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1

Introduction

This thesis deals with the singularities of perfect cone toroidal compactifications of arithmetic quotients of hermitian symmetric domains of type IV. We shall introduce the objects involved, motivate this problem and give an overview of analogous results known for similar spaces.
The search for compactifications for moduli spaces is a long-standing problem that drew the attention of many algebraic geometers. After the constructions of good compactifications for the moduli spaces of curves and of abelian varieties, also the moduli space of polarized irreducible holomorphically symplectic varieties has been a hot research topic, we cite the following just to give a taste: [Sha80] [Ma18], [GHS07], [AET19] and [FM20].
We focus for a moment on K3 surfaces, which are a fundamental class of compact complex surfaces and a generalisation of which in higher dimension are holomorphically symplectic manifolds. Much work has been done in recent years to describe the moduli space of polarised K3 surfaces. Except for low degree, where the moduli space can be identified with a GIT quotient (for instance see [Sha80] for degree 2), the moduli space of polarised K3 surfaces is constructed as an arithmetic quotient of an hermitian symmetric domain. Let us briefly review the construction, given any K3 surface X its second cohomology group equipped with the intersection product is isomorphic to the lattice $\Lambda_{K3} := U^{\oplus 3} \oplus E_8^{\oplus 2}(-1)$, where U is the hyperbolic plane and E_8 is the lattice associated to the corresponding Coxeter diagram. If L is an ample line bundle on X of degree $2d$ with primitive class, its orthogonal complement in $H^2(X,\mathbb{Z})$ is isomorphic to $\Lambda_{2d} := U^{\oplus 2} \oplus E_8^{\oplus 2}(-1) \oplus \langle -2d \rangle$. The moduli space of $2d$ polarised K3 surfaces is then given by the quotient of the hermitian symmetric domain

$$\mathscr{D}_{2d} := \{x \in \mathbb{P}(\Lambda_{2d} \otimes \mathbb{C}) |\ (x,x) = 0 \text{ and } (x,\bar{x}) > 0\}^+$$

by the action of an arithmetic group (here the symbol + denotes the choice of a connected component). In the spirit of [GHS07], we consider a more general problem, treating not only lattices of the form Λ_{2d} but any non degenerate lattice of signature $(2,n)$ and arithmetic quotients of the associated hermitian symmetric domains, which are known as domains of type IV. We notice that also moduli spaces of irreducible holomorphically symplectic varieties are strictly related to this construction, see [GHS13]. Several compactifications are known for locally symmetric varieties (see [BJ06]). A canonical compactification is given by the Baily-Borel one (see [BB66]), but because of the high codimension of the boundary

it is hardly expected to have a meaningful modular interpretation. On the other hand, infinitely many toroidal compactifications can be potentially constructed. Unfortunately, the construction is rather involved and it is not an easy task to describe the combinatorial data defining them. Indeed, a toroidal compactification depends on a choice of a collection of fans for certain cones (see Definition 4.3.2 for more details). The fans consist of infinitely many cones and are finite only modulo the action of a group. If for the moduli space of principally polarised abelian varieties several compactifications have been described and compared (for the second Voronoi compactification it is also known a modular interpretation, see [Ale02]) the same cannot be said for the moduli space of K3 surfaces. However, there are some results in this direction: the collection given by the reflection fan, a somewhat natural choice, has been extensively analysed in [AET19]. Nonetheless, toroidal compactifications turned out to be very useful to study the locally symmetric varieties: in [GHS07] and [Ma18] the authors determined the Kodaira dimension of several locally symmetric varieties by proving the existence of compactifications with mild singularities and by studying modular forms extending on them.
The starting point of this project was to bridge this gap by studying a specific toroidal compactification for quotients of type IV domains. In [SB06] Shepherd-Barron showed that the aforementioned first Voronoi compactification of the moduli space of principally polarised abelian varieties is a canonical model in the sense of MMP. The first Voronoi compactification builds on the so-called perfect cone decomposition for cones. The latter is a versatile construction that can be employed to define toroidal compactifications for all locally symmetric varieties (see Section 4.4), which we will refer to as the **perfect cone compactification**.

What there is in this thesis:

First of all, let us spend few words about what there is not: there is neither a finite list of the perfect cones (modulo an action of a group) nor a nice characterisation of them as it is known for the perfect cone decomposition of the positive definite symmetric matrices (see Paragraph 1.4 in [CMGHL17] for a brief account).
Let D denote a type IV domain, $\Gamma \leq \mathrm{Aut}(D)$ be an arithmetic group and Σ_p be the collection of perfect cone fans, in this work there is a study of the singularities of the associated perfect cone compactification $\overline{D/\Gamma}^{\Sigma_p}$. The first step is the observation that even if unable to describe the perfect cones, we are capable to state something about the toric varieties associated to them:

Theorem 1.0.1 (Theorem 3.3.3)**.** *The toric varieties associated to the perfect cones are $\mathbb{Q}$-Gorenstein and have canonical singularities.*

The boundary of $\overline{D/\Gamma}^{\Sigma_p}$ is locally described as quotient of these toric varieties by the action of a finite group. We point out that a significant difference between this representation and

the analogue for the toroidal compactifications in the case of abelian varieties is that in the latter the action on the toric variety is free in codimension one, while in the former the presence of reflections must be taken into account. In fact, because of Example 3.4.6 it seems unlikely to exclude that $\overline{D/\Gamma}^{\Sigma_p}$ has not canonical singularities without a more precise knowledge of perfect cones.

In order to study the singularities of a compactifcation $\overline{D/\Gamma}^{\Sigma}$, for a general collection of fans Σ it is customary to consider the compactification induced on the quotient by a neat normal subgroup $\Gamma' \lhd \Gamma$ of finite index. A Γ-collection of fans is also Γ'-admissible and in the literature the existence of a finite morphism $\overline{D/\Gamma'}^{\Sigma} \to \overline{D/\Gamma}^{\Sigma}$ is often used. Because we were not able to trace a proof of this fact we wrote up the details in Proposition 4.3.16.

We then focus on the stable orthogonal group $\widetilde{O}^+(L)$, the definition will be recalled in Chapter 5, because of its importance in the study of moduli of K3 surfaces. If $\Gamma' \lhd \widetilde{O}^+(L)$ is a normal subgroup, we denote by Δ the branch divisor of $\overline{D/\Gamma'}^{\Sigma} \to \overline{D/\widetilde{O}^+(L)}^{\Sigma}$. Our main result is:

Theorem 1.0.2 (Theorem 5.2.7)**.** *Under the technical assumption (5.2.1) the pair*

$$\left(\overline{D_L/\widetilde{O}^+(L)}^{\Sigma_p}, \frac{1}{2}\Delta\right)$$

has Kawamata log terminal singularities.

We give now a brief overview of each chapter of the thesis. Chapter 2 deals with the classical theory of hermitian symmetric spaces in order to introduce hermitian symmetric spaces of type IV. In Chapter 3 the perfect cone decomposition of a cone is introduced and we treat the singularities relevant to this work: canonical and klt singularities, singularities of toric varieties and of their quotients. The construction of toroidal compactifications takes up Chapter 4, the perfect cone compactification is introduced in Section 4.4. Finally, Chapter 5 contains the main theorem, the results on the singularities of toric varieties developed in the second chapter are applied to the perfect cone compactification.

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2

A domain of type IV

The scope of this chapter is to introduce the protagonist of the whole story, namely

$$D_L = \{[x] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (x,x) = 0\ (x,\bar{x}) > 0\}^+$$

where L is a non-degenerate lattice of signature $(2,n)$ for an integer $n \geq 3$; here the symbol $+$ denotes the choice of a connected component.
Our protagonist is referred to with a lot of names: hermitian symmetric space of non-compact type, period domain, domain of type IV. As the plethora of locutions which are attributed to it may suggest, the object D_L happens to carry a lot of structure. In this chapter we will recall the main aspects and properties with the aim of putting our work in the right setting. We have no presumption of completeness nor of originality in that.
In Section 2.1 we treat the topic of hermitian symmetric spaces, in Section 2.2 we focus on their description in terms of Lie algebras and Lie groups. Section 2.3 is devoted to the duality between hermitian symmetric spaces of compact and of non-compact type. In Section 2.4 we present the domain D_L within the theory developed in the previous sections, while in Section 2.5 we recall the Harish-Candra and Borel embeddings for hermitian symmetric spaces, and give formulas for the specific case of D_L.

2.1 Hermitian symmetric domains

We start by recalling some basic definitions. In what follows all the manifolds we treat are assumed to be connected.

Definition 2.1.1. A Riemannian manifold is a pair (M,g), where M is a smooth manifold and g is a smooth 2-tensor such that:

$$\forall p \in M,\ g_p : T_pM \times T_pM \to \mathbb{R}$$

is a positive definite symmetric bilinear form. The tensor g is called Riemannian metric.

Recall that if (V,g,J), is a real vector space with scalar product g with a complex structure J acting by g-isometries, then the association

$$h : V \times V \to C$$

$$(x,y) \mapsto g(x,y) - ig(Jx,y)$$

defines a hermitian metric on the complex vector space (V,J).
Let M be a complex manifold, we denote by M^∞ the underlying smooth manifold. If M is a complex manifold, the multiplication by i on the tangent space at every point induces a vector bundle isomorphism on the tangent bundle of the underlying smooth manifold:

$$J\colon TM^\infty \to TM^\infty$$

which we refer to as the complex structure of M.

Definition 2.1.2. A hermitian manifold is a complex manifold together with a Riemannian metric g on the underlying smooth manifold such that the complex structure $J \in \mathrm{End}(TM^\infty)$ acts by g-isometries. In what follows we will write a hermitian manifold as a triple (M,J,g) or as (M,J,h) for a smooth manifold M with a complex structure J and hermitian respectively Riemannian metric h respectively g.

Given a hermitian manifold (M,J,g) we consider the automorphism group $\mathrm{Aut}(M)$ of holomorphic isometries. To be precise, if $\mathrm{Is}(M^\infty,g)$ denotes the group of isometries of the underlying smooth manifold and $\mathrm{Hol}(M)$ the group of holomorphic automorphisms, we have:

$$\mathrm{Aut}(M) := \mathrm{Is}(M^\infty,g) \cap \mathrm{Hol}(M)$$

Definition 2.1.3. A connected hermitian manifold M is said to be a hermitian symmetric space if the group $\mathrm{Aut}(M)$ acts transitively on M and for some point $p \in M$ there exists an involutive[1] automorphism $s_p \in \mathrm{Aut}(M)$ with p as isolated fixed point.

Notice that, by transitivity of the action of $\mathrm{Aut}(M)$, if such an involution exists for a point, then it exists for any point in M. Moreover such a symmetry is unique as this lemma shows:

Lemma 2.1.4. *Let M be a hermitian symmetric space, let $\mathrm{Stab}(p) \subseteq \mathrm{Aut}(M)^\circ$ be the stabiliser of a the point $p \in M$, then for an element $s_p \in \mathrm{Stab}(p)$*

$$s_p \text{ is a symmetry if and only if } (ds_p)_p = -\mathrm{id}_{T_pM}.$$

Proof. First of all, thanks to [Hel78], Lemma I.11.2 the representation of $\mathrm{Stab}(p)$ given by the differential is faithful, that is the morphism

$$d\colon \mathrm{Stab}(p) \to O(T_pM, g_p) \tag{2.1.1}$$
$$\varphi \mapsto (d\varphi)_p \tag{2.1.2}$$

[1] here and in the following by an "involution" or by an "involutive automorphism" we mean a non trivial morphism whose square is the identity.

is injective. Hence, it suffices to show that for a symmetry s_p we have $(ds_p)_p = -\mathrm{id}_{T_pM}$. For this, notice that from $s_p^2 = \mathrm{id}$ it follows that $(ds_p)_p^2 = \mathrm{id}_{T_pM}$. Let $0 \neq X \in T_pM$ be a vector such that $(ds_p)_p X = X$. Let $\gamma = \gamma(X)$ be the geodesic such that $\gamma(0) = p$ and $\frac{d\gamma}{dt}(0) = X$, then thanks to Theorem I.6.1 of [Hel78] γ is pointwise fixed by s_p contradicting the hypothesis that p is an isolated fixed point for s_p. □

In particular, for any element $g \in \mathrm{Stab}(p)$:

$$d(gs_pg^{-1}) = dg\ ds_p\ dg^{-1} = -\mathrm{id}_{T_pM}$$

and from the faithfulness of 2.1.1 it follows that $gs_pg^{-1} = s_p$, in other words $\mathrm{Stab}(p)$ lies in the centraliser of s_p in $\mathrm{Aut}(M)^\circ$. We will get back to this in Theorem 2.2.2

Example 2.1.5. In dimension 1, irreducible symmetric spaces are isomorphic to either the complex plane $\mathbb{C}$ or the upper half-plane $\mathbb{H}$ or to a complex torus $\mathbb{C}/\Lambda$, where Λ is a discrete rank 2 free abelian group in $\mathbb{C}$.

These examples reflect some aspects that are present in higher dimensions and that lead to the following classification:

Definition 2.1.6. A hermitian symmetric space is said to be:

- *euclidean* if it is isomorphic to $\mathbb{C}^n/\Lambda$ for a discrete additive subgroup $\Lambda \subset \mathbb{C}^n$;
- *irreducible* if it is not euclidean and cannot be written as product of hermitian symmetric spaces of lower dimension;
- *non-euclidean* if it can be written as a product of irreducible hermitian symmetric domains;
- *of compact type* if it is isomorphic to the product of irreducible hermitian symmetric that are compact;
- *of non-compact type* if it is isomorphic to the product of irreducible hermitian symmetric spaces that are non-compact.

Remark 2.1.7. The hermitian symmetric spaces of non-compact type are also called domains: thanks to [Hel78] Theorem.VIII.7.1, any of them can be realised as a bounded domain of a complex vector space.

Theorem 2.1.8. *A hermitian symmetric space M decomposes as the product*

$$M = M_0 \times (M_1^+ \times \cdots \times M_r^+) \times (M_1^- \times \cdots \times M_s^-)$$

where:

- *M_0 is a euclidean hermitian symmetric space;*
- *M_i^+ is a simply connected irreducible hermitian symmetric space of compact type;*
- *M_j^- is a simply connected irreducible hermitian symmetric space of non-compact type.*

Moreover the factors are unique up to reordering.

Proof. From [Hel78] Propositions VIII.4.4 and VIII.5.5 and Theorem VIII.4.6 one can deduce the statement with the extra assumption that M is simply connected.
Thanks to [Wol11] Corollary 8.7.11 one can reduce to that case. □

It is worth to notice that the proofs in [Hel78] and in [Wol11] rely on a detailed study of the automorphism groups of hermitian spaces: one could say that we proceed in a kind of opposite order describing properties of the hermitian symmetric spaces as results whilst for the main references these are actually intermediate steps leading to the above theorem.
We conclude this section by recalling a result characterising the differentiable manifolds underlying irreducible hermitian spaces:

Theorem 2.1.9 ([Hel78], Theorem VIII.6.1 and Proposition VIII.6.2)**.** *There is a bijection between:*

- *the smooth manifolds underlying irreducible hermitian symmetric spaces of non-compact type and manifolds G/K where G is a connected non-compact simple Lie group with trivial center, K is a maximal compact subgroup of G whose center has a non-trivial connected component, or equivalently is isomorphic to S^1.*
- *the smooth manifolds underlying irreducible hermitian symmetric spaces of compact type and manifolds G/K where G is a connected compact simple Lie group with trivial center, K is a maximal connected proper subgroup of G whose center has a non-trivial connected component, or equivalently is isomorphic to S^1.*

2.2 Lie groups and Lie algebras of Hermitian symmetric spaces

The theory of Lie groups play an important role in the study of hermitian symmetric spaces, here we present the main results that concern us. The main goal is to endow the quotient of certain real Lie pairs of groups (G,K) with a hermitian structure depending on properties of the groups. In Theorem 2.2.9 we will show that the structure of a hermitian symmetric space of non-compact type can be fully recovered except for the sign complex structure from the automorphism group and the stabiliser of a point. We are mainly interested in spaces of non-compact type, nevertheless we will consider also compact type ones because their theories are deeply intertwined.

