Toeplitz operators and use this to prove the bounds for $\lambda\left(\Delta \,\middle|\, \mathcal{P}_n^N\right)$.

Subsection 4.2 also contains the proof of Theorem 1.2. For that we estimate the largest singular value of the matrix representation with the help of results from [11] and [1]. Here we use the special Toeplitz matrix structure of the representation of the ordinary differential operator D^2, sending a function to its second derivative. To be more concrete, we will estimate the singular values by the eigenvalues of the so-called hermitian part of the representation of D^2, which turns out to be an hermitian Toeplitz matrix with polynomially increasing entries.

In Subsection 4.3 we will prove Theorem 1.3. As in the Laguerre case, we consider the hermitian part of the matrix representation $[D^2]$, now of course with respect to the Hermite polynomials. We will see that a transformation of this matrix converges to a Toeplitz operator and that we can use this to prove the theorem.

In Subsection 4.4 we will present the proof of Theorem 1.4, which is based on the estimation of eigenvalues of hermitian matrices by Rayleigh quotients.

Subsection 4.5 is devoted to the proofs of the Theorems 1.5, 1.6 and 1.7. In the Laguerre and Legendre case the proves use the upper and lower asymptotic bounds known for the operator on $\mathcal{P}_n^N$, while we apply in the Hermite case a theorem presented in [17].

In Section 5 we present results about the asymptotics of norms of the operators Δ^2 and Δ_ν, $\nu \in \mathbb{N}$. These operators are examples of other partial differential operators on on $\mathcal{P}_n(E)$ that can be tackled by the methods from Section 4.

Section 6 finally gives an overview over the maybe most interesting open problems.

2 Historic overview and motivation

The history of Markov-type inequalities started with the work of the Markov brothers [15] and [16] in the late 19th century. They found best constants in inequalities of the form $\|D^\nu f\| \leq C\|f\|$ for $f \in \mathcal{P}_n := \mathcal{P}_n^1$ with the L^∞-norm over a finite interval, where $D^\nu : f \mapsto f^{(\nu)}$ is the operator of taking the νth derivative. Andrei Markov considered the case $\nu = 1$ and Vladimir Markov the case $\nu \geq 2$.

Erhard Schmidt [21] was the first who replaced the L^∞-norm by L^2-norms. In 1944,

he proved in [22] that

$$\lambda\left(D\,|\mathcal{P}_n\right) \sim \frac{2}{\pi}n, \quad \eta\left(D\,|\mathcal{P}_n\right) = \sqrt{2n} \quad \text{and} \quad \gamma\left(D\,|\mathcal{P}_n\right) \sim \frac{1}{\pi}n^2.$$

Shampine [23] showed the result cited above:

$$\lambda\left(D^2\,|\mathcal{P}_n\right) \sim \frac{n^2}{\mu_0^2} \quad \text{and} \quad \gamma\left(D^2\,|\mathcal{P}_n\right) \sim \frac{n^4}{4\mu_0^4}, \tag{2.1}$$

where μ_0 is the smallest positive solution of the equation $1+\cos\mu\cosh\mu = 0$. Note that $1/(\mu_0^2) = 0.28441....$

In the late 1980s, Peter Dörfler considered higher order derivatives on $\mathcal{P}_n$ equipped with the Laguerre norm. In [9] and [10] he showed

$$\frac{1}{2\nu!}\sqrt{\frac{4}{2\nu+1}} \le \liminf_{n\to\infty}\frac{\lambda\left(D^\nu\,|\mathcal{P}_n\right)}{n^\nu} \le \limsup_{n\to\infty}\frac{\lambda\left(D^\nu\,|\mathcal{P}_n\right)}{n^\nu} \le \frac{1}{2\nu!}\sqrt{\frac{2\nu}{2\nu-1}}$$

for $\nu \ge 1$, but it was not known if the limit of $\lambda\left(D^\nu\,|\mathcal{P}_n\right)/n^\nu$ exists. The existence of this limit was proved in 2009 by Böttcher and Dörfler [3], and these authors also showed that this limit is the norm $\|L_\nu\|$ of the Volterra integral operator on $L^2(0,1)$ defined by

$$(L_\nu f)(x) := \frac{1}{\Gamma(\nu)}\int_x^1 (y-x)^{\nu-1} f(y)\,dy$$

for all $x \in (0,1)$.

Böttcher and Dörfler considered higher order derivatives on $\mathcal{P}_n$ equipped with the Legendre norm in [4]. Again, the limit of the appropriately scaled norms was shown to be the norm of a certain integral operator. More concretely, they found

$$\lim_{n\to\infty}\frac{\gamma\left(D^\nu\,|\mathcal{P}_n\right)}{n^{2\nu}} = \|G_\nu\|,$$

where G_ν is the integral operator on $L^2(0,1)$ given by

$$(G_\nu f)(x) = \frac{1}{2^{\nu-1}\Gamma(\nu)}\int_x^1 \sqrt{xy}\left(y^2-x^2\right)^{\nu-1} f(y)\,dy, \quad x \in (0,1).$$

Before [4] it was also in this case not clear whether the limit of $\gamma\left(D^\nu\,|\mathcal{P}_n\right)/n^{2\nu}$ actually exists.

Multivariate Markov-type inequalities have been studied for example in [13] and [18]. A main topic of research so far was considerations for the L^∞-norm. In [5], Böttcher and Dörfler first obtained best constants in Markov-type inequalities for linear partial differential operators of arbitrary orders with constant coefficients acting on a linear space $\mathcal{P}_n(E)$ of polynomials equipped with classical L^2-norms.

More concretely, they considered the following situation. Let n and N be positive integers and let be E a closed subset of $[0,1]^N$ such that E contains a point of the interior of $[0,1]^N$ and that E contains the cube $[0,x_1]\times\ldots\times[0,x_N]$ for each of its points $(x_1,\ldots,x_N)$. Choices for E are for example $E=[0,1]^N$ or

$$E=\Omega_N:=\left\{(x_1,\ldots,x_N)\in[0,1]^N:\ x_1+\ldots+x_N\le 1\right\}.$$

The space $\mathcal{P}_n(E)$ is the linear space of all complex polynomials of the form

$$f(t_1,\ldots,t_N)=\sum_{(i_1/n,\ldots,i_N/n)\in E} f_{i_1,\ldots,i_N}t_1^{i_1}\ldots t_N^{i_N}.$$

For a multi-index $\nu:=(\nu_1,\ldots,\nu_N)$ of non-negative integers we set $|\nu|:=\nu_1+\ldots+\nu_N$. If $\nu_{j_1},\ldots,\nu_{j_k}$, $j_1<\ldots<j_k$, are the positive entries of ν, then we define for a point $(x_1,\ldots,x_k)\in[0,1]^k$ the point $(x_1,\ldots,x_k)_\nu\in[0,1]^N$ in the following way: the lth coordinate is zero if $\nu_l=0$ and the lth coordinate is x_m if $l=j_m$. For example, if $\nu=(\nu_1,0,\nu_3,\nu_4,0,)$ with nonzero ν_1, ν_3 and ν_4, then $k=3$, $j_1=1$, $j_2=3$, $j_3=4$ and $(x_1,x_2,x_3)_\nu=(x_1,0,x_2,x_3,0,0)$. Now set

$$E_\nu:=\left\{(x_1,\ldots,x_k)\in[0,1]^k:\ (x_1,\ldots,x_k)_\nu\in E\right\}.$$

Böttcher and Dörfler showed the following asymptotic behaviour of norms of products of partial derivatives $\partial_j:=\partial/\partial x_j$ on $\mathcal{P}_n(E)$:

$$\lambda\left(\partial_1^{\nu_1}\ldots\partial_N^{\nu_N}\,|\mathcal{P}_n(E)\right)\sim n^{|\nu|}\left\|L_{\nu_{j_1}}\otimes\ldots\otimes L_{\nu_{j_k}}\,|L^2(E_\nu)\right\|,$$

$$\eta\left(\partial_1^{\nu_1}\ldots\partial_N^{\nu_N}\,|\mathcal{P}_n(E)\right)\sim n^{\frac{|\nu|}{2}}\max_{(x_1,\ldots,x_k)\in E_\nu}(2x_1)^{\nu_{j_1}/2}\ldots(2x_k)^{\nu_{j_k}/2}$$

and

$$\gamma\left(\partial_1^{\nu_1}\ldots\partial_N^{\nu_N}\,|\mathcal{P}_n(E)\right)\sim n^{2|\nu|}\left\|G_{\nu_{j_1}}\otimes\ldots\otimes G_{\nu_{j_k}}\,|L^2(E_\nu)\right\|.$$

These formulas deliver results for linear combinations of products of partial derivatives. More concretely, the authors have shown in [5] the following.

For $M\in\mathbb{N}$ and a polynomial

$$p(\chi_1,\ldots,\chi_N)=\sum_{\nu_1+\ldots+\nu_N\le M}p_{\nu_1,\ldots,\nu_N}\chi_1^{\nu_1}\cdots\chi_N^{\nu_N}$$

we denote by

$$p_0(\chi_1,\ldots,\chi_N):=\sum_{\nu_1+\ldots+\nu_N=M}p_{\nu_1,\ldots,\nu_N}\chi_1^{\nu_1}\cdots\chi_N^{\nu_N}$$

the principal part of p. If now C stands for λ, η or γ and if we set $\sigma=1$ in the Laguerre case, $\sigma=1/2$ in the Hermite case and $\sigma=2$ in the Legendre case, then there are constants $0<c_1<c_2<\infty$ such that

$$c_1n^{\sigma M}\le C\left(p_0(\partial_1,\ldots,\partial_N)\,|\mathcal{P}_n(E)\right)\le c_2n^{\sigma M},$$

and further we have

$$C\left(p(\partial_1,\ldots,\partial_N)\,|\mathcal{P}_n(E)\right) \sim C\left(p_0(\partial_1,\ldots,\partial_N)\,|\mathcal{P}_n(E)\right)$$

(see Theorem 1.5. in [5]). This means that the asymptotic behavior of the norm of a linear partial differential operator with constant coefficients is determined by its principal part. So if we want to apply the results of [5] cited above to a linear partial differential operator $p(\partial_1,\ldots,\partial_N)$, we have to assume that p has a single term of maximal order $|\nu|$. In the case of the Laplace operator all terms have the same order 2, so the results of [5] are not applicable. Nevertheless there are some estimates for the norm of the Laplace operator in [5]. So we have in the Hermite case for $N=2$ and $E=\Omega_2$

$$2 \le \liminf_{n\to\infty} \frac{\eta\left(\Delta\,|\mathcal{P}_n\left(\Omega_2\right)\right)}{n} \le \limsup_{n\to\infty} \frac{\eta\left(\Delta\,|\mathcal{P}_n\left(\Omega_2\right)\right)}{n} \le 4.$$

For general N the lower and upper bound in this chain of inequalities become 2 and $2N$, so the estimate gets worser for increasing N. In the Laguerre case the authors have shown

$$0.197 \le \liminf_{n\to\infty} \frac{\lambda\left(\Delta\,|\mathcal{P}_n\left(\Omega_2\right)\right)}{n^2} \le \limsup_{n\to\infty} \frac{\lambda\left(\Delta\,|\mathcal{P}_n\left(\Omega_2\right)\right)}{n^2} < 0.569$$

and in the Legendre case

$$0.091 \le \liminf_{n\to\infty} \frac{\gamma\left(\Delta\,|\mathcal{P}_n\left(\Omega_2\right)\right)}{n^4} \le \limsup_{n\to\infty} \frac{\gamma\left(\Delta\,|\mathcal{P}_n\left(\Omega_2\right)\right)}{n^4} < 0.143,$$

where the upper bounds go to infinity and the lower bounds to zero for increasing N.

These problems in the case of the Laplace operator are the motivation for this work.

3 The matrix representation of the operators

In the following we write $\mathcal{P}_n := \mathcal{P}_n^1$. Then we identify $\mathcal{P}_n^N$ with $\bigotimes_{k=1}^N \mathcal{P}_n$ and therefore have

$$\Delta = D^2 \otimes I \otimes \ldots \otimes I + I \otimes D^2 \otimes I \otimes \ldots \otimes I + \ldots + I \otimes \ldots \otimes I \otimes D^2, \qquad (3.1)$$

where I is the identity operator on $\mathcal{P}_n$ and $D^2 : \mathcal{P}_n \to \mathcal{P}_n$ is the ordinary differential operator $D^2 : f \mapsto f''$. Now we can choose an orthonormal basis in $\mathcal{P}_n$ and construct by tensoring an orthonormal basis on $\mathcal{P}_n^N$. Then we can calculate the matrix representation of Δ in this basis, where the tensor products become Kronecker products.

Due to the associativity of the Kronecker product we can write this representation as

$$\begin{aligned}[\Delta] &= \left[D^2\right]\otimes[I]\otimes\ldots\otimes[I]+[I]\otimes\left[D^2\right]\otimes[I]\otimes\ldots\otimes[I]\\ &\quad +\ldots+[I]\otimes\ldots\otimes[I]\otimes\left[D^2\right]\\ &= \sum_{k=1}^{N} I_{n+1}\otimes\ldots\otimes I_{n+1}\otimes[D^2]\otimes I_{n+1}\otimes\ldots\otimes I_{n+1}\\ &= \sum_{k=1}^{N} I_{(n+1)^{k-1}}\otimes[D^2]\otimes I_{(n+1)^{N-k}},\end{aligned}$$

where I_k denotes the $k\times k$ identity matrix.

3.1 The Laguerre case

We denote by

$$L_k(t)=1-\binom{k}{1}\frac{t}{1!}+\binom{k}{2}\frac{t^2}{2!}-\cdots+(-1)^k\binom{k}{k}\frac{t^k}{k!}$$

the kth normalized Laguerre polynomial. Then the polynomials $\{L_0,L_1,\ldots,L_n\}$ form an orthonormal basis in $\mathcal{P}_n$. As shown in [9, 23] or [3], the matrix representation of the operator $D^2:\mathcal{P}_n\to\mathcal{P}_n, f\mapsto f''$ in this basis is the $(n+1)\times(n+1)$ upper triangular Toeplitz matrix

$$[D^2]=\begin{pmatrix} 0 & 0 & 1 & 2 & \ldots & n-2 & n-1\\ & 0 & 0 & 1 & \ldots & n-3 & n-2\\ & & \ddots & \ddots & \ddots & \ddots & \vdots\\ & & & \ddots & \ddots & \ddots & \vdots\\ & & & & \ddots & 0 & 1\\ & & & & & 0 & 0\\ & & & & & & 0\end{pmatrix}. \tag{3.2}$$

The polynomials $L_{j_1}\otimes\cdots\otimes L_{j_N}$ with $j_1,\ldots,j_N$ ranging from 0 to n form an orthonormal basis in $\mathcal{P}_n^N$, and the matrix representation of the Laplace operator in this basis is

$$[\Delta]=\left[D^2\right]\otimes I\otimes\ldots\otimes I+I\otimes\left[D^2\right]\otimes I\otimes\ldots\otimes I+\ldots+I\otimes\ldots\otimes I\otimes\left[D^2\right], \tag{3.3}$$

where I stands for the $(n+1)\times(n+1)$ identity matrix. The matrix (3.3) induces a bounded linear operator on $\ell^2(\{0,1,\ldots,n\}^N)=\mathbf{C}^{n+1}\otimes\cdots\otimes\mathbf{C}^{n+1}$ in the natural manner. This operator will also be denoted by $[\Delta]$. Now the norm of Δ on $\mathcal{P}_n^N$ is just the norm of the operator $[\Delta]$ on $\ell^2(\{0,1,\ldots,n\}^N)$.

3.2 The Hermite case

By H_k we denote the kth Hermite polynomial

$$H_k(t) = 2^k \Gamma\left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) \sum_{j=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{\left\lceil \frac{k}{2} \right\rceil - \frac{1}{2}}{j} \frac{(-1)^j}{\Gamma\left(\left\lfloor \frac{k}{2} \right\rfloor - j + 1\right)} t^{k-2j}$$

and let

$$h_k(t) = \omega_k H(t), \quad \omega_k := \left(2^k \sqrt{\Gamma\left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) \Gamma\left(\left\lceil \frac{k}{2} \right\rceil + \frac{1}{2}\right)}\right)^{-1}$$

be the kth normalized Hermite polynomial. The polynomials $\{h_0, h_1, \ldots, h_n\}$ form an orthonormal basis in $\mathcal{P}_n$ with respect to the norm (1.3).

As shown for example in [14], the matrix representation of D^2 in this basis is the $(n+1) \times (n+1)$ matrix

$$\left[D^2\right] = \begin{pmatrix} 0 & 0 & 2\sqrt{2} & & & & 0 \\ & & & 2\sqrt{6} & & & \\ \vdots & \vdots & & & 2\sqrt{12} & & \\ & & & & & \ddots & \\ 0 & 0 & 0 & & & & 2\sqrt{n(n-1)} \\ 0 & 0 & 0 & & \ldots & & 0 \\ 0 & 0 & 0 & & \ldots & & 0 \end{pmatrix}. \tag{3.4}$$

The polynomials $h_{j_1} \otimes \cdots \otimes h_{j_N}$ with $j_1, \ldots, j_N \in \{0, \ldots, n\}$ form an orthonormal basis in $\mathcal{P}_n^N$. Thus, by (3.1) the matrix representation $[\Delta]$ of the Laplace operator in this basis is

$$\left[D^2\right] \otimes I \otimes \ldots \otimes I + I \otimes \left[D^2\right] \otimes I \otimes \ldots \otimes I + \cdots + I \otimes \ldots \otimes I \otimes \left[D^2\right]. \tag{3.5}$$

3.3 The Legendre case

Now we equip $\mathcal{P}_n$ with the Legendre norm (1.4). The normalized Legendre polynomials $p_0, \ldots p_n$ with

$$p_k(t) := \sqrt{\frac{2k+1}{2}} \frac{(-1)^k}{2^k \Gamma(k+1)} \frac{d^k}{dt^k} \left[(1-t^2)^k\right]$$

form an orthonormal basis in $\mathcal{P}_n$. From [4] we know that in this basis the differential operator $D : \mathcal{P}_n \to \mathcal{P}_n, f \mapsto f'$ has the representation

$$[D] = \begin{pmatrix} 0 & c_{01} & c_{02} & \dots & c_{0n} \\ & 0 & c_{12} & \dots & c_{1n} \\ & & & & \vdots \\ & & & & c_{n-1,n} \\ & & & & 0 \end{pmatrix}$$

with

$$c_{jk} = \begin{cases} \sqrt{(2j+1)(2k+1)} & : \ j+k \quad \text{odd} \\ 0 & : \ j+k \quad \text{even} \end{cases} . \tag{3.6}$$

If we have $n+1 = 2m$, then $[D]$ is unitarily equivalent to the block matrix

$$\mathbb{D}_1 = \begin{pmatrix} 0 & A_m \\ B_m & 0 \end{pmatrix}$$

through permutations matrices. The $m \times m$ matrices A_m and B_m are

$$\begin{aligned} A_m &= \begin{pmatrix} c_{01} & c_{03} & \dots & c_{0,2m-1} \\ & c_{23} & \dots & c_{2,2m-1} \\ & & \ddots & \vdots \\ & & & c_{2m-2,2m-1} \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{3} & \sqrt{7} & \sqrt{11} & \dots & \sqrt{4m-1} \\ & \sqrt{5}\cdot\sqrt{7} & \sqrt{5}\cdot\sqrt{11} & \dots & \sqrt{5}\cdot\sqrt{4m-1} \\ & & & & \vdots \\ & & & & \sqrt{4m-3}\sqrt{4m-1} \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} B_m &= \begin{pmatrix} 0 & c_{12} & c_{14} & \dots & c_{1,2m-2} \\ & 0 & c_{34} & \dots & c_{3,2m-2} \\ & & \ddots & & \vdots \\ & & & \ddots & c_{2m-3,2m-2} \\ & & & & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \sqrt{3}\cdot\sqrt{5} & \sqrt{3}\cdot\sqrt{9} & \dots & \sqrt{3}\cdot\sqrt{4m-3} \\ & 0 & \sqrt{7}\cdot\sqrt{9} & \dots & \sqrt{7}\cdot\sqrt{4m-3} \\ & & \ddots & & \vdots \\ & & & \ddots & \sqrt{4m-5}\sqrt{4m-3} \\ & & & & 0 \end{pmatrix} . \end{aligned}$$

This implies that $[D^2]$ is unitarily equivalent to

$$\mathbb{D}_2 := \begin{pmatrix} A_m B_m & 0 \\ 0 & B_m A_m \end{pmatrix}. \tag{3.7}$$

The polynomials $p_{j_1} \otimes \ldots \otimes p_{j_N}$ with $j_1, \ldots, j_N \in \{0, \ldots, n\}$ form an orthonormal basis in $\mathcal{P}_n^N$. The Laplace operator has, again due to (3.1), on $\mathcal{P}_n^N$ in this basis the representation

$$[\Delta] = \left[D^2\right] \otimes I \otimes \ldots \otimes I + I \otimes \left[D^2\right] \otimes I \otimes \ldots \otimes I + \cdots + I \otimes \ldots \otimes I \otimes \left[D^2\right]. \tag{3.8}$$

3.4 The Laplace operator on $\mathcal{P}_n(\Omega_N)$ equipped with the Hermite norm

3.4.1 The case $N = 2$

The set

$$B_n^{(N)} := \{h_{i_1} \otimes \ldots \otimes h_{i_N} : \ i_1 + \ldots + i_N \leq n\} \tag{3.9}$$

is an orthonormal basis in $\mathcal{P}_n(\Omega_N)$ with respect to the Hermite norm for all $N \in \mathbb{N}$. To get the matrix representation $[\Delta]$ of the Laplace operator in the case $N = 2$, we apply the operator to the basis elements and develop the resulting vectors in the basis.

For that we set $h_i := 0$ for $i < 0$, and for arbitrary $N \in \mathbb{N}$ and $i_1, \ldots, i_N \in \{0, \ldots, n\}$ we get

$$\begin{aligned}
&(\Delta(h_{i_1} \otimes \ldots \otimes h_{i_N}))(t_1, \ldots, t_N) \\
&\quad = \left(\left(\frac{\partial^2}{\partial t_1^2} + \ldots + \frac{\partial^2}{\partial t_N^2}\right)(h_{i_1} \otimes \ldots \otimes h_{i_N})\right)(t_1, \ldots, t_N) \\
&\quad = h_{i_1}''(t_1) h_{i_2}(t_2) \ldots h_{i_N}(t_N) + h_{i_1}(t_1) h_{i_2}''(t_2) h_{i_3}(t_3) \ldots h_{i_N}(t_N) \\
&\qquad + \ldots + h_{i_1}(t_1) \ldots h_{i_{N-1}}(t_{N-1}) h_{i_N}''(t_N) \\
&\quad = 2\sqrt{\frac{\Gamma(i_1+1)}{\Gamma(i_1-1)}} h_{i_1-2}(t_1) h_{i_2}(t_2) \ldots h_{i_N}(t_N) \\
&\qquad + 2\sqrt{\frac{\Gamma(i_2+1)}{\Gamma(i_2-1)}} h_{i_1}(t_1) h_{i_2-2}(t_2) h_{i_3}(t_3) \ldots h_{i_N}(t_N) \\
&\qquad + \ldots + 2\sqrt{\frac{\Gamma(i_N+1)}{\Gamma(i_N-1)}} h_{i_1}(t_1) \ldots h_{i_{N-1}}(t_{N-1}) h_{i_N-2}(t_N) \\
&\quad = 2\sqrt{i_1(i_1-1)} h_{i_1-2}(t_1) h_{i_2}(t_2) \ldots h_{i_N}(t_N) \\
&\qquad + 2\sqrt{i_2(i_2-1)} h_{i_1}(t_1) h_{i_2-2}(t_2) h_{i_3}(t_3) \ldots h_{i_N}(t_N) \\
&\qquad + \ldots + 2\sqrt{i_N(i_N-1)} h_{i_1}(t_1) \ldots h_{i_{N-1}}(t_{N-1}) h_{i_N-2}(t_N),
\end{aligned} \tag{3.10}$$

where Γ denotes the Gamma function. Now we equip the elements of the orthonormal basis $B_n^{(2)}$ in $\mathcal{P}_n(\Omega_2)$ with the following order:

$$\begin{aligned}\{h_{i_1} \otimes h_{i_2} :\ & i_1 + i_2 \leq n\} \\ = \ & \{h_0 \otimes h_0, \\ & h_0 \otimes h_1,\ h_1 \otimes h_0, \\ & h_0 \otimes h_2,\ h_1 \otimes h_1,\ h_2 \otimes h_0, \\ & h_0 \otimes h_3,\ h_1 \otimes h_2,\ h_2 \otimes h_1,\ h_3 \otimes h_0, \\ & h_0 \otimes h_4,\ h_1 \otimes h_3,\ h_2 \otimes h_2,\ h_3 \otimes h_1,\ h_4 \otimes h_0, \\ & \ldots, h_0 \otimes h_n,\ h_1 \otimes h_{n-1},\ h_2 \otimes h_{n-2}, \ldots, h_{n-1} \otimes h_1,\ h_n \otimes h_0\}.\end{aligned}$$

The polynomials $h_{i_1} \otimes h_{i_2}$ are mapped by the Laplace operator to

$$\Delta(h_{i_1} \otimes h_{i_2}) = 2\sqrt{i_1(i_1-1)}h_{i_1-2} \otimes h_{i_2} + 2\sqrt{i_2(i_2-1)}h_{i_1} \otimes h_{i_2-2}.$$

We develop the images of the basis elements now in the basis and get so the matrix representation

$$[\Delta] = \begin{pmatrix} O_{11} & O_{12} & A_1 & O_{14} & O_{15} & \ldots & O_{1n} & O_{1,n+1} \\ O_{21} & O_{22} & O_{23} & A_2 & O_{25} & \ldots & O_{2n} & O_{2,n+1} \\ O_{31} & O_{32} & O_{33} & O_{34} & A_3 & \ldots & O_{3n} & O_{3,n+1} \\ \vdots & & & & & & \vdots & \vdots \\ O_{n-1,1} & & & \ldots & & & O_{n-1,n} & A_{n-1} \\ O_{n1} & & & \ldots & & & O_{nn} & O_{n,n+1} \\ O_{n+1,1} & & & \ldots & & & O_{n+1,n} & O_{n+1,n+1} \end{pmatrix}, \tag{3.11}$$

where O_{kl} denotes the $k \times l$ zero matrix and A_k stands for the $k \times (k+2)$ matrix

$$A_k = \begin{pmatrix} 2\sqrt{k(k+1)} & 0 & 2\sqrt{2} & & & 0 \\ & 2\sqrt{(k-1)k} & \ddots & \ddots & & \\ & & \ddots & \ddots & 2\sqrt{(k-1)k} & \\ 0 & & & 2\sqrt{2} & 0 & 2\sqrt{k(k+1)} \end{pmatrix}$$

for all $k = 1, \ldots, n-1$.

3.4.2 The case $N = 3$

In the case $N = 3$ we can write down the orthonormal basis in $\mathcal{P}_n(\Omega_3)$ with maintainable effort too and develop the images of the basis elements in the basis. To do this, we write the basis $B_n^{(3)}$ as the disjoint union

$$B_n^{(3)} = \bigcup_{k=0}^{n} B_{n,k}^{(3)}$$

with $B_{n,k}^{(3)} := \{h_{i_1} \otimes h_{i_2} \otimes h_{i_3} : \ i_1 + i_2 + i_3 = k\}$. We give the basis elements in $B_{n,k}^{(3)}$ an order analogously to the order in the case $N = 2$:

$$\begin{aligned} B_{n,k}^{(3)} \ = \ & \{h_0 \otimes h_0 \otimes h_k, h_0 \otimes h_1 \otimes h_{k-1}, h_0 \otimes h_2 \otimes h_{k-2}, \ldots, h_0 \otimes h_k \otimes h_0, \\ & h_1 \otimes h_0 \otimes h_{k-1}, h_1 \otimes h_1 \otimes h_{k-2}, h_1 \otimes h_2 \otimes h_{k-3}, \ldots, h_1 \otimes h_{k-1} \otimes h_0, \\ & h_2 \otimes h_0 \otimes h_{k-2}, h_2 \otimes h_1 \otimes h_{k-3}, h_2 \otimes h_2 \otimes h_{k-4}, \ldots, h_2 \otimes h_{k-2} \otimes h_0, \\ & \ldots, \\ & h_{k-2} \otimes h_0 \otimes h_2, h_{k-2} \otimes h_1 \otimes h_1, h_{k-2} \otimes h_0 \otimes h_2, \\ & h_{k-1} \otimes h_0 \otimes h_1, h_{k-1} \otimes h_1 \otimes h_0, \\ & h_k \otimes h_0 \otimes h_0\} \, . \end{aligned}$$

The cardinality of the set $B_{n,k}^{(3)}$ is $\sum_{j=1}^{k+1} j = \frac{1}{2}(k+1)(k+2)$. Particularly we have due to (3.10)

$$\begin{aligned} \left(\Delta \left(h_{i_1} \otimes h_{i_2} \otimes h_{i_3}\right)\right)(t_1, t_2, t_3) \ = \ & 2\sqrt{i_1(i_1-1)} h_{i_1-2}(t_1) h_{i_2}(t_2) h_{i_3}(t_3) \\ & + 2\sqrt{i_2(i_2-1)} h_{i_1}(t_1) h_{i_2-2}(t_2) h_{i_3}(t_3) \\ & + 2\sqrt{i_3(i_3-1)} h_{i_1}(t_1) h_{i_2}(t_2) h_{i_3-2}(t_3), \end{aligned} \tag{3.12}$$

This implies

$$\Delta : B_{n,k}^{(3)} \to \operatorname{span} B_{n,k-2}^{(3)},$$

so the matrix representation $[\Delta]$ has the form

$$[\Delta] = \begin{pmatrix} 0 & 0 & A_2^{(2)} & & & 0 \\ & & & A_3^{(2)} & & \\ \vdots & \vdots & & & \ddots & \\ & & 0 & & & A_n^{(2)} \\ 0 & 0 & & \ldots & & 0 \\ 0 & 0 & & \ldots & & 0 \end{pmatrix} \tag{3.13}$$

if we arrange the elements of $B_n^{(3)}$ by taking first all elements of $B_{n,0}^{(3)}$, then of $B_{n,1}^{(3)}$, $B_{n,2}^{(3)}$ up to $B_{n,n}^{(3)}$. So the matrix $A_k^{(2)}$ has the size $\frac{1}{2}(k-1)k \times \frac{1}{2}(k+1)(k+2)$ for all $k \in \{2, \ldots, n\}$. Of course the zeros stand for (non-square) zero matrices.