Let $M = (M, J, g)$ be a non-euclidean hermitian symmetric space, then, thanks to Lemma IV.3.2 of [Hel78], the group of isometries $\mathrm{Is}(M^\infty, g)$ equipped with the compact-open topology is a Lie group. As $\mathrm{Aut}(M) \subset \mathrm{Is}(M^\infty, g)$ is a closed subgroup, it acquires the structure of Lie group, too. By connectedness of M[2] it follows that if $\mathrm{Aut}(M)$ acts transitively on M, so does the connected component of the identity $G := \mathrm{Aut}(M)^\circ$. We recollect first some consequences of Theorem 2.1.8 and the proof given in [Hel78].

Corollary 2.2.1. *If M is a non-euclidean hermitian symmetric space, then $\mathrm{Aut}(M)^\circ$ is semisimple.*

Proof. Thanks to Theorem VIII.4.6 of [Hel78] a non-euclidean hermitian symmetric space is simply connected, hence the already cited Proposition VIII.4.4 of [Hel78] is applicable and the statement is part of its proof. □

There is another Lie group that is natural to consider when studying hermitian symmetric spaces: the stabiliser of a point. For this, fix a point $p \in M$ and let $K := \{g \in G : g \cdot p = p\}$ be the subgroup of holomorphic isometries fixing p. By [Hel78], Theorem VIII.4.5 the symmetry s_p lies in the connected component of the identity of $\mathrm{Stab}(p) \subseteq \mathrm{Aut}(M)$ and hence in K. Let σ denote the inner automorphism of G induced by the symmetry s_p relative to the point $p \in M$:

$$\begin{aligned} \sigma : G &\to G \\ g &\mapsto s_p g s_p. \end{aligned}$$

Let G_σ be the subgroup of elements fixed by σ, or equivalently the centraliser of s_p in G, and let $(G_\sigma)^\circ$ be the connected component of its identity. From Theorem 2.1.4 the inclusion $K \subseteq G_\sigma$ follows, here we recollect something more:

Theorem 2.2.2. *The subgroup $K \leq G$ is a compact Lie subgroup and the morphism:*

$$\pi : G/K \to M \tag{2.2.1}$$
$$g \mapsto g \cdot p \tag{2.2.2}$$

is a diffeomorphism.
Moreover the following inclusions hold:

$$(G_\sigma)^\circ \subseteq K \subseteq G_\sigma. \tag{2.2.3}$$

and K contains no normal subgroup of G other than the trivial one $\{e\} \leq G$.

Remark 2.2.3. Notice that the quotient G/K has a canonical structure as differentiable manifold: in general given a closed subgroup H of a Lie group F there exists a unique differentiable structure on the quotient space F/G such that the multiplication map $F \times (F/G) \to F/G$ is differentiable (see Theorem.II.4.2 of [Hel78]).

[2] recall that for us hermitian symmetric spaces are connected by definition

Proof. As recalled in Corollary 2.2.1, the group G is semisimple, hence thanks to Lemma VIII.4.3 of [Hel78] the connected components of the identity of $\mathrm{Is}(M^\infty, g)$ and of $\mathrm{Aut}(M)$ coincide therefore Theorem IV.3.3 of (*op.cit.*), which holds more in general for Riemannian symmetric spaces, is applicable. The compactness of the stabiliser group follows from Theorem VI.2.5 (*op.cit.*), which again holds more in general for Riemannian manifolds. □

This theorem has many consequences and sheds light on the structure of hermitian symmetric spaces. We proceed further by analysing the relation of the metric with this quotient description. Denote with $\mathfrak{g} = \mathrm{Lie}(G)$ the Lie algebra of G, the differential $\theta := d\sigma$ is an involution of $\mathfrak{g}$. Let $\mathfrak{k} = \mathrm{Eig}(\theta, 1)$, respectively $\mathfrak{p} = \mathrm{Eig}(\theta, -1)$ be the eigenspaces of θ with respect to the eigenvalue 1 respectively -1. Then we have a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as a sum of vector spaces. Our choice of notation is justified because from Theorem 2.2.2 it follows that the vector space $\mathfrak{k}$ actually coincides with the Lie algebra $\mathrm{Lie}(K)$ and in particular we see that the morphism in (2.2.1) induces a short exact sequence

$$0 \to \mathfrak{k} \to \mathfrak{g} \xrightarrow{d\pi} T_pM \to 0$$

which in turn induces an isomorphism (of vector spaces)

$$d\pi : \mathfrak{p} \to T_pM. \tag{2.2.4}$$

Since θ is an automorphism of Lie algebras, we have that

$$\forall k_1, k_2 \in \mathfrak{k} \text{ and } \forall p_1, p_2 \in \mathfrak{p} :$$

$$\theta[k_1, p_1] = [\theta k_1, \theta p_1] = [k_1, -p_1] = -[k_1, p_1] \text{ so} \qquad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p} \tag{2.2.5}$$

$$\theta[p_1, p_2] = [\theta p_1, \theta p_2] = [-p_1, -p_2] = [p_1, p_2] \text{ so} \qquad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}. \tag{2.2.6}$$

in particular $\mathfrak{p}$ is not a subalgebra and $\mathfrak{k}$ acts on $\mathfrak{p}$ via restriction of the adjoint representation of $\mathfrak{g}$; the latter can be identified with the representation given by the action of K on the tangent space T_pM.

Corollary 2.2.4 (of Theorem 2.1.8)**.** *If M is irreducible, then the representation of K on T_pM is irreducible and so is the adjoint representation of $\mathfrak{k}$ on $\mathfrak{p} \simeq T_pM$.*

Proof. This is again contained in [Hel78] (see the references given for Theorem 2.1.8): indeed according to loc.cit. page 377, a Riemannian globally symmetric space is irreducible if, in the above notation, K acts irreducibly on T_pM. □

The metric

Lemma 2.2.5. *Let (M,J,g) be an irreducible hermitian symmetric space that is non-euclidean, let $p \in M$ be a fixed point, $G = \mathrm{Aut}(M)$ be the group of holomorphic isometries and $K = \mathrm{Stab}(p)$ be the stabiliser. Let B be Killing form of $\mathfrak{g}$, then there exists a constant $c \in \mathbb{R}$ such that:*

$$\forall X, Y \in T_pM,\ g_p(X,Y) = cB|_{\mathfrak{p}}(X,Y).$$

If M is of compact type (respectively of non-compact type), then c is negative (respectively positive).
Moreover, the metrics on the quotient G/K for which G acts as a group of isometries satisfy the former equality and are uniquely determined by the value of c.

Proof. For the last statement notice that, since M is homogeneous, for every point $x \in M$ there is an automorphism $f \in G$ such that $f(p) = x$, moreover if f acts as isometry, the following holds:

$$\forall X, Y \in T_xM\ \ g_x(X,Y) = g_p(df^{-1}(X), df^{-1}(Y)).$$

Hence the metric g can be recovered by its value on T_pM.
For the first part notice that thanks to Corollary 2.2.4 the group K acts irreducibly via isometries on T_pM, i.e. g_p is a K-invariant positive definite bilinear form.
The statement now follows from the general fact that given a compact linear group acting on a vector space, the latter admits an invariant inner product for such an action; moreover the inner product is unique up to constant if the representation is irreducible. As we will see later in Lemma 2.3.4 the pair $(\mathrm{Lie}(G), \theta)$ is an OSLA (see Definition 2.3.2), in particular the restriction to $\mathfrak{p}$ of the Killing form of $\mathfrak{g}$ is a definite bilinear form. □

Clearly this lemma does not hold without the hypothesis of irreducibility: if g the metric of the reducible hermitian symmetric space $M := M_1 \times M_2$ satisfies the lemma, then for any positive constant $c \in \mathbb{R}_{>0}$, $\tilde{g} := g_{|_{M_1}} \times cg_{|_{M_2}}$ is a metric for which the lemma does not hold.
As it is commonly done in the literature we adopt the following convention:

Definition 2.2.6. The metric of an hermitian symmetric domain (M,J,g) is said to be normalised if the equality in Theorem 2.2.5 holds for $c = 1/2$, that is

$$\forall X, Y \in T_pM,\ g_p(X,Y) = 1/2B|_{\mathfrak{p}}(X,Y).$$

with the notation of Theorem 2.2.5.

The complex structure We pass now to analyse the complex structure of an irreducible hermitian symmetric space (M, J, g). Let $x \in M$ be a fixed point and $K := \mathrm{Stab}(x) \subseteq G := \mathrm{Aut}(M)^\circ$ be its stabiliser. As we have already seen for the metric, also the complex structure of M is completely determined by its value on the tangent space of a point and the action of G. Indeed, given the K-invariant complex structure on T_xM, the complex structure on the all of M can be recovered via translation of elements of G. Indeed, for all $y \in M$, J_y is equal to

$$\begin{aligned} J_y : T_yM &\to T_yM \\ Y &\mapsto df J_x df^{-1}(Y) \end{aligned}$$

where $f \in G$ and $f(x) = y$. Thanks to Proposition VIII.4.2 of [Hel78], given a K-invariant almost complex structure on T_xM, its unique G-invariant extension to the all of M is actually integrable and gives a complex structure. Therefore the complex structure of M is uniquely determined by a complex structure J_x on T_xM, that is K-invariant and compatible with the metric. In virtue of Lemma 2.2.5 and the isomorphism $\mathfrak{p} \simeq T_xM$ the latter conditions can be rewritten as:

$$\forall X \in \mathfrak{k}, Y \in \mathfrak{p} \;\; J_x[X,Y] = [X, J_xY] \tag{2.2.7}$$

$$\forall X, Y \in \mathfrak{p} \;\; B|_{\mathfrak{p}}(J_xX, J_xY) = B|_{\mathfrak{p}}(X,Y) \tag{2.2.8}$$

Lemma 2.2.7. *If J_x is an almost complex structure on T_xM satisfying the identity (2.2.7), then (2.2.8) is equivalent to the following:*

$$\forall X, Y \in \mathfrak{p} \;\; [J_xX, J_xY] = [X,Y] \tag{2.2.9}$$

Proof. For all $X, Y \in \mathfrak{p}, Z \in \mathfrak{k}$ we have:

$$\begin{aligned} B|_{\mathfrak{k}}([X,Y] - [J_xX, J_xY], Z) &= B(Y,[Z,X]) - B(J_xY,[Z,J_xX]) \\ &= B(J_xY, J_x[Z,X]) - B(J_xY,[Z,J_xX]) \\ &= B|_{\mathfrak{p}}(J_xY, J_x[Z,X] - [Z,J_xX]) \end{aligned}$$

where the first equality follows by linearity and by the well-known identity

$$C([a,b],c) = C(a,[b,c]) = C(b,[c,a])$$

which holds for any elements a, b, c of a Lie algebra with Killing form C. The se-cond equality follows by condition 2.2.8 and the third one by linearity. Finally the last expression is identically zero because of 2.2.7. Since the Killing form of $\mathfrak{g} = \mathrm{Lie}(G)$ is non-degenerate by semisimplicity of G, the restriction $B|_{\mathfrak{p}}$ is definite as observed in Lemma 2.2.5, hence $B|_{\mathfrak{k}}$ is non-degenerate and the claim follows. □

The conditions 2.2.8 and 2.2.9 show that J_x can be extended to a derivation of $\mathfrak{g}$ via the following rule:

$$J := \begin{cases} 0 & \text{on } \mathfrak{k} \\ J_x & \text{on } \mathfrak{p} \end{cases}$$

Since all the derivations of a semisimple algebra are inner, there exists a unique $H \in \mathfrak{g}$ such that $J = \mathrm{ad}_{\mathfrak{g}}(H)$. Moreover, since $\mathfrak{p}$ and $\mathfrak{k}$ are eigenspaces for θ one has that hat $\theta J = J\theta$, hence for all $a \in \mathfrak{g}$:

$$[\theta H, \theta a] = \theta([H,a]) = \theta J(a) = J\theta(a) = [H, \theta a] \quad \text{and so} \quad H \in \mathrm{Eig}(\theta, 1) = \mathfrak{k}$$

Furthermore, from $\mathrm{ad}(H)|_{\mathfrak{k}} = J|_{\mathfrak{k}} = 0$ it follows that H lies in the center of $\mathfrak{k}$.
Given a Lie algebra $\mathfrak{g}$ we denote by $\mathfrak{z}(\mathfrak{g})$ its centre.

Corollary 2.2.8. *Let M be an irreducible hermitian symmetric space. Let $G = \mathrm{Aut}(M)^\circ$ and $K = \mathrm{Stab}(p) \leq G$ for $p \in M$ as above. Then there exists a unique $H \in \mathfrak{z}(\mathfrak{k})$ such that $ad(H)_{|_{\mathfrak{p}}} = J_p$.*
Moreover, H and $-H$ are the unique elements in $\mathfrak{g}$ such that:

$$\mathrm{ad}(\pm H)|_{\mathfrak{k}} = 0 \text{ and } \mathrm{ad}(\pm H)|_{\mathfrak{p}} \text{ is an almost complex structure.}$$

Proof. The first part follows from the discussion above. For the second part: The inclusion $[\mathfrak{p},\mathfrak{p}] \subseteq \mathfrak{k}$ (2.2.6) shows that any element satisfying the two conditions lie in $\mathfrak{z}(\mathfrak{k})$. By Theorem 2.1.9 it follows that the connected component of the centre of K is isomorphic to S^1, in particular $\mathfrak{z}(\mathfrak{k}) = \mathbb{R}$ and since the representation of $\mathfrak{k}$ on $\mathfrak{p}$ is faithful the claim follows. □

We now come to the key point of this chapter, that is how to construct a hermitian symmetric domain as the quotient of two Lie groups.
Finally, we sum up all our considerations in the following:

Theorem 2.2.9. *Let (M,J,h) be an irreducible hermitian symmetric space with normalised metric. Let $G = \mathrm{Aut}(M)^\circ$ and $K = \mathrm{Stab}(p) \leq G$ for $p \in M$ be as above. Let $L \in \mathfrak{g}$ be as in the second statement of Corollary 2.2.8 and $\mathfrak{p} = \mathrm{Eig}(\theta, -1)$ as in the discussion following Theorem 2.2.2.*
Then the triple (M',J',h') where:

- *$M' := G/K$, so that $\mathfrak{p} \simeq T_{[e]}(G/K)$.*
- *J' is the unique complex structure on M' for which G acts as biholomorphisms and $J'_{[e]} = \mathrm{ad}(H)|_{\mathfrak{p}}$ for an element $H \in \mathfrak{g}$ such that $\mathrm{ad}(H)|_{\mathfrak{k}} = 0$ and $\mathrm{ad}(H)|_{\mathfrak{p}}$ is an almost complex structure.*
- *h' is the unique metric for which G acts as isometries and $h'_{[e]} = K|_{\mathfrak{p}}$.*

is a hermitian symmetric domain isomorphic to (M,J,h) *or* $(M,-J,h)$.

Remark 2.2.10. In the wake of this analysis of complex structure and metric, it is tempting to conclude that with some more work one can prove that there is a bijection (or an equivalence[3] of categories, once one introduces an appropriate class of morphisms) between the irreducible hermitian symmetric spaces and the pairs of groups described in Theorem 2.1.9, the association being $M \mapsto (G := \mathrm{Aut}(M)^\circ, K)$. Nevertheless that is not part of our statement, and despite a significative investment of time in trying to track the proof of such a statement in the vast literature the author has not found neither that nor a way to prove the statement. In particular a treatment of morphisms of compact hermitian symmetric spaces is not to be found in the literature and it is unclear if the properties of the pair of groups of Theorem 2.1.9 characterise uniquely $\mathrm{Aut}(M)^\circ$ and $\mathrm{Stab}(p)$. It is though interesting to have a look at part of the proof of Theorem 2.1.9 contained in [Hel78]. We describe how to associate to an irreducible hermitian space M a pair of groups with the properties above. This works in both the compact and the non-compact case. Let $G := \mathrm{Aut}(M)^\circ$ be the connected component of the group of holomorphic isometries, and K the stabiliser subgroup of a point in M. Let $\pi\colon \tilde{G} \to G$ be the universal covering group of G, then the pair $(\overline{G} := Ad_{\tilde{G}}(\tilde{G}),\ \overline{K} := Ad_{\tilde{G}}(\pi^{-1}(K))$ is a pair as in the theorem and there are diffeomorphisms $M \simeq G/K \simeq \tilde{G}/\pi^{-1}(K) \simeq \overline{G}/\overline{K}$. The groups fit in the diagram of finite surjective morphisms:

$$\begin{array}{ccc} \tilde{G} & \longrightarrow & \overline{G} \\ & \searrow & \downarrow \\ & & G \end{array}$$

where $\overline{G}$ is adjoint and simple. Therefore all the three groups in the diagram have the same Lie algebra and hence all of them are simple.

2.3 The dual of an irreducible hermitian symmetric domain

In this section we see how to associate to an irreducible hermitian symmetric space of non-compact type a compact one called its dual. In all of this, Lie theory plays a major role.

Remark 2.3.1 (Compact algebras). We recall that a real semi-simple Lie algebra has negative definite Killing form if and only if it is the Lie algebra of a semi-simple compact real Lie group [4] (see [Hel78], Proposition II.6.6). In light of this, we say that a semi-simple real Lie algebra is compact if its Killing form is negative definite.

[3] We point to [Sat80], end of Paragraph II.8, for a result of this kind, showing an equivalence between the category of irreducible hermitian symmetric domains (and appropriate morphisms) and a category of certain semi-simple Lie algebras.

[4] A Lie group is said to be semi-simple if its Lie algebra is. In this way, we exclude tori by our discussion on compactness. See [Hel78], Corollary II.6.7 for something more general.

In the following we make also use of the notion of a compactly embedded subalgebra. Given a Lie algebra $\mathfrak{g}$, the group $\mathrm{GL}(\mathfrak{g})$ of automorphisms of the vector space $\mathfrak{g}$ has Lie algebra $\mathfrak{gl}(\mathfrak{g})$. The image of the adjoint morphism ad_g of $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$. The subgroup of $\mathrm{GL}(\mathfrak{g})$ with Lie algebra $\mathrm{ad}(\mathfrak{g})$ is called the adjoint group of $\mathfrak{g}$ and denoted $\mathrm{Ad}(\mathfrak{g})$.

Definition 2.3.2. Given a Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{k} \subseteq \mathfrak{g}$ is said to be compactly embedded if the subgroup $K \subseteq \mathrm{Ad}(\mathfrak{g})$ with Lie algebra $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$ is compact.

We start by introducing a class of Lie algebras of significative importance in the study of Riemannian and hermitian symmetric spaces. Since we are interested just in non-euclidean symmetric spaces we restrict to algebras corresponding to them[5]:

Definition 2.3.3. An Orthogonal Symmetric Lie Algebra, from now on referred to as an OSLA, consists of a pair $(\mathfrak{g}, \theta)$ where $\mathfrak{g}$ is a semisimple symmetric Lie algebra, θ is an involution and the subalgebra $\mathfrak{k}$ of fixed points of θ is compactly embedded.
Let $\mathfrak{p} := \mathrm{Eig}(\theta, -1)$ be the eigenspace of θ with respect to -1, then $\mathfrak{g}$ decomposes $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The OSLA $(\mathfrak{g}, \theta)$ is said to be:

- irreducible if 0 is the only ideal of $\mathfrak{g}$ contained in $\mathfrak{k}$ and $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$ acts irreducibly on $\mathfrak{p}$;
- of compact type if $\mathfrak{g}$ is compact[6];
- of non-compact type if $\mathfrak{g}$ is non-compact and the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$.

Lemma 2.3.4. *Let M be a non-euclidean hermitian symmetric space, $G = \mathrm{Aut}(M)^\circ$ and θ be the automorphism of $\mathrm{Lie}(G)$ induced by the symmetry s_p of a point $p \in M$. Then the pair $(\mathrm{Lie}(G), \theta)$ is an OSLA.*
Moreover, if M is either of compact type or of non-compact type, so is the OSLA $(\mathrm{Lie}(G), \theta)$.

Proof. By Corollary 2.2.1, if M is non-euclidean, then G is semisimple. Since $s_p^2 = \mathrm{id}$ we have that $\theta^2 = 1$ and by Lemma 2.1.4 $\theta \neq \mathrm{id}$, so it is an involution. We are left to show that $\mathfrak{k}$ is compactly embedded. For this, let $\tilde{G}$ be the unique connected subgroup of $\mathrm{GL}(\mathfrak{g})$ such that $\mathrm{Lie}(G) \simeq \mathfrak{g}$, we have that $\tilde{G} = \mathrm{Ad}_G(G)$: Indeed, $\mathrm{Ad}_G(G)$ is a subgroup of $\mathrm{GL}(\mathfrak{g})$, it is connected because image of the connected group G, and by semisimplicity $\mathrm{Lie}(\mathrm{Ad}(G)) \simeq \mathfrak{g}$. Let K be the stabiliser of p in G. Then $\mathrm{Ad}_G(K)^\circ$ is a connected subgroup of $\tilde{G}$; it is compact because of compactness of K and by (2.2.1) it follows $\mathrm{Lie}(K) \simeq \mathrm{Lie}((G^\sigma))^\circ \simeq \mathfrak{g}^\theta \simeq \mathfrak{k}$. This shows that $(\mathfrak{g}, \theta)$ is an OSLA.
The second part of the statement follows from [Hel78] Proposition VIII.4.4 and Theorem V.1.1 □

[5] compare with [Hel78] page 229
[6] see Remark 2.3.1

Given an OSLA $(\mathfrak{g}, \theta)$, we consider the following construction. Let $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C}$ be the complexification, we set $\mathfrak{g}^* := \mathfrak{k} \oplus i\mathfrak{p}$ and we consider it with the algebra structure inherited by $\mathfrak{g}_{\mathbb{C}}$: as already shown in the previous chapter it is easy to see that $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$ hence the the Lie bracket of $\mathfrak{g}_{\mathbb{C}}$ restricts to $\mathfrak{g}^*$. We set also:

$$\begin{aligned} \theta^* : \mathfrak{g}^* &\to \mathfrak{g}^* \\ (k, p) &\mapsto (k, -p) \end{aligned}$$

for elements $k \in \mathfrak{k}$ and $p \in \mathfrak{p}$.

Definition 2.3.5. Given an OSLA $(\mathfrak{g}, \theta)$, the pair $(\mathfrak{g}^*, \theta^*)$ is called its dual and it is denoted $(\mathfrak{g}, \theta)^*$.

It is clear from the definition that taking dual is an involution, in the sense that the double dual of an OSLA is canonically isomorphic to itself. Furthermore, since for any OSLA $(\mathfrak{g}, \theta)$ the representation $\mathrm{ad}_{\mathfrak{g}} : \mathfrak{k} \to \mathfrak{p}$ is isomorphic to the representation $\mathrm{ad}_{\mathfrak{g}^*} : \mathfrak{k} \to i\mathfrak{p}$, any OSLA $(\mathfrak{g}, \theta)$ is irreducible if and only if its dual is.

Lemma 2.3.6 ([Hel78] Proposition V.2.1)**.** *If $(\mathfrak{g}, \theta)$ is an OSLA, then the dual $(\mathfrak{g}, \theta)^*$ is again an OSLA. Moreover, if $(\mathfrak{g}, \theta)$ is of compact type, the dual is of non-compact type and viceversa.*

Lemma 2.3.7. *If $(\mathfrak{g}, \theta)$ is an irreducible OSLA and $H \in \mathfrak{z}(\mathfrak{k})$ is an element such that $\mathrm{ad}(H)|_{\mathfrak{p}}^2 = -id$, then $\mathfrak{g}$ is simple.*
In particular, if M is an irreducible hermitian symmetric space then $\mathrm{Aut}(M)^\circ$ is simple.

Proof. Indeed, as pointed out in the Remark 2.2.10, this can be deduced by the proof of Theorem VIII.6.1 [Hel78]. Another way to see that is to look at Proposition 5.5 and Theorems 5.3 in Chapter VIII of [Hel78]. For this, one can note that the irreducible Riemannian symmetric spaces of compact type give rise to OSLA's of type I and II in the sense of *loc.cit.*; the ones of type I are simple by definition while the ones of type II cannot be associated to a hermitian space: any such does not contain any non-trivial element in $\mathfrak{z}(\mathfrak{k})$ (see *op.cit.* p.518), whose existence is guaranteed by hermitianity. □

With this last proposition we finally see how to get a compact irreducible space out of a non-compact irreducible hermitian symmetric space, which will be called its dual. For this, let $(\mathfrak{g}, \theta)$ be an OSLA of compact type with $\mathfrak{g}$ simple. We consider the unique connected group G in $\mathrm{GL}(\mathfrak{g})$ whose Lie algebra is isomorphic to $\mathfrak{g}$.

Proposition 2.3.8. *Let $(\mathfrak{g}, \theta)$ be an irreducible OSLA of compact type and let $H \in \mathfrak{z}(\mathfrak{k})$ be such that $\mathrm{ad}(H)|_{\mathfrak{p}}^2 = -id|_{\mathfrak{p}}$. Set G to be the unique connected group in $\mathrm{GL}(\mathfrak{g})$ whose Lie algebra is isomorphic to $\mathfrak{g}$ and K to be the unique Lie group in $\mathrm{GL}(\mathfrak{g})$ whose Lie algebra is isomorphic to $\mathfrak{k}$. Let B denote the Killing form of $\mathfrak{g}$. Then the triple:*

$$(G/K,\ B|_{\mathfrak{p}},\ ad(H)|_{\mathfrak{p}}) \text{ is an irreducible hermitian symmetric space of compact type.}$$

Proof. Let G be the unique connected Lie group in $\mathrm{GL}(\mathfrak{g})$ such that $\mathrm{Lie}(G) \simeq \mathfrak{g}$ and let K be the unique connected Lie subgroup of G whose Lie algebra is isomorphic to $\mathfrak{k}$. Thanks to Lemma 2.3.7 G is simple. Thanks to Proposition VIII.5.1 of [Hel78] the irreducibility of $(\mathfrak{g}, \theta)$ implies that the Lie algebra $\mathfrak{k}$ is a proper maximal subalgebra of $\mathfrak{g}$, so K is a maximal proper connected subgroup of G. The existence of H guarantees the non-triviality of the center of K and Theorem 2.1.9 is therefore applicable. The restriction of the Killing form to $\mathfrak{p} \simeq T_{[e]}(G/K)$ defines a positive bilinear form which can be extended to a G-invariant metric on the manifold G/K and analogously $\mathrm{ad}(H)|_{\mathfrak{p}}$ defines a compatible G-invariant complex structure. Hence the group G acts transitively via holomorphic isometries. Finally, the fact that $ad(H)^2|_{\mathfrak{p}} = -\mathrm{id}_{T_{[e]}(G/K)}$ assure thanks to Lemma 2.1.4 the existence of the symmetry. □

In light of the above discussion, we introduce the notion of dual space of a hermitian symmetric domain. Let (M, J, h) be an irreducible hermitian symmetric space of non-compact type and let $p \in M$ be a point. Let $G := \mathrm{Aut}(M)^\circ$ and θ be the endomorphism of $\mathfrak{g} := \mathrm{Lie}(G)$ induced by the symmetry s_p. Thanks to Lemma 2.3.4 the pair $(\mathfrak{g}, \theta)$ is an OSLA of non-compact type. By Lemma 2.3.6 its dual $(\mathfrak{g}, \theta)^*$ is an OSLA of compact type and Proposition 2.3.8 associates to it an irreducible hermitian space of compact type.

Definition 2.3.9. Given an irreducible hermitian symmetric space of non-compact type (M, J, h), we denote by $M^\vee$ the irreducible hermitian symmetric domain of compact type associated to it via the construction described above.

2.4 The domain D_L and its dual

The irreducible hermitian symmetric domains were classified first by Cartan after he proved the statements in Theorem 2.1.9 and 2.2.9: they are divided in six types, a classification can be found in [Hel78] section X.6. We are interested in the so called domains of type IV, as we may show the space D_L, introduced in the beginning of Chapter 2 is one of those. Let $\mathrm{SO}(n, 2)$ be the group of real invertible matrices preserving the standard bilinear symmetric form of signature $(n, 2)$:

$$\mathrm{SO}(n,2) = \{M \in \mathrm{GL}(n, \mathbb{R}) |\ M^t I_{n,2} M = I_{n,2}\} \text{ where } I_{n,2} = \begin{pmatrix} I_n & 0 \\ 0 & -I_2 \end{pmatrix},$$

It has two connected components and we denote by $\mathrm{SO}(n, 2)^\circ$ the connected component of the identity. The hermitian symmetric domain of type IV of dimension n, denoted D_{IV_n}, is constructed as the quotient:

$$\mathrm{SO}(n,2)^\circ / (\mathrm{SO}(n) \times \mathrm{SO}(2))$$

where[7] $2 \neq n \geq 1$ equals the dimension of the domain D_{IV_n}. In particular, the domain D_{IV_n} can be described as a Grassmannian variety. Indeed, let V be a real space equipped with a symmetric bilinear form S of signature $(n,2)$. We consider the Grassmannian of 2-dimensional oriented planes E, on which the restriction of S is negative definite (we write $S|_E \prec 0$):

$$Gr_{no}(2,V) = \{(E,a) : E \subseteq V,\ \dim(E) = 2,\ S|_E \prec 0,\ a \in (\Lambda^2 E \setminus 0)/\mathbb{R}^*_{>0}\}.$$

Let $O(V,S)$ the orthogonal group. As stated in [Sat80], Appendix 6 it has 4 connected components. Indeed choosing a basis $e_1,...,e_n,e_{n+1},e_{n+2}$, for which the space $\langle e_1,...,e_n\rangle$, respectively $E := \langle e_{n+1},e_{n+2}\rangle$, is positive, respectively negative, definite gives an isomorphism with $\mathbb{R}^{n,2}$, the real vector space equipped with the standard bilinear form of signature $(n,2)$ and in turn an isomorphism between $O(V,S)$ and $O(n,2)$. For any $g \in O(V,S)$ the composition with the inclusion of and the orthogonal projection onto E gives a morphism:

$$g_E : E \hookrightarrow V \xrightarrow{g} V \to E.$$

The morphisms $g \in O(V,S)$ with $\det(g_E) > 0$ preserve the orientation of E. The connected components of $O(V,S)$ consist of:

$$\begin{aligned} C_0 &:= \{g \in O(V,S) : \det(g) > 0,\ \det(g_E) > 0\} \\ C_1 &:= \{g \in O(V,S) : \det(g) > 0,\ \det(g_E) < 0\} \\ C_2 &:= \{g \in O(V,S) : \det(g) < 0,\ \det(g_E) > 0\} \\ C_3 &:= \{g \in O(V,S) : \det(g) < 0,\ \det(g_E) < 0\} \end{aligned}$$

The first two sets form the group $\mathrm{SO}(n,2)$, which has 2 connected components. The group $\mathrm{SO}(n,2)$ acts transitively on Gr_{no} and the stabiliser of the point corresponding to E (equipped with the orientation given by the class of $e_{n+1} \wedge e_{n+2}$) is the compact and connected group $\mathrm{SO}(n) \times \mathrm{SO}(2)$. Thus Gr_{no} has 2 connected components and letting $\mathrm{SO}(n,2)^\circ$ act on E gives an isomorphism of $\mathrm{SO}(n,2)^\circ/(\mathrm{SO}(n) \times \mathrm{SO}(2))$ with one of its connected components which is therefore identified with D_{IV_n}. Finally, the group of holomorphic isometries of D_{IV_n} is:

$$\mathrm{Aut}(D_{IV_n}) = (C_0 \cup C_2)/\{\pm \mathrm{id}\},$$

while the identity connected component is

$$\mathrm{Aut}(D_{IV_n})^\circ = C_0/\mathrm{Z}(\mathrm{C}_0).$$

So, when n is even, the identity connected component is a subgroup of index 2.

[7] notice that for $n = 2$ one gets the space $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ which is not irreducible (see for instance [Wol72], page 353).

Following [Sat80] we present another incarnation of the domain D_{IV_n} for which the complex structure is more evident. For this, we consider the complexification $V \otimes \mathbb{C}$ and the space of 1-dimensional complex subspaces that are isotropic for the extension of S to $V \otimes \mathbb{C}$ and negative definite for the associated hermitian form:

$$\tilde{D} = \{[w] \in \mathbb{P}(V \otimes \mathbb{C}) : S(w,w) = 0 \text{ and } S(\bar{w},w) < 0\}.$$

Some important trivialities. First of all notice that the set $\tilde{D}$ is well-defined, indeed $\forall \lambda \in \mathbb{C}^*$:

- $S(\lambda w, \lambda w) = \lambda^2 S(w,w)$ so that for every $v \in [w] \in \mathbb{P}(V \otimes \mathbb{C})$ it holds
$$S(w,w) = 0 \Leftrightarrow S(v,v) = 0;$$
- $S(\lambda w, \overline{\lambda w}) = |\lambda|^2 S(w,\bar{w})$ with $|\lambda|^2 \in \mathbb{R}_{>0}$ so that for every $v \in [w] \in \mathbb{P}(V \otimes \mathbb{C})$ we have
$$S(w,w) > 0 \Leftrightarrow S(v,v) > 0.$$

The following shows that $\tilde{D}$ is the Grassmannian Gr_{no} in disguise:

Proposition 2.4.1. *The association:*

$$\begin{aligned} \varphi : \tilde{D} &\to Gr_{no} \\ [w] &\mapsto (\langle \mathrm{Re}(w), \mathrm{Im}(w) \rangle, [\mathrm{Re}(w) \wedge \mathrm{Im}(w)]) \end{aligned}$$

defines a diffeomorphism.