In the next step we have to give a more detailed description of the matrices $A_k^{(2)}$. To do this, we calculate $\Delta \left(h_{1_1} \otimes h_{i_2} \otimes h_{i_3}\right)$ by (3.12) and get after developing in $B_n^{(3)}$ for $k \geq 2$ the matrix

$A_k^{(2)} =$

$$\begin{pmatrix} A_{k-1} & 0 & 2\sqrt{2}I_{k-1} & & & & 0 \\ 0 & A_{k-2} & \ddots & \ddots & & & \\ & & \ddots & 0 & 2\sqrt{(k-2)(k-3)}I_3 & & \\ & & & A_2 & 0 & 2\sqrt{(k-1)(k-2)}I_2 & \\ 0 & & & & A_1 & 0 & \sqrt{k(k-1)} \end{pmatrix}. \tag{3.14}$$

4 Proofs of the main results

4.1 A general estimate for the norm of the Laplace operator

Before we focus on the results for the different norms on $\mathcal{P}_n^N$, some general considerations will be helpful. As seen in Section 3, the matrix representation of the Laplace operator has in all three cases the structure

$$[\Delta] = \sum_{k=1}^{N} I_{(n+1)^{k-1}} \otimes \left[D^2\right] \otimes I_{(n+1)^{N-k}}, \tag{4.1}$$

just differing in the concrete form of $[D^2]$.

We know, that the operator norm of Δ is nothing but the spectral norm of a matrix representation with respect to an orthonormal basis, and the spectral norm is just the largest singular value. So in all three cases, the proofs of the (asymptotic) bounds are based on estimating the largest singular value of the particular representations. For that we will use the structure in (4.1) and a theorem from [11] which provides us with an inequality for the singular values of an arbitrary matrix.

First we want to formulate the result used from [11]. We denote by M_n the set of all (complex) $n \times n$ matrices. For arbitrary $A \in M_n$ we define the hermitian part $H(A)$ of A by

$$H(A) := \frac{1}{2}\left(A + A^*\right)$$

and the skew-hermitian part $S(A)$ by

$$S(A) := \frac{1}{2}\left(A - A^*\right).$$

Moreover, for hermitian A, we denote by $\lambda_{\max}(A) = \lambda_1(A) \geq \ldots \geq \lambda_n(A)$ the ordered eigenvalues and by $\sigma_{\max}(A) = \sigma_1(A) \geq \ldots \geq \sigma_n(A)$ the ordered singular values of A. We have the following general estimates for singular values, formulated as Corollary 3.1.5 in [11]

Corollary 4.1. *Let A be an arbitrary $n \times n$ matrix and let U and V be unitary $n \times n$ matrices. Then we have*

$$\sigma_k(A) \geq \lambda_k(H(UAV)) \tag{4.2}$$

for all $k = 1, \ldots, n$.

With good choices for U and V we deduce from (4.2) estimates for the largest singular value, which can be applied for $A = [D^2]$. Below we will see, why this is useful to calculate lower bounds for $\sigma_1([\Delta])$.

Corollary 4.2. *Let be A an arbitrary $n \times n$ matrix. Then the singular values of A satisfy the following inequality:*

$$\sigma_k(A) \geq \max\{\lambda_k(H(A)), -\lambda_{n-k+1}(H(A)), \lambda_k(\mathbf{i}S(A)), -\lambda_{n-k+1}(\mathbf{i}S(A))\} \tag{4.3}$$

for all $k = 1, \ldots, n$.

Proof. This corollary is just a consequence of Corollary 4.1 and is formulated in [11] as exercise 36 on page 160, but nevertheless we want to give a short proof of this inequality.

We have to apply (4.2) with the right choices of U and V. First we set $U = V = I_n$ and get immediatly $\sigma_k(A) \geq \lambda_k(H(A))$. We now set $U := \mathbf{i}I_n$ and $V := I_n$. These matrices are of course unitary and we have

$$H\left((\mathbf{i}I_n)AI_n\right) = \frac{1}{2}\left[(\mathbf{i}I_n)AI_n + I_nA^*(-\mathbf{i}I_n)\right] = \frac{1}{2}\left[\mathbf{i}A - \mathbf{i}A^*\right] = \mathbf{i}S(A).$$

Due to (4.2) we get $\sigma_k(A) \geq \lambda_k(\mathbf{i}S(A))$ for all $k \in \{1, \ldots, n\}$.

The matrices $H(A)$ and $-H(A)$ are selfadjoint and it holds

$$-\lambda_{n-k+1}(H(A)) = \lambda_k(-H(A)).$$

Now we have

$$\begin{aligned} -H(A) &= -\frac{1}{2}(A + A^*) = \frac{1}{2}\left[(-I_n)AI_n + I_nA(-I_n)\right] \\ &= \frac{1}{2}\left[(-I_n)AI_n + ((-I_n)AI_n)^*\right] = H((-I_n)AI_n), \end{aligned}$$

and again due to (4.2) it follows

$$-\lambda_{n-k+1}(H(A)) = \lambda_k(-H(A)) = \lambda_k\left(H((-I_n)AI_n)\right) \leq \sigma_k(A)$$

for all $k \in \{1, \ldots, n\}$.

For $k \in \{1, \ldots, n\}$ we finally have $-\lambda_{n-k+1}(\mathbf{i}S(A)) = \lambda_k(-\mathbf{i}S(A))$ and it is easy to see that $-\mathbf{i}S(A) = H((-\mathbf{i}I_n)AI_n)$ holds, which, because of (4.2), implies

$$-\lambda_{n-k+1}(\mathbf{i}S(A)) = \lambda_k(-\mathbf{i}S(A)) = \lambda_k\left(H((-\mathbf{i}I_n)AI_n)\right) \leq \sigma_k(A).$$

With these four estimates we get the assertion. □

We will apply Corollary 4.2 to $[\Delta]$ in all three cases. For that, we need so-called Kronecker sums.

Let A be an arbitrary $n \times n$ matrix and let B be an arbitrary $m \times m$ matrix. Then the Kronecker sum $A \oplus B$ is defined as the $nm \times nm$ matrix

$$A \oplus B := (I_m \otimes A) + (B \otimes I_n)\,.$$

For the spectrum of Kronecker sums the following theorem holds.

Theorem 4.3. *Let $A \in M_n$ and $B \in M_m$ be given. If λ is an eigenvalue of A with corresponding eigenvector $x \in \mathbb{C}^n$ and if μ is an eigenvalue of B with corresponding eigenvector $y \in \mathbb{C}^m$, then $\lambda + \mu$ is an eigenvalue of $A \oplus B$ and $x \otimes y \in \mathbb{C}^{nm}$ is a corresponding eigenvector.*

On the other hand, every eigenvalue of $A \oplus B$ is the sum of an eigenvalue of A and an eigenvalue of B.

This is Theorem 4.4.5 in [11], where one can also find a proof. This theorem as well its proof given in [11] can be extended to $N \in \mathbb{N}$ instead of two summands. For matrices $A_1, \ldots, A_N$ of order $n_1, \ldots, n_N$ we define the Kronecker sum $A_1 \oplus \ldots \oplus A_N$ as

$$A_1 \oplus \ldots \oplus A_N := \sum_{j=1}^{N} I_{\prod_{k=j+1}^{N} n_k} \otimes A_j \otimes I_{\prod_{k=1}^{j-1} n_k}, \tag{4.4}$$

where we set $\prod_{k=j}^{l} n_k = 1$ for $l < j$. Then we have the following.

Corollary 4.4. *Let $A_1, \ldots, A_N$, $N \in \mathbb{N}$, be matrices of order $n_1, \ldots, n_N$ and let $A := A_1 \oplus \ldots \oplus A_N$ be their Kronecker sum. If $\lambda_i \in \sigma(A_i)$ for all $i \in \{1, \ldots, N\}$, then we have $\lambda_1 + \ldots + \lambda_N \in \sigma(A)$.*

Every $\lambda \in \sigma(A)$ is of the form $\lambda = \lambda_1 + \ldots + \lambda_N$ with $\lambda_i \in \sigma(A_i)$ for all $i \in \{1, \ldots, N\}$.

Proof. Induction by N shows easily

$$A_1 \oplus A_2 \oplus \ldots \oplus A_N = A_1 \oplus (A_2 \oplus (\ldots (A_{N-1} \oplus A_N) \ldots)).$$

Now repeated application of Theorem 4.3 provides us with the asserted result. □

For $N \geq 2$ we will use the notation

$$A^{\oplus N} := A \oplus A^{\oplus N-1}, \quad A^{\oplus 1} := A.$$

Now we leave the general setting and focus on the concrete norms on $\mathcal{P}_n^N$.

4.2 The Laguerre case

In this section we first want to derive safe upper and lower bounds for $\lambda\left(\Delta\left|\mathcal{P}_n^N\right.\right)$, that is we want to prove Theorem 1.1. As seen in (3.2), the matrix representation of the operator D^2 is a Toeplitz matrix, so we can use methods from the theory of (multilevel) Toeplitz operators to get the upper and the lower bound of the operator norm.

The method used here is presented in [6] for the case of ordinary Toeplitz matrices. In our situation, we need a multilevel version of this method, especially of Theorem 5.10 in [6]. First of all we should introduce some notation for so called N-dimensional multilevel Toeplitz operators (see, for example, Section 6.9 of [8]).

Let $b = (b_{j_1,\ldots,j_N})_{j_1,\ldots,j_N=-\infty}^{\infty} \subset \mathbb{C}$ be an array of complex numbers. We consider the operator on $\ell^2\left(\mathbb{N}^N\right)$ given by

$$\sum_{k_1,\ldots,k_N\geq 1} b_{m_1-k_1,\ldots,m_N-k_N} x_{k_1,\ldots,k_N}, \quad m_1,\ldots,m_N \geq 1, \quad (x_{k_1,\ldots,k_N}) \in \ell^2\left(\mathbb{N}^N\right). \tag{4.5}$$

If this operator is bounded, we denote it by $T_{+\ldots+}(b)$ and call it an N-dimensional multilevel Toeplitz operator. It is well known that in the case $N = 1$ the operator defined by (4.5) is bounded if and only if there exists a function $a \in L^\infty(\mathbb{T})$, the so called symbol of the operator, such that $b_n = a_n$ holds for all n, where a_n is the nth Fourier coefficient

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} a\left(e^{\mathbf{i}\theta}\right) e^{-\mathbf{i}n\theta}\, d\theta$$

of the function a. For a proof of this result we could for example refer to [8], Theorem 1.9. Moreover, for a Toeplitz operator $T_+(a)$ with $a \in L^\infty(\mathbb{T})$ we have the equality

$$\|T_+(a)\| = \|a\|_\infty$$

(see Equation 1.14, page 10, in [8]).

We can extend these results to multilevel Toeplitz operators. Our symbols are now functions $a \in L^\infty\left(\mathbb{T}^N\right)$ on the N-dimensional torus $\mathbb{T}^N$, that is, the entries $a_{m_1,\ldots,m_N}$ of the operator $T_{+\ldots+}(a)$, $a \in L^\infty\left(\mathbb{T}^N\right)$, are the Fourier coefficients of a, given by

$$a_{m_1,\ldots,m_N} = \frac{1}{(2\pi)^N}\int_0^{2\pi}\cdots\int_0^{2\pi} a\left(e^{\mathbf{i}\theta_1},\ldots,e^{\mathbf{i}\theta_N}\right) e^{-\mathbf{i}m_1\theta_1}\ldots e^{-\mathbf{i}m_N\theta_N}\, d\theta_1\ldots d\theta_N$$

(see also Chapter 6.9 in [8]).

We introduce some further notation. For $n \in \mathbb{N}$ we define projectors $\bigotimes_{k=1}^N P_n$ on $\ell^2\left(\mathbb{N}^N\right)$ by

$$\left(\left(\bigotimes_{k=1}^N P_n\right) x\right)_{k_1,\ldots,k_N} = \begin{cases} x_{k_1,\ldots,k_N} & \text{if} \quad 1 \leq k_1 \leq n \wedge \ldots \wedge 1 \leq k_N \leq n \\ 0 & \text{otherwise} \end{cases}.$$

Now we set

$$T_{n,\dots,n}(a) := \left(\bigotimes_{k=1}^{N} P_n\right) T_{+\dots+}(a) \left(\bigotimes_{k=1}^{N} P_n\right)\Bigg|_{\operatorname{im}\left(\bigotimes_{k=1}^{N} P_n\right)}.$$

We can identify $T_{n,\dots,n}(a)$ with an operator on $\bigotimes_{k=1}^{N} \mathbb{C}^n$.

The next step is, analogously to [6], the construction of Laurent polynomials on $\mathbb{T}^N$ whose norms are upper and lower bounds for the norm of the operator $T_{n,\dots,n}(a)$. For that we consider again an array $b = (b_{j_1,\dots,j_N})_{j_1,\dots,j_N=-\infty}^{\infty} \subset \mathbb{C}$ of complex numbers. Then we get a "matrix" $T_{n,\dots,n}(b)$, generated by b with entries $b_{j_1-k_1,\dots,j_N-k_N}$.
Then we define functions

$$S_{n,\dots,n}(b),\ \sigma_{n,\dots,n}(b) : \mathbb{T}^N \to \mathbb{C}$$

on the torus $\mathbb{T}^N$ by

$$S_{n,\dots,n}(b)\left(e^{\mathbf{i}\phi_1},\dots,e^{\mathbf{i}\phi_N}\right) := \sum_{|j_1|\le n-1,\dots,|j_N|\le n-1} b_{j_1,\dots,j_N} e^{\mathbf{i}j_1\phi_1}\cdot\ldots\cdot e^{\mathbf{i}j_N\phi_N}$$

and

$$\sigma_{n,\dots,n}(b)\left(e^{\mathbf{i}\phi_1},\dots,e^{\mathbf{i}\phi_N}\right) := \sum_{|j_1|\le n-1,\dots,|j_N|\le n-1} b_{j_1,\dots,j_N} \prod_{k=1}^{N}\left(1-\frac{|j_k|}{n}\right) e^{\mathbf{i}j_k\phi_k}.$$

We can show now the following estimates.

Theorem 4.5. *Let $n \in \mathbb{N}$ and $b = (b_{j_1,\dots,j_N})_{j_1,\dots,j_N=-\infty}^{\infty} \subset \mathbb{C}$ be an array of complex numbers. Then we have the estimate*

$$\|\sigma_{n,\dots,n}(b)\|_\infty \le \|T_{n,\dots,n}(b)\| \le \|S_{n,\dots,n}(b)\|_\infty. \tag{4.6}$$

Proof. First we consider $S_{n,\dots,n}(b)$. For $(m_1,\dots,m_N) \in \mathbb{Z}^N$ with $|m_k| \le n-1$ for all $k \in \{1,\dots,N\}$ we will denote by $s_{m_1,\dots,m_N}$ the Fourier coefficients of $S_{n,\dots,n}(b)$. If we set

$$A_{m_1,\dots,m_N} := \left\{(j_1,\dots,j_N) \in \mathbb{Z}^N :\ |j_k| \le n-1\ \forall k \in \{1,\dots,N\}\right\} \setminus \{(m_1,\dots,m_N)\},$$

we get

$s_{m_1,\dots,m_N}$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^N}\int_0^{2\pi}\dots\int_0^{2\pi} S_{n,\dots,n}(b)\left(e^{\mathbf{i}\phi_1},\dots,e^{\mathbf{i}\phi_N}\right)\prod_{k=1}^{N} e^{-\mathbf{i}m_k\phi_k}\; d\phi_1\dots d\phi_N\\
&= \frac{1}{(2\pi)^N}\int_0^{2\pi}\dots\int_0^{2\pi}\sum_{|j_1|\le n-1,\dots,|j_N|\le n-1} b_{j_1,\dots,j_N}\prod_{k=1}^{N} e^{\mathbf{i}j_k\phi_k}\prod_{k=1}^{N} e^{-\mathbf{i}m_k\phi_k}\; d\phi_1\dots d\phi_N\\
&= \frac{1}{(2\pi)^N}\int_0^{2\pi}\dots\int_0^{2\pi} b_{m_1,\dots,m_N}\; d\phi_1\dots d\phi_N\\
&\quad +\frac{1}{(2\pi)^N}\int_0^{2\pi}\dots\int_0^{2\pi}\sum_{j_1,\dots,j_N\in A_{m_1,\dots,m_N}} b_{j_1,\dots,j_N}\prod_{k=1}^{N} e^{\mathbf{i}(j_k-m_k)\phi_k}\; d\phi_1\dots d\phi_n\\
&= b_{m_1,\dots,m_N}\\
&\quad +\frac{1}{(2\pi)^N}\sum_{j_1,\dots,j_N\in A_{m_1,\dots,m_N}} b_{j_1,\dots,j_N}\int_0^{2\pi}\dots\int_0^{2\pi}\prod_{k=1}^{N} e^{\mathbf{i}(j_k-m_k)\phi_k}\; d\phi_1\dots d\phi_N\\
&= b_{m_1,\dots,m_N}+\frac{1}{(2\pi)^N}\sum_{j_1,\dots,j_N\in A_{m_1,\dots,m_N}} b_{j_1,\dots,j_N}\prod_{k=1}^{N}\int_0^{2\pi} e^{\mathbf{i}(j_k-m_k)\phi_k}\; d\phi_k\\
&= b_{m_1,\dots,m_N},
\end{aligned}
$$

because

$$
\int_0^{2\pi} e^{\mathbf{i}(j-m)\phi}\; d\phi = \begin{cases} 2\pi & : \; j=m\\ 0 & : \; j\neq m.\end{cases}
$$

So we get the equality $T_{n,\dots,n}(b) = T_{n,\dots,n}\left(S_{n,\dots,n}(b)\right)$. This leads to the estimate

$$
\begin{aligned}
\|T_{n,\dots,n}(b)\| &= \|T_{n,\dots,n}\left(S_{n,\dots,n}(b)\right)\|\\
&= \left\|\left(\bigotimes_{k=1}^{N} P_n\right) T_{+\dots+}\left(S_{n,\dots,n}(b)\right)\left(\bigotimes_{k=0}^{N} P_n\right)\Big|_{\mathrm{im}\left(\bigotimes_{k=0}^{N} P_n\right)}\right\|\\
&\le \left\|\left(\bigotimes_{k=1}^{N} P_n\right) T_{+\dots+}\left(S_{n,\dots,n}(b)\right)\right\| \le \|T_{+\dots+}\left(S_{n,\dots,n}(b)\right)\|\\
&= \|S_{n,\dots,n}(b)\|_\infty .
\end{aligned}
$$

Now we want to derive the lower bound. For that we fix

$$
t_1 = e^{\mathbf{i}\phi_1},\dots,t_N = e^{\mathbf{i}\phi_N}\in\mathbb{T}
$$

and set

$$
x := x_{t_1,\dots,t_N} := \frac{1}{n^{\frac{N}{2}}}\left(\overline{t}_1^{h_1}\cdot\ldots\cdot\overline{t}_N^{h_N}\right)_{h_1,\dots h_N=0}^{n-1} = \frac{1}{n^{\frac{N}{2}}}\left(\prod_{k=1}^{N}\overline{t}_k^{h_k}\right)_{h_1,\dots h_N=0}^{n-1}.
$$

Then the $\ell^2\left(\{0,\dots,n-1\}^N\right)$ norm of x is

$$\begin{aligned}\|x\| &= \sqrt{\left(\frac{1}{n^{\frac{N}{2}}}\right)^2 \sum_{h_1,\dots,h_N=0}^{n-1}\left|\prod_{k=1}^{N}\bar{t}_k^{h_k}\right|^2} = \sqrt{\frac{1}{n^N}\sum_{h_1,\dots,h_N=0}^{n-1}1} = \sqrt{\frac{n^N}{n^N}} \\ &= 1.\end{aligned}$$

In the next step we calculate

$$\begin{aligned}&\langle T_{n,\dots,n}(b)x_{t_1,\dots,t_N}, x_{t_1,\dots,t_N}\rangle \\ &= \left\langle \left(\sum_{k_1,\dots,k_N=0}^{n-1} b_{h_1-k_1,\dots,h_N-k_N}\frac{1}{n^{\frac{N}{2}}}\prod_{j=1}^{N}\bar{t}_j^{k_j}\right)_{h_1,\dots,h_N=0}, \frac{1}{n^{\frac{N}{2}}}\left(\prod_{l=1}^{N}t_l^{h_l}\right)_{h_1,\dots,h_N=0}\right\rangle \\ &= \sum_{h_1,\dots,h_N=0}^{n-1}\sum_{k_1,\dots,k_N=0}^{n-1}\frac{1}{n^N}b_{h_1-k_1,\dots,h_N-k_N}\prod_{l=1}^{N}\bar{t}_l^{k_l}\prod_{l=1}^{N}t_l^{h_l} \\ &= \frac{1}{n^N}\sum_{h_1,\dots,h_N=0}^{n-1}\sum_{k_1,\dots,k_N=0}^{n-1} b_{h_1-k_1,\dots,h_N-k_N}\prod_{l=1}^{N}e^{-\mathbf{i}k_l\phi_l}\prod_{l=1}^{N}e^{\mathbf{i}h_l\phi_l} \\ &= \frac{1}{n^N}\sum_{h_1,\dots,h_N=0}^{n-1}\sum_{k_1,\dots,k_N=0}^{n-1} b_{h_1-k_1,\dots,h_N-k_N}\prod_{l=1}^{N}e^{\mathbf{i}(h_l-k_l)\phi_l}.\end{aligned}$$

We are looking for a better expression for the last formula in this calculation. For that we use the equality (4.8),

$$\begin{aligned}&\sum_{h_1,\dots,h_N=0}^{n-1}\sum_{k_1,\dots,k_N=0}^{n-1} b_{h_1-k_1,\dots,h_N-k_N}\prod_{l=1}^{N}e^{\mathbf{i}(h_l-k_l)\phi_l} \\ &= \sum_{|k_1|,\dots,|k_N|\le n-1} b_{k_1,\dots,k_N}\prod_{l=1}^{N}(n-|k_l|)e^{\mathbf{i}k_l\phi_l},\end{aligned}$$

from Lemma 4.6, what we will prove below. Equation (4.8) implies

$$\begin{aligned}\langle T_{n,\dots,n}(b)x_{t_1,\dots,t_N}, x_{t_1,\dots,t_N}\rangle &= \frac{1}{n^N}\sum_{|k_1|,\dots,|k_N|\le n-1} b_{k_1,\dots,k_N}\prod_{l=1}^{N}(n-|k_l|)\,e^{\mathbf{i}k_l\phi_l} \\ &= \sum_{|k_1|,\dots,|k_N|\le n-1} b_{k_1,\dots,k_N}\prod_{l=1}^{N}\left(1-\frac{|k_l|}{n}\right)e^{\mathbf{i}k_l\phi_l} \\ &= \sigma_{n,\dots,n}(b)\left(e^{\mathbf{i}\phi_1},\dots,e^{\mathbf{i}\phi_N}\right).\end{aligned}$$

With that we now have

$$\begin{aligned}\left|\sigma_{n,\dots,n}(b)\left(e^{\mathbf{i}\phi_1},\dots,e^{\mathbf{i}\phi_N}\right)\right| &= \left|\langle T_{n,\dots,n}(b)x_{t_1,\dots,t_N}, x_{t_1,\dots,t_N}\rangle\right| \\ &\le \|T_{n,\dots,n}(b)\|\,\|x_{t_1,\dots,t_N}\|^2 = \|T_{n,\dots,n}(b)\|.\end{aligned}$$

Finally we get the estimate

$$\|\sigma_{n,\dots,n}(b)\|_\infty = \sup_{(t_1,\dots,t_N)\in\mathbb{T}^N} |\sigma_{n,\dots,n}(b)(t_1,\dots,t_N)| \le \|T_{n,\dots,n}(b)\|\,, \tag{4.7}$$

as desired. □

Lemma 4.6. *For $n, N \in \mathbb{N}$, $b = (b_{j_1,\dots,j_N})_{j_1,\dots,j_N=-\infty}^{\infty} \subset \mathbb{C}$ and $\phi_1,\dots,\phi_N \in [0,2\pi]$ it holds*

$$\begin{aligned}&\sum_{h_1,\dots,h_N=0}^{n-1}\sum_{k_1,\dots,k_N=0}^{n-1} b_{h_1-k_1,\dots,h_N-k_N}\prod_{l=1}^{N} e^{\mathbf{i}(h_l-k_l)\phi_l}\\ &= \sum_{|k_1|,\dots,|k_N|\le n-1} b_{k_1,\dots,k_N}\prod_{l=1}^{N}(n-|k_l|)e^{\mathbf{i}k_l\phi_l}.\end{aligned} \tag{4.8}$$

Proof. The assertion will be proved by induction. We begin with $N = 1$. By induction by n we will show that

$$\sum_{j,k=0}^{n-1} b_{j-k}e^{\mathbf{i}(j-k)\theta} = \sum_{|j|\le n-1}(n-|j|)b_j e^{\mathbf{i}j\theta} \tag{4.9}$$

holds. For $n = 1$ this is obvious. If (4.9) holds for n, it follows for $n+1$:

$$\begin{aligned}\sum_{j,k=0}^{n} b_{j-k}e^{\mathbf{i}(j-k)\theta} &= \sum_{j,k=0}^{n-1} b_{j-k}e^{\mathbf{i}(j-k)\theta} + \sum_{j=0}^{n-1} b_{j-n}e^{\mathbf{i}(j-n)\theta} + \sum_{j=0}^{n} b_{n-j}e^{\mathbf{i}(n-j)\theta}\\ &= \sum_{|j|\le n-1}(n-|j|)b_j e^{\mathbf{i}j\theta} + \sum_{j=-n}^{-1} b_j e^{\mathbf{i}j\theta} + \sum_{j=0}^{n} b_j e^{\mathbf{i}j\theta}\\ &= \sum_{j=-(n-1)}^{-1}\left[(n-|j|)b_j e^{\mathbf{i}j\theta} + b_j e^{\mathbf{i}j\theta}\right] + b_{-n}e^{\mathbf{i}(-n)\theta}\\ &\quad + \sum_{j=0}^{n-1}\left[(n-j)b_j e^{\mathbf{i}j\theta} + b_j e^{\mathbf{i}j\theta}\right] + b_n e^{\mathbf{i}n\theta}\\ &= \sum_{j=-(n-1)}^{-1}(n+1-|j|)b_j e^{\mathbf{i}j\theta} + (n+1-n)b_{-n}e^{\mathbf{i}(-n)\theta}\\ &\quad + \sum_{j=0}^{n-1}(n+1-j)b_j e^{\mathbf{i}j\theta} + (n+1-n)b_n e^{\mathbf{i}n\theta}\\ &= \sum_{|j|\le n}(n+1-|j|)b_j e^{\mathbf{i}j\theta}.\end{aligned}$$

This shows (4.9). Now we can prove (4.8) by induction by N. Let us assume that (4.8) is true for $N-1$. Then we have

$$\begin{aligned}
&\sum_{\substack{h_1,\dots,h_N=0\\k_1,\dots,k_N=0}}^{n-1} b_{h_1-k_1,\dots,h_N-k_N} \prod_{l=1}^{N} e^{\mathbf{i}(h_l-k_l)\phi_l} \\
&= \sum_{h_N,k_N=0}^{n-1} e^{\mathbf{i}(h_N-k_N)\phi_N} \left(\sum_{h_1,k_1,\dots,h_{N-1},k_{N-1}=0}^{n-1} b_{h_1-k_1,\dots,h_{N-1}-k_{N-1},h_N-k_N} \prod_{l=1}^{N-1} e^{\mathbf{i}(h_l-k_l)\phi_l} \right) \\
&= \sum_{h_N,k_N=0}^{n-1} e^{\mathbf{i}(h_N-k_N)\phi_N} \sum_{|k_1|,\dots,|k_{N-1}|\le n-1} b_{k_1,\dots,k_{N-1},h_N-k_N} \prod_{l=1}^{N-1} (n-|k_l|)\, e^{\mathbf{i}k_l\phi_l} \\
&= \sum_{h_N,k_N=0}^{n-1} e^{\mathbf{i}(h_N-k_N)\phi_N} a_{h_N-k_N}
\end{aligned}$$

with

$$a_{h_N-k_N} := \sum_{|k_1|,\dots,|k_{N-1}|\le n-1} b_{k_1,\dots,k_{N-1},h_N-k_N} \prod_{l=1}^{N-1} (n-|k_l|)\, e^{\mathbf{i}k_l\phi_l}.$$

Due to (4.9) we get

$$\begin{aligned}
&\sum_{\substack{h_1,\dots,h_N=0\\k_1,\dots,k_N=0}}^{n-1} b_{h_1-k_1,\dots,h_N-k_N} \prod_{l=1}^{N} e^{\mathbf{i}(h_l-k_l)\phi_l} \\
&= \sum_{|k_N|\le n-1} (n-|k_N|)\, a_{k_N} e^{\mathbf{i}k_N\phi_N} \\
&= \sum_{|k_N|\le n-1} (n-|k_N|)\, e^{\mathbf{i}k_N\phi_N} \sum_{|k_1|,\dots,|k_{N-1}|\le n-1} b_{k_1,\dots,k_N} \prod_{l=1}^{N-1} (n-|k_l|)\, e^{\mathbf{i}k_l\phi_l} \\
&= \sum_{|k_1|,\dots,|k_N|\le n-1} b_{k_1,\dots,k_N} \prod_{l=1}^{N} (n-|k_l|)\, e^{\mathbf{i}k_l\phi_l}
\end{aligned}$$

which completes the proof. □

Now this result will be applied to the Laplace operator on $\mathcal{P}_n^N$.

Proof of Theorem 1.1. Due to (3.3) we know the matrix representation of the Laplace operator on $\mathcal{P}_n^N$ with respect to the orthonormal basis of the Laguerre polynomials, namely

$$[\Delta] = \sum_{k=1}^{N} I_{(n+1)^{k-1}} \otimes \left[D^2\right] \otimes I_{(n+1)^{N-k}}$$

with

$$\left[D^2\right] = (d_{ij})_{i,j=0}^{n} = \begin{pmatrix} 0 & 0 & 1 & 2 & \dots & n-1 \\ 0 & 0 & 0 & 1 & \dots & n-2 \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & & 1 \\ 0 & 0 & 0 & \dots & & 0 \\ 0 & 0 & 0 & \dots & & 0 \end{pmatrix}.$$

We want to consider $[\Delta]$ as an operator

$$[\Delta] : \bigotimes_{k=1}^{N} \mathbb{C}^{n+1} \to \bigotimes_{k=1}^{N} \mathbb{C}^{n+1}.$$

By $\{e_i\}_{i=0}^{n}$ we denote the standard basis in $\mathbb{C}^{n+1}$. We write $C \in \bigotimes_{k=1}^{N} \mathbb{C}^{n+1}$ as

$$C = \sum_{i_1,\dots,i_N=0}^{n} c_{i_1,\dots,i_N}(e_{i_1} \otimes \dots \otimes e_{i_N}), \quad (e_{i_1} \otimes \dots \otimes e_{i_N})_{k_1,\dots,k_N} = \prod_{l=1}^{N} \delta_{i_l,k_l},$$

with $c_{i_1,\dots,i_N} \in \mathbb{C}$ and get

$$\begin{aligned}
[\Delta]C &= \sum_{k=1}^{N} I_{(n+1)^{k-1}} \otimes \left[D^2\right] \otimes I_{(n+1)^{N-k}} \left(\sum_{i_1,\dots,i_N=0}^{n} c_{i_1,\dots,i_N}(e_{i_1} \otimes \dots \otimes e_{i_N}) \right) \\
&= \sum_{k=1}^{N} \sum_{i_1,\dots,i_N=0}^{n} c_{i_1,\dots,i_N} \left(I_{(n+1)^{k-1}} \otimes \left[D^2\right] \otimes I_{(n+1)^{N-k}} \right) (e_{i_1} \otimes \dots \otimes e_{i_N}) \\
&= \sum_{k=1}^{N} \sum_{i_1,\dots,i_N=0}^{n} c_{i_1,\dots,i_N} \left(I_{n+1}e_{i_1} \otimes \dots \otimes I_{n+1}e_{i_{k-1}} \otimes \left[D^2\right] e_{i_k} \otimes I_{n+1}e_{i_{k+1}} \right. \\
&\quad \left. \otimes \dots \otimes I_{n+1}e_{i_N} \right) \\
&= \sum_{k=1}^{N} \sum_{i_1,\dots,i_N=0}^{n} c_{i_1,\dots,i_N} \left(\begin{pmatrix} \delta_{0,i_1} \\ \vdots \\ \delta_{n,i_1} \end{pmatrix} \otimes \dots \otimes \begin{pmatrix} \delta_{0,i_{k-1}} \\ \vdots \\ \delta_{n,i_{k-1}} \end{pmatrix} \otimes \begin{pmatrix} d_{0,i_k} \\ \vdots \\ d_{n,i_k} \end{pmatrix} \otimes \begin{pmatrix} \delta_{0,i_{k+1}} \\ \vdots \\ \delta_{n,i_{k+1}} \end{pmatrix} \right. \\
&\quad \left. \otimes \dots \otimes \begin{pmatrix} \delta_{0,i_N} \\ \vdots \\ \delta_{n,i_N} \end{pmatrix} \right) \\
&= \sum_{k=1}^{N} \sum_{i_1,\dots,i_N=0}^{n} c_{i_1,\dots,i_N} \left(\delta_{j_1,i_1} \cdot \dots \cdot \delta_{j_{k-1},i_{k-1}} d_{j_k,i_k} \delta_{j_{k+1},i_{k+1}} \cdot \dots \cdot \delta_{j_N,i_N} \right)_{j_1,\dots,j_N=0}^{n} \\
&= \sum_{i_1,\dots,i_N=0}^{n} c_{i_1,\dots,i_N} \left(\sum_{k=1}^{N} \delta_{j_1,i_1} \cdot \dots \cdot \delta_{j_{k-1},i_{k-1}} d_{j_k,i_k} \delta_{j_{k+1},i_{k+1}} \cdot \dots \cdot \delta_{j_N,i_N} \right)_{j_1,\dots,j_N=0}^{n}.
\end{aligned}$$

Now we want to interprete $[\Delta]$ as a multilevel Toeplitz operator, which means that we are looking for a set of complex numbers $(b_{i_1,\ldots,i_N})_{i_1,\ldots,i_N}$, such that

$$T_{n+1,\ldots,n+1}(b)C = [\Delta]C$$

holds for all $C \in \bigotimes_{k=1}^{N} \mathbb{C}^{n+1}$. We have

$$T_{n+1,\ldots,n+1}(b)C = \sum_{i_1,\ldots,i_N=0}^{n} (b_{k_1-i_1,\ldots,k_N-i_N} c_{i_1,\ldots,i_N})_{k_1,\ldots,k_N=0}^{n}$$

and so we get

$$\begin{aligned} b_{k_1-i_1,\ldots,k_N-i_N} &= \sum_{j=1}^{N} \delta_{k_1,i_1} \cdot \ldots \cdot \delta_{k_{j-1},i_{j-1}} d_{k_j,i_j} \delta_{k_{j+1},i_{j+1}} \cdot \ldots \cdot \delta_{k_N,i_N} \\ &= d_{k_1,i_1}\delta_{k_2,i_2} \cdot \ldots \cdot \delta_{k_N,i_N} + \delta_{k_1,i_1} d_{k_2,i_2} \delta_{k_3,i_3} \cdot \ldots \cdot \delta_{k_N,i_N} \\ &\quad + \ldots + \delta_{k_1,i_1} \cdot \ldots \cdot \delta_{k_{N-1},i_{N-1}} d_{k_N,i_N}. \end{aligned}$$

It follows, for $m_1, \ldots, m_N \in \mathbb{Z}$,

$$\begin{aligned} b_{m_1,\ldots,m_N} &= b_{m_1+i_1-i_1,\ldots,m_N+i_N-i_N} \\ &= \sum_{j=1}^{N} \delta_{m_1+i_1,i_1} \cdot \ldots \cdot \delta_{m_{j-1}+i_{j-1},i_{j-1}} d_{m_j+i_j,i_j} \delta_{m_{j+1}+i_{j+1},i_{j+1}} \cdot \ldots \cdot \delta_{m_N+i_N,i_N}. \end{aligned}$$

This leads to

$$b_{m_1,\ldots,m_N} = \begin{cases} 0 & : \ \#\{i \in \{1,\ldots,N\} |\ m_i = 0\} \in \{0,\ldots,N-2\} \cup \{N\} \\ d_{m_1+i_1,i_1} & : \ m_2 = \ldots = m_N = 0 \wedge m_1 \neq 0 \\ d_{m_2+i_2,i_2} & : \ m_1 = m_3 = \ldots = m_N = 0 \wedge m_2 \neq 0 \\ \vdots & \\ d_{m_N+i_N,i_N} & : \ m_1 = \ldots = m_{N-1} = 0 \wedge m_N \neq 0 \end{cases}.$$

Further we have

$$d_{m+i,i} = \begin{cases} 0 & : \ m \geq -1 \\ -m-1 & : \ m \leq -2 \end{cases}.$$

This implies

$$b_{m_1,\ldots,m_N} = \begin{cases} -m_1 - 1 & : \ m_1 \leq -2 \wedge m_2 = \ldots = m_N = 0 \\ -m_2 - 1 & : \ m_2 \leq -2 \wedge m_1 = m_3 = \ldots = m_N = 0 \\ \vdots & \\ -m_N - 1 & : \ m_N \leq -2 \wedge m_1 = \ldots = m_{N-1} = 0 \\ 0 & : \ \text{otherwise} \end{cases}.$$

Now we can determine the functions $\sigma_{n,\dots,n}(b)$ and $S_{n,\dots,n}(b)$.

For arbitrary $\phi_1, \dots, \phi_N \in [0, 2\pi)$ we have

$$\begin{aligned}
&\sigma_{n,\dots,n}(b)\left(e^{\mathbf{i}\phi_1}, \dots, e^{\mathbf{i}\phi_N}\right) \\
&= \sum_{|j_1|,\dots,|j_N|\le n-1} b_{j_1,\dots,j_N} \prod_{k=1}^{N} \left(1 - \frac{|j_k|}{n}\right) e^{\mathbf{i}j_k\phi_k} \\
&= \sum_{|j_1|\le n-1} b_{j_1,0,\dots,0} \left(1 - \frac{|j_1|}{n}\right) e^{\mathbf{i}j_1\phi_1} + \sum_{|j_2|\le n-1} b_{0,j_2,0,\dots,0} \left(1 - \frac{|j_2|}{n}\right) e^{\mathbf{i}j_2\phi_2} \\
&+ \dots + \sum_{|j_N|\le n-1} b_{0,\dots,0,j_N} \left(1 - \frac{|j_N|}{n}\right) e^{\mathbf{i}j_N\phi_N}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j_1=-(n-1)}^{-2} (-j_1 - 1)\left(1 - \frac{|j_1|}{n}\right) e^{\mathbf{i}j_1\phi_1} + \dots + \sum_{j_N=-(n-1)}^{-2} (-j_N - 1)\left(1 - \frac{|j_N|}{n}\right) e^{\mathbf{i}j_N\phi_N} \\
&= \sum_{j=2}^{n-1} (j-1)\left(1 - \frac{j}{n}\right) e^{-\mathbf{i}j\phi_1} + \dots + \sum_{j=2}^{n-1} (j-1)\left(1 - \frac{j}{n}\right) e^{-\mathbf{i}j\phi_N} \\
&= \sum_{j=2}^{n-1} (j-1)\left(1 - \frac{j}{n}\right) \sum_{k=1}^{N} e^{-\mathbf{i}j\phi_k}.
\end{aligned}$$

Moreover, for all $\phi_1, \dots, \phi_N \in [0, 2\pi)$,

$$\begin{aligned}
S_{n,\dots,n}(b)\left(e^{\mathbf{i}\phi_1}, \dots, e^{\mathbf{i}\phi_N}\right) &= \sum_{|j_1|,\dots,|j_N|\le n-1} b_{j_1,\dots,j_N} \prod_{k=1}^{N} e^{\mathbf{i}j_k\phi_k} \\
&= \sum_{|j_1|\le n-1} b_{j_1,0,\dots,0} e^{\mathbf{i}j_1\phi_1} + \dots + \sum_{|j_N|\le n-1} b_{0,\dots,0,j_N} e^{\mathbf{i}j_N\phi_N} \\
&= \sum_{j=-(n-1)}^{-2} (-j-1) e^{\mathbf{i}j\phi_1} + \dots + \sum_{j=-(n-1)}^{-2} (-j-1) e^{\mathbf{i}j\phi_N} \\
&= \sum_{j=2}^{n-1} (j-1) e^{-\mathbf{i}j\phi_1} + \dots + \sum_{j=2}^{n-1} (j-1) e^{-\mathbf{i}j\phi_N} \\
&= \sum_{j=2}^{n-1} (j-1) \sum_{k=1}^{N} e^{-\mathbf{i}j\phi_k}.
\end{aligned}$$

The next step is the calculation of the $\|\cdot\|_\infty$ norms of these functions. For that we

chose arbitrary $e^{\mathbf{i}\phi_1}, \ldots, e^{\mathbf{i}\phi_N} \in \mathbb{T}$ and get

$$\begin{aligned}\left|\sigma_{n,\ldots,n}(b)\left(e^{\mathbf{i}\phi_1}, \ldots, e^{\mathbf{i}\phi_N}\right)\right| &\leq \sum_{j=2}^{n-1}(j-1)\left(1-\frac{j}{n}\right)\left|\sum_{k=1}^{N} e^{-\mathbf{i}j\phi_k}\right| \\ &\leq N\sum_{j=2}^{n-1}(j-1)\left(1-\frac{j}{n}\right).\end{aligned}$$

We claim that

$$\sum_{k=2}^{n-1}(k-1)\left(1-\frac{k}{n}\right) = \frac{n^2-3n+2}{6} \tag{4.10}$$

for all $n \in \mathbb{N}$, $n \geq 3$.

Indeed, for $n = 3$, this is obvious. If (4.10) holds for n, then it follows for $n+1$:

$$\begin{aligned}\sum_{k=2}^{n}(k-1)\left(1-\frac{k}{n+1}\right) &= \sum_{k=2}^{n-1}(k-1)\left(1-\frac{k}{n+1}\right) + (n-1)\left(1-\frac{n}{n+1}\right) \\ &= \frac{1}{n+1}\sum_{k=2}^{n-1}(k-1)(n+1-k) + \frac{n-1}{n+1} \\ &= \frac{1}{n+1}\left[\sum_{k=2}^{n-1}(k-1)(n-k) + \sum_{k=2}^{n-1}(k-1)\right] + \frac{n-1}{n+1} \\ &= \frac{1}{n+1}\left[n\sum_{k=2}^{n-1}(k-1)\left(1-\frac{k}{n}\right) + \sum_{k=1}^{n-2}k\right] + \frac{n-1}{n+1} \\ &= \frac{1}{n+1}\left[n\frac{n^2-3n+2}{6} + \frac{(n-2)(n-1)}{2}\right] + \frac{n-1}{n+1} \\ &= \frac{n^3-n}{6(n+1)} = \frac{n(n^2-1)}{6(n+1)} = \frac{n(n-1)}{6} \\ &= \frac{(n+1)^2-3(n+1)+2}{6}.\end{aligned}$$

So (4.10) is shown. We therefore get the estimate

$$\left|\sigma_{n,\ldots,n}(b)\left(e^{\mathbf{i}\phi_1}, \ldots, e^{\mathbf{i}\phi_N}\right)\right| \leq N\frac{n^2-3n+2}{6}.$$

This inequality is even an equality if we chose $\phi_1 = \ldots = \phi_N = 0$. It follows for $n \geq 3$ that

$$\|\sigma_{n,\ldots,n}(b)\|_\infty = N\frac{n^2-3n+2}{6}$$

and

$$\|\sigma_{n+1,\ldots,n+1}(b)\|_\infty = N\frac{n^2-n}{6}. \tag{4.11}$$

For $S_{n,\dots,n}(b)$ we get

$$\begin{aligned}\left|S_{n,\dots,n}(b)\left(e^{\mathbf{i}\phi_1},\dots,e^{\mathbf{i}\phi_N}\right)\right| &\leq \sum_{j=2}^{n-1}(j-1)\left|\sum_{k=1}^{N}e^{\mathbf{i}j\phi_k}\right| \leq N\sum_{j=2}^{n-1}(j-1) = N\sum_{j=1}^{n-2}j \\ &= \frac{N}{2}(n-2)(n-1) = \frac{N}{2}\left(n^2-3n+2\right).\end{aligned}$$

Here we have an equality if we chose $\phi_1 = \ldots = \phi_N = 0$. This leads to

$$\|S_{n+1,\dots,n+1}(b)\|_\infty = \frac{N}{2}\left(n^2-n\right). \tag{4.12}$$

With (4.11), (4.12) and Theorem 4.5 we finally get for the norm of Δ on $\mathcal{P}_n^N$ the upper and the lower bounds

$$\frac{N}{6}(n^2-n) \leq \lambda\left(\Delta\left|\mathcal{P}_n^N\right.\right) \leq \frac{N}{2}(n^2-n).$$

This completes the proof of Theorem 1.1. □

Now we want to derive the sharper asymptotic bounds for $\lambda\left(\Delta\left|\mathcal{P}_n^N\right.\right)$, that is, we want to prove Theorem 1.2. Before we prove Theorem 1.2, we need some results about eigenvalues of hermitian Toeplitz matrices with polynomialy increasing entries provided by [1].

For $\alpha > 0$ we denote by K_α the compact and selfadjoint integral operator on $L^2(0,1)$ defined by

$$(K_\alpha f)(x) := \int_0^1 |x-y|^\alpha f(y)\, dy$$

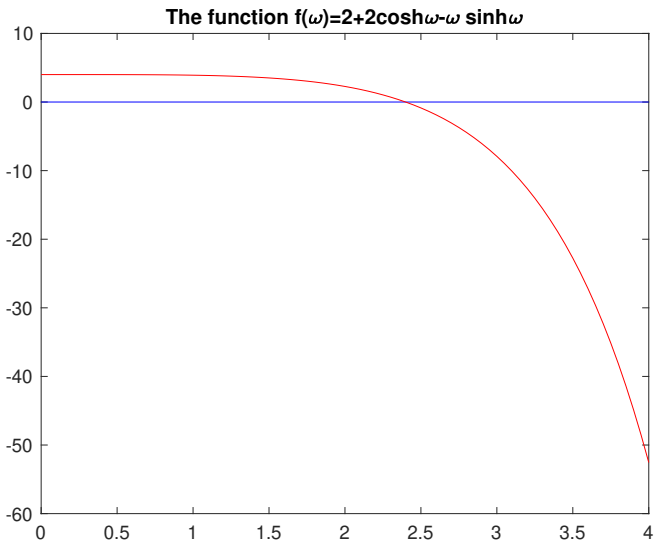
for $x \in (0,1)$. Let be $\mu_1(K_\alpha) \geq \mu_2(K_\alpha) \geq \ldots > 0$ the positive eigenvalues of K_α and $\mathscr{L}_+$ the index set of the positive eigenvalues of K_α. That is $\mathscr{L}_+ = \{1,\dots,m\}$, if K_α has exactly $m < \infty$ eigenvalues, and $\mathscr{L}_+ = \mathbb{N}$ if K_α has countably many positive eigenvalues. By Theorem 1.2 from [1], the operator K_1 has exactly one positive eigenvalue,

$$\mu_1(K_1) = \frac{2}{\omega_0^2} = 0,3471\ldots,$$

where $\omega_0 = 2,3993\ldots$ is the positive solution of the equation
$2 + 2\cosh(\omega) - \omega\sinh(\omega) = 0$. Let $(a_n)_n$ be a series of complex numbers. We write $T_n[a_0,\dots,a_{n-1}]$ for the hermitian Toeplitz matrix

$$T_n = T_n[a_0,\dots a_{n-1}] = \begin{pmatrix} a_0 & a_1 & \dots & a_{n-1} \\ \overline{a_1} & a_0 & \dots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a_{n-1}} & \overline{a_{n-2}} & \dots & a_0 \end{pmatrix}.$$

For the eigenvalues $\widetilde{\lambda}_1(T_n) \leq \widetilde{\lambda}_2(T_n) \leq \ldots \leq \widetilde{\lambda}_n(T_n)$ of T_n we have the following estimates (Theorem 1.1 in [1]).



Theorem 4.7. *Let $\alpha > 0$ be a real number. If $a_k = k^\alpha + o(k^\alpha)$ for $k \to \infty$, then the eigenvalues of $T_n = T_n[a_0, \dots, a_{n-1}]$ satisfy*

$$\widetilde{\lambda}_{n+1-l}(T_n) = \mu_l(K_\alpha) n^{\alpha+1} + o(n^{\alpha+1}) \tag{4.13}$$

as $n \to \infty$ and $l \in \mathscr{L}_+$. For $l \in \mathbb{N} \setminus \mathscr{L}_+$ we have

$$\widetilde{\lambda}_{n+1-l}(T_n) = o(n^{\alpha+1}) \tag{4.14}$$

as $n \to \infty$.

With this results we are now able to prove Theorem 1.2.

Proof of Theorem 1.2. The operator norm $\lambda\left(\Delta \,\middle|\, \mathcal{P}_n^N\right)$ of the Laplace operators is equal to the largest singular value $\sigma_1([\Delta])$ of its matrix representation $[\Delta]$ with respect to the Laguerre basis on $\mathcal{P}_n^N$, given by (3.3). With Corollary 4.2, equation

(4.4) and Corollary 4.4 we get the estimate

$$\begin{aligned}
\lambda\left(\Delta\,\middle|\mathcal{P}_n^N\right) &= \sigma_1\left([\Delta]\right) \geq \lambda_1\left(\frac{1}{2}\left([\Delta]+[\Delta]^T\right)\right)\\
&= \frac{1}{2}\lambda_1\left(\sum_{k=1}^{N} I_{(n+1)^{k-1}} \otimes \left[D^2\right] \otimes I_{(n+1)^{N-k}}\right.\\
&\quad \left. + \sum_{k=1}^{N} I_{(n+1)^{k-1}} \otimes \left[D^2\right]^T \otimes I_{(n+1)^{N-k}}\right)\\
&= \frac{1}{2}\lambda_1\left(\sum_{k=1}^{N} I_{(n+1)^{k-1}} \otimes \left(\left[D^2\right]+\left[D^2\right]^T\right) \otimes I_{(n+1)^{N-k}}\right)\\
&= \frac{1}{2}\lambda_1\left(\left(\left[D^2\right]+\left[D^2\right]^T\right)^{\oplus N}\right)\\
&= \frac{N}{2}\lambda_1\left(\left[D^2\right]+\left[D^2\right]^T\right).
\end{aligned}$$

So we need a nice expression for the largest eigenvalue of

$$\left[D^2\right]+\left[D^2\right]^T = \begin{pmatrix}
0 & 0 & 1 & 2 & & \dots & n-2 & n-1\\
0 & 0 & 0 & 1 & & \dots & & n-2\\
1 & 0 & & & & & \vdots & \vdots\\
2 & 1 & & \ddots & \ddots & & & \\
 & & & \ddots & \ddots & & 1 & 2\\
\vdots & \vdots & & & & & 0 & 1\\
n-2 & & \dots & & 1 & 0 & 0 & 0\\
n-1 & n-2 & \dots & & 2 & 1 & 0 & 0
\end{pmatrix}. \tag{4.15}$$

For that we will use Theorem 4.7. We write $T_{n+1} = T_{n+1}[a_0, \dots, a_n]$ with $a_0 = a_1 = 0$, $a_2 = 1, \dots, a_n = n-1$. The entries of this matrix can be written as

$$a_k = f(k), \quad f(k) = \begin{cases} k-1 & : \ k \geq 2 \\ 0 & : \ k \leq 1 \end{cases}.$$

For f we have

$$f(k) = (k-1) - (k-1)\chi_{(-\infty,1]}(k) = k + (-1-(k-1)\chi_{(-\infty,1]}(k)).$$

Now it holds

$$\frac{-1-(k-1)\chi_{(-\infty,1]}(k)}{k} = \begin{cases} \frac{-1-(k-1)}{k} = -1 & : \ k \leq 1 \\ -\frac{1}{k} & : \ k > 1 \end{cases} \to 0$$

for $k \to \infty$. So the entries of T_{n+1} are of the form $a_k = k + o(k)$ for $k \to \infty$. Hence we can apply Theorem 4.7. For the largest eigenvalue λ_1 of $[D^2] + [D^2]^T$ we conclude

$$\begin{aligned}
&\lambda_1 \left(\left[D^2\right] + \left[D^2\right]^T \right) \\
&= \widetilde{\lambda}_{n+1}\left(T_{n+1}\right) = \widetilde{\lambda}_{n+2-1}\left(T_{n+1}\right) = \mu_1(K_1)(n+1)^2 + o\left(n^2\right) \\
&= \frac{2}{\omega_0^2}(n+1)^2 + o\left(n^2\right). \qquad (n \to \infty)
\end{aligned}$$

We get for arbitrary $N \in \mathbb{N}$ and $n \to \infty$ the estimate

$$\lambda\left(\Delta\left|\mathcal{P}_n^N\right.\right) \geq \frac{N}{2}\lambda_1\left(\left[D^2\right] + \left[D^2\right]^T\right) = \frac{N}{\omega_0^2}(n+1)^2 + o\left(n^2\right).$$

for $n \to \infty$. This implies

$$\liminf_{n\to\infty} \frac{\lambda\left(\Delta\left|\mathcal{P}_n^N\right.\right)}{Nn^2} \geq \lim_{n\to\infty}\left(\frac{1}{Nn^2}\frac{N}{\omega_0^2}(n+1)^2 + \frac{o\left(n^2\right)}{Nn^2}\right) = \frac{1}{\omega_0^2} \geq 0.173.$$

The proposed upper bound is just a consequence of the Minkowski inequality. Namely, from (3.3) we derive $\lambda\left(\Delta\left|\mathcal{P}_n^N\right.\right) \leq N\lambda\left(D^2\left|\mathcal{P}_n\right.\right)$ and with (2.1) we conclude

$$\limsup_{n\to\infty} \frac{\lambda\left(\Delta\left|\mathcal{P}_n^N\right.\right)}{Nn^2} \leq \lim_{n\to\infty} \frac{N\lambda\left(D^2\left|\mathcal{P}_n\right.\right)}{Nn^2} = \frac{1}{\mu_0^2} \leq 0.285,$$

which completes the proof. □

Thus, we have shown the asserted asymptotic lower bound. However, from Corollary 4.2 we know further lower bounds for the singular values of a given matrix and it is natural to ask if these estimates can be used to get sharper lower bounds.

The answer is no, but we think it is nevertheless of interest to check what these other bounds would be. We sketch the idea by considering the largest eigenvalues of the matrix $\mathbf{i}S([\Delta])$.

Due to estimate (4.3) and Corollary 4.4 we have

$$\lambda\left(\Delta\left|\mathcal{P}_n^N\right.\right) \geq N\lambda_1\left(\mathbf{i}S\left(\left[D^2\right]\right)\right). \tag{4.16}$$

Now we ask for the asymptotic behavior of $\lambda\left(\Delta\left|\mathcal{P}_n^N\right.\right)$ provided by (4.16). For that we have to determine the largest eigenvalue of

$$\mathbf{i}S\left(\left[D^2\right]\right) = \frac{\mathbf{i}}{2}\begin{pmatrix}
0 & 0 & 1 & 2 & \dots & \dots & n-1 \\
0 & 0 & 0 & 1 & \dots & \dots & n-2 \\
-1 & 0 & 0 & 0 & \dots & \dots & n-3 \\
-2 & -1 & 0 & 0 & \dots & \dots & n-4 \\
\vdots & \vdots & \vdots & \vdots & & & \vdots \\
 & & & & & 0 & 1 \\
\vdots & \vdots & \vdots & \vdots & & 0 & 0 & 0 \\
-(n-1) & -(n-2) & -(n-3) & -(n-4) & \dots & -1 & 0 & 0
\end{pmatrix}.$$

This matrix is an hermitian Toeplitz matrix and again we want to apply the results from [1].