Proof. We consider the real structure of $V \otimes \mathbb{C}$ given by the inclusion of V and we write $w = w_1 + iw_2$ for elements $w \in V \otimes \mathbb{C}$ and $w_1, w_2 \in V$. If w represents an element in $\tilde{D}$ we have:

$$\begin{aligned} 0 = S(w,w) &= S(w_1,w_1) - S(w_2,w_2) + 2iS(w_1,w_2), \\ 0 > S(w,\bar{w}) &= S(w_1,w_1) + S(w_2,w_2). \end{aligned}$$

In particular the plane $\langle w_1, w_2 \rangle$ is negative definite.
Clearly if $\lambda \in \mathbb{C}^*$ the elements w and λw define the same plane. In order to conclude that φ is well-defined, it is then sufficient to notice that multiplication by a complex number is a conformal map, in particular it defines an orientation preserving morphism on the plane defined by w.
It is also clear that the following association defines the inverse morphism of φ: a negative definite oriented plane $E \subseteq V$ maps to the class of $v_1 + iv_2$ in $\mathbb{P}(V \otimes \mathbb{C})$, where $\{v_1, v_2\}$ is a consistently oriented orthonormal basis of E. □

Some other triviality: the domain D_L is of type IV. Indeed, if L is a non-degenerate lattice of signature $(2,n)$ for an integer $n \geq 3$, we denote by T the bilinear form on $W \otimes \mathbb{R}$, then $S := -T$ is a symmetric bilinear form of signature $(n,2)$. Moreover, for every $w \in W \otimes \mathbb{C}$ it holds:

- $T(w,w) = 0$ if and only if $S(w,w) = 0$;
- $T(w,w) > 0$ if and only if $S(w,\bar{w}) < 0$.

This shows that the set

$$D_W := \{[w] \in \mathbb{P}(W \otimes \mathbb{C}) | T(w,w) = 0 \text{ and } T(\bar{w},w) > 0\}$$

actually coincides with the realisation of a type IV domain given by Satake and can therefore be identified with the quotient $\mathrm{SO}(n,2)^\circ/(\mathrm{SO}(n) \times \mathrm{SO}(2))$. As D_L is isomorphic as Hermitian symmetric domain to the realisation proposed by Satake as Grassmannian or as domain in $\mathbb{P}^{n+1}_{\mathbb{C}}$, we choose the representation we prefer for studying its structure, but a remark is in order: toroidal compactifications, as we explain in the next chapters, depend on a rational structure of the automorphism group $\mathrm{Aut}(D_{IV})^\circ$ and we will have to fix a such. We determine now what the compact dual of D_{IV} introduced in Definition 2.3.9.
For this, a standard computation shows that:

$$\mathfrak{so}(n,2) := \mathrm{Lie}(\mathrm{SO}(n,2)) = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_2^t & X_3 \end{pmatrix} : \begin{array}{r} X_1 = -X_1^t \in \mathrm{Mat}(n \times n, \mathbb{R}) \\ X_3 = -X_3^t \in \mathrm{Mat}(2 \times 2, \mathbb{R}) \\ X_2 \in \mathrm{Mat}(n \times 2, \mathbb{R}) \end{array} \right\}$$

Moreover, the association:

$$\begin{aligned} \theta : \mathfrak{so}(n,2) &\to \mathfrak{so}(n,2) \\ X &\mapsto I_{n,2} X I_{n,2} \end{aligned}$$

defines an involution which fixes the subalgebra:

$$\left\{ X \in \mathfrak{so}(n,2) : X = \begin{pmatrix} X_1 & 0 \\ 0 & X_3 \end{pmatrix} \right\}$$

that is the Lie algebra of the subgroup $\mathrm{SO}(n) \times \mathrm{SO}(2) \leq \mathrm{SO}(n,2)$.
In order to determine the compact dual in the way we presented it in Section2.3 we take a different perspective to describe $\mathfrak{so}(n,2)$, which reflects the realisation of D_{IV_n} as complex subspace of $\mathbb{P}^{n+1}_{\mathbb{C}}$ opposed to realisation inside the Grassmannian of oriented planes. For this, we consider the hermitian form defined on $\mathbb{C}^{n+2}$ defined by:

$$\begin{aligned} h : \mathbb{C}^{n+2} \times \mathbb{C}^{n+2} &\to \mathbb{C} \\ (x,y) &\mapsto x^t I_{n,2} \bar{y} \end{aligned}$$

and we denote with $\mathrm{U}(n,2) \leq \mathrm{GL}(n+2,\mathbb{C})$ the subgroup of matrices preserving h:

$$\mathrm{U}(n,2) := \{g \in \mathrm{GL}(n+2,\mathbb{C}) : \ g^t I_{n,2} g = I_{n,2}\}$$

Moreover, we consider the group $G :=\mathrm{U}(n,2) \cap \mathrm{SO}(n+2,\mathbb{C})$. A standard computation allows to determine its Lie algebra:

$$\begin{aligned}\mathfrak{g} &= \left\{M \in \mathfrak{gl}(n+2,\mathbb{C}) : \ M^t = -M \text{ and } \overline{M}^t I_{n,2} = I_{n,2} M\right\} \\ &= \left\{ \begin{pmatrix} X_1 & iX_2 \\ -iX_2^t & X_3 \end{pmatrix} : \begin{array}{r} \bar{X}_1 = X_1,\ X_1^t = -X_1 \\ \bar{X}_3 = X_3,\ X_3^t = -X_3 \\ \bar{X}_2 = X_2 \end{array} \right\}\end{aligned}$$

and we notice that the morphism:

$$\begin{aligned}\mathfrak{so}(n,2) &\to \mathfrak{gl}(n+2,\mathbb{C}) \\ \begin{pmatrix} X_1 & X_2 \\ X_2^t & X_3 \end{pmatrix} &\mapsto \begin{pmatrix} X_1 & iX_2 \\ -iX_2^t & X_3 \end{pmatrix}\end{aligned}$$

gives an isomorphism[8] $\mathfrak{so}(n,2) \simeq \mathfrak{g}$. Under this isomorphism the involution θ corresponds to an involution $\tilde{\theta} : \mathfrak{g} \to \mathfrak{g}$ whose associated eigenspace decomposition $g = \mathfrak{k} \oplus i\mathfrak{p}$ reads:

$$\begin{aligned}\mathfrak{k} &= \left\{X \in \mathfrak{g} : X = \begin{pmatrix} X_1 & 0 \\ 0 & X_3 \end{pmatrix}\right\} \\ \mathfrak{p} &= \left\{X \in \mathfrak{g} : X = \begin{pmatrix} 0 & iX_2 \\ -iX_2^t & 0 \end{pmatrix}\right\}.\end{aligned}$$

The dual algebra $\mathfrak{g}^*$ can be therefore computed straightforwardly:

$$\begin{aligned}\mathfrak{g}^* &= \left\{X \in \mathfrak{gl}(n+2,\mathbb{R}) : X = \begin{pmatrix} X_1 & -X_2 \\ X_2^t & X_3 \end{pmatrix}\ X_1 = -X_1^t,\ X_3 = -X_3^t\right\} \\ &= \{M \in \mathfrak{gl}(n+2,\mathbb{R}) : M^t = -M\},\end{aligned}$$

that is the algebra $\mathfrak{g}^*$ is isomorphic to $\mathfrak{so}(n+2)$ and the compact dual can be constructed as the quotient $D_{IV_n}^\vee \simeq \mathrm{SO}(n+2)/(\mathrm{SO}(n)\times\mathrm{SO}(2))$. In this way, $D_{IV_n}^\vee$ can be identified with the Grassmannian of oriented planes of an $n+2$-dimensional real vector space equipped with a definite bilinear form.

So far we have been imprecise about the structure of $D_{IV_n} \simeq \mathrm{SO}(2,n)^\circ/(\mathrm{SO}(n)\times\mathrm{SO}(2))$: as exploited in Theorem 2.2.9, the complex structure making such a quotient a complex manifold is not uniquely determined, but there are two possible choices. With the content of next section we describe the compact dual in another form and we give an embedding $D_{IV_n} \hookrightarrow D_{IV_n}^\vee$ which will make the complex structures of both spaces clearer.

[8] The isomorphism of Proposition 2.4.1 gives a natural action of the group G on D_{IV_n}, this isomorphism shows that up to restricting to the connected component of the identity and quotienting by the center, G is the group of holomorphic isometries of D_{IV_n}.

2.5 Borel and Harish-Chandra embeddings

Let X_0 be a hermitian symmetric space of non-compact type. In this section, we review the Harish-Chandra and Borel embedding through which X can be embedded in the tangent space at one of its points and respectively in its compact dual, say X. Let $G = \mathrm{Aut}(X_0)^\circ$ and $K = \mathrm{Stab}_G(x)$ be the stabiliser of a fixed point $x \in X$, then we know thanks to Theorem 2.2.2 that there is a diffeomorphism $X_0 \simeq G/K$; as before we write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{k} = \mathrm{Lie}(K)$ and we denote by H the unique element in $\mathfrak{z}(\mathfrak{k})$ such that $\mathrm{ad}(H)_{\mathfrak{p}}$ coincides with the complex structure J_x on $T_xX \simeq \mathfrak{p}$. Moreover, thanks to [Sat80] page 222, if s_x denotes the symmetry at the point x and σ the associated inner automorphism of G, the inclusions in Theorem 2.2.2 reduce actually to the equality $K = G^\sigma$. In particular, we deduce that G is an adjoint group: the center $Z(G)$ is a normal subgroup that lies in the centraliser of σ, therefore $Z(G) \leq G^\sigma = K$ and thanks to Theorem 2.2.2 K has no non trivial proper subgroups. Hence G admits a natural complexification: if $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{g}_C = \mathfrak{g} \otimes \mathbb{C}$, let $G_{\mathbb{C}}$ the only connected subgroup of $\mathrm{GL}(\mathfrak{g}_{\mathbb{C}})$ whose Lie algebra coincides with the image of $\mathfrak{g}_{\mathbb{C}}$ inside $\mathfrak{gl}(\mathfrak{g}_{\mathbb{C}})$ under the adjoint representation $\mathrm{ad} : \mathfrak{g}_{\mathbb{C}} \hookrightarrow \mathfrak{gl}(\mathfrak{g}_{\mathbb{C}})$. Similarly, let $K_{\mathbb{C}}$ be the unique connected subgroup of $\mathrm{GL}(\mathfrak{g}_{\mathbb{C}})$ whose Lie algebra is $\mathrm{ad}(\mathfrak{k} \otimes \mathbb{C})$. Furthermore we consider the eigenspace decomposition of $\mathfrak{p} \otimes \mathbb{C}$ for the complex structure J_x:

$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_- \text{ where } \mathfrak{p}_\pm = \mathrm{Eig}(J_x \otimes \mathbb{C}, \pm i)$$

The relations (2.2.5) and (2.2.6) give the following ones:

Lemma 2.5.1. *Given $\mathfrak{p}_+, \mathfrak{p}_-$, and $\mathfrak{k}_{\mathbb{C}}$ as above the following inclusions hold:*

$$[\mathfrak{k}_{\mathbb{C}}, \mathfrak{p}_\pm] \subseteq \mathfrak{p}_\pm, \qquad [\mathfrak{p}_+, \mathfrak{p}_+] = [\mathfrak{p}_-, \mathfrak{p}_-] = 0, \qquad [\mathfrak{p}_+, \mathfrak{p}_-] \subseteq \mathfrak{k}_{\mathbb{C}}$$

Proof. We show just the first inclusion, the other ones can be derived similarly. Let θ be the involution of $\mathfrak{g}$ induced by the symmetry at the point $x \in X$, by the definition of $\mathfrak{k}$ and $\mathfrak{p}$ for any two elements $k \in \mathfrak{k},\ p \in \mathfrak{p}_+$ we have:

$$\theta([k,p]) = [\theta(k), \theta(p)] = [k, -p] = -[k,p] \text{ i.e. } [k,p] \in \mathrm{Eig}(\theta, -1) = \mathfrak{p}_{\mathbb{C}},$$

while applying the Leibniz rule for the derivation $\delta := \mathrm{ad}(H) \otimes \mathbb{C}$ we get:

$$\delta([k,p]) = [\delta(k), p] + [k, \delta(p)] = 0 + [k, ip] = i[k,p] \text{ i.e. } [k,p] \in \mathfrak{p}_+$$

and the claim follows. □

In particular $\mathfrak{p}_+, \mathfrak{p}_- \subseteq \mathfrak{g}_{\mathbb{C}}$ are abelian subalgebras. Let $K_{\mathbb{C}}, P_+, P_-$ be the unique connected subgroups of $G_{\mathbb{C}}$ with Lie algebras $\mathfrak{k}_{\mathbb{C}}, \mathfrak{p}_+$ and $\mathfrak{p}_-$. From the lemma it follows that $K_{\mathbb{C}}$ acts on $P_\pm$ by conjugation and $P_\pm \rtimes K_{\mathbb{C}}$ is a subgroup of $G_{\mathbb{C}}$. Furthermore, let G_c be the compact group with Lie algebra $\mathfrak{g}^*$ giving the compact dual $X \simeq G_c/K$ of X_0.

Lemma 2.5.2. *Let $G_{\mathbb{C}}, K_{\mathbb{C}}, P_+, P_-, G, G_c$ be as above, then*

- *The exponential map gives an isomorphism:*

$$\exp : \mathfrak{p}_{\pm} \to P_{\pm};$$

- *the multiplication map gives an injection:*

$$m : (P_+ \times P_-) \rtimes K_{\mathbb{C}} \to G_{\mathbb{C}} \text{ such that } G \subseteq \mathrm{Im}(m);$$

- *the following equalities hold:*

$$K = G \cap (K_{\mathbb{C}} P_-) = G_c \cap (K_{\mathbb{C}} P_-)$$

Proof. See [Hel78] Chapter VIII, Lemma 7.8, 7.9, 7.10 □

Hence it follows that there is a diagram with the following morphisms:

$$\begin{array}{ccccccc} X_0 \simeq G/K & \hookrightarrow & m\left((P_+ \times P_-) \rtimes K_{\mathbb{C}}\right) / \left(P_- \rtimes K_{\mathbb{C}}\right) & \hookrightarrow & G_{\mathbb{C}} / \left(P_- \rtimes K_{\mathbb{C}}\right) & \hookleftarrow & G_c/K \simeq X \\ & & \| & & & & \\ & & P_+ & & & & \\ & & \cong \uparrow \exp & & & & \\ & & \mathfrak{p}_+ & & & & \end{array}$$

And actually something more holds:

Theorem 2.5.3 (Harish-Chandra and Borel embedding)**.** *In the above diagram the injection* $X \simeq G_c/K \to G_{\mathbb{C}}/\left(P_- \rtimes K_{\mathbb{C}}\right)$ *is an isomorphism and realises the compact dual as a complex projective variety.*
The morphism obtained as composition in the above diagram:

$$i_{HC} : X_0 \hookrightarrow \mathfrak{p}_+$$

is a holomorphic open embedding and realises X_0 *as a bounded domain.*
The morphism $\iota : \mathfrak{p}_+ \hookrightarrow X$ *has a Zariski open image. The holomorphic injection* $i_B := \iota \circ i_{HC} : X_0 \to X$ *is known as Borel embedding.*

The following class of morphisms between hermitian symmetric domains has a natural extension along the relative Harish-Chandra and Borel embeddings.

Definition 2.5.4. A holomorphic morphism $f : D_1 \to D_2$ between two hermitian symmetric domains is said to be symmetric if for each $x \in D_1$ the equality:

$$f \circ s_x = s_{f(x)} \circ f$$

holds where s_x denotes the symmetry at the point x.