First we should fix some notation. For $a_0 = 0$ and $a_1, \ldots, a_{n-1} \in \mathbb{C}$ we set

$$T_n := T_n[a_0, \ldots, a_{n-1}] := \mathbf{i} \begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_{n-1} \\ -\overline{a_1} & a_0 & a_1 & \ldots & a_{n-2} \\ -\overline{a_2} & -\overline{a_1} & a_0 & \ldots & a_{n-3} \\ \vdots & \vdots & \vdots & & \vdots \\ -\overline{a_{n-1}} & -\overline{a_{n-2}} & -\overline{a_{n-3}} & \ldots & a_0 \end{pmatrix}.$$

We are interested in the particular case

$$T_n := T_n[0, 1^\alpha, 2^\alpha, \ldots, (n-1)^\alpha].$$

We denote the eigenvalues of T_n by

$$\widetilde{\lambda}_1(T_n) \leq \widetilde{\lambda}_2(T_n) \leq \ldots \leq \widetilde{\lambda}_n(T_n).$$

Furthermore we define the integral operator

$$\widetilde{K}_\alpha : L^2(0,1) \to L^2(0,1), \quad (\widetilde{K}_\alpha f)(x) := \int_0^1 \widetilde{k}_\alpha(x,y) f(y)\, dy,$$

$x \in (0,1)$, with the kernel

$$\widetilde{k}_\alpha(x,y) = \mathbf{i} \begin{cases} -|y-x|^\alpha & : \ x \in (0,1),\ y \in (0,x] \\ |y-x|^\alpha & : \ y \in (0,1),\ x \in (0,y) \end{cases}.$$

As in [1] we denote by

$$\mu_1(\widetilde{K}_\alpha) \geq \mu_2(\widetilde{K}_\alpha) \geq \ldots > 0$$

the positive eigenvalues of $\widetilde{K}_\alpha$ and we write $\mathscr{L}_+$ for the index set of the positive eigenvalues. Then we can show similar to Theorem 1.1 from [1] the following: Let a real number $\alpha > 0$ and numbers a_k of the form $a_k = k^\alpha + o(k^\alpha)$ for $k \to \infty$ be given. *Then for the matrix* $T_n = T_n[a_0, \ldots, a_{n-1}]$ *it holds for* $n \to \infty$ *that*

$$\widetilde{\lambda}_{n+1-l}(T_n) = \mu_l(\widetilde{K}_\alpha) n^{\alpha+1} + o(n^{\alpha+1}) \tag{4.17}$$

for all $l \in \mathscr{L}_+$.

This result can be used to determine the largest eigenvalue of $\mathbf{i}S([D^2])$. We know so far from the proof of Theorem 1.2 that the entries a_k of the first row of $2S([D^2])$ are of the desired form $a_k = k + o(k)$, so that we can apply (4.17). We are left with asking for the eigenvalues of the integral operator $\widetilde{K}_1$.

One can show,that the operator $\widetilde{K}_1$ has the eigenvalues $\lambda_{1,2} = \pm\sqrt{\frac{1}{12}}$. With this result we obtain the lower bound

$$\begin{aligned}\lambda\left(\Delta\left|\mathcal{P}_n^N\right.\right) &\geq \frac{N}{2}\sqrt{\frac{1}{12}}(n+1)^2 + o\left((n+1)^2\right) \\ &\approx \frac{N}{2}\cdot 0,2887(n+1)^2 + o\left((n+1)^2\right),\end{aligned}$$

which is worse than the bound

$$\lambda\left(\Delta\left|\mathcal{P}_n^N\right.\right) \geq \frac{N}{\omega_0^2}(n+1)^2 + o\left((n+1)^2\right) \approx \frac{N}{2}\cdot 0,3471(n+1)^2 + o\left((n+1)^2\right)$$

proposed in Theorem 1.2.

4.3 The Hermite case

In this section we consider the Laplace operator on $\mathcal{P}_n^N$ equipped with the Hermite norm (1.3), so our aim is to prove Theorem 1.3.

We will proceed as in the proof of Theorem 1.2. First we mention that by (3.5) the representation $[\Delta]$ is again the Kronecker sum of N copies of $[D^2]$, differing from the representation in the Laguerre case only in the matrix $[D^2]$, provided by (3.4). Now we can apply Corollary 4.2 and Corollary 4.4. In fact it is sufficient to use the estimate $\sigma_1([\Delta]) \geq \lambda_1(H([\Delta]))$ to get the asymptotic value of the norm, so we will not apply the other three estimates from Corollary 4.2.

We are interested in the largest eigenvalue of the $(n+1)\times(n+1)$ matrix

$$H_n := H\left(\left[D^2\right]\right) = \begin{pmatrix} 0 & 0 & \sqrt{2} & & & & 0 \\ 0 & 0 & 0 & \sqrt{6} & & & \\ \sqrt{2} & 0 & 0 & 0 & \sqrt{12} & & \\ & \sqrt{6} & 0 & 0 & & \ddots & \\ & & \sqrt{12} & & \ddots & & \sqrt{n(n-1)} \\ & & & \ddots & & 0 & 0 \\ 0 & & & & \sqrt{n(n-1)} & 0 & 0 \end{pmatrix}. \quad (4.18)$$

For compact subsets $E_1, E_2, E_3, \ldots$ of $\mathbb{C}$, we define

$$\liminf E_n := \left\{\lambda\in\mathbb{C}:\ \forall n\in\mathbb{N}\ \exists\lambda_n\in E_n \text{ such that } \lim_{n\to\infty}\lambda_n = \lambda\right\}.$$

Proof of Theorem 1.3. An upper bound for the norm of the Laplace operator follows from a simple application of the triangle inequality and the fact that the norm of

each summand in (3.5) is $\|[D^2]\|$:

$$\eta\left(\Delta\,|\mathcal{P}_n^N\right) = \left\|\bigoplus_{k=1}^{N}\left[D^2\right]\right\| \leq \sum_{k=1}^{N}\left\|\left[D^2\right]\right\| = N\left\|\left[D^2\right]\right\|.$$

Due to (3.4), the norm $\|[D^2]\|$ is just the largest entry of the matrix $[D^2]$, which implies the asymptotic upper bound

$$\limsup_{n\to\infty}\frac{\eta\left(\Delta\,|\mathcal{P}_n^N\right)}{Nn} \leq \limsup_{n\to\infty}\frac{2N\sqrt{n(n-1)}}{Nn} = 2. \tag{4.19}$$

Now we have to determine a lower bound. Let W_n denote the flip matrix

$$W_n := \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$$

and consider the matrix

$$A_n := \frac{1}{\sqrt{n(n-1)}}W_nH_nW_n$$

$$= \frac{1}{\sqrt{n(n-1)}} \cdot \begin{pmatrix} 0 & 0 & \sqrt{n(n-1)} & & & 0 \\ 0 & 0 & 0 & \sqrt{(n-1)(n-2)} & & \\ \sqrt{n(n-1)} & 0 & 0 & & \ddots & \\ & \sqrt{(n-1)(n-2)} & & \ddots & & \sqrt{2} \\ & & \ddots & & 0 & 0 \\ & & & \sqrt{2} & 0 & 0 \end{pmatrix}.$$

This matrix is $1/\sqrt{n(n-1)}$ times the matrix that results from (4.18) by writing the entries on the diagonals in reverse order. We denote by P_n the operator on $\ell^2(\mathbb{N})$ that projects a sequence to its first $n+1$ components. So A_nP_n is for each $n \in \mathbb{N}$ a well-defined bounded and linear operator on $\ell^2(\mathbb{N})$. We have

$$\|A_n\| = \frac{\|H_n\|}{\sqrt{n(n-1)}} \leq \frac{\|[D^2]\| + \left\|[D^2]^T\right\|}{\sqrt{n(n-1)}} = \frac{\sqrt{n(n-1)}+\sqrt{n(n-1)}}{\sqrt{n(n-1)}} = 2,$$

so the norms $\|A_nP_n\|$ are uniformly bounded by 2. If $x \in \ell^2(\mathbb{N})$ is a sequence with a finite number of non-vanishing entries, then A_nP_nx converges to $T(a)x$ as

$n \to \infty$, where $T(a)$ is the bounded Toeplitz operator given by the infinite matrix that has ones on the second subdiagonal, ones on the second superdiagonal, and zeros elsewhere. Consequently, $A_n P_n$ converges to $T(a)$ pointwise. We write $\sigma(T)$ for the spectrum of a bounded linear operator T on $\ell^2(\mathbb{N})$. Proposition 3.3 of [2] says that if A_n and A are selfadjoint and $A_n P_n \to A$ pointwise, then $\sigma(A) \subset \liminf \sigma(A_n)$. Thus, $\sigma\left(T(a)\right) \subset \liminf \sigma(A_n)$. The symbol of $T(a)$ is

$$a(e^{\mathbf{i}\theta}) = e^{2\mathbf{i}\theta} + e^{-2\mathbf{i}\theta} = 2\cos(2\theta),$$

because the Fourier coefficients $\widehat{a}_k$ of a and so the entries of $T(a)$ are

$$\widehat{a}_k = \frac{1}{2\pi}\int_0^{2\pi} a\left(e^{\mathbf{i}\theta}\right) e^{-\mathbf{i}k\theta}\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \left(e^{(2-k)\mathbf{i}\theta} + e^{-(2+k)\mathbf{i}\theta}\right)\, d\theta.$$

So we have

$$\widehat{a}_2 = \frac{1}{2\pi}\int_0^{2\pi} \left(1 + e^{-4\mathbf{i}\theta}\right)\, d\theta = 1 + \frac{1}{2\pi}\left[\frac{e^{-4\mathbf{i}\theta}}{-4\mathbf{i}}\right]_0^{2\pi} = 1,$$

$$\widehat{a}_{-2} = \frac{1}{2\pi}\int_0^{2\pi} \left(1 + e^{4\mathbf{i}\theta}\right)\, d\theta = 1 + \frac{1}{2\pi}\left[\frac{e^{4\mathbf{i}\theta}}{4\mathbf{i}}\right]_0^{2\pi} = 1$$

and

$$\widehat{a}_k = \frac{1}{2\pi}\left[\frac{e^{(2-k)\mathbf{i}\theta}}{(2-k)\mathbf{i}}\right]_0^{2\pi} + \frac{1}{2\pi}\left[\frac{e^{-(2+k)\mathbf{i}\theta}}{-(2+k)\mathbf{i}}\right]_0^{2\pi} = 0$$

for all $k \in \mathbb{Z} \setminus \{-2, 2\}$. Hence $\sigma\left(T(a)\right) = [-2, 2]$; see for example Corollary 1.12 in [6]. This implies, that for every $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$, such that $\lambda_{\max}(A_n) > 2 - \varepsilon$ for all $n > n_0$. With $\sigma(H_n) = \sqrt{n(n-1)}\sigma(A_n)$ we now get

$$\lambda_{\max}(H_n) = \sqrt{n(n-1)}\lambda_{\max}(A_n) > (2-\varepsilon)\sqrt{n(n-1)}$$

for all $n > n_0$, which leads to $\lambda_{\max}(H_n)/\sqrt{n(n-1)} \geq 2 - \varepsilon$ for all $n > n_0$. As we can choose $\varepsilon > 0$ arbitrarily small, we conclude that

$$\liminf_{n\to\infty} \lambda_{\max}(H_n)/\sqrt{n(n-1)} \geq 2,$$

which gives us

$$\liminf_{n\to\infty} \frac{\eta\left(\Delta \left| \mathcal{P}_n^N \right.\right)}{Nn} \geq \liminf_{n\to\infty} \frac{\lambda_{\max}\left(H([D^2])\right)}{n} = \liminf_{n\to\infty} \frac{\lambda_{\max}(H_n)}{\sqrt{n(n-1)}} \geq 2.$$

This estimate together with (4.19) completes the proof. □

4.4 The Legendre case

Finally we equip $\mathcal{P}_n^N$ with the Legendre norm, given by (1.4), and want to prove Theorem 1.4. The following theorem provides an exact lower bound for the norm of the Laplace operator in the case where n is odd.

Theorem 4.8. *For $n, N \in \mathbb{N}$ with $n+1 = 2m$ and for all $x = (x_0, \dots, x_{m-1})^\top \in \mathbb{C}^m \setminus \{0\}$ the norm of $\Delta : \mathcal{P}_n^N \to \mathcal{P}_n^N$ satisfies*

$$\gamma\left(\Delta \,|\mathcal{P}_n^N\right) \geq \frac{N}{2} \max\{L_1(x), L_2(x)\}$$

with

$$L_1(x) := \frac{1}{\|x\|^2} \sum_{j=1}^{m-1} \sum_{k=0}^{j-1} \sqrt{(4k+1)(4j+1)}[j(2j+1) - k(2k+1)]\,(x_j\overline{x_k} + x_k\overline{x_j})$$

and

$$L_2(x) := \frac{1}{\|x\|^2} \sum_{j=1}^{m-1} \sum_{k=0}^{j-1} \sqrt{(4k+3)(4j+3)}[2j^2 + 3j - 2k^2 - 3k]\,(x_k\overline{x_j} + x_j\overline{x_k})\,.$$

Proof. Again due to (4.3), we will look for a lower bound of $\lambda_{\max}\left(H\left([D^2]\right)\right)$. This will be done by application of the Rayleigh-Ritz theorem, which says that if A is a Hermitian $n \times n$ matrix, then

$$\lambda_{\min}(A) \leq \frac{1}{\overline{x}^\top x} \overline{x}^\top A x \leq \lambda_{\max}(A). \tag{4.20}$$

for all $x \in \mathbb{C}^n \setminus \{0\}$; see, e.g., Theorem 4.2.2 in [12].

As mentioned in Section 3.3, $[D^2]$ is unitarily equivalent to $\mathbb{D}_2$ from (3.7). This implies that $[D^2] + [D^2]^\top$ is unitarily equivalent to $\mathbb{D}_2 + \mathbb{D}_2^\top$. It follows that

$$\lambda_{\max}\left(\left[D^2\right] + \left[D^2\right]^T\right) = \lambda_{\max}\left(\mathbb{D}_2 + \mathbb{D}_2^\top\right). \tag{4.21}$$

Due to the block structure of

$$\mathbb{D}_2 + \mathbb{D}_2^\top = \begin{pmatrix} A_m B_m + B_m^\top A_m^\top & 0 \\ 0 & B_m A_m + A_m^\top B_m^\top \end{pmatrix}$$

we get

$$\lambda_{\max}\left(\mathbb{D}_2 + \mathbb{D}_2^\top\right) = \max\left\{\lambda_{\max}\left(A_m B_m + B_m^\top A_m^\top\right), \lambda_{\max}\left(B_m A_m + A_m^\top B_m^\top\right)\right\}.$$

To apply (4.20), we first want to calculate the entries of the matrices A_mB_m and B_mA_m. We have

$$A_mB_m = \begin{pmatrix} c_{01} & c_{03} & \dots & c_{0,2m-1} \\ & c_{23} & \dots & c_{2,2m-1} \\ & & \ddots & \vdots \\ & & & c_{2m-2,2m-1} \end{pmatrix} \cdot \begin{pmatrix} 0 & c_{12} & c_{14} & \dots & c_{1,2m-2} \\ & 0 & c_{34} & \dots & c_{3,2m-2} \\ & & \ddots & & \vdots \\ & & & \ddots & c_{2m-3,2m-2} \\ & & & & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & c_{01}c_{12} & c_{01}c_{14}+c_{03}c_{34} & \dots & \sum_{k=0}^{m-2} c_{0,2k+1}c_{2k+1,2m-2} \\ & 0 & c_{23}c_{34} & \dots & \sum_{k=1}^{m-2} c_{2,2k+1}c_{2k+1,2m-2} \\ & & \ddots & & \vdots \\ & & & \ddots & \sum_{k=m-2}^{m-2} c_{2m-4,2k+1}c_{2k+1,2m-2} \\ & & & & 0 \end{pmatrix}.$$

We denote the entries of A_mB_m by a_{ij}, $i,j \in \{0,\dots,m-1\}$. Then we have

$$a_{ij} = \begin{cases} \sum_{k=i}^{j-1} c_{2i,2k+1}c_{2k+1,2j} & : i = 0,\dots,j-1 \text{ and } j = 1,\dots,m-1 \\ 0 & : \text{otherwise} \end{cases}.$$

With (3.6) we get for $j = 1,\dots,m-1$ and $i = 0,\dots,j-1$ the equalities

$$\begin{aligned} a_{ij} &= \sum_{k=i}^{j-1} c_{2i,2k+1}c_{2k+1,2j} \\ &= \sum_{k=i}^{j-1} \sqrt{(4i+1)(4k+3)}\sqrt{(4k+3)(4j+1)} = \sqrt{(4i+1)(4j+1)}\sum_{k=i}^{j-1}(4k+3) \\ &= \sqrt{(4i+1)(4j+1)}\left[4\sum_{k=i}^{j-1} k + 3(j-i)\right] \\ &= \sqrt{(4i+1)(4j+1)}\left[4\sum_{k=0}^{j-1} k - 4\sum_{k=0}^{i-1} k + 3(j-i)\right] \\ &= \sqrt{(4i+1)(4j+1)}\,[2j(j-1) - 2i(i-1) + 3(j-i)] \\ &= \sqrt{(4i+1)(4j+1)}[j(2j+1) - i(2i+1)], \end{aligned}$$

where $\sum_{k=s}^{t} k := 0$ if $t < s$. So we obtain for A_mB_m the entries

$$a_{ij} = \begin{cases} \sqrt{(4i+1)(4j+1)}[j(2j+1) - i(2i+1)] & : i = 0,\dots,j-1 \\ & \quad \text{and } j = 1,\dots,m-1\,. \\ 0 & : \text{otherwise} \end{cases} \quad (4.22)$$

Now let us consider the entries of

$$B_m A_m = \begin{pmatrix} 0 & c_{12}c_{23} & c_{12}c_{25}+c_{14}c_{45} & \dots & \sum_{k=1}^{m-1} c_{1,2k}c_{2k,2m-1} \\ & 0 & c_{34}c_{45} & \dots & \sum_{k=2}^{m-1} c_{3,2k}c_{2k,2m-1} \\ & & \ddots & & \vdots \\ & & & \ddots & \sum_{k=m-1}^{m-1} c_{2m-3,2k}c_{2k,2m-1} \\ & & & & 0 \end{pmatrix}.$$

We denote the entries of this matrix by b_{ij}, $i,j = 0,\dots,m-1$ and have

$$b_{ij} = \begin{cases} \sum_{k=i+1}^{j} c_{2i+1,2k}c_{2k,2j+1} & : i = 0,\dots,j-1 \\ & \text{and } j = 1,\dots,m-1\,. \\ 0 & : \text{otherwise} \end{cases}$$

A direct computation shows

$$b_{ij} = \begin{cases} \sqrt{(4i+3)(4j+3)}\,[2j^2+3j-2i^2-3i] & : i = 0,\dots,j-1 \\ & \text{and } j = 1,\dots,m-1\,. \\ 0 & : \text{otherwise} \end{cases} \quad (4.23)$$

Now we are interested in $\lambda_{\max}\left(A_m B_m + B_m^\top A_m^\top\right)$ and $\lambda_{\max}\left(B_m A_m + A_m^\top B_m^\top\right)$. Due to (4.20) these eigenvalues are bounded from below for all $x \in \mathbb{C}^m \setminus \{0\}$ by

$$L_1(x) := \frac{1}{\overline{x}^\top x}\overline{x}^\top (A_m B_m + B_m^\top A_m^\top)x \quad \text{and} \quad L_2(x) := \frac{1}{\overline{x}^\top x}\overline{x}^\top (B_m A_m + A_m^\top B_m^\top)x.$$

For arbitrary $(x_0,\dots,x_{m-1})^\top \in \mathbb{C}^m \setminus \{0\}$ we deduce from (4.22)

$$\begin{aligned} \frac{1}{\overline{x}^\top x}\overline{x}^\top A_m B_m x &= \frac{1}{\|x\|^2}\sum_{j=0}^{m-1}\overline{x_j}\sum_{i=0}^{m-1} a_{ji}x_i = \frac{1}{\|x\|^2}\sum_{j=0}^{m-1}\sum_{i=0}^{m-1} a_{ij}x_j\overline{x_i} \\ &= \frac{1}{\|x\|^2}\sum_{j=1}^{m-1}\sum_{i=0}^{j-1}\sqrt{(4i+1)(4j+1)}[j(2j+1)-i(2i+1)]x_j\overline{x_i}. \end{aligned}$$

Further direct calculations yield

$$\frac{1}{\overline{x}^\top x}\overline{x}^\top B_m^\top A_m^\top x = \frac{1}{\|x\|^2}\sum_{j=1}^{m-1}\sum_{i=0}^{j-1}\sqrt{(4i+1)(4j+1)}[j(2j+1)-i(2i+1)]x_i\overline{x_j},$$

which leads to

$$L_1(x) = \frac{1}{\|x\|^2}\sum_{j=1}^{m-1}\sum_{i=0}^{j-1}\sqrt{(4i+1)(4j+1)}[j(2j+1)-i(2i+1)]\left(x_j\overline{x_i}+x_i\overline{x_j}\right). \quad (4.24)$$

Another direct computation gives us

$$L_2(x) = \frac{1}{\|x\|^2} \sum_{j=1}^{m-1} \sum_{i=0}^{j-1} \sqrt{(4i+3)(4j+3)}[2j^2+3j-2i^2-3i]\left(x_i\overline{x_j} + x_j\overline{x_i}\right). \quad (4.25)$$

This completes the proof of the theorem. □

Proof of Theorem 1.4. We will apply the inequality from Theorem 4.8 with a concrete vector $x \in \mathbb{C}^m$ to deduce the asymptotic lower bound of the operator norm. The upper bound will finally be shown with the triangle inequality.

The Faulhaber formula states that

$$\sum_{k=1}^{n} k^l = \frac{1}{l+1} \sum_{m=0}^{l} \binom{l+1}{m} (n+1)^{l-m+1} B_m \quad (4.26)$$

for all $n, l \in \mathbb{N}$, where B_m denotes the m-th Bernoulli number. Of special interest are the cases $l \in \{1, \ldots, 5\}$:

$$\begin{aligned}
\sum_{k=1}^{n} k^1 &= \frac{n(n+1)}{2} \\
\sum_{k=1}^{n} k^2 &= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \\
\sum_{k=1}^{n} k^3 &= \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 \\
\sum_{k=1}^{n} k^4 &= \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n \\
\sum_{k=1}^{n} k^5 &= \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2.
\end{aligned}$$

A full proof can be found in Section 4.5 of [19], for example.

We set $x := \left(\sqrt{4i+1}\right)_{i=0}^{m-1}$. Repeated application of (4.26) leads to

$$\begin{aligned}
&L_1(x) \\
&= \frac{1}{\|x\|^2} \sum_{j=1}^{m-1} \sum_{i=0}^{j-1} \sqrt{(4i+1)(4j+1)}[j(2j+1) - i(2i+1)] 2\sqrt{(4i+1)(4j+1)} \\
&= \frac{2}{\|x\|^2} \sum_{j=1}^{m-1} \sum_{i=0}^{j-1} (4i+1)(4j+1)[j(2j+1) - i(2i+1)]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{\|x\|^2} \sum_{j=1}^{m-1} (4j+1) \left[j(2j+1) \sum_{i=0}^{j-1} (4i+1) - \sum_{i=0}^{j-1} i(2i+1)(4i+1) \right] \\
&= \frac{2}{\|x\|^2} \sum_{j=1}^{m-1} (4j+1) \left[j(2j+1)(2j(j-1)+j) - \sum_{i=0}^{j-1} (8i^3+6i^2+i) \right] \\
&= \frac{2}{\|x\|^2} \sum_{j=1}^{m-1} (4j+1) \left[4j^4 - \frac{8}{4} j^4 + o\left(j^4\right) \right] = \frac{2}{\|x\|^2} \sum_{j=1}^{m-1} \left[8j^5 + o\left(j^5\right) \right] \\
&= \frac{2}{\|x\|^2} \left[\frac{8}{6} m^6 + o\left(m^6\right) \right].
\end{aligned}
$$

With

$$
\|x\|^2 = \sum_{i=0}^{m-1} |x_i|^2 = \sum_{i=0}^{m-1} (4i+1) = 2m(m-1) + m = 2m^2 - m
$$

we finally get

$$
L_1(x) = \frac{1}{2m-1} \left(\frac{8}{3} m^5 + o\left(m^5\right) \right). \tag{4.27}
$$

Theorem 4.8 requires that n is odd. Let us denote the norm $\gamma\left(\Delta \left| \mathcal{P}_n^N \right.\right)$ by α_n if n is odd and by β_n if n is even. From Theorem 4.8 we infer that

$$
\begin{aligned}
\liminf_{n\to\infty} \frac{\alpha_n}{Nn^4} &= \liminf_{m\to\infty} \frac{\alpha_{2m-1}}{N(2m-1)^4} = \liminf_{m\to\infty} \frac{\alpha_{2m-1}}{N(2m)^4} = \frac{1}{16} \liminf_{m\to\infty} \frac{\alpha_{2m-1}}{Nm^4} \\
&\geq \frac{1}{16} \liminf_{m\to\infty} \frac{N}{2} \frac{L_1(x)}{Nm^4} \geq \frac{1}{16} \liminf_{m\to\infty} \frac{1}{2} \left(\frac{8}{3} \frac{m^5}{2m-1} + \frac{o\left(m^5\right)}{2m-1} \right) \frac{1}{m^4} \\
&= \frac{1}{16} \lim_{m\to\infty} \left(\frac{4}{3} \frac{m}{2m-1} + \frac{o\left(m^5\right)}{(2m-1)m^4} \right) = \frac{1}{16} \cdot \frac{2}{3} = \frac{1}{24}.
\end{aligned}
$$

Since $\mathcal{P}_{2m-1}^N$ is a linear subspace of $\mathcal{P}_{2m}^N$, we have $\alpha_{2m-1} \leq \beta_{2m}$ and thus

$$
\frac{\beta_n}{Nn^4} = \frac{\beta_{2m}}{N(2m)^4} \geq \frac{(2m-1)^4}{(2m)^4} \frac{\alpha_{2m-1}}{N(2m-1)^4}.
$$

This leads to

$$
\liminf_{n\to\infty} \frac{\beta_n}{Nn^4} \geq \liminf_{n\to\infty} \frac{\alpha_n}{Nn^4} \geq \frac{1}{24}
$$

and completes the proof of the lower bound in Theorem 1.4.

The upper bound is just a consequence of the triangle inequality. Indeed, we have

$$
\begin{aligned}
&\gamma\left(\Delta \left| \mathcal{P}_n^N \right.\right) \\
&= \left\| \left[D^2\right] \otimes I \otimes \ldots \otimes I + I \otimes \left[D^2\right] \otimes I \otimes \ldots \otimes I + \ldots + I \otimes \ldots \otimes I \otimes \left[D^2\right] \right\| \\
&\leq \sum_{k=1}^{N} \left\| \left[D^2\right] \right\| = N \left\| \left[D^2\right] \right\|
\end{aligned}
$$

and as mentioned above, Shampine [23] has shown that $\|[D^2]\|/n^4$ converges to $1/(4\mu_0^2)$, where $\mu_0 \approx 1.8751041$ is the smallest positive solution of the equation $1 + \cosh\mu\cos\mu = 0$. It follows that

$$\limsup_{n\to\infty} \frac{\gamma\left(\Delta\left|\mathcal{P}_n^N\right.\right)}{Nn^4} \leq \limsup_{n\to\infty} \frac{\|[D^2]\|}{n^4} = \frac{1}{4\mu_0^2},$$

which is the asserted upper bound. □

Remark 4.9. We have seen that the proof of Theorem 1.4 is based on Theorem 4.8, which provides a lower bound depending on an arbitrary vector $x \in \mathbb{C}^m \setminus \{0\}$. Thus it is natural to ask what happens if we choose other vectors for x or calculate a lower bound with L_2 instead of L_1.

For example we can try for $l \in \mathbb{N}_0$ the vector $x^{(l)}$ given by

$$x^{(l)} := \left((4i+1)^{\frac{2l+1}{2}}\right)_{i=0}^{m-1}.$$

The advantage of this vector is that we can use again the Faulhaber formula to calculate $\left\|x^{(l)}\right\|$ and $L_1\left(x^{(l)}\right)$ and that we perhaps could improve the results by a good choice of the parameter l. We get

$$\left\|x^{(l)}\right\| = \frac{2\cdot 4^{2l}}{l+1} m^{2l+2} + o\left(m^{2l+2}\right)$$

and

$$L_1\left(x^{(l)}\right) = \frac{2}{\|x^{(l)}\|^2}\left[\frac{4^{2l+3}}{2(l+2)(l+3)(l+4)} m^{2l+6} + o\left(m^{2l+6}\right)\right].$$

This leads to the asymptotic lower bound $\frac{l+1}{(l+2)(l+3)(l+4)}$. It is easy to see, that the function

$$f : [0,\infty) \to \mathbb{R}, \quad x \mapsto \frac{x+1}{(x+2)(x+3)(x+4)}$$

is decreasing, so the asymptotic lower bound takes its maximum value for $l = 0$, what is just $\frac{1}{24}$, that is the bound from Theorem 1.4. Analogously we can insert

$$x^{(l)} := \left((4i+3)^{\frac{2l+1}{2}}\right)_{i=0}^{m-1},$$

$l \in \mathbb{N}_0$, in L_2, but this leads again to the already known asymptotic lower bound $\frac{l+1}{(l+2)(l+3)(l+4)}$.

From Corollary 4.2 we know that we have also the estimate

$$\sigma_{\max}([\Delta]) \geq \lambda_{\max}(\mathbf{i}S([\Delta])),$$

so we should end this chapter with some remarks on this estimate.

The well known estimate $\gamma\left(\Delta\left|\mathcal{P}_n^N\right.\right) \geq N\lambda_{\max}\left(\mathbf{i}S\left([D^2]\right)\right)$ and calculations completely analogous to the calculations in the proof of Theorem 4.8 give us for $n+1 = 2m$ and an arbitrary $x = (x_0, \ldots, x_{m-1})^T \in \mathbb{C}^m \setminus \{0\}$ for $\gamma\left(\Delta\left|\mathcal{P}_n^N\right.\right)$ the lower bound

$$\gamma\left(\Delta\left|\mathcal{P}_n^N\right.\right) \geq \frac{N}{2}\max\left\{\widetilde{L}_1(x), \widetilde{L}_2(x)\right\} \tag{4.28}$$

with

$$\widetilde{L}_1(x) = \frac{1}{\|x\|^2}\mathbf{i}\sum_{j=1}^{m-1}\sum_{i=0}^{j-1}\sqrt{4i+1}\sqrt{4j+1}\left(2j^2+j-2i^2-i\right)\left(\overline{x_i}x_j - x_i\overline{x_j}\right)$$

and

$$\widetilde{L}_2(x) = \frac{1}{\|x\|^2}\mathbf{i}\sum_{j=1}^{m-1}\sum_{i=0}^{j-1}\sqrt{4i+3}\sqrt{4j+3}\left(2j^2+3j-2i^2-3i\right)\left(\overline{x_i}x_j - x_i\overline{x_j}\right).$$

A natural choice for x is

$$x := \left((4i+1)^{\frac{2l+1}{2}} + \mathbf{i}(4i+1)^{\frac{2\widetilde{l}+1}{2}}\right)_{i=0}^{m-1} \tag{4.29}$$

with $l, \widetilde{l} \in \mathbb{N}_0$. For $l = \widetilde{l}$, the application of the Faulhaber formula leads to

$$\widetilde{L}_1(X) = \frac{2}{\|x\|^2}\left(\frac{4^{2l+3}}{2(l+2)(l+3)(l+4)}m^{2l+6} + o\left(m^{2l+6}\right)\right)$$

with

$$\|x\|^2 = \frac{4^{2l+1}}{l+1}m^{2l+2} + o\left(m^{2l+2}\right).$$

We get then by (4.28) the asymptotic bound

$$\liminf_{n\to\infty}\frac{\gamma\left(\Delta\left|\mathcal{P}_n^N\right.\right)}{Nn^4} \geq \frac{l+1}{2(l+2)(l+3)(l+4)}.$$

For $l \neq \widetilde{l}$ we get only the trivial lower bound zero, so x from (4.29) cannot improve the results from Theorem 1.4 and it is not clear so far if we find an x such that $\widetilde{L}_1(x)$ or $\widetilde{L}_2(x)$ provide a sharper lower bound.

4.5 The Laplace operator on $\mathcal{P}_n(\Omega_N)$

4.5.1 The Hermite case

In this section we want to prove Theorem 1.6.

Of course the norm of the operator is again just the largest singular value of the matrix representation $[\Delta]$ of the Laplace operator in the basis $B_n^{(N)}$ from (3.9). We use the following general singular value estimate to get a lower bound for this value. If A is an arbitrary $m \times n$ matrix with rows $a_{i\cdot}$ and columns $a_{\cdot j}$, then the largest singular value of A satisfies the inequality

$$\sigma_{\max}(A) \geq \max\left\{ \max_{0\leq i\leq m-1}\{\|a_{i\cdot}\|\}, \max_{0\leq j\leq n-1}\{\|a_{\cdot j}\|\} \right\}, \tag{4.30}$$

where $\|\cdot\|$ denotes the euclidian norm. This result can be found as Theorem 5 in [17].

Proof of Theorem 1.6. Before we derive the lower bound in the case $N \in \{2,3\}$, we want to calculate the asymptotic value of the operator norm for general N. The orthonormal basis $B_n^{(N)}$ in $\mathcal{P}_n(\Omega_N)$ particularly includes the vector $h_n \otimes h_0 \otimes \ldots \otimes h_0$. We apply the Laplace operator to this vector and get due to (3.10) that

$$\Delta(h_n \otimes h_0 \otimes \ldots \otimes h_0) = 2\sqrt{n(n-1)} h_{n-2} \otimes h_0 \otimes \ldots \otimes h_0.$$

This means, that the matrix representation of the Laplace operator with respect to $B_n^{(N)}$ contains a column of the form

$$\left(0, \ldots, 0, 2\sqrt{n(n-1)}, 0, \ldots, 0\right)^T.$$

Due to (4.30), the maximal singular value $\sigma_{\max}([\Delta])$ is bounded from below by the maximum of the euclidian norms of all columns of $[\Delta]$, so we have in particular

$$\eta(\Delta|\mathcal{P}_n(\Omega_N)) = \sigma_{\max}([\Delta]) \geq \left\| \left(0, \ldots, 0, 2\sqrt{n(n-1)}, 0, \ldots, 0\right)^T \right\| = 2\sqrt{n(n-1)}.$$

Obviously, the space $\mathcal{P}_n(\Omega_N)$ is a linear subspace of $\mathcal{P}_n^N$. This provides us with the upper bound

$$\frac{2\sqrt{n(n-1)}}{Nn} \leq \frac{\eta(\Delta|\mathcal{P}_n(\Omega_N))}{Nn} \leq \frac{\eta\left(\Delta\left|\mathcal{P}_n^N\right.\right)}{Nn}.$$

The left quotient in this chain of inequalities goes for $n \to \infty$ to $2/N$ and we know from Theorem 1.3 the asymptotic behavior $\eta\left(\Delta\left|\mathcal{P}_n^N\right.\right) \sim 2Nn$ which proves the asymptotic upper and lower bounds for $\eta(\Delta|\mathcal{P}_n(\Omega_N))$ for arbitrary $N \in \mathbb{N}$.

Now we want to show the lower bounds from Theorem 1.6.

We start with $N = 2$ and denote for $k = 1, \ldots, n-1$ the rows and respectively columns of A_k from 3.11 by $a_{i\cdot}^{(k)}$, $i = 0, \ldots, k-1$, and $a_{\cdot j}^{(k)}$, $j = 0, \ldots, k+1$. The estimate (4.30) implies due to the block structure of $[\Delta]$ that

$$\sigma_{\max}([\Delta]) \geq \max\left\{ \max_{1\leq k\leq n-1} \max_{0\leq i\leq k-1}\{\|a_{i\cdot}^{(k)}\|\}, \max_{1\leq k\leq n-1} \max_{0\leq j\leq k+1}\{\|a_{\cdot j}^{(k)}\|\} \right\}. \tag{4.31}$$

Clearly we have $\max_{0\le j\le 2}\{\|a_{\cdot j}^{(1)}\|\} = 2\sqrt{2}$ and $\max_{0\le j\le 3}\{\|a_{\cdot j}^{(2)}\|\} = 2\sqrt{6}$. The columns of A_k have for $k \ge 3$ the form

$$(2\sqrt{k(k+1)}, 0, \ldots, 0)^T, (0, 2\sqrt{(k-1)k}, 0, \ldots, 0)^T,$$

$$(0, \ldots, 0, 2\sqrt{k(k+1)})^T, (0, \ldots, 0, 2\sqrt{(k-1)k}, 0)^T$$

or

$$(0, \ldots, 0, 2\sqrt{j(j-1)}, 0, \ldots, 0, 2\sqrt{(k+1-j)(k-j)}, 0, \ldots, 0)^T$$

with $j = 2, \ldots, k-1$. So the euclidian norms of the columns are $\|a_{\cdot 0}^{(k)}\| = \|a_{\cdot k+1}^{(k)}\| = 2\sqrt{k(k+1)}$, $\|a_{\cdot 1}^{(k)}\| = \|a_{\cdot k}^{(k)}\| = 2\sqrt{k(k-1)}$ and

$$\left\|a_{\cdot j}^{(k)}\right\| = \sqrt{4j(j-1) + 4(k+1-j)(k-j)} = 2\sqrt{2j^2 + (-2-2k)j + k^2 + k}$$

for $j = 2, \ldots, k-1$.

The convex function $f : [2, k-1] \to \mathbb{R},\ x \mapsto 2x^2 + (-2-2k)x + k^2 + k$ describes a parabola, so the maximum of f must be realized on the boundary of the interval $[2, k-1]$. At these points, the function takes the value $f(2) = f(k-1) = k^2 - 3k + 4$, so we conclude

$$\max_{2\le j\le k-1} \left\|a_{\cdot j}^{(k)}\right\| = 2\sqrt{k^2 - 3k + 4}.$$

This leads to

$$\max_{0\le j\le k+1} \left\|a_{\cdot j}^{(k)}\right\| = \max\left\{2\sqrt{k(k+1)},\ 2\sqrt{k^2-3k+4}\right\} = 2\sqrt{k^2+k}$$

and so we get

$$\sigma_{\max}([\Delta]) \ge \max_{1\le k\le n-1} 2\sqrt{k^2+k}$$

due to (4.31). The function $g : [1, n-1] \to \mathbb{R}, x \mapsto x^2 + x$ takes its maximum also on the boundary of the interval and it holds $g(1) = 2 \le n^2 - n = g(n-1)$ für $n \ge 2$. With that we get finally

$$\sigma_{\max}([\Delta]) \ge 2\sqrt{n^2 - n}. \tag{4.32}$$

Now we want to consider the lower bound of largest singular value provided by the maximum of the norms of the rows. We have

$$\max_{0\le i\le 0} \left\|a_{i\cdot}^{(1)}\right\| = 4, \quad \max_{0\le i\le 1} \left\|a_{i\cdot}^{(2)}\right\| = 4\sqrt{2}$$

and

$$\begin{aligned}
&\max_{0\le i\le k-1} \left\|a_{i\cdot}^{(k)}\right\| \\
&= \max_{2\le i\le k+1} \left\|\left(0, \ldots, 0, 2\sqrt{i(i-1)}, 0, \ldots, 0, 2\sqrt{(k+3-i)(k+2-i)}, 0, \ldots, 0\right)\right\| \\
&= \max_{2\le i\le k+1} 2\sqrt{2i^2 + (-6-2k)i + k^2 + 5k + 6}.
\end{aligned}$$

The function $f : [2, k+1] \to \mathbb{R}, x \mapsto 2x^2 + (-6-2k)x + k^2 + 5k + 6$ becomes, by the same argument as above, maximal on $\{2, k+1\}$ and the maximum value is $f(2) = f(k+1) = k^2 + k + 2$. It follows

$$\max_{1\le k\le n-1} \max_{0\le i\le k-1} \left\| a_{i\cdot}^{(k)} \right\| = \max_{1\le k\le n-1} 2\sqrt{k^2+k+2}.$$

The function $g : [1, n-1] \to \mathbb{R}, x \mapsto x^2 + x + 2$ assumes its maximum on $\{1, n-1\}$, and the corresponding values are $g(1) = 4 \le n^2 - n + 2 = g(n-1)$ for $n \ge 5$. With (4.31) the lower bound

$$\sigma_{\max}([\Delta]) \ge \max\left\{2\sqrt{n^2-n}, 2\sqrt{n^2-n+2}\right\} = 2\sqrt{n^2-n+2}$$

follows.

Now we consider the case $N = 3$. We will denote the entries of $A_k^{(2)}$ from (3.14) by $d_{ij}^{(k)}$. We calculate the lower bound of the largest singular value provided by the maximum of the euclidian norms of all rows of the matrix representation.

From (3.13) and (4.30) we infer that

$$\sigma_{\max}([\Delta]) \ge \max_{k=2,\dots,n} \max_{i=0,\dots,\frac{1}{2}k(k-1)-1} \left\| d_{i\cdot}^{(k)} \right\|. \tag{4.33}$$

It holds

$$\max_{i=0,\dots,\frac{1}{2}k(k-1)-1} \left\| d_{i\cdot}^{(k)} \right\| = \max_{l=1,\dots,k-1} \max_{i=0,\dots,l-1} \left\| \widetilde{a}_{i\cdot}^{l} \right\|,$$

where we define $\widetilde{a}_{i\cdot}^{(l)} := \left(a_{i\cdot}^{(l)}, 2\sqrt{(l-k+1)(l-k)}\right)$ as the row vector $a_{i\cdot}^{(l)}$, extended by the entry $2\sqrt{(l-k+1)(l-k)}$. Thus we have

$$\begin{aligned}
&\max_{i=0,\dots,l-1} \left\| \widetilde{a}_{i\cdot}^{(l)} \right\| \\
&= \max_{i=2,\dots,l+1} \sqrt{\left[2\sqrt{i(i-1)}\right]^2 + \left[2\sqrt{(l+2-i)(l+3-i)}\right]^2 + \left[2\sqrt{(l-k+1)(l-k)}\right]^2} \\
&= \max_{i=2,\dots,l+1} 2\sqrt{2i^2 + (-6-2l)i + 2l^2 + (6-2k)l + k^2 - k + 6}.
\end{aligned}$$

When we set $i = 2$ or $i = l+1$, the last square root becomes maximal. So the maximal norm of the extended row vectors is

$$\max_{i=0,\dots,l-1} \left\| \widetilde{a}_{i\cdot}^{(l)} \right\| = 2\sqrt{2l^2 + (2-2k)l + k^2 - k + 2},$$

which shows

$$\max_{i=0,\dots,\frac{1}{2}k(k-1)-1} \left\| d_{i\cdot}^{(k)} \right\| = \max_{l=1,\dots,k-1} 2\sqrt{2l^2 + (2-2k)l + k^2 - k + 2}.$$

The term $2l^2 + (2-2k)l + k^2 - k + 2$, considered as a function in l on the interval $[1, k-1]$, becomes maximal for $l = 1$ or $l = k-1$, so we get now for $k \geq 2$

$$\max_{l=1,\ldots,k-1} 2\sqrt{2l^2 + (2-2k)l + k^2 - k + 2} = 2\sqrt{k^2 - k + 2}.$$

From (4.33) we infer

$$\sigma_{\max}([\Delta]) \geq \max_{k=2,\ldots,n} 2\sqrt{k^2 - k + 2} = 2\sqrt{n^2 - n + 2},$$

which completes the proof. □

Remark 4.10. One may ask what happens if we estimate $\sigma_{\max}([\Delta])$ by the maximum of the euclidian norms of all columns.

Due to the structure of the matrix representation (3.13) and due to (4.30) we get

$$\sigma_{\max}([\Delta]) \geq \max_{k=2,\ldots,n} \max_{j=0,\ldots,\frac{1}{2}(k+1)(k+2)-1} \left\| d_{\cdot j}^{(k)} \right\|. \tag{4.34}$$

The calculation of the right side of this inequality is very similar to the proof of Theorem 1.6. First we have

$$\max_{j=0,\ldots,\frac{1}{2}(k+1)(k+2)-1} \left\| d_{\cdot j}^{(k)} \right\| = \max \left\{ 2\sqrt{k^2 - k}, \max_{l=1,\ldots,k-3} \max_{i=0,\ldots,l+1} \left\| \widetilde{a}_{\cdot i}^{(l)} \right\| \right\},$$

where the vector $\widetilde{a}_{\cdot i}^{(l)}$ is defined as the column $a_{\cdot i}^{(l)}$ extended by

$$2\sqrt{(k-l-1)(k-l-2)}.$$

For these vectors we obtain

$$\left\| \widetilde{a}_{\cdot 0}^{(l)} \right\| = \left\| \widetilde{a}_{\cdot l+1}^{(l)} \right\| = 2\sqrt{k^2 - 3k + 2 + (4-2k)l + 2l^2},$$

$$\left\| \widetilde{a}_{\cdot 1}^{(l)} \right\| = \left\| \widetilde{a}_{\cdot l}^{(l)} \right\| = 2\sqrt{k^2 - 3k + 2 + (2-2k)l + 2l^2}$$

and

$$\left\| \widetilde{a}_{\cdot i}^{(l)} \right\| = 2\sqrt{2i^2 + (-2-2l)i + k^2 - 3k + 2 + (4-2k)l + 2l^2}$$

for $i = 2, \ldots, l-1$. We maximize the function

$$f : [2, l-1] \to \mathbb{R}, \;\; x \mapsto 2x^2 + (-2-2l)x + k^2 - 3k + 2 + (4-2k)l + 2l^2$$

and get

$$\max_{i=0,\ldots,l+1} \left\| \widetilde{a}_{\cdot i}^{(l)} \right\| = 2\sqrt{k^2 - 3k + 2 + (4-2k)l + 2l^2}.$$

The maximum of the function

$$g : [1, k-3] \to \mathbb{R}, \;\; x \mapsto 2x^2 + (4-2k)x + k^2 - 3k + 2$$

is $g(1) = g(k-3) = k^2 - 5k + 8$, thus we have

$$\max_{l=1,\ldots,k-3} \max_{i=0,\ldots,l+1} \left\| \widetilde{a}_{\cdot i}^{(l)} \right\| = 2\sqrt{k^2 - 5k + 8}.$$

Since $k \geq 2$, we now conclude

$$\max_{j=0,\ldots,\frac{1}{2}(k+1)(k+2)-1} \left\| d_{\cdot j}^{(k)} \right\| = 2\sqrt{k^2 - k}.$$

With that and (4.34) we get

$$\sigma_{\max}([\Delta]) \geq \max_{k=2,\ldots,n} 2\sqrt{k^2 - k} = 2\sqrt{n^2 - n}$$

for $n \geq 2$, that is, the estimate is, as for $N = 2$, weaker than the one provided by the norms of the rows.

4.5.2 The Laguerre and the Legendre case

Now we consider the Laplace operator on the space $\mathcal{P}_n(\Omega_N)$ equipped with the Laguerre or respectively the Legendre norm. Here we cannot proceed as in the cases above, where every variable of the polynomials occurs with the power of at most n.

The problems starts with finding a matrix representation for Δ for arbitrary N, because $\mathcal{P}_n(\Omega_N)$ is not a simple tensor product of copies of $\mathcal{P}_n$. Nevertheless we can get at least an idea of the representations in the case $N = 2$. Recall that we denote by

$$r_k(t) := \frac{1}{\Gamma(k+1)} e^t \frac{d^k}{dt^k} t^k e^{-t}$$

the kth normalized Laguerre polynomial and by

$$p_k(t) = \sqrt{\frac{2k+1}{2}} \frac{(-1)^k}{2\Gamma(k+1)} \frac{d^k}{dt^k} (1 - t^2)^k$$

the kth normalized Legendre polynomial. For $q \in \{r, p\}$, the set

$$B_n^{(N)} := \{q_{i_1} \otimes \ldots \otimes q_{i_N} : \ i_1 + \ldots + i_N \leq n\}$$

is an orthonormal basis in $\mathcal{P}_n(\Omega_N)$ with respect to the appropriate norm. In the case $N = 2$ we give the elements of this basis the order

$$\begin{aligned} B_n^{(2)} \ = \ & \{q_0 \otimes q_0, \\ & q_0 \otimes q_1, \ q_1 \otimes q_0, \\ & q_0 \otimes q_2, \ q_1 \otimes q_1, \ q_2 \otimes q_0, \\ & q_0 \otimes q_3, \ q_1 \otimes q_2, \ q_2 \otimes q_1, \ q_3 \otimes q_0, \\ & q_0 \otimes q_4, \ q_1 \otimes q_3, \ q_2 \otimes q_2, \ q_3 \otimes q_1, \ q_4 \otimes q_0, \\ & \ldots, q_0 \otimes q_n, \ q_1 \otimes q_{n-1}, \ q_2 \otimes q_{n-2}, \ldots, q_{n-1} \otimes q_1, \ q_n \otimes q_0\}. \end{aligned}$$

So the cardinality of this basis and therefore the dimension of $\mathcal{P}_n(\Omega_2)$ is $\sum_{k=1}^{n+1} k = (n+1)(n+2)/2$.

Now we consider the matrix representation of the Laplace operator in the Laguerre case. For the polynomials r_k we have

$$r_k'(t) = \begin{cases} \sum_{j=0}^{k-1} r_j(t) & : \ k \geq 1 \\ 0 & : \ k = 0 \end{cases}.$$

For $k \geq 2$, this implies

$$r_k''(t) = r_0'(t) + r_1'(t) + \ldots + r_{k-1}'(t) = \sum_{j=0}^{0} r_j(t) + \sum_{j=0}^{1} r_j(t) + \ldots + \sum_{j=0}^{k-2} r_j(t) = \sum_{i=0}^{k-2} \sum_{j=0}^{i} r_j.$$

As a consequence we get for $r_{i_1} \otimes r_{i_2} \in B_n^{(2)}$ that

$$\begin{aligned} \Delta(r_{i_1} \otimes r_{i_2}) &= r_{i_1}'' \otimes r_{i_2} + r_{i_1} \otimes r_{i_2}'' \\ &= \begin{cases} \sum_{j=0}^{i_1-2} \sum_{l=0}^{j} r_l \otimes r_{i_2} + \sum_{j=0}^{i_2-2} \sum_{l=0}^{j} r_{i_1} \otimes r_l & : \ i_1, i_2 \geq 2 \\ \sum_{j=0}^{i_1-2} \sum_{l=0}^{j} r_l \otimes r_{i_2} & : \ i_1 \geq 2, i_2 < 2 \\ \sum_{j=0}^{i_2-2} \sum_{l=0}^{j} r_{i_1} \otimes r_l & : \ i_1 < 2, i_2 \geq 2 \\ 0 & : \ i_1, i_2 < 2 \end{cases}. \end{aligned}$$

The equality

$$\begin{aligned} \sum_{j=0}^{k-2} \sum_{l=0}^{j} r_l &= \sum_{j=0}^{k-2} (r_0 + r_1 + \ldots + r_j) \\ &= r_0 + (r_0 + r_1) + (r_0 + r_1 + r_2) + \ldots + (r_0 + r_1 + \ldots + r_{k-2}) \\ &= (k-1) r_0 + (k-2) r_1 + (k-3) r_2 + \ldots + r_{k-2} \\ &= \sum_{l=0}^{k-2} (k-l-1) r_l \end{aligned}$$

implies that $\Delta(r_{i_1} \otimes r_{i_2})$ equals

$$\begin{cases} \sum_{l=0}^{i_1-2} (i_1 - l - 1) r_l \otimes r_{i_2} + \sum_{l=0}^{i_2-2} (i_2 - l - 1) r_{i_1} \otimes r_l & : \ i_1, i_2 \geq 2 \\ \sum_{l=0}^{i_1-2} (i_1 - l - 1) r_l \otimes r_{i_2} & : \ i_1 \geq 2, i_2 < 2 \\ \sum_{l=0}^{i_2-2} (i_2 - l - 1) r_{i_1} \otimes r_l & : \ i_1 < 2, i_2 \geq 2 \\ 0 & : \ i_1, i_2 < 2 \end{cases}. \tag{4.35}$$

To get the matrix representation of Δ, we consider the representation of the polynomials $\Delta(r_{i_1} \otimes r_{i_2})$ for $i_1 + i_2 = k$, $k \geq 2$ in our basis. The following table contains the columns of the matrix representation belonging to these elements $r_{i_1} \otimes r_{i_2}$ of the basis. The pairs (i_1, i_2) represent here the basis elements $r_{i_1} \otimes r_{i_2}$. The entries of the columns are the coefficients of the basis elements labelled in the left column in the representation of the elements labelled in the first row.

	$(0,k)$	$(1,k-1)$	$(2,k-2)$	$(3,k-3)$	$\dots$	$(k-3,3)$	$(k-2,2)$	$(k-1,1)$	$(k,0)$
$(0,0)$	$k-1$	0	0	0	$\dots$	0	0	0	$k-1$
$(0,1)$	$k-2$	0	0	0	$\dots$	0	0	$k-2$	0
$(1,0)$	0	$k-2$	0	0	$\dots$	0	0	0	$k-2$
$(0,2)$	$k-3$	0	0	0	$\dots$	0	$k-3$	0	0
$(1,1)$	0	$k-3$	0	0	$\dots$	0	0	$k-3$	0
$(2,0)$	0	0	$k-3$	0	$\dots$	0	0	0	$k-3$
$(0,3)$	$k-4$	0	0	0	$\dots$	$k-4$	0	0	0
$(1,2)$	0	$k-4$	0	0	$\dots$	0	$k-4$	0	0
$(2,1)$	0	0	$k-4$	0	$\dots$	0	0	$k-4$	0
$(3,0)$	0	0	0	$k-4$	$\dots$	0	0	0	$k-4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(0,k-3)$	2	0	0	2	$\dots$	0	0	0	0
$(1,k-4)$	0	2	0	0	$\dots$	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(k-4,1)$	0	0	0	0	$\dots$	0	0	2	0
$(k-3,0)$	0	0	0	0	$\dots$	2	0	0	2
$(0,k-2)$	1	0	1	0	$\dots$	0	0	0	2
$(1,k-3)$	0	1	0	1	$\dots$	0	0	0	0
$(2,k-4)$	0	0	1	0	$\dots$	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(k-4,2)$	0	0	0	0	$\dots$	0	1	0	0
$(k-3,1)$	0	0	0	0	$\dots$	1	0	1	0
$(k-2,0)$	0	0	0	0	$\dots$	0	1	0	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(0,n)$	0	0	0	0	$\dots$	0	0	0	0
$(1,n-1)$	0	0	0	0	$\dots$	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(n-1,1)$	0	0	0	0	$\dots$	0	0	0	0
$(n,0)$	0	0	0	0	$\dots$	0	0	0	0

For a better illustration of what $[\Delta]$ looks like, we quote the matrix in the special

case $n = 5$:

$$[\Delta] = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 1 & 2 & 0 & 0 & 2 & 3 & 0 & 0 & 0 & 3 & 4 & 0 & 0 & 0 & 0 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 0 & 2 & 0 & 3 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 0 & 2 & 0 & 3 & 0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & & & & \dots & & & & & & & & & & & & \dots & & & & 0 \\
\vdots & & & & & & & & & & & & & & & & & & & & \vdots \\
0 & & & & \dots & & & & & & & & & & & & \dots & & & & 0
\end{pmatrix}.$$

Remark 4.11. The maybe most simple idea to find a lower bound for the largest singular value of this matrix is the application of (4.30). This gives us by a direct computation

$$\sigma_{\max}([\Delta]) \geq \sqrt{\frac{2}{3}}\sqrt{n^3 + o(n^3)}.$$

So we have

$$\liminf_{n\to\infty} \frac{\lambda(\Delta|\mathcal{P}_n(\Omega_2))}{n^{3/2}} \geq \sqrt{\frac{2}{3}}.$$

But then we cannot find an non-trivial upper bound by the estimate $\lambda(\Delta|\mathcal{P}_n(\Omega_2)) \leq \lambda(\Delta|\mathcal{P}_n^2)$, because $\lambda(\Delta|\mathcal{P}_n^2)$ is bounded by a quadratic polynomial in n, which implies only

$$\limsup_{n\to\infty} \frac{\lambda(\Delta|\mathcal{P}_n(\Omega_2))}{n^{3/2}} \leq \infty.$$

The following argument to obtain lower bounds was kindly communicated to me by Stefan Kunis.

Proof of Theorem 1.5. For all $N \in \mathbb{N}$ we have $\frac{1}{N}[0,1]^N \subset \Omega_N \subset [0,1]^N$. This implies for the related spaces of polynomials the relation

$$\mathcal{P}_n\left(\frac{1}{N}[0,1]^N\right) \subset \mathcal{P}_n(\Omega_N) \subset \mathcal{P}_n^N.$$

With that we get for the norms of the Laplace operator the estimate

$$\lambda\left(\Delta\left|\mathcal{P}_n\left(\frac{1}{N}[0,1]^N\right)\right.\right) \leq \lambda(\Delta|\mathcal{P}_n(\Omega_N)) \leq \lambda\left(\Delta\left|\mathcal{P}_n^N\right.\right). \tag{4.36}$$

The space $\mathcal{P}_n\left(\frac{1}{N}[0,1]^N\right)$ is just the linear space of complex polynomials in n variables in which every variable occurs with power of at most $\lfloor n/N \rfloor$, where $\lfloor x \rfloor$ denotes for an arbitrary $x \in \mathbb{R}$ the largest integer smaller than or equal to x. That is, we have $\mathcal{P}_n\left(\frac{1}{N}[0,1]^N\right) = \mathcal{P}^N_{\lfloor n/N \rfloor}$. Now we apply Theorem 1.2 with $\lfloor n/N \rfloor$ instead of n and get

$$\begin{aligned}\liminf_{n\to\infty} \frac{\lambda\left(\Delta\left|\mathcal{P}_n\left(\Omega_N\right)\right.\right)}{Nn^2} &\geq \liminf_{n\to\infty} \frac{\lambda\left(\Delta\left|\mathcal{P}^N_{\lfloor n/N \rfloor}\right.\right)}{Nn^2} \\ &= \liminf_{n\to\infty} \frac{\lfloor n/N \rfloor^2}{n^2} \frac{\lambda\left(\Delta\left|\mathcal{P}^N_{\lfloor n/N \rfloor}\right.\right)}{N\lfloor n/N \rfloor^2} \geq 0.173 \cdot \frac{1}{N^2},\end{aligned}$$

where we use that $\frac{1}{n^2}\lfloor n/N \rfloor^2$ goes to $1/N^2$ if n goes to infinity.

The upper bound is also a simple consequence of (4.36) und Theorem 1.2, namely

$$\limsup_{n\to\infty} \frac{\lambda\left(\Delta\left|\mathcal{P}_n\left(\Omega_N\right)\right.\right)}{Nn^2} \leq \limsup_{n\to\infty} \frac{\lambda\left(\Delta\left|\mathcal{P}^N_n\right.\right)}{Nn^2} \leq 0.285,$$

which completes the proof. □

Now we turn to the Legendre case.