Lemma 2.5.5. *Let (D_1, x_1) (D_2, x_2) be two pointed hermitian symmetric domains and let $f : D_1 \to D_2$ be a symmetric morphism mapping x_1 to x_2, then f extends naturally to holomorphic morphisms of the corresponding Harish-Chandra and Borel embedding:*

$$\begin{array}{ccc} D_1 & \xrightarrow{f} & D_2 \\ {\scriptstyle i_{HC}}\downarrow & & \downarrow{\scriptstyle i_{HC}} \\ \mathfrak{p}_{1,+} & \longrightarrow & \mathfrak{p}_{2,+} \\ {\scriptstyle i_B}\downarrow & & \downarrow{\scriptstyle i_B} \\ D_1^\vee & \longrightarrow & D_2^\vee. \end{array}$$

Proof. See [AMRT10] Paragraph III.2.2. □

Borel and Harish-Chandra embeddings of D_{IV_n} We have already seen that the compact dual of D_{IV_n} can be realised as the quotient of real groups $D_{IV_n}^\vee = \mathrm{SO}(n+2)/(\mathrm{SO}(n) \times \mathrm{SO}(2))$. Thanks to Theorem 2.5.3 of the previous section this can also be described as a quotient of the complex Lie group, namely the complexification[9] $\mathrm{SO}(n+2)_{\mathbb{C}} = \mathrm{SO}(n+2, \mathbb{C})$. Moreover, if V is a real vector space equipped with a non-degenerate bilinear form S of signature $(n, 2)$ and $Q = \{[w] \in \mathbb{P}(V \otimes \mathbb{C}) : S(w, w) = 0\}$ denotes the quadric of isotropic lines in $V \otimes \mathbb{C}$, the proof of Proposition 2.4.1 shows that the association

$$\begin{aligned} \varphi : Q &\to Gr_o(2, V) \\ [w] &\mapsto (\langle \mathrm{Re}(w), \mathrm{Im}(w) \rangle, [\mathrm{Re}(w) \wedge \mathrm{Im}(w)]) \end{aligned}$$

gives a diffeomorphism, in particular D_{IV_n} is diffeomorphic to Q. Now, let $w \in L \otimes \mathbb{C}$ be a non-trivial isotropic vector representing a point in D_L. We let the complex group $\mathrm{SO}(n+2)_{\mathbb{C}}$ act on it. Its orbit is clearly the whole of Q, we deduce by Theorem 2.5.3 that the complex structure making Q a hermitian symmetric space is the one given by the embedding in $\mathbb{P}(V \otimes \mathbb{C})$, which in turn induces the complex structure on D_L. In conclusion, identifying D_{IV_n} with the a connected component of $\tilde{D}$ we have that the Borel embedding is given by the natural inclusion:

$$D_L \hookrightarrow Q.$$

For completeness we recall also the Harish-Chandra embedding of D_{IV_n}. Our main reference is [Wol72]. Let $\{e_1, ..., e_{n+2}\}$ be a basis of V for which the bilinear form is represented by the matrix $I_{n,2}$ and let D_V be the domain of type IV associated to V, which sits via the Borel embedding in the quadric $Q \subseteq \mathbb{P}(V \otimes \mathbb{C})$. We fix the point $P := [e_{n+1} + ie_{n+2}] \in D_V$. After

[9] we note that $\mathrm{SO}(n+2)_{\mathbb{C}} \simeq \mathrm{SO}(n, 2)$, since symmetric bilinear forms on a complex vector space are isomorphic if and only if they have the same rank.

a suitable identification of $\mathfrak{p} \simeq T_P D_V$ with $\mathbb{C}^n$, the injection of the tangent space of D_V at P in Q is described via the map:

$$\begin{aligned} \zeta : \mathbb{C}^n &\to \mathbb{P}(V \otimes \mathbb{C}) \\ Z &\mapsto \left[\begin{pmatrix} 2Z \\ 1 + Z^t Z \\ i(1 - Z^t Z) \end{pmatrix} \right] \end{aligned}$$

The preimage in $\mathbb{C}^n$ consists of the domain:

$$\{Z \in \mathbb{C}^n : 2\bar{Z}^t Z < 1 + |Z^t Z|^2,\ \bar{Z}^t Z < 1\}$$

and gives the Harish-Chandra embedding in $T_P D_V$.

3

Perfect cone decomposition, singularities and toric varieties

The purpose of this chapter is twofold: On the one side we introduce the perfect cone decomposition, which is the main theme of this work; on the other side we study singularities of toric varieties and of their quotients. The ingredient common to both are cones. In Section 3.1 we define the perfect cone decomposition, we will see in Chapter 4 how this can be used to define a toroidal compactification of any arithmetic quotient of any hermitian symmetric domain. In Section 3.2 we review certain singularities that are of importance in the Minimal Model Program. Section 3.3 is devoted to prove that the singularities of toric varieties coming from the perfect cones are canonical. Finally, in Section 3.4 we focus on the singularities of quotients of toric varieties, these give a local description of the boundary of toroidal compactifications.
We conclude this introduction with a result that builds the bridge between this chapter, which focuses on cones, with the remaining ones, which treat hermitian symmetric spaces. A cone in a real finite-dimensional vector space V is a set closed under multiplication by positive scalars. Let $C \subseteq V \setminus \{0\}$ be an open cone whose closure does not contain any line, we denote by $\overline{C}^* \subseteq V^*$ the linear forms that are non-negative on C; we define the dual cone C^* to be the interior of $\overline{C}^*$. We say that a cone is self-adjoint if there is a positive definite form on V whose induced isomorphism between V and V^* transforms C in C^*. A self-adjoint cone is necessarily convex. We say that C is homogeneous if the group $\mathrm{Aut}(V, C)$ of automorphisms of V preserving the cone C acts transitively on C. Self-adjoint homogeneous cones are of importance to us because of the following

Theorem 3.0.1. *Let W be an affine space whose translation space is a real vector space V and let $C \subseteq V$ be a homogeneous self-adjoint cone. Then the open subset*

$$U := W + iC = \{x + iy : x \in W,\ y \in C\}$$

of $W \otimes \mathbb{C}$ is a hermitian symmetric space of non-compact type. The group of automorphisms that preserve the closure of U in the natural topology of $W \otimes \mathbb{C}$ is

$$G = V \ltimes \mathrm{Aut}(V, C)$$

where V acts by real translation and an element $g \in \mathrm{Aut}(V, C)$ acts via its complex-linear extension.

Proof. We refer to [AMRT10], Proposition 1.2 for a detailed proof of the first statement. The second statement is the content of [PS69] Theorem 1. We just point out that V acts on U by real translations and since C is homogeneous it is easy to show that the complex-linear extensions of elements in $\mathrm{Aut}(C,V)^\circ$ together with V act transitively on U: Indeed, let $x_0+iy_0, x_1+iy_1 \in U$ be two points, by homogeneity of C there exists a morphism $g_1 \in \mathrm{Aut}(C,V)^\circ$ such that $g_1 \cdot y_0 = y_1$, then, if t denotes the translation by $y_0 - g\cdot x_0$, the composite $t \circ g$ maps x_0+iy_0 to x_1+iy_1. □

3.1 The perfect cone decomposition

As already mentioned, a toroidal compactification of an arithmetic quotient of a hermitian symmetric space depends on a certain integral structure of a hermitian symmetric space induced by the arithmetic group considered. The counterpart in the language of cones is given by the following.
Assume now that $V \simeq L \otimes \mathbb{R}$ for a finitely generated free abelian group L, in particular V has a rational structure. Let G be an arithmetic subgroup of the group of automorphisms of V fixing the cone C. Recall that a rational polyhedral cone $\sigma \subseteq V$ is a convex set of the form $\sigma = \mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_2 + ... + \mathbb{R}_{\geq 0}v_n$ for vectors $v_1, ..., v_n \in L \otimes \mathbb{Q}$. We remark that, in contrast to the homogeneous self-adjoint cones that we consider, a rational polyhedral cone is a closed subset of a real vector space.
In the following chapter we explain how to define a toroidal compactification. Here we introduce the necessary ingredients we will use: These consist of a family of fans for certain homogeneous cones C with an action of G which are a G-admissible polyhedral decomposition in the sense of the following

Definition 3.1.1. A G-admissible polyhedral decomposition for the cone C is a family $\Sigma = \{\sigma_\alpha\}$ of rational polyhedral cones $\sigma_\alpha \subseteq \overline{C}$ such that:

- the family Σ is closed under taking faces, i.e. for every $\sigma_\alpha \in \Sigma$, every face of σ_α is a cone in Σ;
- for every pair of cones $\sigma_\alpha,\ \sigma_\beta \in \Sigma$ the intersection is a face of both;
- the group G acts on Σ, i.e. for every element $g \in G$ and for every cone $\sigma_\alpha \in \Sigma$, $g\sigma_\alpha$ is a cone in Σ;
- the family Σ is finite modulo the action of the group G;
- $C = \cup_\alpha(\sigma_\alpha \cap C)$.

The exact data that define a toroidal compactification (essentially, roughly speaking, which cones have to be considered and which compatibility conditions must be satisfied by the various fans) is the subject of the next chapter. Here we recall one of the most important

results of the monograph [AMRT10]. Thanks to this, we know that admissible decompositions, and hence toroidal compactifications (see Section 4.4 for details), exist for any locally symmetric variety.

Proposition 3.1.2 ([AMRT10] Corollary II.5.23)**.** *Let C be a self-adjoint homogeneous cone in a real vector space V with integral structure $L \subseteq V$ and let $G \subseteq \mathrm{Aut}(V, C)$ be an arithmetic subgroup. Taking cones over the faces of the closed convex hull of $\overline{C} \cap L \setminus \{0\}$ yields a G-admissible polyhedral decomposition of C.*

Remark 3.1.3. We note that, as G is arithmetic, it acts on $L \otimes \mathbb{Q}$ and therefore, on the decomposition defined in the Proposition above.

Definition 3.1.4. Given a self-adjoint homogeneous cone C in a real vector space V with integral structure $L \subseteq V$, we refer to the decomposition given above as the perfect cone decomposition $\Sigma_{p,L}(C)$.
In order to make the notation lighter we write just Σ_p if the cone and the integral structures are clear from the context.

As we will see later (4.3.6) the perfect cone decomposition provides the data necessary to define the toroidal compactification that is at the centre of the study of this work, the perfect cone toroidal compactification. Even if at this point we have not yet explained how to define a toroidal compactification starting from a decomposition, we can remark one of its peculiar properties that makes it important: The perfect cone decomposition is defined for every convex homogeneous self-adjoint cone, as a consequence we will see later (4.4.1) that the perfect cone toroidal compactification can be defined for every locally symmetric variety.

3.2 Canonical and klt singularities

Aim of this section is to introduce the notion of singularities that are relevant to us, namely canonical and Kawamata log terminal (klt) singularities. The former were introduced by Reid in [Rei80] and have ever since played an important role in the Minimal Model Program in higher dimension. Among their generalisations to pairs we focus on klt singularities. Indeed we prove that quotients of toric varieties by certain finite groups happen to have klt singularities (see 3.4.7 for a precise statement).
We firstly introduce notions and notations. For this, we recall that if X is a non-singular variety of dimension n, the canonical line bundle $\Omega_X^n := \Lambda^n \Omega_X^1$ is an invertible sheaf. One often refers to any divisor D defining the same class of Ω_X in the Picard group, i.e. a divisor D for which $\mathcal{O}_X(D) \simeq \Omega_X^n$ holds, as the canonical divisor of X and denote it K_X. If the variety X is not regular, then the sheaf Ω_X^n need not be invertible. However, if X is assumed to be at least normal so that the regular locus $j : X^{reg} \hookrightarrow X$ has complement of codimension greater than 2, then the double dual $\Omega_X^{[n]} := (\Omega^n)^{\vee\vee}$ is a reflexive sheaf that equals the push-forward

of the canonical line bundle of the regular locus: $\Omega_X^{[n]} := j_*\Omega^n_{X^{reg}}$. We refer to it equally as the canonical sheaf of X. Being a reflexive sheaf, the sheaf $\Omega_X^{[n]}$ is isomorphic to $\mathcal{O}_X(D)$ for a Weil divisor D. We refer to any such divisor as the canonical divisor of X and denote it K_X. Finally, we recall that if for a Weil divisor D there exists a natural number n such that the sheaf $\mathcal{O}_X(nD)$ is invertible, then D is said to be $\mathbb{Q}$-Cartier. A normal variety X with $\mathbb{Q}$-Cartier canonical divisor K_X is said to be $\mathbb{Q}$-Gorenstein.
$\mathbb{Q}$-Cartier divisors play an important role for the study of singularities because their pullback is well-defined along any morphism. Indeed, let $f : Y \to X$ be any morphism and let D be a prime Weil divisor on X such that $\mathcal{O}_X(nD)$ is Cartier, then the pullback of the class of D is defined to be the $\mathbb{Q}$-divisor class $f^*(D)$ for which the following equality holds:

$$O_Y(f^*D) = \frac{1}{n} f^*\mathcal{O}_X(nD).$$

Finally, we recall a few things about valuations, whose language is well-suited to treat singularities. Given a totally ordered group G, a valuation of a field K is a map $v : K \setminus \{0\} \to G$ such that:

$$\begin{aligned} \forall x, y \in K \setminus \{0\} : v(xy) = v(x) + v(y) \text{ and} \\ v(x+y) \geq \min\{v(x), v(y)\}. \end{aligned}$$

When G is isomorphic to $\mathbb{Z}$ the valuation is said to be discrete. We will be interested only in valuations on the function field $K(X)$ of a complex variety X for which $v(\mathbb{C}) = 0$. We refer to these as valuations of $K(X)/\mathbb{C}$. In particular we will consider the following

Definition 3.2.1. Let X be an integral scheme of finite type over $\mathbb{C}$ with function field $K(X)$. A valuation v of $K(X)/\mathbb{C}$ is said to have center x on X if its valuation ring dominates $\mathcal{O}_{X,x}$.

Remark 3.2.2. As a direct application of the valuative criterions of separatedness and properness ([Har77] Theorem 4.3 and Theorem 4.7) we have that: if X is separated, then the center of any valuation of $K(X)/\mathbb{C}$ is unique (if it exists); whilst if X is proper, then any valuation of $K(X)/\mathbb{C}$ has a unique center in X.

Remark 3.2.3. We note that divisors on a normal variety X give rise to discrete valuations of $K(X)$. Indeed, if X is normal, it is regular in codimension 1. If $D \subseteq X$ is a prime Weil divisor, then $\mathcal{O}_{X,D} \subseteq K(X)$ is a discrete valuation ring with fraction field $K(X)$.

If $f : Y \to X$ is a proper birational morphism of normal varieties and $E \subseteq Y$ is a prime divisor, then E defines a discrete valuation on the function field $K(Y)$ and consequently on $K(X)$. We refer to the center of this valuation on X as $\text{Center}_X(E)$.

Definition 3.2.4. We say that a normal $\mathbb{Q}$-Gorenstein variety X over $\mathbb{C}$ has canonical (respectively terminal) singularities if there exists a proper birational morphism $f : X' \to X$ from a smooth variety X' such that:

$$K_{X'} = f^*K_X + \sum_{i \in I} a_i E_i$$

with $a_i \in \mathbb{Q}_{\geq 0}$ (respectively $a_i \in \mathbb{Q}_{>0}$) where $\{E_i\}_{i\in I}$ is the set of all irreducible prime divisors lying in the exceptional locus of f.