First we want to get an idea about the structure of the matrix representation for $N = 2$. For the normalized Legendre polynomials we have

$$p'_k = \sum_{j=0}^{k-1} c_{ji} p_j$$

with

$$c_{jk} = \begin{cases} 0 & : \ j+k \quad \text{even} \\ \sqrt{(2j+1)(2k+1)} & : \ j+k \quad \text{odd} \end{cases}.$$

Therefore we calculate the second derivatives of the polynomials as

$$\begin{aligned}
p_k'' &= \sum_{j=1}^{k-1} c_{jk} p_j' = \sum_{j=1}^{k-1} c_{jk} \sum_{i=0}^{j-1} c_{ij} p_i \\
&= \sum_{j=1}^{k-1} c_{jk} \left(c_{0j} p_0 + c_{1j} p_1 + \ldots + c_{j-1,j} p_{j-1} \right) \\
&= c_{1k} c_{01} p_0 + c_{2k} (c_{02} p_0 + c_{12} p_1) + c_{3k} (c_{03} p_0 + c_{13} p_1 + c_{23} p_2) \\
&\quad + c_{4k} (c_{04} p_0 + c_{14} p_1 + c_{24} p_2 + c_{34} p_3) \\
&\quad + \ldots + c_{k-1,k} (c_{0,k-1} p_0 + c_{1,k-1} p_1 + \ldots + c_{k-2,k-1} p_{k-2}) \\
&= \sum_{i=1}^{k-1} c_{ik} c_{0i} p_0 + \sum_{i=2}^{k-1} c_{ik} c_{1i} p_1 + \sum_{i=3}^{k-1} c_{ik} c_{2i} p_2 + \ldots + \sum_{i=k-1}^{k-1} c_{ik} c_{k-2,i} p_{k-2} \\
&= \sum_{j=0}^{k-2} \sum_{i=j+1}^{k-1} c_{ik} c_{ji} p_j .
\end{aligned}$$

The application of the Laplace operator on an arbitrary basis element $p_{i_1} \otimes p_{i_2} \in B_n^{(2)}$ gives us

$$\begin{aligned}
\Delta \left(p_{i_1} \otimes p_{i_2} \right) &= p_{i_1}'' \otimes p_{i_2} + p_{i_1} \otimes p_{i_2}'' \\
&= \sum_{j=0}^{i_1-2} \sum_{i=j+1}^{i_1-1} c_{i,i_1} c_{ji} p_j \otimes p_{i_2} + \sum_{j=0}^{i_2-2} \sum_{i=j+1}^{i_2-1} c_{i,i_2} c_{ji} p_{i_1} \otimes p_j
\end{aligned}$$

(if $i_1, i_2 \geq 2$ holds, otherwise the double sum, belonging to the index, vanishes). The following example shows the structure of the matrix $[\Delta]$ for $n = 6$; only the entries denoted by red stars are not zero:

$$\begin{pmatrix}
0 & 0 & 0 & * & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * \\
0 & & & & & & & \ldots & & & & & & & & & & & & \ldots & & & & & & 0 \\
\vdots & & & & & & & & & & & & & & & & & & & & & & & & & & & \vdots \\
0 & & & & & & & \ldots & & & & & & & & & & & & \ldots & & & & & & 0
\end{pmatrix}$$

The matrix has obviously not a chessboard structure as for $N = 1$, but only a

block-chessboard structure,

$$[\Delta] = \begin{pmatrix} O_{1,1} & O_{1,2} & A_{1,3} & O_{1,4} & A_{1,5} & O_{1,6} & A_{1,7} \\ O_{2,1} & O_{2,2} & & A_{2,4} & O_{2,5} & A_{2,6} & O_{2,7} \\ & & & & A_{3,5} & O_{3,6} & A_{3,7} \\ \vdots & \vdots & & & & A_{4,6} & O_{4,7} \\ & & & & & & A_{5,7} \\ O_{6,1} & O_{6,2} & & \dots & \dots & & O_{6,7} \\ O_{7,1} & O_{7,2} & & \dots & \dots & & O_{7,7} \end{pmatrix},$$

where $O_{k,l}$ are zero matrices of size $k \times l$ and where we denote by $A_{k,l}$ the blocks with two non-vanishing diagonals. Of course we can transform $[\Delta]$ into a unitarily equivalent block matrix

$$\widetilde{[\Delta]} = \begin{pmatrix} \mathbb{A} & \\ & \mathbb{B} \end{pmatrix}$$

with

$$\mathbb{A} = \begin{pmatrix} 0 & A_{1,3} & A_{1,5} & A_{1,7} \\ \vdots & & A_{3,5} & A_{3,7} \\ \vdots & & & A_{5,7} \\ 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix} \quad \text{and} \quad \mathbb{B} = \begin{pmatrix} 0 & 0 & A_{2,4} & A_{2,6} \\ \vdots & \vdots & & A_{4,6} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

but $\mathbb{A}$ has again not a chessboard structure, so it becomes problematic to find a non-zero complex vector x to calculate a lower bound via the Rayleigh quotient

$$\frac{x^* \widetilde{[\Delta]} x}{x^* x} \leq \lambda_{\max} \left(H \left(\widetilde{[\Delta]} \right) \right).$$

Nevertheless we can proceed as in the Laguerre case by the argument of Stefan Kunis.

Proof of Theorem 1.7. The proof is just a simple application of Theorem 1.4 with $\lfloor n/N \rfloor$ instead of n and (4.36) with γ instead of λ. More concretely we have

$$\begin{aligned} \liminf_{n\to\infty} \frac{\gamma\left(\Delta\left|\mathcal{P}_n\left(\Omega_N\right)\right.\right)}{Nn^4} &\geq \liminf_{n\to\infty} \frac{\gamma\left(\Delta\left|\mathcal{P}^N_{\lfloor n/N \rfloor}\right.\right)}{Nn^4} \\ &= \liminf_{n\to\infty} \frac{\lfloor n/N \rfloor^4}{n^4} \frac{\gamma\left(\Delta\left|\mathcal{P}^N_{\lfloor n/N \rfloor}\right.\right)}{N\lfloor n/N \rfloor^4} \geq \frac{1}{24N^4} \end{aligned}$$

and

$$\limsup_{n\to\infty} \frac{\gamma\left(\Delta\left|\mathcal{P}_n\left(\Omega_N\right)\right.\right)}{Nn^4} \leq \limsup_{n\to\infty} \frac{\gamma\left(\Delta\left|\mathcal{P}^N_n\right.\right)}{Nn^4} \leq \frac{1}{4\mu_0^2}.$$

□

5 Other differential operators

In the following chapter we will focus on $E = [0,1]^N$. The methods used in the proofs of Theorem 1.1, Theorem 1.2 and Theorem 1.3 can be applied also to other differential operators to which results from [5] cannot be applied. We will focus on the following examples.

For $\nu \in \mathbb{N}$ we consider the operator

$$\Delta_\nu : \mathcal{P}_n^N \to \mathcal{P}_n^N,\ f \mapsto \partial_1^\nu f + \ldots + \partial_N^\nu f.$$

Further we are interested in the operator

$$\Delta^2 : \mathcal{P}_n^N \to \mathcal{P}_n^N,\ f \mapsto \Delta(\Delta f).$$

According to our notation used for the Laplace operator we will denote the operator norms of these operators by $\lambda\left(\Delta_\nu \left| \mathcal{P}_n^N \right.\right)$, $\eta\left(\Delta_\nu \left| \mathcal{P}_n^N \right.\right)$, and $\lambda\left(\Delta^2 \left| \mathcal{P}_n^N \right.\right)$, respectively.

By application of the method of multi-level Toeplitz operators, used in the proof of Theorem 1.1, we get save and asymptotic bounds for Δ_ν and Δ^2 on $\mathcal{P}_n^N$, equipped with the Laguerre norm, as seen in [7]. As a further result, we will see that we can improve the asymptotic lower bound for $\lambda\left(\Delta_\nu \left| \mathcal{P}_n^N \right.\right)$ be applying Lemma 4.2. The idea of the proof of Theorem 1.3 can also be used to determine the asymptotic value of $\eta\left(\Delta_\nu \left| \mathcal{P}_n^N \right.\right)$ for the Hermite norm.

We have to mention here that unfortunately we have not been able get satisfactory results on the norms of Δ_ν or Δ^2 in the case of the Legendre norm.

5.1 The operator Δ_ν

5.1.1 The Laguerre case

For $\nu \in \mathbb{N}$, we denote by D^ν the ordinary differential operator on $\mathcal{P}_n$, sending an polynomial to its νth derivative. We have the following safe and asymptotic bounds for the norm of the operator Δ_ν on $\mathcal{P}_n^N$.

Theorem 5.1. *For all natural numbers $n, N, \nu \geq 1$ we have for the norm of the operator Δ_ν on $\mathcal{P}_n^N$, equipped with the Laguerre norm, the safe bounds*

$$\frac{N}{\nu+1}\binom{n}{\nu} \leq \lambda\left(\Delta_\nu \left| \mathcal{P}_n^N \right.\right) \leq N\binom{n}{\nu}$$

and the asymptotic bounds

$$\frac{1}{(\nu+1)!} \leq \liminf_{n\to\infty} \frac{\lambda\left(\Delta_\nu \left| \mathcal{P}_n^N \right.\right)}{Nn^\nu} \leq \limsup_{n\to\infty} \frac{\lambda\left(\Delta_\nu \left| \mathcal{P}_n^N \right.\right)}{Nn^\nu} \leq \frac{1}{(\nu-1)!}\frac{1}{\sqrt{2\nu(2\nu-1)}}.$$

Before we prove this theorem, we need a small lemma.

Lemma 5.2. *For positive integers n, m and q the equation*

$$\sum_{k=m}^{n} \binom{k}{q} = \binom{n+1}{q+1} - \binom{m}{q+1} \tag{5.1}$$

holds.

Proof. We prove this formula by induction on n. For $n = m$ we have

$$\begin{aligned}
\sum_{k=m}^{m} \binom{k}{q} &= \binom{m}{q} = \binom{m}{q}\left[\frac{m+1}{q+1} - \frac{m-q}{q+1}\right] \\
&= \frac{m!}{q!}\left[\frac{m+1}{(q+1)(m-q)!} - \frac{1}{(q+1)(m-q-1)!}\right] \\
&= \frac{(m+1)!}{(q+1)!(m-q)!} - \frac{m!}{(q+1)!(m-q-1)!} = \binom{m+1}{q+1} - \binom{m}{q+1}.
\end{aligned}$$

Let us asume that (5.1) holds for n. Then we get for $n+1$ that

$$\begin{aligned}
\sum_{k=m}^{n+1} \binom{k}{q} &= \sum_{k=m}^{n} \binom{k}{q} + \binom{n+1}{q} = \binom{n+1}{q+1} - \binom{m}{q+1} + \binom{n+1}{q} \\
&= \frac{(n+1)!}{(q+1)!(n-q)!} + \frac{(n+1)!}{q!(n+1-q)!} - \binom{m}{q+1} \\
&= \frac{(n+1)!(n+1-q) + (n+1)!(q+1)}{(q+1)!(n+1-q)!} - \binom{m}{q+1} \\
&= \frac{(n+1)!(n+1-q+q+1)}{(q+1)!(n+2-(q+1))!} - \binom{m}{q+1} \\
&= \frac{(n+2)!}{(q+1)!(n+2-(q+1))!} - \binom{m}{q+1} \\
&= \binom{n+2}{q+1} - \binom{m}{q+1},
\end{aligned}$$

which completes the proof. □

Proof of Theorem 5.1. To prove this result, we first ask for the matrix representation $[\Delta_\nu]$ with respect to an orthonormal basis in $\mathcal{P}_n^N$. Analogously to the Chapter 3, we identify $\mathcal{P}_n^N$ with $\bigotimes_{k=1}^N \mathcal{P}_n$ and equip this space with the Laguerre basis

$$\{L_{j_1} \otimes \ldots \otimes L_{j_N} : \; j_1, \ldots, j_N \in \{0, \ldots, n\}\}.$$

We write Δ_ν as

$$\Delta_\nu = D^\nu \otimes I \otimes \ldots \otimes I + I \otimes D^\nu \otimes I \otimes \ldots \otimes I + \ldots + I \otimes \ldots \otimes I \otimes D^\nu$$

and get the matrix representation

$$[\Delta_\nu] = \sum_{k=1}^{N} I_{(n+1)^{k-1}} \otimes [D^\nu] \otimes I_{(n+1)^{N-k}}. \tag{5.2}$$

The matrix representation of D^ν with respect to the Laguerre basis on $\mathcal{P}_n$ is

$$[D^\nu] = \left(d_{ij}^{(\nu)}\right)_{i,j=0}^{n} = (-1)^\nu \begin{pmatrix} 0 & \binom{0}{\nu-1} & \binom{1}{\nu-1} & \dots & \binom{n-1}{\nu-1} \\ & 0 & \binom{0}{\nu-1} & \dots & \binom{n-2}{\nu-1} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & \binom{0}{\nu-1} \\ & & & & 0 \end{pmatrix}, \tag{5.3}$$

as shown in [9] and [10]. We will now proceed as in the proof of Theorem 1.1. The representation of Δ_ν induces in a natural manner an operator

$$[\Delta_\nu] : \bigotimes_{k=1}^{N} \mathbb{C}^{n+1} \to \bigotimes_{k=1}^{N} \mathbb{C}^{n+1}$$

with the same norm as Δ_ν. A straightforward computation as in the proof of Theorem 1.1 shows, that $[\Delta_\nu]$ acts on an arbitrary element

$$C = \sum_{i_1,\dots,i_N=0}^{n} c_{i_1,\dots,i_N}(e_{i_1} \otimes \dots \otimes e_{i_N}) \in \bigotimes_{k=1}^{N} \mathbb{C}^{n+1}$$

by

$$\begin{aligned} [\Delta_\nu]\, C \;\; = \;\; & \sum_{i_1,\dots,i_N=0}^{n} c_{i_1,\dots,i_N} \\ & \cdot \left(\sum_{k=1}^{N} \delta_{j_1,i_1} \cdot \ldots \cdot \delta_{j_{k-1},i_{k-1}} \cdot d_{j_k,i_k}^{(\nu)} \cdot \delta_{j_{k+1},i_{k+1}} \cdot \ldots \cdot \delta_{j_N,i_N} \right)_{j_1,\dots,j_N=0}^{n}. \end{aligned}$$

We look for a complex array $(b_{i_1,\dots,i_N})_{i_1,\dots,i_N}$ such that $[\Delta_\nu]$ is equal to the multilevel Toeplitz operator $T_{n+1,\dots,n+1}(b)$. The same calculation as in the proof of Theorem 1.1 gives us for all $m_1, \dots, m_N \in \mathbb{Z}$ that

$$b_{m_1,\dots,m_N} = \begin{cases} d_{m_1+i_1,i_1}^{(\nu)} & : \; m_2 = \dots = m_N = 0, m_1 \neq 0 \\ d_{m_2+i_2,i_2}^{(\nu)} & : \; m_1 = m_3 = \dots = m_N = 0, m_2 \neq 0 \\ \vdots & \\ d_{m_N+i_N,i_N}^{(\nu)} & : \; m_1 = \dots = m_{N-1} = 0, m_N \neq 0 \\ 0 & \text{otherwise} \end{cases},$$

but now we have

$$d^{(\nu)}_{m+i,i} = (-1)^\nu \cdot \begin{cases} \binom{-m-1}{\nu-1} & : \ m \leq -1 \\ 0 & \text{otherwise} \end{cases} \tag{5.4}$$

and so we here get

$$b_{m_1,\ldots,m_N} = (-1)^\nu \cdot \begin{cases} \binom{-m_1-1}{\nu-1} & : \ m_2 = \ldots = m_N = 0, m_1 \leq -1 \\ \binom{-m_2-1}{\nu-1} & : \ m_1 = m_3 = \ldots = m_N = 0, m_2 \leq -1 \\ \vdots & \\ \binom{-m_N-1}{\nu-1} & : \ m_1 = \ldots = m_{N-1} = 0, m_N \leq -1 \\ 0 & \text{otherwise} \end{cases} . \tag{5.5}$$

Now we can compute $\sigma_{n,\ldots,n}(b)$ and $S_{n,\ldots,n}(b)$ for this case. For $\phi_1, \ldots, \phi_N \in [0, 2\pi)$, computations analogous to those for $\nu = 2$ show that

$$\begin{aligned}
&\sigma_{n,\ldots,n}(b)\left(e^{\mathbf{i}\phi_1}, \ldots, e^{\mathbf{i}\phi_N}\right) \\
&= \sum_{|j_1|,\ldots,|j_N| \leq n-1} b_{j_1,\ldots,j_N} \prod_{k=1}^{N} \left(1 - \frac{|j_k|}{n}\right) e^{\mathbf{i}j_k\phi_k} \\
&= \sum_{|j_1| \leq n-1} b_{j_1,0,\ldots,0} \left(1 - \frac{|j_1|}{n}\right) e^{\mathbf{i}j_1\phi_1} + \sum_{|j_2| \leq n-1} b_{0,j_2,0,\ldots,0} \left(1 - \frac{|j_2|}{n}\right) e^{\mathbf{i}j_2\phi_2} \\
&+ \ldots + \sum_{|j_N| \leq n-1} b_{0,\ldots,0,j_N} \left(1 - \frac{|j_N|}{n}\right) e^{\mathbf{i}j_N\phi_N} \\
&= (-1)^\nu \sum_{j=1}^{n-1} \binom{j-1}{\nu-1} \left(1 - \frac{j}{n}\right) e^{-\mathbf{i}j\phi_1} + \ldots + (-1)^\nu \sum_{j=1}^{n-1} \binom{j-1}{\nu-1} \left(1 - \frac{j}{n}\right) e^{-\mathbf{i}j\phi_N} \\
&= (-1)^\nu \sum_{j=\nu}^{n-1} \binom{j-1}{\nu-1} \left(1 - \frac{j}{n}\right) \sum_{k=1}^{N} e^{-\mathbf{i}j\phi_k}
\end{aligned}$$

and

$$\begin{aligned}
&S_{n,\ldots,n}(b)\left(e^{\mathbf{i}\phi_1}, \ldots, e^{\mathbf{i}\phi_N}\right) \\
&= \sum_{|j_1|,\ldots,|j_N| \leq n-1} b_{j_1,\ldots,j_N} \prod_{k=1}^{N} e^{\mathbf{i}j_k\phi_k} \\
&= \sum_{|j_1| \leq n-1} b_{j_1,0,\ldots,0} e^{\mathbf{i}j_1\phi_1} + \ldots + \sum_{|j_N| \leq n-1} b_{0,\ldots,0,j_N} e^{\mathbf{i}j_N\phi_N}
\end{aligned}$$

$$= (-1)^\nu \sum_{j=1}^{n-1} \binom{j-1}{\nu-1} e^{-\mathbf{i}j\phi_1} + \ldots + (-1)^\nu \sum_{j=1}^{n-1} \binom{j-1}{\nu-1} e^{-\mathbf{i}j\phi_N}$$
$$= (-1)^\nu \sum_{j=\nu}^{n-1} \binom{j-1}{\nu-1} \sum_{k=1}^{N} e^{-\mathbf{i}j\phi_k}.$$

With equality (5.1) we get

$$\sum_{j=\nu}^{n} \binom{j-1}{\nu-1} = \binom{n}{\nu} \quad \text{and} \quad \sum_{j=\nu}^{n} \binom{j}{\nu} = \binom{n+1}{\nu+1} = \frac{n+1}{\nu+1}\binom{n}{\nu}. \tag{5.6}$$

Of course $\sigma_{n,\ldots,n}(b)\left(e^{\mathbf{i}\phi_1}, \ldots, e^{\mathbf{i}\phi_N}\right)$ is a lower bound for $\|\sigma_{n,\ldots,n}(b)\|_\infty$ for all values $\phi_1, \ldots, \phi_N \in [0, \pi)$. We choose now $\phi_1 = \ldots = \phi_N = 0$ and obtain, with (5.6),

$$\begin{aligned}
&|\sigma_{n,\ldots,n}(b)(1,\ldots,1)| \\
&= \sum_{j=\nu}^{n-1} \binom{j-1}{\nu-1} \left(1 - \frac{j}{n}\right) \sum_{k=1}^{N} 1 = N \sum_{j=\nu}^{n-1} \binom{j-1}{\nu-1} \left(1 - \frac{j}{n}\right) \\
&= N \sum_{j=\nu}^{n-1} \binom{j-1}{\nu-1} - N \sum_{j=\nu}^{n-1} \frac{j}{n} \binom{j-1}{\nu-1} \\
&= N \sum_{j=\nu}^{n-1} \binom{j-1}{\nu-1} - \frac{N\nu}{n} \sum_{j=\nu}^{n-1} \binom{j}{\nu} \\
&= N \binom{n-1}{\nu} - \frac{N\nu}{n} \cdot \frac{n}{\nu+1} \binom{n-1}{\nu} \\
&= \frac{N}{\nu+1} \binom{n-1}{\nu}.
\end{aligned}$$

A similar calculation shows that, for all $\phi_1, \ldots, \phi_N$ due to (5.6),

$$\left|S_{n,\ldots,n}\left(e^{\mathbf{i}\phi_1}, \ldots, e^{\mathbf{i}\phi_N}\right)\right| \leq \sum_{j=\nu}^{n-1} \binom{j-1}{\nu-1} \left|\sum_{k=1}^{N} e^{-\mathbf{i}j\phi_k}\right| = N \binom{n-1}{\nu}$$

and so we conclude

$$\|\sigma_{n,\ldots,n}(b)\|_\infty \geq \frac{N}{\nu+1} \binom{n-1}{\nu} \quad \text{and} \quad \|S_{n,\ldots,n}(b)\|_\infty \leq N \binom{n-1}{\nu}.$$

This together with Theorem 4.5 implies the safe bounds

$$\begin{aligned}
\frac{N}{\nu+1}\binom{n}{\nu} &= \|\sigma_{n+1,\ldots,n+1}(b)\|_\infty \\
&\leq \|T_{n+1,\ldots,n+1}(b)\| = \lambda\left(\Delta_\nu \left| \mathcal{P}_n^N\right.\right) \leq \|S_{n+1,\ldots,n+1}(b)\|_\infty = N\binom{n}{\nu}.
\end{aligned}$$

To turn to the asymptotic bounds, we consider

$$\frac{1}{Nn^\nu}\frac{N}{\nu+1}\binom{n}{\nu} = \frac{n(n-1)\cdot\ldots\cdot(n-\nu+1)}{n^\nu(\nu+1)!}.$$

This quotient goes to $1/(\nu+1)!$, if n goes to infinity, which implies the asymptotic lower bound from Theorem 5.1. From the safe upper bound we get the asymptotic upper bound

$$\limsup_{n\to\infty}\frac{\lambda\left(\Delta_\nu\left|\mathcal{P}_n^N\right.\right)}{Nn^\nu} \le \frac{1}{\nu!},$$

but the Minkowski inequality provides a sharper asymptotic upper bound. From (5.2) we get $\lambda\left(\Delta_\nu\left|\mathcal{P}_n^N\right.\right) \le N\left\|D^\nu\right\|$ and from [3] we know that $\left\|D^\nu\right\|/n^\nu$ converges to the norm of the Volterra integral operator L_ν defined by

$$L_\nu : L^2(0,1) \to L^2(0,1),\ (L_\nu f)(x) = \frac{1}{(\nu-1)!}\int_0^x (x-y)^{\nu-1} f(y)\ dy,$$

$x \in (0,1)$, and that the norm of the operator satisfies

$$\|L_\nu\| \le \frac{1}{(\nu-1)!}\frac{1}{\sqrt{2\nu(2\nu-1)}},$$

which shows the asserted asymptotic upper bound from Theorem 5.1. The right hand side in the last inequality is smaller than or equal to $1/\nu!$ if and only if

$$1 \le \sqrt{2}\sqrt{(2\nu-1)/\nu}.$$

But this is true for all $\nu \ge 1$, because the function $f : [1,\infty) \to \mathbb{R},\ x \mapsto (2x-1)/x$ is increasing and $f(1) = 1$, so the Minkowski inequality provides indeed the sharper upper bound. □

We have have seen in Theorem 1.2, that we can improve the asymptotic bounds from Theorem 1.1 with the help of Lemma 4.2 and a result from [1] about the eigenvalues of hermitian Toeplitz matrices with polynomially increasing entries. We can apply this techniques for Δ_ν and get a sharper asymptotic lower bound.

Theorem 5.3. *For all natural numbers N and ν, $\nu \ge 2$, the norm of the operator $\Delta_\nu : \mathcal{P}_n^N \to \mathcal{P}_n^N$, where $\mathcal{P}_n^N$ is equipped with the Laguerre norm, satisfies*

$$\liminf_{n\to\infty}\frac{\lambda\left(\Delta_\nu\left|\mathcal{P}_n^N\right.\right)}{Nn^\nu} \ge \frac{\sqrt{2}}{2}\cdot\frac{1}{\nu!}\sqrt{\frac{(2\nu)!+(\nu!)^2}{(2\nu+1)!}}.$$

Proof. We proceed as in the proof of Theorem 1.2.

The norm of the operator Δ_ν is nothing but the spectral norm, that is, the largest singular value, of the matrix $[\Delta_\nu]$ from (5.2). We know from Corollary 4.2 that the

largest singular value of a matrix A is bounded below by the largest eigenvalue of its hermitian part $H(A)$. As in the case $\nu = 2$, the hermitian part of $[\Delta_\nu]$ is just the Kronecker sum of N copies of $H([D^\nu])$, that is,

$$H([\Delta_\nu]) = (H([D^\nu]))^{\oplus N}.$$

Therefore we get with Corollary 4.2 and Corollary 4.4 that

$$\begin{aligned}\lambda\left(\Delta_\nu \middle| \mathcal{P}_n^N\right) &= \sigma_{\max}([\Delta_\nu]) \geq \lambda_{\max}\left((H([D^\nu]))^{\oplus N}\right) \\ &= \frac{N}{2}\lambda_{\max}\left([D^\nu] + [D^\nu]^T\right).\end{aligned}$$

By 5.3, the matrix $[D^\nu] + [D^\nu]^T$ is obviously an hermitian Toeplitz matrix. We are interested now in the entries of the first row of $[D^\nu] + [D^\nu]^T$. Clearly,

$$\begin{aligned}\binom{k}{\nu-1}\frac{1}{k^{\nu-1}} &= \frac{1}{(\nu-1)!}\cdot\frac{k!}{(k-\nu+1)!\cdot k^{\nu-1}} \\ &= \frac{1}{(\nu-1)!}\cdot\frac{k(k-1)\cdot\ldots\cdot(k-\nu+2)}{k^{\nu-1}} \\ &= \frac{1}{(\nu-1)!}\frac{k^{\nu-1}+o\left(k^{\nu-1}\right)}{k^{\nu-1}} \to \frac{1}{(\nu-1)!},\end{aligned}$$

that is,

$$\binom{k}{\nu-1} = \frac{k^{\nu-1}}{(\nu-1)!} + \binom{k}{\nu-1} - \frac{k^{\nu-1}}{(\nu-1)!} = \frac{k^{\nu-1}}{(\nu-1)!} + o\left(k^{\nu-1}\right).$$

For the entries $a_0, \ldots, a_n$ of the first row of $(-1)^\nu [D^\nu]$, this implies

$$\begin{aligned}a_k &= \begin{cases}\binom{k}{\nu-1} & : \ k \geq \nu-1 \\ 0 & \text{otherwise}\end{cases} = \chi_{[\nu-1,\infty)}(k)\binom{k}{\nu-1} \\ &= \left(1-\chi_{(-\infty,\nu-1)}(k)\right)\binom{k}{\nu-1} = \binom{k}{\nu-1} - \chi_{(-\infty,\nu-1)}(k)\binom{k}{\nu-1} \\ &= \frac{k^{\nu-1}}{(\nu-1)!} + o\left(k^{\nu-1}\right) - \chi_{(-\infty,\nu-1)}(k)\left(\frac{k^{\nu-1}}{(\nu-1)!} + o\left(k^{\nu-1}\right)\right)\end{aligned}$$

with

$$\begin{aligned}&\chi_{(-\infty,\nu-1)}(k)\left(\frac{k^{\nu-1}}{(\nu-1)!} + o\left(k^{\nu-1}\right)\right)\frac{1}{k^{\nu-1}} \\ &= \chi_{(-\infty,\nu-1)}(k)\left(\frac{1}{(\nu-1)!} + \frac{o\left(k^{\nu-1}\right)}{k^{\nu-1}}\right) \to 0\end{aligned}$$

for $k \to \infty$, that is, we conclude

$$a_k = \frac{k^{\nu-1}}{(\nu-1)!} + o\left(k^{\nu-1}\right).$$

We set now $\mathbb{D}^\nu := (-1)^\nu(\nu-1)!\,[D^\nu]$. The matrix $\mathbb{D}^\nu + (\mathbb{D}^\nu)^T$ is of course an hermitian Toeplitz matrix and the entries $\widetilde{a_k}$ of its first row are

$$\widetilde{a_k} = (\nu-1)! \cdot a_k = k^{\nu-1} + o\left(k^{\nu-1}\right)$$

for $k \to \infty$, that is, $\mathbb{D}^\nu + (\mathbb{D}^\nu)^T$ satisfies the assumptions of Theorem 4.7. From this theorem and the Corollaries 4.2 and and 4.4 we deduce

$$\begin{aligned}
\lambda\left(\Delta_\nu \left| \mathcal{P}_n^N \right.\right) &= \lambda\left((-1)^\nu \Delta_\nu \left| \mathcal{P}_n^N \right.\right) && (5.7)\\
&\geq (-1)^\nu \frac{N}{2} \lambda_{\max}\left([D^\nu] + [D^\nu]^T\right) = \frac{N}{2(\nu-1)!} \lambda_{\max}\left(\mathbb{D}^\nu + (\mathbb{D}^\nu)^T\right)\\
&= \frac{N}{2(\nu-1)!} \mu_1\left(K_{\nu-1}\right)(n+1)^\nu + o\left((n+1)^\nu\right). && (5.8)
\end{aligned}$$

Theorem 1.5 in [1] provides a lower bound for the eigenvalue $\mu_1(K_\alpha)$ of the integral operator K_α, $\alpha > 0$, namely

$$\mu_1(K_\alpha) \geq C_\alpha := \frac{\sqrt{2}}{\alpha+1}\sqrt{\frac{1}{2\alpha+3} + \frac{\Gamma(\alpha+2)^2}{\Gamma(2\alpha+4)}}.$$

In our case we have

$$C_{\nu-1} = \frac{\sqrt{2}}{\nu-1}\sqrt{\frac{1}{2\nu+1} + \frac{\Gamma(\nu+1)^2}{\Gamma(2\nu+2)}} = \frac{\sqrt{2}}{\nu}\sqrt{\frac{(2\nu)! + (\nu!)^2}{(2\nu+1)!}}.$$

With (5.8) we get

$$\lambda\left(\Delta_\nu \left| \mathcal{P}_n^N \right.\right) \geq \frac{N}{2(\nu-1)!}\frac{\sqrt{2}}{\nu}\sqrt{\frac{(2\nu)! + (\nu!)^2}{(2\nu+1)!}}(n+1)^\nu + o\left((n+1)^\nu\right),$$

which finally implies the asymptotic lower bound

$$\begin{aligned}
\liminf_{n\to\infty} \frac{\lambda\left(\Delta_\nu \left| \mathcal{P}_n^N \right.\right)}{Nn^\nu} &\geq \liminf_{n\to\infty} \frac{1}{2(\nu-1)!}\frac{(n+1)^\nu}{n^\nu}\frac{\sqrt{2}}{\nu}\sqrt{\frac{(2\nu)! + (\nu!)^2}{(2\nu+1)!}}\\
&= \frac{\sqrt{2}}{2} \cdot \frac{1}{\nu!}\sqrt{\frac{(2\nu)! + (\nu!)^2}{(2\nu+1)!}}
\end{aligned}$$

and completes the proof. □

We remark that for $\nu = 2$ we know the bound

$$\liminf_{n\to\infty} \frac{\lambda\left(\Delta_\nu \left| \mathcal{P}_n^N \right.\right)}{Nn^\nu} \geq 0.174$$

from Theorem 1.2, which is better than the estimate

$$\liminf_{n\to\infty} \frac{\lambda\left(\Delta_\nu \left| \mathcal{P}_n^N\right.\right)}{Nn^\nu} \geq 0.165$$

provided by Theorem 5.3. As the following proposition shows, the asymptotic lower bound ensured by Theorem 5.3 is indeed sharper than the bound from Theorem 5.1 if ν is large enough.

Proposition 5.4. *For an integer $\nu \geq 3$, put*

$$\Lambda_1(\nu) := \frac{\sqrt{2}}{2} \cdot \frac{1}{\nu!}\sqrt{\frac{(2\nu)! + (\nu!)^2}{(2\nu+1)!}} \quad \textit{and} \quad \Lambda_2(\nu) := \frac{1}{(\nu+1)!}.$$

Then $\Lambda_1(\nu) \geq \Lambda_2(\nu)$.

Proof. We have

$$\begin{aligned}
&\Lambda_1(\nu) \geq \Lambda_2(\nu) \\
&\Leftrightarrow \frac{\sqrt{2}}{2} \cdot \frac{1}{\nu!}\sqrt{\frac{(2\nu)! + (\nu!)^2}{(2\nu+1)!}} \geq \frac{1}{(\nu+1)!} \Leftrightarrow \frac{1}{2}\frac{(2\nu)! + (\nu!)^2}{(2\nu+1)!} \geq \left(\frac{1}{\nu+1}\right)^2 \\
&\Leftrightarrow \frac{(\nu!)^2}{(2\nu+1)!} \geq \frac{2}{(\nu+1)^2} - \frac{1}{2\nu+1} \Leftrightarrow \frac{(\nu!)^2}{(2\nu+1)!} \geq \frac{1+2\nu-\nu^2}{(2\nu+1)(\nu+1)^2} \\
&\Leftrightarrow \frac{(\nu!)^2}{(2\nu)!} \geq \frac{1+2\nu-\nu^2}{(\nu+1)^2}.
\end{aligned}$$

The last inequality holds for all $\nu \geq 3$, because the zeros of the quadratic function $\nu \mapsto -\nu^2+2\nu+1$ are $1 \pm \sqrt{2}$, so the right-hand side becomes negative for $\nu \geq 3$. □

Now we consider for arbitrary $N, \nu \in \mathbb{N}$ the operator Δ_ν on $\mathcal{P}_n(\Omega_N)$. The following corollary is a simple consequence of Theorem 5.1 and Theorem 5.3.

Corollary 5.5. *The norm operator*

$$\Delta_\nu : \mathcal{P}_n(\Omega_N) \to \mathcal{P}_n(\Omega_N),$$

where we equip $\mathcal{P}_n(\Omega_N)$ with the Laguerre norm, satisfies for arbitrary $N, \nu \in \mathbb{N}$

$$\begin{aligned}
\frac{\sqrt{2}}{2}\frac{1}{N^\nu \cdot \nu!}\sqrt{\frac{(2\nu)! + (\nu!)^2}{(2\nu+1)!}} &\leq \liminf_{n\to\infty} \frac{\lambda\left(\Delta_\nu \left| \mathcal{P}_n(\Omega_N)\right.\right)}{Nn^\nu} \leq \limsup_{n\to\infty} \frac{\lambda\left(\Delta_\nu \left| \mathcal{P}_n(\Omega_N)\right.\right)}{Nn^\nu} \\
&\leq \frac{1}{(\nu-1)!}\frac{1}{\sqrt{2\nu(2\nu-1)}}.
\end{aligned}$$

Proof. The proof uses the same argument as the proof of Theorem 1.5. We replace Δ in (4.36) by Δ_ν. Then we apply Theorem 5.1 with $\lfloor n/N \rfloor$ instead of n and get the stated asymptotic lower bound, where we use

$$\lim_{n\to\infty} \frac{\lfloor n/N \rfloor^\nu}{n^\nu} = \frac{1}{N^\nu}.$$

The asymptotic upper bound is just a consequence of Theorem 5.3. □

5.1.2 The Hermite case

For the Hermite norm, we get the following asymptotic value for the operator norm of Δ_ν.

Theorem 5.6. *For all $N, \nu \in \mathbb{N}$, the norm of the operator Δ_ν on $\mathcal{P}_n^N$, equipped with the Hermite norm, satisfies*

$$\lim_{n\to\infty} \frac{\eta\left(\Delta_\nu \,\middle|\, \mathcal{P}_n^N\right)}{N n^{\nu/2}} = 2^{\nu/2}.$$

Proof. As in the case $\nu = 2$, we consider the matrix representation $[\Delta_\nu]$ of the differential operator with respect to the orthonormal basis on $\mathcal{P}_n^N \simeq \bigotimes_{k=1}^N \mathcal{P}_n$ whose elements are tensor products of normalized Hermite polynomials. Again this representation has the structure of a Kronecker sum, namely,

$$[\Delta_\nu] = \sum_{k=1}^{N} I_{(n+1)^{k-1}} \otimes [D^\nu] \otimes I_{(n+1)^{N-k}} = [D^\nu]^{\oplus N},$$

where $[D^\nu]$ denotes the matrix representation of the ordinary differential operator $D^\nu : \mathcal{P}_n \to \mathcal{P}_n$ with respect to the Hermite basis. For the normalized Hermite polynomials h_i we have the well-known formula

$$D^\nu h_i = \begin{cases} 2^{\nu/2}\sqrt{\frac{\Gamma(i+1)}{\Gamma(i-\nu+1)}}\, h_{i-\nu} & : \nu \le i \\ 0 & \text{otherwise} \end{cases} \tag{5.9}$$

and so we get

$$[D^\nu] = 2^{\nu/2} \begin{pmatrix} 0 & \dots & 0 & \sqrt{\frac{\Gamma(\nu+1)}{\Gamma(1)}} & & & 0 \\ & & & & \sqrt{\frac{\Gamma(\nu+2)}{\Gamma(2)}} & & \\ \vdots & & \vdots & & & \ddots & \\ & & & 0 & & & \sqrt{\frac{\Gamma(n+1)}{\Gamma(n-\nu+1)}} \\ 0 & \dots & 0 & & \dots & & 0 \\ \vdots & & \vdots & & & & \vdots \\ 0 & \dots & 0 & & \dots & & 0 \end{pmatrix},$$

where the first ν columns and the last ν rows are zero. Due to Corollary 4.2 we can estimate the largest singular value of $[\Delta_\nu]$ again by the largest eigenvalue of the hermitian part $H([\Delta_\nu])$ and obtain due to Corollary 4.4 that

$$\begin{aligned}&\eta\left(\Delta_\nu \,\middle|\mathcal{P}_n^N\right) = \sigma_{\max}\left([\Delta_\nu]\right)\\ &\geq \lambda_{\max}\left(H\left([\Delta_\nu]\right)\right) = \lambda_{\max}\left(H\left(\left[D^\nu\right]^{\oplus N}\right)\right) = \frac{N}{2}\lambda_{\max}\left(\left[D^\nu\right] + \left[D^\nu\right]^T\right).\end{aligned}$$

Now we denote by H_n the matrix

$$H_n = 2^{-\nu/2}\left(\left[D^\nu\right] + \left[D^\nu\right]^T\right)$$

$$= \begin{pmatrix} 0 & \dots & 0 & \sqrt{\frac{\Gamma(\nu+1)}{\Gamma(1)}} & & & 0 \\ \vdots & & \vdots & & \sqrt{\frac{\Gamma(\nu+2)}{\Gamma(2)}} & & \\ 0 & \dots & 0 & \ddots & & \ddots & \\ \sqrt{\frac{\Gamma(\nu+1)}{\Gamma(1)}} & & \ddots & \ddots & \ddots & & \sqrt{\frac{\Gamma(n+1)}{\Gamma(n-\nu+1)}} \\ & \sqrt{\frac{\Gamma(\nu+2)}{\Gamma(2)}} & & \ddots & 0 & \dots & 0 \\ & & \ddots & & \dots & & \dots \\ 0 & & & \sqrt{\frac{\Gamma(n+1)}{\Gamma(n-\nu+1)}} & 0 & \dots & 0 \end{pmatrix}$$

and set $A_n := F_n\sqrt{\frac{\Gamma(n-\nu+1)}{\Gamma(n+1)}}H_nF_n$, where F_n refers to the flip matrix.

The function $f : \mathbb{R} \to \mathbb{R}, x \mapsto (x+\nu-1)(x+\nu-2)\dots x$ is a polynomial which becomes zero if and only if $x \in \{-(\nu-1), -(\nu-2), \dots, 0\}$. As the leading coefficient of f is positive, the function is increasing on $[0,\infty)$. Consequently,

$$\begin{aligned}\max_{k=1,\dots,n-\nu+1}\sqrt{\frac{\Gamma(\nu+k)}{\Gamma(k)}} &= \max_{k=1,\dots,n-\nu+1}\sqrt{\frac{(\nu+k-1)!}{(k-1)!}} = \max_{k=1,\dots,n-\nu+1}\sqrt{f(k)}\\ &= \sqrt{f(n-\nu+1)} = \sqrt{\frac{\Gamma(n+1)}{\Gamma(n-\nu+1)}}.\end{aligned}$$

This implies

$$\begin{aligned}\|A_n\| &= \sqrt{\frac{\Gamma(n-\nu+1)}{\Gamma(n+1)}}\|H_n\|\\ &\leq 2\sqrt{\frac{\Gamma(n-\nu+1)}{\Gamma(n+1)}}\max_{k=1,\dots,n-\nu+1}\sqrt{\frac{\Gamma(\nu+k)}{\Gamma(k)}} = 2,\end{aligned}$$

that is, the norms of the matrices A_n are uniformly bounded by 2. Now we define a function $a : \mathbb{T} \to \mathbb{C}$ by $a\left(e^{\mathbf{i}\theta}\right) := e^{\nu\mathbf{i}\theta} + e^{-\nu\mathbf{i}\theta} = 2\cos(\nu\theta)$. A direct computation shows that the Fourier coefficients $\widehat{a}_n$ of a have the form

$$\widehat{a}_n = \begin{cases} 1 & : \ n \in \{-\nu, \nu\} \\ 0 & \text{otherwise} \end{cases}.$$

As in the case $\nu = 2$, $A_n x$ converges to $T(a)x$ for all finite sequences $x \in \ell^2$, where

$$T(a) = \begin{pmatrix} 0 & \dots & 0 & 1 & & \\ \vdots & & \vdots & & 1 & \\ 0 & \dots & 0 & & & \ddots \\ 1 & & & \ddots & & \\ & 1 & & & & \\ & & \ddots & & & \end{pmatrix}$$

is the infinite Toeplitz matrix generated by a. If we denote by P_n the projector on the first n entries of a sequence, then $A_n P_n$ converges pointwise to $T(a)$. As in the proof of Theorem 1.3 we conclude that

$$[-2, 2] = a(\mathbb{T}) = \sigma\left(T(a)\right) \subset \liminf_{n\to\infty} \sigma(A_n)$$

and hence

$$\liminf_{n\to\infty} \sqrt{\frac{\Gamma(n-\nu+1)}{\Gamma(n+1)}}\lambda_{\max}(H_n) \geq 2.$$

Furthermore we have

$$\limsup_{n\to\infty} \sqrt{\frac{\Gamma(n-\nu+1)}{\Gamma(n+1)}}\lambda_{\max}(H_n) = \limsup_{n\to\infty} \lambda_{\max}(A_n) = \limsup_{n\to\infty} \|A_n\| \leq 2$$

and get

$$\lim_{n\to\infty} \sqrt{\frac{\Gamma(n-\nu+1)}{\Gamma(n+1)}}\lambda_{\max}(H_n) = 2. \tag{5.10}$$

The estimate

$$\eta\left(\Delta_\nu \left| \mathcal{P}_n^N \right.\right) \geq \frac{N}{2} \cdot 2^{\nu/2}\lambda_{\max}(H_n).$$

leads to

$$\begin{aligned}
\frac{\eta\left(\Delta_\nu\left|\mathcal{P}_n^N\right.\right)}{Nn^{\nu/2}} &\geq \frac{2^{\nu/2}}{2n^{\nu/2}}\lambda_{\max}(H_n) = \frac{2^{\nu/2}}{2n^{\nu/2}}\sqrt{\frac{\Gamma(n+1)}{\Gamma(n-\nu+1)}}\sqrt{\frac{\Gamma(n-\nu+1)}{\Gamma(n+1)}}\lambda_{\max}(H_n) \\
&= \frac{2^{\nu/2}}{2n^{\nu/2}}\sqrt{\frac{n!}{(n-\nu)!}}\sqrt{\frac{\Gamma(n-\nu+1)}{\Gamma(n+1)}}\lambda_{\max}(H_n) \\
&= \frac{2^{\nu/2}}{2n^{\nu/2}}\sqrt{n(n-1)\ldots(n-\nu+1)}\sqrt{\frac{\Gamma(n-\nu+1)}{\Gamma(n+1)}}\lambda_{\max}(H_n) \\
&= \frac{2^{\nu/2}}{2}\sqrt{\frac{n^\nu - o\left(n^\nu\right)}{n^\nu}}\sqrt{\frac{\Gamma(n-\nu+1)}{\Gamma(n+1)}}\lambda_{\max}(H_n).
\end{aligned}$$

Due to (5.10) the last expression converges to $2^{\nu/2}$, that is, we have

$$\liminf_{n\to\infty}\frac{\eta\left(\Delta_\nu\left|\mathcal{P}_n^N\right.\right)}{Nn^{\nu/2}} \geq 2^{\nu/2}. \tag{5.11}$$

From the Minkowski inequality it follows that $\eta\left(\Delta_\nu\left|\mathcal{P}_n^N\right.\right) \leq N\left\|D^\nu\right\|$, and from [4] we know that $\left\|D^\nu\right\| \sim 2^{\nu/2}n^{\nu/2}$. This implies

$$\limsup_{n\to\infty}\frac{\eta\left(\Delta_\nu\left|\mathcal{P}_n^N\right.\right)}{Nn^{\nu/2}} \leq \lim_{n\to\infty}\frac{\left\|D^\nu\right\|}{n^{\nu/2}} = 2^{\nu/2},$$

which together with (5.11) completes the proof. □

Finally we want to consider the operator Δ_ν on $\mathcal{P}_n(\Omega_N)$ equipped with the Hermite norm. We have the following asymptotic behavior.

Theorem 5.7. *For $\nu, N \in \mathbb{N}$ it holds*

$$\frac{2^{\nu/2}}{N} \leq \liminf_{n\to\infty}\frac{\eta\left(\Delta_\nu\left|\mathcal{P}_n(\Omega_N)\right.\right)}{Nn^{\nu/2}} \leq \limsup_{n\to\infty}\frac{\eta\left(\Delta_\nu\left|\mathcal{P}_n(\Omega_N)\right.\right)}{Nn^{\nu/2}} \leq 2^{\nu/2}.$$

Proof. We remember that an orthonormal basis in $\mathcal{P}_n(\Omega_N)$ is given by

$$\left\{h_{i_1}\otimes\ldots\otimes h_{i_N} \mid i_1+\ldots+i_N \leq n\right\}.$$

This basis contains especially the function $h_n\otimes h_0\otimes\ldots\otimes h_0$. Due to (5.9) we have

$$\begin{aligned}
&\left(\Delta_\nu\left(h_n\otimes h_0\otimes\ldots\otimes h_0\right)\right)(t_1,\ldots,t_N) \\
&= \left(h_n^{(\nu)}\otimes h_0\otimes\ldots\otimes h_0\right)\right)(t_1,\ldots,t_N) + \left(h_n\otimes h_0^{(\nu)}\otimes h_0\otimes\ldots\otimes h_0\right)\right)(t_1,\ldots,t_N) \\
&\quad +\ldots+\left(h_n\otimes h_0\otimes\ldots\otimes h_0\otimes h_0^{(\nu)})\right)(t_1,\ldots,t_N) \\
&= h_n^{(\nu)}(t_1) = 2^{\nu/2}\sqrt{\frac{\Gamma(n+1)}{\Gamma(n-\nu+1)}}h_{n-\nu}(t_1).
\end{aligned}$$

Due to estimate (4.30) we get

$$\begin{aligned}\eta\left(\Delta_\nu\left|\mathcal{P}_n\left(\Omega_N\right)\right.\right) &\geq \left\|2^{\nu/2}\sqrt{\frac{\Gamma(n+1)}{\Gamma(n-\nu+1)}}h_{n-\nu}\otimes h_0\otimes\ldots\otimes h_0\right\| \\ &= 2^{\nu/2}\sqrt{\frac{\Gamma(n+1)}{\Gamma(n-\nu+1)}}.\end{aligned}$$

This implies

$$\begin{aligned}\frac{\eta\left(\Delta_\nu\left|\mathcal{P}_n\left(\Omega_N\right)\right.\right)}{Nn^{\nu/2}} &\geq \frac{2^{\nu/2}}{Nn^{\nu/2}}\sqrt{\frac{\Gamma(n+1)}{\Gamma(n-\nu+1)}} = \frac{2^{\nu/2}}{Nn^{\nu/2}}\sqrt{\frac{n!}{(n-\nu)!}} \\ &= \frac{2^{\nu/2}}{Nn^{\nu/2}}\sqrt{n(n-1)\ldots(n-\nu+1)} = \frac{2^{\nu/2}}{N}\sqrt{\frac{n^\nu+o\left(n^\nu\right)}{n^\nu}} \\ &\to \frac{2^{\nu/2}}{N}\end{aligned}$$

for $n\to\infty$, so we get the asserted lower bound.

The space $\mathcal{P}_n\left(\Omega_N\right)$ is a linear subspace of $\mathcal{P}_n^N$ and so we have

$$\frac{\eta\left(\Delta_\nu\left|\mathcal{P}_n\left(\Omega_N\right)\right.\right)}{Nn^{\nu/2}} \leq \frac{\eta\left(\Delta_\nu\left|\mathcal{P}_n^N\right.\right)}{Nn^{\nu/2}},$$

where the last term converges to $2^{\nu/2}$ by Theorem 5.6. This completes the proof. □

Remark 5.8. We can ask what happens if we apply the method from the proof of Theorem 1.5 to the operator $\Delta_\nu:\mathcal{P}_n\left(\Omega_N\right)\to\mathcal{P}_n\left(\Omega_N\right)$, if we equip $\mathcal{P}_n\left(\Omega_N\right)$ with the Hermite norm. We replace in (4.36) λ by η and Δ by Δ_ν. Now we apply Theorem 5.6 with $\lfloor n/N\rfloor$ instead of n and get

$$\frac{2^{\nu/2}}{N^{\nu/2}} \leq \liminf_{n\to\infty}\frac{\eta\left(\Delta_\nu\left|\mathcal{P}_n\left(\Omega_N\right)\right.\right)}{Nn^{\nu/2}},$$

so the lower asymptotic bound is for $N>2$ worse than the bound known from Theorem 5.7.

5.2 The operator Δ^2

For the operator Δ^2 we get the following asymptotic bounds.

Theorem 5.9. *If we consider Δ^2 on $\mathcal{P}_n^N$, then we have for all $n,N\in\mathbb{N}$ the safe bounds*

$$\frac{N}{5}\binom{n}{4}+2\binom{N}{2}\left(\frac{1}{3}\binom{n}{2}\right)^2 \leq \lambda\left(\Delta^2\left|\mathcal{P}_n^N\right.\right) \leq N\binom{n}{4}+2\binom{N}{2}\binom{n-1}{2}^2$$

and the asymptotic bounds

$$\frac{1}{36}-\frac{7}{360N}\leq\liminf_{n\to\infty}\frac{\lambda\left(\Delta^2\left|\mathcal{P}_n^N\right.\right)}{N^2n^4}\leq\limsup_{n\to\infty}\frac{\lambda\left(\Delta^2\left|\mathcal{P}_n^N\right.\right)}{N^2n^4}\leq\frac{1}{\mu_0^4}$$

with $0.0277<1/36$, $7/360<0.0195$ *and* $1/\mu_0^4<0.0810$, *where* $\mu_0=1.8746\ldots$ *is the smallest positive solution of the equation* $1+\cosh\mu\cos\mu=0$.

Proof. The upper asymptotic bound is an obvious consequence of the Minkowski inequality $\lambda\left(\Delta^2\left|\mathcal{P}_n^N\right.\right)\leq\lambda\left(\Delta\left|\mathcal{P}_n^N\right.\right)^2$ and the upper bound from Theorem 1.2.

To prove the safe bounds, we will again apply Theorem 4.5. Due to (3.3), the matrix representation of Δ^2 with respect to the Laguerre basis is

$$\begin{aligned}\left[\Delta^2\right] &= [\Delta]^2=\left(\sum_{k=1}^N I_{(n+1)^{k-1}}\otimes\left[D^2\right]\otimes I_{(n+1)^{N-k}}\right)^2\\ &= \sum_{k=1}^N I_{(n+1)^{k-1}}\otimes\left[D^2\right]^2\otimes I_{(n+1)^{N-k}}\\ &\quad +2\sum_{1\leq j<l\leq N} I\otimes\ldots\otimes I\otimes\left[D^2\right]\otimes I\otimes\ldots\otimes I\otimes\left[D^2\right]\otimes I\otimes\ldots\otimes I,\end{aligned}$$

where $[D^2]^2=[D^4]=\left(d_{ij}^{(4)}\right)_{i,j}$ and where in the second sum $[D^2]$ is the jth and lth factor in the summands. By I we denote of course I_{n+1}. The operator $[\Delta^2]$ induces again an operator

$$\left[\Delta^2\right]:\bigotimes_{k=1}^N\mathbb{C}^{n+1}\to\bigotimes_{k=1}^N\mathbb{C}^{n+1}$$

of the same norm. We are looking for a complex array $(b_{i_1,\ldots,i_N})_{i_1,\ldots,i_N\in\mathbb{Z}}$ such that $[\Delta^2]=T_{n+1,\ldots,n+1}(b)$ holds on $\bigotimes_{k=1}^N\mathbb{C}^{n+1}$. For an arbitrary

$$C=\sum_{i_1,\ldots,i_N=0}^n c_{i_1,\ldots,i_N}\left(e_{i_1}\otimes\ldots\otimes e_{i_N}\right)\in\bigotimes_{k=1}^N\mathbb{C}^{n+1},$$

we have

$$\begin{aligned}\left[\Delta^2\right]C &= \sum_{k=1}^N\sum_{i_1,\ldots,i_N=0}^n c_{i_1,\ldots,i_N}e_{i_1}\otimes\ldots\otimes e_{i_{k-1}}\otimes\left[D^4\right]e_{i_k}\otimes e_{i_{k+1}}\otimes\ldots\otimes e_{i_N}\\ &\quad +2\sum_{1\leq j<l\leq N}\sum_{i_1,\ldots,i_N=0}^n c_{i_1,\ldots,i_N}e_{i_1}\otimes\ldots\otimes e_{i_{j-1}}\otimes\left[D^2\right]e_{i_j}\\ &\quad \otimes e_{i_{j+1}}\otimes\ldots\otimes e_{i_{l-1}}\otimes\left[D^2\right]e_{i_l}\otimes e_{i_{l+1}}\otimes\ldots\otimes e_{i_N}.\end{aligned}$$

This equals

$$\sum_{k=1}^{N}\sum_{i_1,\ldots,i_N=0}^{n} c_{i_1,\ldots,i_N}$$
$$\cdot\left(\begin{pmatrix}\delta_{0,i_1}\\ \vdots\\ \delta_{n,i_1}\end{pmatrix}\otimes\ldots\otimes\begin{pmatrix}\delta_{0,i_{k-1}}\\ \vdots\\ \delta_{n,i_{k-1}}\end{pmatrix}\otimes\begin{pmatrix}d^{(4)}_{0,i_k}\\ \vdots\\ d^{(4)}_{n,i_k}\end{pmatrix}\otimes\begin{pmatrix}\delta_{0,i_{k+1}}\\ \vdots\\ \delta_{n,i_{k+1}}\end{pmatrix}\otimes\ldots\otimes\begin{pmatrix}\delta_{0,i_N}\\ \vdots\\ \delta_{n,i_N}\end{pmatrix}\right)$$
$$+2\sum_{1\le j<l\le N}\sum_{i_1,\ldots,i_N=0}^{n} c_{i_1,\ldots,i_N}\left(\begin{pmatrix}\delta_{0,i_1}\\ \vdots\\ \delta_{n,i_1}\end{pmatrix}\otimes\ldots\otimes\begin{pmatrix}\delta_{0,i_{j-1}}\\ \vdots\\ \delta_{n,i_{j-1}}\end{pmatrix}\otimes\begin{pmatrix}d^{(2)}_{0,i_j}\\ \vdots\\ d^{(2)}_{n,i_j}\end{pmatrix}\right.$$
$$\left.\otimes\begin{pmatrix}\delta_{0,i_{j+1}}\\ \vdots\\ \delta_{n,i_{j+1}}\end{pmatrix}\otimes\ldots\otimes\begin{pmatrix}\delta_{0,i_{l-1}}\\ \vdots\\ \delta_{n,i_{l-1}}\end{pmatrix}\otimes\begin{pmatrix}d^{(2)}_{0,i_l}\\ \vdots\\ d^{(2)}_{n,i_l}\end{pmatrix}\otimes\begin{pmatrix}\delta_{0,i_{l+1}}\\ \vdots\\ \delta_{n,i_{l+1}}\end{pmatrix}\otimes\ldots\otimes\begin{pmatrix}\delta_{0,i_N}\\ \vdots\\ \delta_{n,i_N}\end{pmatrix}\right),$$

which in turn is equal to

$$\sum_{k=1}^{N}\sum_{i_1,\ldots,i_N=0}^{n} c_{i_1,\ldots,i_N}\left(\delta_{j_1,i_1}\cdot\ldots\cdot\delta_{j_{k-1},i_{k-1}}d^{(4)}_{j_k,i_k}\delta_{j_{k+1},i_{k+1}}\cdot\ldots\cdot\delta_{j_N,i_N}\right)^{n}_{j_1,\ldots,j_N=0}$$
$$+2\sum_{1\le k<l\le N}\sum_{i_1,\ldots,i_N=0}^{n} c_{i_1,\ldots,i_N}\left(\delta_{j_1,i_1}\cdot\ldots\cdot\delta_{j_{k-1},i_{k-1}}d^{(2)}_{j_k,i_k}\delta_{j_{k+1},i_{k+1}}\cdot\right.$$
$$\left.\ldots\cdot\delta_{j_{l-1},i_{l-1}}d^{(2)}_{j_l,i_l}\delta_{j_{l+1},i_{l+1}}\cdot\ldots\cdot\delta_{j_N,i_N}\right)^{n}_{j_1,\ldots,j_N=0}$$
$$=\sum_{i_1,\ldots,i_N=0}^{n} c_{i_1,\ldots,i_N}\left(\sum_{k=1}^{N}\delta_{j_1,i_1}\cdot\ldots\cdot\delta_{j_{k-1},i_{k-1}}d^{(4)}_{j_k,i_k}\delta_{j_{k+1},i_{k+1}}\cdot\ldots\cdot\delta_{j_N,i_N}\right.$$
$$+2\sum_{1\le k<l\le N}\delta_{j_1,i_1}\cdot\ldots\cdot\delta_{j_{k-1},i_{k-1}}d^{(2)}_{j_k,i_k}\delta_{j_{k+1},i_{k+1}}$$
$$\left.\cdot\ldots\cdot\delta_{j_{l-1},i_{l-1}}d^{(2)}_{j_l,i_l}\delta_{j_{l+1},i_{l+1}}\cdot\ldots\cdot\delta_{j_N,i_N}\right)^{n}_{j_1,\ldots,j_N=0}.$$

We compare this expression with

$$T_{n+1,\ldots,n+1}(b)C=\sum_{i_1,\ldots,i_N=0}^{n}(b_{j_1-i_1,\ldots,j_N-i_N}c_{i_1,\ldots,i_N})^{n}_{j_1,\ldots,j_N=0}$$

and conclude

$$\begin{aligned} b_{j_1-i_1,\ldots,j_N-i_N} &= \sum_{k=1}^{N} \delta_{j_1,i_1} \cdot \ldots \cdot \delta_{j_{k-1},i_{k-1}} d^{(4)}_{j_k,i_k} \delta_{j_{k+1},i_{k+1}} \cdot \ldots \cdot \delta_{j_N,i_N} \\ &\quad +2 \sum_{1\le k<l\le N} \delta_{j_1,i_1} \cdot \ldots \cdot \delta_{j_{k-1},i_{k-1}} d^{(2)}_{j_k,i_k} \delta_{j_{k+1},i_{k+1}} \cdot \\ &\quad \ldots \cdot \delta_{j_{l-1},i_{l-1}} d^{(2)}_{j_l,i_l} \delta_{j_{l+1},i_{l+1}} \cdot \ldots \cdot \delta_{j_N,i_N}. \end{aligned}$$

So we get for all $m_1, \ldots, m_N \in \mathbb{Z}$ and $i \in \mathbb{Z}$ that

$$\begin{aligned} b_{m_1,\ldots,m_N} &= b_{m_1+i-i,\ldots,m_N+i-i} \\ &= \sum_{k=1}^{N} \delta_{m_1+i,i} \cdot \ldots \cdot \delta_{m_{k-1}+i,i} d^{(4)}_{m_k+i,i} \delta_{m_{k+1}+i,i} \cdot \ldots \cdot \delta_{m_N+i,i} \\ &\quad +2 \sum_{1\le k<l\le N} \delta_{m_1+i,i} \cdot \ldots \cdot \delta_{m_{k-1}+i,i} d^{(2)}_{m_k+i,i} \delta_{m_{k+1}+i,i} \cdot \\ &\quad \ldots \cdot \delta_{m_{l-1}+i,i} d^{(2)}_{m_l+i,i} \delta_{m_{l+1}+i,i} \cdot \ldots \cdot \delta_{m_N+i,i}. \end{aligned} \tag{5.12}$$

First we mention that

$$b_{0,\ldots,0} = \sum_{k=1}^{N} d^{(4)}_{i,i} + 2 \sum_{1\le k<l\le N} d^{(2)}_{i,i} \cdot d^{(2)}_{i,i} \cdot \delta_{m_{l+1}+i,i} = 0.$$

If the number non-zero entries of $(m_1, \ldots, m_N)$ is larger than or equal to three, then every summand in (5.12) contains at least one factor $\delta_{m_k+i,i} = 0$ for $m_k \neq 0$ and all i, that is, in this case $b_{m_1,\ldots,m_N}$ vanishes. We only have to consider the cases where exactly one or exactly two indices of $b_{m_1,\ldots,m_N}$ are not zero.