If the condition $a_i \geq 0$ holds for a resolution, then for any proper birational morphism $g : V \to X$ from a smooth variety V, in the formula $K_V = g^*K_X + \sum_i a_i E_i$ all the coefficients of the exceptional divisors are non-negative (see [Mat02] Remark 4-2-2 item (iii)).
Kawamata log terminal singularities are a class of singularities of pairs. By a pair (X, B) we mean a normal variety X together with a boundary $B := \sum d_i D_i$, i.e. a $\mathbb{Q}$-divisor for which the coefficient of each prime divisor D_i satisfies $0 \leq d_i \leq 1$.
If $K_X + B$ is $\mathbb{Q}$-Cartier, its pullback along any morphism is well-defined and for a birational morphism $f : Y \to X$ from a smooth variety Y, one can write the ramification formula:

$$K_Y = f^*(K_X + B) + \sum c_i E_i$$

for some rational coefficients c_i and some divisors E_i. We notice however that if B is non trivial, the divisors appearing in the sum need not be exceptional. In particular the respective coefficients c_i are not uniquely determined by this formula. To deal with this problem we rewrite the ramification formula as:

$$K_Y = f^*(K_X + B) - f_*^{-1}(B) + \sum a_i E_i$$

where all the summands E_i are exceptional for f. The coefficient $a_i := a(E_i, X, B)$ is independent of the model Y, but it depends only on the center of E_i ([Mat02], Proposition-Definition 4-4-1), it is called discrepancy of (X, B) at E_i. When working with pairs it is generally convenient to work with log discrepancies. The log discrepancy of f at an exceptional divisor E_i is defined to be $b(E_i, X, B) := a(E_i, X, B) + 1$ (in the following we refer to these simply as b_i when no confusion arises from the context).
By setting $B_Y := f_*^{-1}(B) + \sum E_i$ the relation between the canonical divisors can be conveniently expressed in the form:

$$K_Y + B_Y = f^*(K_X + B) + \sum b_i E_i.$$

We define the discrepancy of (X, B) to be:

$$\operatorname{disc}(X, B) := \inf\{a(E, X, B) : E \text{ corresponds to a discrete valuation of } K(X) \text{ such that } \operatorname{Center}_X(E) \neq \emptyset \text{ and } \operatorname{codim}_X(\operatorname{Center}_X(E)) \geq 2\}.$$

Remark 3.2.5. This means that we consider the infimum over all the discrepancies at all divisors that can be realised as exceptional for some morphism $f : V \to X$. Moreover, we note that if X is assumed to be proper, the condition $\operatorname{Center}_X(E) \neq \emptyset$ is always fulfilled.

Definition 3.2.6. The pair (X, B) with $B = \sum d_i D_i$ is said to have

- canonical singularities if $K_X + B$ is $\mathbb{Q}$-Cartier and $\operatorname{disc}(X, B) \geq 0$;

- Kawamata log terminal (klt) singularities if $K_X + B$ is $\mathbb{Q}$-Cartier, $d_i < 1$ for any i and $\mathrm{disc}(X, B) > -1$.

Remark 3.2.7. It is immediate from the definition that the variety X has canonical singularities if and only if the pair $(X, 0)$ has canonical singularities.

We analyse now the behaviour of singularities under finite morphisms, being interested in pairs we start by recalling some properties of pullback of divisors along a finite morphism between normal varieties of the same dimension. For the rest of this section we fix such a morphism $f : Y \to X$. Let $U \subseteq X$ denote the regular locus of X and $V := f^{-1}(U)$ its preimage. The complement of U has codimension at least 2 and so does the complement of V.

First of all, we notice that the pullback of Weil divisors is well-defined. If $D \subseteq X$ is any Weil divisor, its restriction $D|_U$ is a Cartier divisor on U and f^*D is defined to be the extension to Y of the pullback $f|_V^* D_U$.

Secondly, the pullback of a $\mathbb{Q}$-Cartier divisor is again $\mathbb{Q}$-Cartier. Indeed, let $D \subseteq X$ be a divisor with mD Cartier for an integer, then mf^*D coincides with f^*mD on the open V, which has complement of codimension 2. Thus they agree and f^*D is $\mathbb{Q}$-Cartier.

Finally, thanks to [KM98], Lemma 5.16 the image of a prime $\mathbb{Q}$-Cartier divisor is again $\mathbb{Q}$-Cartier.

In order to study singularities of pairs we need a ramification formula relating canonical divisors along finite morphisms, we follow mainly [Sho93], Section 2. Let $f : Y \to X$ be a finite morphism between normal varieties of the same dimension. Given a prime divisor $D \subseteq X$, we write $f^*D = \sum m_f(E_i)E_i$ for its pullback with $E_i \subseteq Y$ prime divisors mapped on D and $m_f(E_i)$ positive integers. The coefficient $m_f(E_i)$ is called multiplicity of f at E_i and $r_f(E_i) := m_f(E_i) - 1$ is called the ramification index of f at E_i. We also set $R := \sum r_f(E)E$ to be the ramification divisor of f. Given a boundary $B := \sum d_i D_i$ on X where the sum runs over all prime divisors in X and $b_i = 0$ for all but finitely many divisors, the ramification formula can be conveniently written as:

$$K_Y + B_Y = f^*(B_X + K_X) \text{ where } B_Y := \sum_{D_i \subseteq X} \sum_{\substack{E_j \subseteq Y \\ f(E_j) = D_i}} (d_i m_j - r_j) E_j. \tag{3.2.1}$$

Here to shorten the notation $m_j := m_f(E_j)$ and $r_j := r_f(E_j)$ denote the multiplicity and the ramification indices.

Since for all i we assume that $d_i \leq 1$ it follows for all j such that $f(E_j) = D_i$ that $d_i m_j - r_j \leq 1$ so the divisor B_Y is a boundary if and only if:

$$d_i m_j - r_j \geq 0 \quad \Leftrightarrow \quad d_i \geq r_j - m_j = \frac{(m_j - 1)}{m_j}.$$

Remark 3.2.8. In particular, if $f : Y \to X$ is a finite morphism and B_X is a boundary, a necessary condition for B_Y to be a boundary is that the branch divisor lies in the support of B_X.

Lemma 3.2.9. *Let $f : Y \to X$ be a finite morphism between normal varieties of the same dimension, let $B_X \subseteq X$ be a boundary and let B_Y be as in formula 3.2.1. If B_Y is a boundary, then:*

The pair (X, B_X) has klt singularities if and only if (Y, B_Y) has.

Proof. If $K_X + B_X$ is $\mathbb{Q}$-Cartier, then its pullback $f^*(K_X + B_X) = K_Y + B_Y$ is $\mathbb{Q}$-Cartier, too. On the other side, if $K_Y + B_Y$ is $\mathbb{Q}$-Cartier, then

$$\deg(f)(K_X + B_X) = f_* f^*(K_X + B_X) = f_*(K_Y + B_Y)$$

is Cartier and therefore $K_X + B_X$ is $\mathbb{Q}$-Cartier.
Let $g : \widetilde{X} \to X$ be a proper birational morphism, we consider its pullback $\widetilde{Y}$ along f:

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{\tilde{f}} & \widetilde{X} \\ {\scriptstyle \tilde{g}}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X. \end{array}$$

Let $B_{\widetilde{X}}$ be the boundary on $\widetilde{X}$ for which holds $K_{\widetilde{X}} + B_{\widetilde{X}} = g^*(K_X + B_X) + \sum b_i E_i$ where b_i are the log discrepancies of g at E_i, and let $B_{\widetilde{Y}}$ be the divisor on $\widetilde{Y}$ for which $K_{\widetilde{Y}} + B_{\widetilde{Y}} = \tilde{f}^*(K_{\widetilde{X}} + B_{\widetilde{X}})$ holds. A quick computation shows that $B_{\widetilde{Y}}$ also satisfies $K_{\widetilde{Y}} + B_{\widetilde{Y}} = \tilde{g}^*(K_Y + B_Y) + \sum b_{ij} E_{ij}$, where $\{E_{i,j}\}$ is a collection of $\tilde{g}$-exceptional divisors such that for all i, j the divisor $E_{i,j}$ maps to E_i under $\tilde{f}$ and b_{ij} are the log discrepancies of $\tilde{g}$ at the $\tilde{g}$-exceptional divisors E_{ij}. It follows that:

$$\begin{aligned} K_{\widetilde{Y}} + B_{\widetilde{Y}} &= \tilde{f}^*(K_Y + B_Y) \\ &= \tilde{f}^*(g^*(K_X + B_X) + \sum b_i E_i \\ &= \tilde{g}^* f^*(K_X + B_X) + \sum b_i \left(\sum_{\tilde{f}(E_{ij}) = E_i} m_{ij}(\tilde{f}) E_{ij} \right) \end{aligned}$$

and in particular the discrepancies of g are computed in terms of the discrepancies of $\tilde{g}$:

$$b_{ij} = b_i m_{ji}.$$

This shows the "if" part. Not all the proper birational morphisms with target Y fit in a diagram as above, nonetheless those which fit in such a diagram are enough to compute the discrepancy of (Y, B_Y): see [Kol96], Chapter VI. Theorem 1.3. □

3.3 Perfect cone toric varieties have canonical singularities

Toric varieties with canonical singularities can be characterised in terms of the cones defining them. In this section we recall a useful criterion for singularities of toric varieties and we show that it applies to the varieties arising from the cones of the perfect cone decomposition. Firstly, we recall the basics of toric varietites, for a detailed treatment we point to [CLS11].
Let $N \simeq \mathbb{Z}^n$ be a finitely generated free abelian group and let $T := \mathbb{C}^* \otimes N$ be the associated torus whose lattice of 1-parameter subgroups is given by N; we denote by M the lattice of characters of T and by $\langle\ ,\rangle : M \times N \to \mathbb{Z}$ the natural pairing. A rational convex polyhedral cone $\sigma \subseteq N \otimes \mathbb{R}$ defines a toric affine variety T_σ by setting $T_\sigma := \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M])$, where $\sigma^\vee$ is the dual cone of σ for the pairing $\langle\ ,\rangle$. A fan $\Sigma \subseteq N \otimes \mathbb{R}$ of convex polyhedral cones defines a toric variety, which we denote by T_Σ, constructed by gluing the affine toric varieties T_σ with $\sigma \in \Sigma$. In what follows we denote by $\Sigma(k)$ the subset of Σ consisting of all the k-dimensional cones and we abuse notation by denoting with the same letter σ the smallest fan containing σ as the only maximal cone.
Let $\sigma \subset N \otimes \mathbb{R}$ be a rational convex polyhedral cone and let T_σ be the associated affine toric variety. If $\rho \in \sigma(1)$ is a ray, we denote by u_ρ the unique non-trivial primitive vector in $\rho \cap N$. We consider the polytope

$$\Pi_\sigma := \mathrm{convex-hull}(\{0\} \cup \{u_\rho | \rho \in \sigma(1)\}) \subset N \otimes \mathbb{R}.$$

We recall the characterisation given in Proposition 11.4.12 of [CLS11]:

Proposition 3.3.1. *With the above notation,*

- *the following are equivalent:*
  - *the variety T_σ is $\mathbb{Q}$-Gorenstein;*
  - *there exists an $m \in M \otimes \mathbb{Q}$ such that $\forall \rho \in \sigma(1)\ \langle m, u_\rho \rangle = 1$;*
  - *the polytope Π_σ has a unique facet F not containing the origin.*
- *If the variety T_σ is $\mathbb{Q}$-Gorenstein, then the variety T_σ has canonical singularities if and only if the only non-zero lattice points of Π_σ lie in the facet not containing the origin (or, more concisely, $\Pi_\sigma \cap N \setminus \{0\} \subset F \cap N$).*

Remark 3.3.2. If $\Sigma = \{\sigma_\alpha\}_\alpha$ is a fan and T_Σ is the associated toric variety, then T_Σ has canonical (respectively terminal) singularities if and only if for every cone $\sigma \in \Sigma$ the variety T_σ has canonical (respectively terminal) singularities.
We can further restrict ourselves to consider only maximal cones, in the sense that the variety T_Σ is $\mathbb{Q}$-Gorenstein (respectively has canonical singularities) if and only if for every maximal cone $\sigma \in \Sigma$ the variety T_σ is $\mathbb{Q}$-Gorenstein (respectively has canonical singularities).

We focus now on the toric varieties appearing in the definition of the perfect cone compactification. Let $L \simeq \mathbb{Z}^n$ be a free abelian group and $C \subset L \otimes \mathbb{R}$ an open convex cone that is homogeneous and self-adjoint. We consider the set

$$K_p = \text{convex} - \text{hull}\left(\overline{C} \cap L \setminus \{0\}\right)$$

and we denote by Σ_p the fan given by taking cones over the faces of K_p and the trivial cone, that is:

$$\Sigma_p = \{\sigma \subset L_{\mathbb{R}} \text{ cone} \mid \text{there is a face } F \subset K_p \text{ such that } \sigma = \mathbb{R}_{\geq 0}F\}.$$

Theorem 3.3.3 ([Gio19], Theorem 3.4)**.** *The toric variety T_{Σ_p} associated to the fan Σ_p is $\mathbb{Q}$-Gorenstein and has canonical singularities.*

Proof. Thanks to Remark 3.3.2, in order to prove the theorem it is enough to show that for every $\sigma \in (\Sigma_p)_{max}$ the variety T_σ is $\mathbb{Q}$-Gorenstein and has canonical singularities. Let $\sigma = \mathbb{R}_{\geq 0}F$ be such a cone, where F is a facet of K_p. Let H denote the supporting hyperplane of F, i.e. $F = H \cap K_p$, and let H^- be the closed half-space such that $H^- \cap K_p = F$. The origin lies in the interior of H^-: since σ is of maximal dimension, the origin does not lie on H. On the other hand, if 0 is in the complement of H^-, then $\mathbb{R}_{>1}F \subseteq H^-$ and $\mathbb{R}_{>1}F \subseteq \mathbb{R}_{>1}K_p \subseteq K_p$, which is absurd.
Recall the definition of the polytope Π_σ :=convex-hull$(\{0\} \cup \{u_\rho | \rho \in \sigma(1)\})$ and let $0 \neq v \in L \cap \Pi_\sigma \subseteq L \cap \sigma \cap H^-$; there exists an $n \in \mathbb{N}$ such that $nv \in K_p \subseteq \overline{C}$. Since $\overline{C}$ is a cone it follows that $v \in \overline{C}$; therefore $v \in L \cap \overline{C} \setminus \{0\} \subseteq K_p$. We conclude that $v \in K_p \cap H^- = F$ and, thanks to Proposition 3.3.1, that T_σ is $\mathbb{Q}$-Gorenstein and has canonical singularities.
□

3.4 Singularities of quotients of toric varieties

In this section we analyse quotients of toric varieties by the action of finite groups with a stress on the singularities of the Minimal Model Program that might appear. We fix the following notation that we use for the rest of the section: Let N denote the a free abelian group of rank n and let $T \simeq N \otimes \mathbb{C}^*$ be the associated torus. For a fan $\Sigma \subseteq N_{\mathbb{R}}$ of strongly convex rational polyhedral cones we denote by T_Σ the associated toric variety. We recall that the automorphism group[1] of the torus is given by:

$$\text{Aut}(T) \simeq \text{GL}(N) \ltimes T$$

[1] By [CLS11] Theorem 3.3.4 the group of automorphisms that respect the group structure of the torus is GL(N). By [Ros61] Theorem 3, if an automorphism of T sends the neutral element to the neutral element, then it is a group homomorphism. Considering translations by elements of the torus one gets the claim.

where the torus acts by translation on itself while $\mathrm{GL}(N)$ consists of group automorphisms. Furthermore, for an element $g \in \mathrm{Aut}(T)$ we write $g = (\gamma, a) \in \mathrm{Aut}(T) \simeq \mathrm{GL}(N) \ltimes T$ and we denote by pr the projection:

$$\begin{aligned} pr : \mathrm{Aut}(T) &\to \mathrm{GL}(N) \\ (\gamma, a) &\mapsto \gamma. \end{aligned}$$

We are interested in studying quotients of toric varieties by the action of finite groups. For this, we notice that if $G \leq T$ is a finite subgroup of the torus, it acts on any toric variety T_Σ. On the other hand, a finite group $G \leq \mathrm{GL}(N)$ acts on the toric variety T_Σ if and only the action of G on $N \otimes \mathbb{R}$ induces an action on the fan Σ.
Before starting with the study of quotients we review a couple of facts about divisors on a toric variety, some of these are probably not strictly necessary to the discussion below, but they give a picture that makes the results we are going to show definitely more meaningful. Given a fan $\Sigma \subseteq N \otimes \mathbb{R}$, divisors of the associated toric variety $T \hookrightarrow T_\Sigma$ invariant under the action of the torus play a special role: Their classes generate not only the divisor class group but they also provide a nice presentation of it. First of all we notice that torus invariant prime divisors are exactly the irreducible components of the complement $T_\Sigma \setminus T$ and thanks to the orbit-cone correspondence (see [CLS11] Theorem 3.2.6) they are in bijection with the one dimensional cones $\rho \in \Sigma(1)$. Indeed, given a ray $\rho \in \Sigma$ its orbit $O(\rho)$ has codimension one and its closure D_ρ is clearly torus invariant. A torus invariant divisor is therefore of the form:

$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$$

where a_ρ are integers. We denote by $\mathrm{Div}_T(T_\Sigma)$ the free group of torus invariant divisors. A source of torus invariant divisors is given by characters: For any element $m \in M$ the associated character χ^m defines the torus invariant divisor

$$\mathrm{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle u_\rho, m \rangle D_\rho$$

where u_ρ is the primitive vector in $\rho \cap N$. The divisor class group $\mathrm{Cl}(\Sigma)$ fits in the following exact sequence ([CLS11] Theorem 4.1.3):

$$M \to \mathrm{Div}_T(T_\Sigma) \to \mathrm{Cl}(T_\Sigma) \to 0$$

where the first morphism maps m to $\mathrm{div}(\chi^m)$ and the second one maps a divisor to its class. Analogously, if $\mathrm{CaDiv}_T(T_\Sigma)$ is the free group of Cartier torus invariant divisors, then the Cartier class $\mathrm{CaCl}(T_\Sigma)$ fits in the following exact sequence ([CLS11] Theorem 4.2.1):

$$M \to \mathrm{CaDiv}_T(T_\Sigma) \to \mathrm{CaCl}(T_\Sigma) \to 0.$$

Finally, the canonical divisor class of a toric variety contains a distinguished divisor. Therefore there is a natural choice for the canonical divisor. This is torus invariant and consists of the whole boundary taken with negative sign:

$$K_{T_\Sigma} = -\sum_{\rho\in\Sigma(1)} D_\rho.$$

The first point we want to make is the following

Lemma 3.4.1. *Let $i : G \hookrightarrow \mathrm{Aut}(T)$ be the inclusion of a finite group such that the composite $pr \circ i : G \to \mathrm{GL}(N)$ is injective. Let G act on the toric variety T_Σ. If $D \subseteq T_\Sigma$ is a divisor pointwise fixed by a non-trivial element $g \in G$, then D is not a torus invariant divisor, i.e. $D \cap T \neq \emptyset$*

Remark 3.4.2. This is essentially a reformulation of [Ma18], Lemma A.12 and is a generalisation of [GHS07], Lemma 2.20: Here the variety is not assumed to be smooth and the group G is not assumed to be contained in $\mathrm{Aut}(T)$.