Let the multi-index be of the form $(m_1, \ldots, m_N) = (0, \ldots, 0, m_k, 0, \ldots, 0)$ with arbitrary $k \in \{1, \ldots, N\}$. Then, due to (5.12), $b_{m_1,\ldots,m_N}$ becomes

$$b_{0,\ldots,0,m_k,0,\ldots,0} = d^{(4)}_{m_k+i,i} + 2 \sum_{l=1,l\neq k}^{N} d^{(2)}_{m_k+i,i} \cdot d^{(2)}_{m_l+i,i} = d^{(4)}_{m_k+i,i}.$$

We realize that $b_{0,\ldots,0,m_k,0,\ldots,0}$ depends on the value m_k but not on the position of this value in the multi-index. So due to (5.4) we get

$$b_{0,\ldots,0,j,0,\ldots,0} = d^{(4)}_{j+i,i} = \begin{cases} \binom{-j-1}{3} & : \ j \le -1 \\ 0 & \text{otherwise} \end{cases}. \tag{5.13}$$

If $(m_1, \ldots, m_N)$ includes exactly two non-zero entries, we get

$$b_{0,\ldots,0,m_j,0,\ldots,0,m_l,0,\ldots,0} = 2 d_{m_j+i,i} \cdot d_{m_l+i,i}.$$

This value also depends only on the values m_j and m_l but not on their positions in the multi-index. With (5.4) we now get

$$b_{0,\dots,0,j,0,\dots,0,l,0,\dots,0} = 2d_{j+i,i} \cdot d_{l+i,i} = \begin{cases} 2\binom{-j-1}{1}\binom{-l-1}{1} & : \ j,l \leq -1 \\ 0 & \text{otherwise} \end{cases}. \tag{5.14}$$

Now we can compute $\|\sigma_{n,\dots,n}(b)\|_\infty$. For arbitrary $\phi_1, \dots, \phi_N \in [0, 2\pi)$ we have

$$\begin{aligned}
&\left|\sigma_{n,\dots,n}(b)\left(e^{\mathbf{i}\phi_1}, \dots, e^{\mathbf{i}\phi_N}\right)\right| \\
&= \left| \sum_{|j_1|,\dots,|j_N| \leq n-1} b_{j_1,\dots,j_N} \prod_{k=1}^{N} \left(1 - \frac{|j_k|}{n}\right) e^{\mathbf{i}j_k\phi_k} \right| \\
&\leq \sum_{|j_1|,\dots,|j_N| \leq n-1} b_{j_1,\dots,j_N} \prod_{k=1}^{N} \left(1 - \frac{|j_k|}{n}\right) \\
&= \sum_{|j| \leq n-1} b_{j,0,\dots,0} \left(1 - \frac{|j|}{n}\right) + \dots + \sum_{|j| \leq n-1} b_{0,\dots,0,j} \left(1 - \frac{|j|}{n}\right) \\
&+ \sum \sum_{|j|,|l| \leq n-1} b_{0,\dots,0,j,0,\dots,0,l,0,\dots,0} \left(1 - \frac{|j|}{n}\right)\left(1 - \frac{|l|}{n}\right),
\end{aligned}$$

where the double sum should be interpreted as the sum over all multi-indices with exactly two non-zero entries and absolute values of all entries smaller n. There are N possibilities to choose one element of N and $\binom{N}{2}$ to choose two, so due to (5.13) and (5.14) we have now

$$\begin{aligned}
&\left|\sigma_{n,\dots,n}(b)\left(e^{\mathbf{i}\phi_1}, \dots, e^{\mathbf{i}\phi_N}\right)\right| \\
&\leq \sum_{|j| \leq n-1} N b_{j,0,\dots,0} \left(1 - \frac{|j|}{n}\right) + \sum_{|j|,|l| \leq n-1} \binom{N}{2} b_{j,l,0,\dots,0} \left(1 - \frac{|j|}{n}\right)\left(1 - \frac{|l|}{n}\right) \\
&= N \sum_{j=4}^{n-1} \binom{j-1}{3} \left(1 - \frac{j}{n}\right) + 2\binom{N}{2} \sum_{j,l=2}^{n-1} (j-1)(l-1) \left(1 - \frac{j}{n}\right)\left(1 - \frac{l}{n}\right) \\
&\overset{(5.6)}{=} N\binom{n-1}{4} - \frac{4N}{n} \cdot \frac{n}{5}\binom{n-1}{4} + 2\binom{N}{2} \left(\sum_{j=2}^{n-1} (j-1)\left(1 - \frac{j}{n}\right)\right)^2 \\
&= \frac{N}{5}\binom{n-1}{4} + 2\binom{N}{2} \left(\sum_{j=2}^{n-1} \binom{j-1}{1} - \frac{2}{n}\sum_{j=2}^{n-1} \binom{j}{2}\right)^2 \\
&\overset{(5.6)}{=} \frac{N}{5}\binom{n-1}{4} + 2\binom{N}{2} \left(\binom{n-1}{2} - \frac{2}{3}\binom{n-1}{2}\right)^2 \\
&= \frac{N}{5}\binom{n-1}{4} + 2\binom{N}{2} \left(\frac{1}{3}\binom{n-1}{2}\right)^2
\end{aligned}$$

For $\phi_1 = \ldots = \phi_N$ we get by the same calculation

$$\sigma_{n,\ldots,n}(b)(1,\ldots,1) = \frac{N}{5}\binom{n-1}{4} + 2\binom{N}{2}\left(\frac{1}{3}\binom{n-1}{2}\right)^2,$$

which implies

$$\|\sigma_{n,\ldots,n}(b)\|_\infty = \frac{N}{5}\binom{n-1}{4} + 2\binom{N}{2}\left(\frac{1}{3}\binom{n-1}{2}\right)^2.$$

So we obtain

$$\lambda\left(\Delta^2 \,|\mathcal{P}_n^N\right) \geq \|\sigma_{n+1,\ldots,n+1}(b)\|_\infty = \frac{N}{5}\binom{n}{4} + 2\binom{N}{2}\left(\frac{1}{3}\binom{n}{2}\right)^2$$

by Theorem 4.5.

The calculation of $\|S_{n,\ldots,n}(b)\|_\infty$ is similar. For $\phi_1, \ldots, \phi_N \in [0, 2\pi)$ we infer from (5.13) and (5.14) that

$$\begin{aligned}
\left|S_{n,\ldots,n}(b)\left(e^{\mathbf{i}\phi_1}, \ldots, e^{\mathbf{i}\phi_N}\right)\right| &= \left|\sum_{|j_1|,\ldots,|j_N|\leq n-1} b_{j_1,\ldots,j_N} \prod_{k=1}^N e^{\mathbf{i}j_k\phi_k}\right| \\
&\leq \sum_{|j_1|,\ldots,|j_N|\leq n-1} b_{j_1,\ldots,j_N} \\
&= N\sum_{|j|\leq n-1} b_{j,0,\ldots,0} + \binom{N}{2}\sum_{|j|,|l|\leq n-1} b_{j,l,0,\ldots,0} \\
&= N\sum_{j=4}^{n-1}\binom{j-1}{3} + 2\binom{N}{2}\sum_{j,l=2}^{n-1}(j-1)(l-1) \\
&\overset{(5.6)}{=} N\binom{n-1}{4} + 2\binom{N}{2}\binom{n-1}{2}^2.
\end{aligned}$$

The last expression is equal to $S_{n,\ldots,n}(b)(1,\ldots,1)$, so we have

$$\|S_{n,\ldots,n}(b)\|_\infty = N\binom{n-1}{4} + 2\binom{N}{2}\binom{n-1}{2}^2.$$

If we replace n by $n+1$, Theorem 4.5 gives us the asserted safe upper bound.

The asymptotic lower bound is now a simple consequence of the safe lower bound.

We have

$$\begin{aligned}
\liminf_{n\to\infty} \frac{\lambda\left(\Delta^2 \left| \mathcal{P}_n^N\right.\right)}{N^2 n^4} &\geq \lim_{n\to\infty} \left(\frac{1}{N^2 n^4} \left(\frac{N}{5} \binom{n}{4} + 2 \binom{N}{2} \left(\frac{1}{3} \binom{n}{2} \right)^2 \right) \right) \\
&= \lim_{n\to\infty} \left(\frac{1}{120 N n^3} \frac{(n-1)!}{(n-4)!} + \frac{N-1}{36N} \left(\frac{n!}{n^2 (n-2)!} \right)^2 \right) \\
&= \lim_{n\to\infty} \left(\frac{1}{120N} \frac{(n-1)(n-2)(n-3)}{n^3} + \frac{N-1}{36N} \left(\frac{n-1}{n} \right)^2 \right) \\
&= \frac{1}{120N} + \frac{N-1}{36N} = \frac{1}{36} - \frac{7}{360N},
\end{aligned}$$

which completes the proof. □

Remark 5.10. It is natural to ask what happens if we derive an asymptotic upper bound from the safe upper bound from Theorem 5.9. We then have

$$\begin{aligned}
\frac{\lambda\left(\Delta^2 \left| \mathcal{P}_n^N\right.\right)}{N^2 n^4} &\leq \frac{1}{N^2 n^4} \frac{N \cdot n!}{24 (n-4)!} + \frac{1}{N^2 n^4} \frac{2 \cdot N!}{2 (N-2)!} \left(\frac{n!}{2 (n-2)!} \right)^2 \\
&= \frac{1}{24N} \frac{(n-1)(n-2)(n-3)}{n^3} + \frac{N-1}{4N} \left(\frac{n-1}{n} \right)^2 \\
&\to \frac{1}{4} - \frac{5}{24N}
\end{aligned}$$

for $n \to \infty$, which is now our asymptotic upper bound. This bound becomes worse with growing N and is ≥ 0.14853 for $N \geq 2$ and < 0.042 for $N = 1$, so the Minkowski inequality provides us for $N \geq 2$ with the better asymptotic upper bound.

We end this section with an estimate for the norm of the operator Δ^2 on $\mathcal{P}_n(\Omega_N)$ in the Laguerre case.

Corollary 5.11. *The norm of the operator*

$$\Delta^2 : \mathcal{P}_n(\Omega_N) \to \mathcal{P}_n(\Omega_N),$$

where we equip $\mathcal{P}_n(\Omega_N)$ with the Laguerre norm, satisfies for arbitrary $N \in \mathbb{N}$

$$\frac{1}{N^4} \left(\frac{1}{36} - \frac{7}{360N} \right) \leq \liminf_{n\to\infty} \frac{\lambda\left(\Delta^2 \left| \mathcal{P}_n(\Omega_N) \right.\right)}{N^2 n^4} \leq \limsup_{n\to\infty} \frac{\lambda\left(\Delta^2 \left| \mathcal{P}_n(\Omega_N) \right.\right)}{N^2 n^4} \leq \frac{1}{\mu_0^4}.$$

Proof. We proceed as in the proof of Theorem 1.5. Here we have to replace Δ in (4.36) by Δ^2. Then we apply Theorem 5.9 with $\lfloor n/N \rfloor$ instead of n and get the result. □

6 Open Problems

6.1 The Laplace operator on $\mathcal{P}_n(E)$

If we ask for the norm of the Laplace operator

$$\Delta : \mathcal{P}_n(E) \to \mathcal{P}_n(E)$$

we get, as seen in this work, an asymptotic formula for the norm if we equip $\mathcal{P}_n(E)$ with the Hermite norm and set $E = [0,1]^N$ (Theorem 1.3) and asymptotic bounds for $E = \Omega_N$ (Theorem 1.6). For the Laguerre and Legendre norms we get at least safe and asymptotic bounds if we choose $E = [0,1]^N$ (Theorem 1.1, Theorem 1.2 and Theorem 1.4) or $E = \Omega_N$ (Theorem 1.5 and Theorem 1.7). So the maybe most obvious question is: what happens for the three norms for general E?

Unfortunately we have no answer so far.

6.2 Norms with parameters on $\mathcal{P}_n(E)$

Another open problem is the question what happens if we allow parameters in the norms on $\mathcal{P}_n(E)$. The Laguerre, Hermite, and Legendre norms are then

$$\|f\|^2 = \int_{(0,\infty)^N} |f(t_1,\ldots,t_N)|^2 \, t_1^{\alpha_1} \ldots t_N^{\alpha_N} e^{-t_1} \ldots e^{-t_N} \, dt_1 \ldots dt_N,$$

$$\|f\|^2 = \int_{\mathbb{R}^N} |f(t_1,\ldots,t_N)|^2 \, |t_1|^{2\alpha_1} \ldots |t_N|^{2\alpha_N} \, dt_1 \ldots dt_N,$$

and

$$\|f\|^2 = \int_{(-1,1)^N} |f(t_1,\ldots,t_N)|^2 \, (1-t_1^2)^{\alpha_1} \ldots (1-t_N^2)^{\alpha_N} \, dt_1 \ldots dt_N,$$

where $\alpha_1,\ldots,\alpha_N > -1$ are fixed parameters. We can try to apply Lemma 4.2 as in the case $\alpha_1 = \ldots = \alpha_N = 0$, if $E = [0,1]^N$. For that we make the identification

$$\mathcal{P}_n^N = \bigotimes_{k=1}^{N} \mathcal{P}_{n,\alpha_k},$$

where the notation $\mathcal{P}_{n,\alpha_k}$ indicates that $\mathcal{P}_n$ is equipped with the one-dimensional Laguerre, Hermite, or Legendre norm with parameter α_k. We endow $\mathcal{P}_{n,\alpha_k}$ with the orthonormal basis of weigthed normalized Laguerre, Hermite, or Legendre polynomials and get by tensoring an orthonormal basis on $\mathcal{P}_n^N$. The matrix representation of Δ is now the Kronecker sum

$$[\Delta] = \left[D^2\right]_{\alpha_1} \oplus \ldots \oplus \left[D^2\right]_{\alpha_N},$$

where $[D^2]_{\alpha_k}$ is the matrix representation of $D^2 : \mathcal{P}_{n,\alpha_k} \to \mathcal{P}_{n,\alpha_k}$. If we want to apply Lemma 4.2, we have, by Corollary 4.4, to consider the maximal eigenvalues of $H\left([D^2]_{\alpha_k}\right)$, but this causes problems, because we loose now structure in these matrices.

For example, let us consider the Laguerre case. Then the representation $[D^2]_\alpha$ of D^2 with respect to the norm with parameter α becomes

$$\left[D^2\right]_\alpha = \mathbb{D}_n^{-1}(\alpha)\left[D^2\right]\mathbb{D}_n(\alpha),$$

where $[D^2]$ is the well known represention in the case $\alpha = 0$, while $\mathbb{D}_n(\alpha)$ is

$$\mathbb{D}_n(\alpha) := \operatorname{diag}\left(\omega_0(\alpha), \ldots, \omega_N(\alpha)\right), \quad \omega_k(\alpha) := \sqrt{\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}}.$$

So $[D^2]_\alpha$ is no longer a Toeplitz matrix and we cannot apply the results from [1] as in the proof of Theorem 1.2 to estimate the largest eigenvalue of $H\left([D^2]_\alpha\right)$. Due to the same reason it is not clear if we can interpret $[\Delta]$ as a multi-level Toeplitz operator to apply Theorem 4.5.

6.3 The operator Δ_ν in the Legendre case

We end this section with some remarks on the generalized Laplace operator Δ_ν on $\mathcal{P}_n^N$ equipped with the Legendre norm. The matrix representation of Δ_ν with respect to the normalized Legendre polynomials on $\mathcal{P}_n^N$ is

$$[\Delta_\nu] = \sum_{k=1}^{N} I_{(n+1)^{k-1}} \otimes [D^\nu] \otimes I_{(n+1)^{N-k}}.$$

As for the Laguerre or Hermite norm we get the lower bound

$$\gamma\left(\Delta_\nu \left| \mathcal{P}_n^N \right.\right) \geq \frac{N}{2}\lambda_{\max}\left([D^\nu] + [D^\nu]^T\right). \tag{6.1}$$

We know for example from [4] that the ordinary differential operator D on $\mathcal{P}_n$ has with respect to the Legendre polynomials the representation

$$[D] = \begin{pmatrix} 0 & c_{01} & c_{02} & \ldots & c_{0n} \\ & \ddots & c_{12} & \ldots & c_{1n} \\ & & & \ddots & \vdots \\ & & & \ddots & c_{nn} \\ 0 & & & & 0 \end{pmatrix}$$

with

$$c_{jk} = \begin{cases} \sqrt{(2j+1)(2k+1)} & : \ j+k \quad \text{odd} \\ 0 & : \quad \text{even} \end{cases}.$$

For odd n and $n+1 = 2m$, the matrix $[D]$ is unitarily equivalent to the block matrix

$$\mathbb{D}_1 := \begin{pmatrix} 0 & A_m \\ B_m & 0 \end{pmatrix},$$

where we use the notation from Chapter 3.3. This implies that $[D^\nu]$ is unitarily equivalent to

$$\mathbb{D}_\nu = \begin{pmatrix} A_m B_m \dots A_m B_m & 0 \\ 0 & B_m A_m \dots B_m A_m \end{pmatrix}$$

for even ν and and to

$$\mathbb{D}_\nu = \begin{pmatrix} 0 & A_m B_m \dots A_m \\ B_m A_m \dots B_m & 0 \end{pmatrix}$$

for odd ν. The products in the non-vanishing blocks consist of ν factors.

In the following we assume ν to be even. From (6.1) we conclude

$$\gamma\left(\Delta_\nu \left| \mathcal{P}_n^N \right.\right) \geq \frac{N}{2} \lambda_{\max}\left(\mathbb{D}_\nu + \mathbb{D}_\nu^T\right).$$

The matrix $\mathbb{D}_\nu + \mathbb{D}_\nu^T$ is obviously hermitian and due to its block structure we have

$$\begin{aligned} \lambda_{\max}\left(\mathbb{D}_\nu + \mathbb{D}_\nu^T\right) \quad = \quad & \max\left\{ \lambda_{\max}\left(A_m B_m \dots A_m B_m + B_m^T A_m^T \dots B_m^T A_m^T\right), \right. \\ & \left. \lambda_{\max}\left(B_m A_m \dots B_m A_m + A_m^T B_m^T \dots A_m^T B_m^T\right) \right\}. \end{aligned}$$

Again, as for $\nu = 2$, we can use the Rayleigh quotient to get a lower bound for these eigenvalues and get the estimates

$$\gamma\left(\Delta_\nu \left| \mathcal{P}_n^N \right.\right) \geq \frac{N}{2} \frac{\overline{x}^T \left(A_m B_m \dots A_m B_m + B_m^T A_m^T \dots B_m^T A_m^T\right) x}{\overline{x}^T x} \tag{6.2}$$

and

$$\gamma\left(\Delta_\nu \left| \mathcal{P}_n^N \right.\right) \geq \frac{N}{2} \frac{\overline{x}^T \left(B_m A_m \dots B_m A_m + A_m^T B_m^T \dots A_m^T B_m^T\right) x}{\overline{x}^T x}$$

for all $x \in \mathbb{C}^{n+1} \setminus \{0\}$.

To get concrete lower bounds we have to calculate the Rayleigh quotients for some good vectors x. For example, we can choose $(x_j)_{j=0}^{m-1}$ with $x_j = \sqrt{4j+1}$ for $j = 0, \dots, m-1$ for (6.2). Then the quotient becomes a polynomial in m. As in the case $\nu = 2$, we can try to calculate this polynomial with the Faulhaber formula, but for general ν this seems to be disproportionately complicated, even if we ask only for the leading terms of this polynomials.

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Theses

on the thesis

"Asymptotic bounds and values for the norm of the Laplace operator and other partial differential operators on spaces of polynomials"

by Christian Rebs.

1. Markov-type inequalities give upper bounds for partial derivatives of algebraic polynomials in terms of the norms of the polynomials. For a partial differential operator $\mathcal{D} : \mathcal{P}_n(E) \to \mathcal{P}_n(E)$, such an inequality has the form

$$\|\mathcal{D}f\| \leq C\|f\| \quad \text{for all} \quad f \in \mathcal{P}_n(E), \tag{A}$$

   where E is a closed subset of $[0,1]^N$ and $\mathcal{P}_n(E)$ is the linear space of all complex algebraic polynomials of the form

$$f(t_1, \ldots, t_N) := \sum_{i_1/n, \ldots, i_N/n \in E} c_{t_1,\ldots,i_N} t_1^{i_1} \ldots t_N^{i_N}.$$

   The constant C depends on the operator $\mathcal{D}$, the positive integers n and N, the norm chosen on $\mathcal{P}_n(E)$, and the set E. In this work we equip $\mathcal{P}_n(E)$ with one of the norms

$$\|f\|^2 := \int_{(0,\infty)^N} |f(t_1, \ldots, t_N)|^2 e^{-t_1} \ldots e^{-t_N} \, dt_1 \ldots dt_N \quad \text{(Laguerre norm)},$$

$$\|f\|^2 := \int_{\mathbb{R}^N} |f(t_1, \ldots, t_N)|^2 e^{-t_1^2} \ldots e^{-t_N^2} \, dt_1 \ldots dt_N \quad \text{(Hermite norm)},$$

   or

$$\|f\|^2 := \int_{[-1,1]^N} |f(t_1, \ldots, t_N)|^2 \, dt_1 \ldots dt_N \quad \text{(Legendre norm)}.$$

2. The best constant C in (A) is just the operator norm of $\mathcal{D}$. We choose $\mathcal{D}$ as $\Delta := \partial^2/\partial t_1^2 + \ldots + \partial^2/\partial t_N^2$, $\Delta_\nu := \partial^\nu/\partial t_1^\nu + \ldots + \partial^\nu/\partial t_N^\nu$, $\nu \in \mathbb{N}$, or Δ^2. The spaces $\mathcal{P}_n(E)$ are finite dimensional and so the operator norm of $\mathcal{D}$ is the spectral norm of its matrix representation $[\mathcal{D}]$ with respect to an orthonormal basis on $\mathcal{P}_n(E)$.

3. If we choose $E = [0,1]^N$ and equip $\mathcal{P}_n^N := \mathcal{P}_n\left([0,1]^N\right)$ with the the Laguerre norm then we can interpret $[\mathcal{D}]$ for all the operators Δ, Δ_ν and Δ^2 as a multilevel Toeplitz operator. We generalize estimates for the norm of an ordinary Toeplitz operator to the multilevel case. Application of this generalization provides us with safe and asymptotic bounds for the norms of the differential operators considered.

4. For $E = [0,1]^N$ and all the norms of interest, the matrix $[\Delta]$ is the Kronecker sum of N copies of the representation $[D^2]$ of the ordinary differential operator on $\mathcal{P}_n := \mathcal{P}_n^1$ sending a polynomial to its second derivative. Due to this structure, we get a lower bound for the norm of Δ by the largest eigenvalue of the hermitian part $H([D^2])$ of $[D^2]$.

5. We use different techniques to get results about this largest eigenvalue for the different norms.

   In the Laguerre case, $H([D^2])$ is an hermitian Toeplitz matrix with polynomially increasing entries and we can apply results about the eigenvalues of such matrices. This leads to an improvement of the asymptotic bounds known so far.

   In the Hermite case we use that the flipped matrix gives us an operator on ℓ^2 which converges (after scaling) to an infinite Toeplitz operator with well-known spectrum. Here we find even an asymptotic value for the norm of the Laplace operator and not only upper and lower bounds.

   Finally, in the Legendre case we find a lower bound for the maximal eigenvalue of $H([D^2])$ by a Rayleigh quotient. This lower bound and the Minkowski inequality provide us with asymptotic bounds for the norm of Δ.

6. The more general operator Δ_ν has on $\mathcal{P}_n^N$ also the structure of a Kronecker sum. For the Laguerre and the Hermite norms, we can proceed as in the case $\nu = 2$ to find asymptotic bounds in the Laguerre case and an asymptotic value in the Hermite case.

7. Things becomes much more difficult in the case $E \neq [0,1]^N$, because $[\mathcal{D}]$ has then no longer the structure of a Kronecker sum. If E is the simplex $\Omega_N := \left\{(x_1, \ldots, x_N)^T \in [0,1]^N : \; x_1 + \cdots + x_N \leq 1\right\}$ and we equip $\mathcal{P}_n(\Omega_N)$ with the Hermite norm, we get at least asymptotic bounds for the norm of Δ_ν for arbitrary $\nu, N \in \mathbb{N}$ and safe bounds for $N \in \{2,3\}$ and $\nu = 2$ by estimating the norm by the maximum of the euclidian norms of the rows and columns of $[\Delta_\nu]$. In this case we also find bounds for the norm of Δ^2.

   In the Laguerre case we find asymptotic bounds for the norms of Δ_ν and Δ^2, where we use the relation

   $$\mathcal{P}_n\left(\frac{1}{N}[0,1]^N\right) \subset \mathcal{P}_n(\Omega_N) \subset \mathcal{P}_n^N.$$

   We apply this method also in the Legendre case to find a result for Δ on $\mathcal{P}_n(\Omega_N)$.

Lebenslauf

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Publikationen

1. Albrecht Böttcher, Christian Rebs, *On the constants in Markov inequalities for the Laplace operator on polynomials with the Laguerre norm*, Asymptotic Analysis 101 (2017), 227-239.

2. Christian Rebs, *On the best constants in Markov inequalities for the Laplace operator on polynomials with the Hermite and Legendre norm*, Linear Algebra and its Applications 572 (2019), 32-45.

Selbstständigkeitserklärung

Hiermit erkläre ich gegenüber der Technischen Universität Chemnitz, dass ich die vorliegende Arbeit selbstständig und unter ausschließlicher Verwendung der angegebenen Quellen und Hilfsmittel angefertigt habe.

Chemnitz, den 12. Dezember 2019

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Christian Rebs