Proof. Looking for a contradiction, we assume that D is a torus invariant prime divisor that is pointwise fixed by an element $1 \neq g \in G$, in particular D is the closure $\overline{O(\rho)}$ of the orbit of a ray $\rho \in \Sigma$. Since $g = (\gamma, a)$ fixes D, the element γ fixes ρ. In particular, it acts as the identity on the sublattice $N_0 := \mathbb{Z}(N \cap \rho)$ generated by $N \cap \rho$. Let $\overline{N} = N/N_0$ be the quotient by N_0. Acting with T on any point $x \in O(\rho) \subseteq D$ gives an isomorphism $T_{\overline{N}} \simeq O(\rho)$.
Let $\bar{g} = (\bar{\gamma}, \bar{a})$ represent the action of g on D where $\bar{\gamma} \in \mathrm{GL}(\overline{N})$ is induced by γ and $\bar{a} \in T_{\bar{N}}$. By assumption g acts as the identity on D, hence $\bar{\gamma} = \mathrm{id}_{\overline{N}}$. One may choose a γ-equivariant splitting $N \otimes \mathbb{Q} \simeq (\bar{N} \otimes \mathbb{Q}) \oplus (N_0 \otimes \mathbb{Q})$ and conclude that $\gamma \in \mathrm{GL}(N) \hookrightarrow \mathrm{GL}(N_\mathbb{Q})$ acts as the identity. This gives a contradiction. □

Conversely, any translation by an element of the torus fixes no point on the torus while it can fix a torus invariant divisor as the following basic example shows:

Example 3.4.3. Let ζ_d denote the d-th root of unity and let $G := \langle (1, \zeta_d) \rangle \leq (\mathbb{C}^*)^2$ act on the toric variety $\mathbb{C}^2$, then the divisor $\mathbb{C} \times \{0\} \subseteq \mathbb{C}^2$ is pointwise fixed by any element of G and lies in the boundary.

Recall that an element of finite order $s \in \mathrm{GL}(n, \mathbb{C})$ is said to be a quasi-reflection if all but one of its eigenvalues are equal to 1 and the eigenspace associated to the eigenvalue different from 1 has dimension 1. It is said to be a reflection if it is a quasi-reflection with an eigenvalue equal to -1.

Proposition 3.4.4. *Let $G \leq \mathrm{GL}(N)$ be a finite group acting on T_Σ and let $\pi : T_\Sigma \to T_\Sigma/G$ denote the quotient morphism, then the ramification divisor is given by:*

$$R(\pi) = \sum_{\substack{r\in G \\ \textit{reflection}}} \mathrm{Fix}(r)$$

where $\mathrm{Fix}(r)$ denotes the fixed locus of r equipped with the reduced structure.

Proof. By Lemma 3.4.1, no component of the ramification divisor is torus-invariant, hence if $R(\pi)$ has support on a prime divisor D, then $D = \overline{D \cap T}$ and we are left to prove the statement for the restriction $\pi|_T : T \to T/G$.
For this, we assume that $T_\Sigma = T$ is the torus and we consider a point $x \in D$ for a prime divisor D fixed by an element $g \in G$. The multiplication by x induces a g-invariant morphism between the tangent spaces $m_x : T_eT \to T_xT$. The tangent space to D at x is a g-invariant codimension 1 subspace $T_xD \subseteq T_xT$ and so its preimage $m_x^{-1}(T_xD) \subseteq T_eT \simeq N \otimes \mathbb{C}$. In particular, being g the identity on $m_x^{-1}(T_xD)$, all but one of its eigenvalues are equal to one, i.e. g is a quasi-reflection. Let λ be the unique non-trivial eigenvalue of g, since $g \in \mathrm{GL}(N)$, its trace $tr(g) = n - 1 + \lambda$ is an integer, so g is a reflection and the claim is shown. □

Remark 3.4.5. In the context of singularities reflections are normally defined in a broader way. A reflection is an involution whose fixed point set has codimension 1. One does not normally require that this comes from a global reflection of a vector space.
We present here an example where G is a group as in Lemma 3.4.1 acting on the toric variety T_Σ, for which the ramification need not come from an element in $GL(N)$:
Let G be the group generated by

$$g = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, (1,-1) \right) \in \mathrm{GL}(\mathbb{C}^2) \ltimes (\mathbb{C}^*)^2$$

so that the action of g on $\mathbb{C}^2$ is given by

$$\begin{aligned} \mathbb{C}^2 &\to \mathbb{C}^2 \\ (a,b) &\mapsto (a,-1/b). \end{aligned}$$

Then the locus fixed by g is

$$\mathrm{Fix}(g) = \{(a,b) \in \mathbb{C}^2 \mid b = -1/b\} = \mathbb{C} \times \{\pm i\},$$

in particular g is an involution with fixed point set of codimension 1. Anyhow, it is worth noting that in this example $g = (\gamma, a) \in \mathrm{GL}(\mathbb{C}^2) \ltimes (\mathbb{C}^*)^2$ with a reflection γ.

The following example shows that the quotient of a toric variety with canonical singularities by a finite group of group automorphisms need not be $\mathbb{Q}$-Gorenstein. This is of relevance to us because it suggests that without further knowledge of the cones of the fan Σ_p it is not possible to conclude that the toroidal compactification that we define in Chapter 4 has canonical singularities.

Example 3.4.6. Let $\sigma \subset \mathbb{R}^3 = N_\mathbb{R}$ be the cone over the polytope

$$P := \text{convex-hull}((-1,-1,1),(1,1,1),(0,2,1),(-1,1,1),(1,-1,1)).$$

The corresponding toric variety $T \hookrightarrow T_\sigma$ has, thanks to Proposition 3.3.1, canonical singularities. Let r be the reflection fixing the coordinate plane $\{(0,y,z) \in N_\mathbb{R} \mid y,z \in \mathbb{R}\}$.

The cone σ is fixed by the action of r on $N_{\mathbb{R}}$, therefore the group $G = \langle r \rangle$ generated by the reflection r acts on the toric variety T_σ. Thanks to Proposition 3.4.4 the ramification divisor R consists of the fixed locus of the reflection r:

$$R = \overline{\{p \in T \ : \ x(p) = x^{-1}(p)\}}.$$

The ramification formula (3.2.1) reads:

$$K_{T_\sigma} = f^*(K_{T_\sigma/G} + 1/2 f(R))$$

where $R \in \mathrm{Div}(T_\sigma)$ is the ramification divisor of the quotient morphism $f : T_\sigma \to T_\sigma/G$. Thanks to Lemma 3.2.9 it follows that $K_{T_\sigma/G}$ is $\mathbb{Q}$-Cartier if and only if R is. Thanks to Proposition 4.1.2 of [CLS11] it can be computed that:

$$R \sim_{lin} \mathrm{div}(x - x^{-1}) - \sum_{\rho_i \neq (0,2,1)} D_{\rho_i}$$

where D_{ρ_i} is the torus-invariant divisor corresponding to the ray $\rho_i \subseteq \sigma$. In particular from the characterisation of Cartier divisors given in Theorem 4.2.8 of [CLS11] it follows that neither R nor $K_{T_\sigma/G}$ is $\mathbb{Q}$-Cartier.
The same argument carries over to many other polytopes that are symmetric with respect to an axis.

Even if the quotient might not be $\mathbb{Q}$-Gorenstein, we can prove a weaker statement about the singularities of quotients of toric varieties by the action of groups lying in $\mathrm{GL}(N)$:

Proposition 3.4.7. *Let T be a torus with associated 1-parameter subgroup lattice N. Let $T \hookrightarrow X$ be a toric variety with canonical singularities acted on by a finite group $G < GL(N)$. Let $\pi : X \to X/G$ be the quotient morphism and R the ramification divisor, then the pair $(X/G, \frac{1}{2}\pi(R))$ has klt singularities.*

Proof. Thanks to Proposition 3.4.4 the ramification divisor equals $R = \sum_r D_r$, where the sum runs over all the reflections of G and D_r denotes the divisor pointwise fixed by the reflection r. The ramification formula 3.2.1 reads $K_X = \pi^*(K_{X/G} + \frac{1}{2}\pi(R))$. The claim follows then from Lemma 3.2.9. □

4

Toroidal compactifications

The main goal of this chapter is to present the construction of toroidal compactifications for arithmetic quotients of a general hermitian symmetric space. From now on we assume that D is a hermitian symmetric domain isomorphic to the quotient $\mathscr{G}(\mathbb{R})/K$ where $\mathscr{G}$ is a semisimple algebraic group defined over $\mathbb{Q}$ and $K \leq \mathscr{G}(\mathbb{R})$ is a maximally compact subgroup[1]. We consider the action of an arithmetic group $\Gamma \leq \mathscr{G}(\mathbb{Q})$, that is for any faithful representation $\rho : \mathscr{G} \to \mathrm{GL}_n$ the group $\rho(\Gamma) \cap \mathrm{GL}_n(\mathbb{Z})$ is of finite index both in $\rho(\Gamma)$ and in $\mathrm{GL}_n(\mathbb{Z})$.
Toroidal compactifications are defined for quotients $\mathscr{F}_\Gamma := D/\Gamma$. In Section 4.1 we study the closure of the domain D in its compact dual. The upshot is that the boundary decomposes into a union of so-called boundary components which in turn are hermitian symmetric domains. Some of these boundary components, the rational ones, are then used to define the Baily-Borel compactification $\overline{D/\Gamma}$. In Section 4.2 we study how D decomposes along the projections onto any of its boundary components. The description relies on an analysis of the normaliser of a boundary component. In Section 4.3 we describe the construction of toroidal compactifications. It relies on certain combinatorial data partly introduced in the previous chapter. Section 4.4 is devoted to the perfect cone compactification, the toroidal compactification defined by the perfect cone decomposition introduced in Section 3.1. The description of the tube domain realisation of D_L takes up the Section 4.5.

4.1 Boundary components and Baily-Borel compactification

Given a hermitian symmetric space D, let $\bar{D}$ denote the euclidean closure in $\mathfrak{p}_+ \simeq \mathbb{C}^n$ of the image of the Harish-Chandra embedding $i_{HC} : D \hookrightarrow \mathfrak{p}_+$. A boundary component consists of a maximally holomorphic arc connected component of $\bar{D}$. To be precise, we consider the equivalence relation Λ defined on $\bar{D}$ generated by:

$x \sim y$ if there exists a holomorphic map $\lambda : \Delta \to \mathfrak{p}_+$ such that $x, y \in \lambda(\Delta)$ and $\lambda(\Delta) \subseteq \bar{D}$.

Definition 4.1.1. A boundary component of a hermitian symmetric domain D is an equivalence class for the equivalence relation Λ.

[1] According to [AMRT10], page 122 this means that either $\mathrm{Aut}(D)^\circ = \mathscr{G}(\mathbb{R})^\circ/(\text{centre})$ or $\mathrm{Aut}(D)^\circ = \mathscr{G}(\mathbb{R})^\circ/N$ for a normal compact subgroup $N \leq \mathscr{G}(\mathbb{R})^\circ$.

In particular, we notice that D is a boundary component of itself.
Let $F_1, F_2 \subseteq \bar{D}$ be two boundary components, one says that F_1 dominates F_2, denoted by $F_2 \prec F_1$, if $F_2 \subseteq \overline{F_1}$
The following result holds for boundary components:

Theorem 4.1.2. *Let D be a hermitian symmetric domain, then*

- *the closure $\bar{D}$ is the disjoint union of boundary components: $\bar{D} = D \sqcup \bigsqcup_{D \neq F} F$;*
- *a boundary component has a natural structure of a hermitian symmetric domain;*
- *if F_1 is a boundary component of D and F_2 is a boundary component of F_2, then F_2 is a boundary component of D;*
- *The group G acts transitively on the set of boundary components of the same dimension.*

Proof. For the first three statements see [AMRT10] Theorem III.3.3. The last statement is proven in [Wol72] page 292, but a proof is also contained in [AMRT10]: By Theorem III.3.3 (ii), boundary components are K translates of standard boundary components. These are in bijection with subsets of the real roots of G with respect to a maximal $\mathbb{R}$-split torus. And standard boundary components of same dimension are G-equivalent because by Proposition III.2.8 the group G induces all the permutations on the set of roots. □

In order to define the Baily-Borel compactification one has to restrict oneself to consider just certain boundary components that satisfy a compatibility property with the rational structure of $\mathcal{G}$. The latter is defined associating to F its normaliser:

Definition 4.1.3. Given a boundary component $F \subseteq \bar{D}$, its normaliser is the group

$$N(F) := \{g \in G : gF = F\}$$

Definition 4.1.4. A boundary component $F \subseteq \bar{D}$ is said to be rational if $N(F)$ is defined over $\mathbb{Q}$, i.e. if there exists an algebraic group $\mathcal{N}$ defined over $\mathbb{Q}$ such that $N(F) = \mathcal{N}(\mathbb{R})$.

We define the rational closure $\overline{D}^{rat}$ of D to be the set

$$\overline{D}^{rat} := D \sqcup \bigsqcup_{F \text{ rational}} F$$

endowed with the so-called Satake topology (see [BB66] Theorem 4.9 for its definition) and we recall the following

Theorem 4.1.5 ([BB66] Theorem 10.11)**.** *Let D be a hermitian symmetric domain and Γ an arithmetic group acting on it, then the quotient $\overline{D}^{rat}/\Gamma$ is a projective variety.*
The space D/Γ embeds in $\overline{D}^{rat}/\Gamma$ as a Zariski open subset and is therefore a quasi-projective variety.

4.2 Normaliser of a boundary component

The goal of this section is to analyse the structure of the normaliser of a boundary component. In particular, the 5-term decomposition allows one to realise the domain D as a Siegel domain of the third kind along the projection onto any boundary component, a crucial step in the construction of a toroidal compactification.
First of all we introduce the following maps.

Definition 4.2.1. Given an irreducible hermitian symmetric domain D with $G = \mathrm{Aut}(D)^\circ$ we fix a point $x \in D$, we set

$$u_x : \mathbb{S}^1 \to G$$

to be the unique morphism such that for $\theta \in \mathbb{S}^1$ the element $u_x(e^{i\theta})$ fixes x and acts on T_xD as the multiplication by $e^{i\theta}$. Moreover, we denote by h_x the square of u_x, so that $\forall \theta \in \mathbb{S}^1$ we have $h_x(e^{i\theta}) = u_x(e^{i\theta})^2$.

Notice that the morphism u_x is well-defined: if D is irreducible it amounts to fixing an isomorphism between $\mathbb{S}^1$ and the center of K that is compatible with the complex structure on T_xD. When D is not irreducible the decomposition Theorem 2.1.8 gives a description of u_0 in terms of the irreducible components of D.

Example 4.2.2. Let $X = \Delta$ be the complex unit disc and fix the point $0 \in \Delta$. Recall that Δ is isomorphic to $\mathbb{H}$ via the Cayley transform:

$$\begin{aligned} \varphi : \mathbb{H} &\to \Delta & \qquad \varphi^{-1} : \Delta &\to \mathbb{H} \\ \tau &\mapsto \frac{\tau - i}{\tau + i} & \qquad w &\mapsto -i\frac{w+1}{w-1} \end{aligned}$$

The group $\mathrm{SL}(2,\mathbb{R})$ acts on $\mathbb{H}$ and the quotient by the center can be identified with the isomorphism group $\mathrm{Aut}(\mathbb{H}) = \mathrm{SL}(2,\mathbb{R})/Z(\mathrm{SL}(2,\mathbb{R}))$.
The action of the group $\mathrm{SL}(2,\mathbb{R})$ on Δ gives a surjection $\pi : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{Aut}(\Delta)$:

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \Delta &\to \Delta \\ z &\mapsto \frac{[i(a+d)-(b-c)]z + [i(a-d)+(b+c)]}{[i(a-d)-(b+c)]z + [i(a+d)+(b-c)]} \end{aligned}$$

We set

$$\begin{aligned} h^{\mathrm{SL}} : \mathbb{S}^1 &\to \mathrm{SL}(2,\mathbb{R}) \\ e^{i\theta} &\mapsto \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix} \end{aligned}$$

so that for $z \in \Delta$ one has $h^{\mathrm{SL}}(e^{i\theta}) \cdot z = e^{i2\theta} z$ and $h_{0,\Delta} = \pi \circ h^{SL}$.

The following lemma is necessary to introduce the decomposition of $N(F)$ relevant to the construction of toroidal compactifications.

Lemma 4.2.3. *Let F be a boundary component of a hermitian symmetric domain X_0 with automorphism group $G := \mathrm{Aut}(D)^\circ$ and let $x \in X_0$ be a point, then there exists a unique pair (f_F, φ_F) of morphisms such that:*

- *$f_F : \mathbb{H} \to X_0$ is symmetric[2] holomorphic, $f_F(i) = x$ and $x_F := \lim_{z\to\infty} z \in F$.*
- *$\varphi_F : \mathbb{S}^1 \times SL(2,\mathbb{R}) \to G$ is a morphism such that $\varphi_F(e^{i\theta}, h^{\mathrm{SL}}(e^{i\theta})) = h_x(e^{i\theta})$.*
- *f_F is φ_F-equivariant, i.e. the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{S}^1 \times \mathrm{SL}(2,\mathbb{R}) \times \mathbb{H} & \xrightarrow{\varphi_F \times f_F} & G \times X_0 \\ \downarrow{\scriptstyle a_{\mathbb{H}}} & & \downarrow{\scriptstyle a_X} \\ \mathbb{H} & \xrightarrow{f_F} & X_0 \end{array}$$

where a_X is the action on X_0 and $a_{\mathbb{H}}$ is the composition of the natural action of $\mathrm{SL}(2,\mathbb{R})$ and the projection $\mathbb{S}^1 \times \mathrm{SL}(2,\mathbb{R}) \times \mathbb{H} \to \mathbb{H}$.

Proof. The existence of such a pair is proven in [AMRT10] Theorem III.3.3.(v), while the proof of its uniqueness is taken up in loc.cit. Theorem III.3.7. □

Theorem 4.2.4. *Let X_0 be an irreducible hermitian symmetric space and $x \in X_0$ a point. Let $F \subseteq X$ be a boundary component. Let w_F denote the one-parameter subgroup:*

$$\begin{aligned} w_F : \mathbb{G}_m &\to \mathscr{G} \\ t &\mapsto \varphi_F\left(1, \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right), \end{aligned}$$

then

$$N(F) \text{ is a parabolic subgroup and it is equal to } \{g \in G : \lim_{t\to 0} w_F(t) g w_F(t)^{-1} \text{ exists}\}.$$

Moreover, the connected component of the identity $N(F)^0$ decomposes as:

$$N(F)^0 = Z(w_F)^0 \ltimes W(F)$$

where:

- *$W(F)$ is the unipotent radical of $N(F)^0$ and can be written as:*

$$W(F) = \{g \in G : \lim_{t\to 0} w_F(t) g w_F(t)^{-1} = 1\};$$

[2] From Definition 2.5.4 this means that for every $p \in \mathbb{H}$ the equality $f_F \circ s_p = s_{f_F(p)} \circ f_F$ holds

- *$Z(w_F)$ is the centraliser of w_F and can be written as:*

$$Z(w_F) = \{g \in G : \lim_{t\to 0} w_F(t) g w_F(t)^{-1} = g\}.$$

Furthermore, the group $Z(w_F)^0$ is equal to

$$Z(w_F)^0 = G_h(F) \cdot G_l(F) \cdot M(F)$$

for subgroups of G where:

- *$G_h(F)$ is semisimple with no compact factors and $G_h(F)/Z(G_h(F)) \simeq \mathrm{Aut}(F)^0$;*
- *$G_l(F)$ is reductive with no compact factors;*
- *$M(F)$ is compact.*

Proof. See [AMRT10], Theorem III.3.10 and Paragraph III.4.1. □

Now, let $U(F)$ denote the centre of $W(F)$ and let $V(F)$ be the quotient $V(F) := W(F)/U(F)$, the exponential map gives an isomorphism with their Lie algebras:

$$\exp\colon \mathrm{Lie}(U(F)) \xrightarrow{\simeq} U(F)$$
$$\exp\colon \mathrm{Lie}(V(F)) \xrightarrow{\simeq} V(F).$$

In particular the groups $U(F)$ and $V(F)$ are real vector spaces. The theorem gives the following decomposition of $N(F)^0$ referred to as the 5-term decomposition:

$$N(F)^0 = [G_h(F) \cdot G_l(F) \cdot M(F)] \times V(F) \times U(F). \tag{4.2.1}$$

We use this to get a decomposition of X along any of its boundary components F.

Proposition 4.2.5. *The group $N(F)^0$ acts transitively on X and $K \cap N(F)^0 = K \cap Z(w_F)$.*

Proof. For the first statement, see [AMRT10] Paragraph III.4.3. For the second one, the proof of [AMRT10] shows that for any $k \in N(F) \cap K$ the equality $k \cdot \varphi_F = \varphi_F \cdot k$ holds, whence the claim follows. □

Set $K_h(F) := G_h(F) \cap K$ and $K_l(F) := G_l(F) \cap K$, letting $N(F)^0$ act on a point $x \in X_0$ we get a diffeomorphism:

$$X_0 = \frac{N(F)^0}{K \cap N(F)^0} \cong \frac{[G_h(F) \cdot G_l(F) \cdot M(F)]}{K \cap Z(w_F)} \cong \frac{G_h(F)}{K_h(F)} \times \frac{G_l(F)}{K_l(F)} \times W(F). \tag{4.2.2}$$

We consider the point $\Omega_F := \varphi_F\left(1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)$, which lies in $U(F)$ and we use it to give a description of the decomposition of X_0:

Theorem 4.2.6. *The group $G_l(F)$ acts on $U(F)$ by conjugation. The orbit of Ω_F under this action*

$$C(F) := \{g\Omega_F g^{-1} : g \in G_l(F)\}$$

is an open homogeneous cone in $U(F)$.
Let B be the Killing form of $G_0 := \mathrm{Aut}(X_0)^\circ$, let θ the involution on $\mathfrak{g} = \mathrm{Lie}(G)$ induced by the symmetry at the point $x \in X_0$ and $\mathfrak{u} = \mathrm{Lie}(U(F))$, then $C(F)$ is self-adjoint with respect the positive-definite quadratic form:

$$\langle x, y\rangle := -B(x, \theta(y))$$

via the isomorphism $\exp : \mathfrak{u}_F \to U(F)$.
The centraliser of Ω_F in $G_l(F)$ is $K_l(F)$, therefore letting $G_l(F)$ acting on Ω_F gives an isomorphism:

$$C(F) \simeq \frac{G_l(F)}{K_l(F)}.$$

Proof. [AMRT10] Theorem III.4.1. □

We deal now with the factor $\frac{G_h(F)}{K_h(F)}$. Recall that $G_0 := \mathrm{Aut}(X_0)^\circ G_{0,\mathbb{C}}$ and the complex Lie Group $G_{0,\mathbb{C}}$ acts on the dual compact X and in particular the group $G_h(F) \subseteq G_{0,\mathbb{C}}$ acts on X.

Theorem 4.2.7. *Let $x_F \in F$ be the point given by Theorem 4.2.3. Its orbit under the action of $G_h(F)$ is F and its stabiliser is $K_h(F)$. In particular, letting $G_k(F)$ act on x_F we obtain a diffeomorphism:*

$$\begin{aligned} \frac{G_h(F)}{K_h(F)} &\xrightarrow{\sim} F \\ [g] &\mapsto g \cdot x_F \end{aligned}$$

Proof. [AMRT10] Lemma.III.4.6. □

In light of this theorem, the diffeomorphism 4.2.2 reads:

$$\begin{aligned} D &\to F \times C(F) \times W(F) \\ x &\mapsto (\pi_F(x), \Phi_F(x), w_F(x)) \end{aligned}$$

where π_F, Φ_F, w_F denote the three projections.
The construction of a toroidal compactification passes through a realisation of D as a Siegel domain of the third kind along the projection onto any boundary component F. The key point is to describe D via a tube domain condition inside a set easier to treat, namely:

$$D(F) := \bigcup_{g \in U(F)_{\mathbb{C}}} g \cdot D = U(F)_{\mathbb{C}} \cdot D$$

where $U(F)_{\mathbb{C}} \subseteq G_{\mathbb{C}}$ acts on the compact dual $\check{D}$ in which D lies and therefore $g \cdot D \subseteq \check{D}$ is well-defined. To give a description of $D(F)$ in term of groups it is meaningful to consider the point:

$$x_F^\circ := s_x(x_F) \in \check{D}$$

where x_F is the point defined in Lemma 4.2.3 and s_x is the symmetry at x of D (and of $\check{D}$ too). We denote by $N(F) \cdot U(F)_{\mathbb{C}}$ the smallest subgroup of $G_{\mathbb{C}}$ that contains both $N(F)$ and $U(F)_{\mathbb{C}}$.

Remark 4.2.8. The smallest subgroup of $G_{\mathbb{C}}$ containing $N(F)$ and $U(F)_{\mathbb{C}}$ is clearly a subgroup of $N(F)_{\mathbb{C}}$. Moreover, $U(F) \leq N(F)$ is normal and therefore $U(F)_{\mathbb{C}} \leq N(F)_{\mathbb{C}}$ is normal, too. In particular, it follows that $U(F)_{\mathbb{C}} \cdot N(F) = N(F) \cdot U(F)_{\mathbb{C}}$ consists of the elements un with $u \in U(F)_{\mathbb{C}}$, $n \in N(F)$.

Proposition 4.2.9. *The group $N(F)^\circ \cdot U(F)_{\mathbb{C}}$ acts transitively on $D(F)$ and the stabiliser of x_F° is $K_h(F) \cdot G_l(F) \cdot M(F)$. In particular:*

$$\begin{aligned} \frac{N(F)^\circ \cdot U(F)_{\mathbb{C}}}{K_h(F) \cdot G_l(F) \cdot M(F)} &\to D(F) \\ [g] &\mapsto g \cdot x_F^\circ \end{aligned}$$

is a diffeomorphism.

Proof. The first statement follows from Proposition 4.2.5 and from the definition of $D(F)$, the second one is [AMRT10] Lemma III.4.6. □

Rewriting $N(F)^\circ$ as in the 5-term decomposition 4.2.1 and using, thanks to Theorem 4.2.7, that $\frac{G_h(F)}{K_h(F)} \simeq F$ we have (where we suppress the F when the notation is clear):

$$D(F) \simeq \frac{(G_h \cdot G_l \cdot M \times V \times U) \cdot U(F)_{\mathbb{C}}}{K_h \cdot G_l \cdot M} \simeq \frac{G_h}{K_h} \times V \times U_{\mathbb{C}} \simeq F \times V \times U_{\mathbb{C}} \tag{4.2.3}$$

This gives a projection on F:

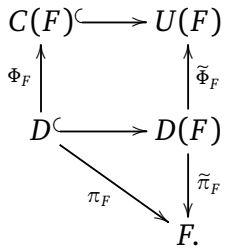
$$\widetilde{\pi_F} : D(F) \simeq \frac{(G_h \cdot G_l \cdot M \times V \times U) \cdot U(F)_{\mathbb{C}}}{K_h \cdot G_l \cdot M} \to \frac{(G_h \cdot G_l \cdot M \times V \times U) \cdot U(F)_{\mathbb{C}}}{K_h \cdot G_l \cdot M \cdot W \cdot U_{\mathbb{C}}} \simeq \frac{G_h(F)}{K_h(F)} \simeq F$$

On the other hand taking the quotient by the group $N(F)^\circ$ gives the projection

$$\widetilde{\Phi_F} : D(F) \to \frac{N(F)^\circ \cdot U(F)_{\mathbb{C}}}{N(F)^\circ} \simeq \frac{U(F)_{\mathbb{C}}}{N(F)^\circ \cap U(F)_{\mathbb{C}}} \xrightarrow{*} iU(F) \simeq U(F)$$

where the arrow * is an isomorphism induced by projecting onto $iU(F)$: the kernel is $U(F) = N(F) \cap U(F)_{\mathbb{C}} \leq U(F)_{\mathbb{C}}$.
The morphisms $\widetilde{\pi}_F$ and $\widetilde{\Phi}_F$ restricted to D coincide with π_F and Φ_F. Furthermore, D is the preimage of C along $\widetilde{\Phi}_F$:

Lemma 4.2.10. *Let $\widetilde{\pi}_F$, $\widetilde{\Phi}_F$, Φ_F and π_F be as above. The following diagram is commutative:*



Moreover, we have:

$$D = \{x \in D(F) | \widetilde{\Phi}_F(x) \in C(F)\}.$$

Proof. [AMRT10] Lemma III.4.7. □

The key point in constructing toroidal compactifications is a factorisation of the projection $\widetilde{\pi}_F$ onto F realising the space $D(F)'$ as a torus bundle over an intermediate space: in this setting toric geometry finally comes into play. For this, we quotient $D(F)$ by the group $U(F)_{\mathbb{C}}$ and use the integral structure given by Γ to build up the torus $U(F)_{\mathbb{C}}/\left(U(F)_{\mathbb{C}} \cap \Gamma\right)$; we treat first the quotient by $U(F)_{\mathbb{C}}$ and defer the other steps to the next section. We notice that thanks to the diffeomorphism (4.2.3) the group $U(F)_{\mathbb{C}}$ acts freely. Hence, if we set $D(F)' := D(F)/U(F)_{\mathbb{C}}$, then the projection

$$p_F : D(F) \to D(F)'$$

realises $D(F)$ as a principal homogeneous space for the complex group $U(F)_{\mathbb{C}}$ and $D(F)'$ comes with a natural complex structure.
The induced morphism $p_F : D(F)' \to F$ is, always thanks to the diffeomorphism (4.2.3), a principal homogeneous space for the group $V(F)$. The latter does not have any natural complex structure, nevertheless each fiber of p_F can be endowed with one:

Theorem 4.2.11. *The projection $\widetilde{\pi}_F : D(F) \to F$ decomposes as*

$$D(F) \xrightarrow{\pi'_F} D(F)' \xrightarrow{p_F} F$$

where π'_F is a principal homogeneous space for $U(F)_{\mathbb{C}}$ and p_F is a principal homogeneous space for $V(F)$.
Each fiber $p_F^{-1}(z)$ can be endowed with a complex structure J_z making $D(F)'$ a trivial complex vector bundle over F. Moreover, there is a holomorphic section $D(F)' \to D(F)$ giving a decomposition:

$$D(F) \simeq U(F)_{\mathbb{C}} \times \mathbb{C}^k \times F$$

where k is half the (real) dimension of $V(F)$.
Under this decomposition, the domain D sitting in $D(F)$ is described as:

$$D = \{(x, y, z) \in U(F)_{\mathbb{C}} \times \mathbb{C}^k \times F : \ \mathrm{Im}(x) - h_z(y, y) \in C(F)\}$$

where $h_z : \mathbb{C}^k \times \mathbb{C}^k \to U(F)$ is a symmetric bilinear form varying real analytically in $z \in F$.

Proof. See the end of paragraph [AMRT10] III.4.3 and references therein. □

Finally, we recall a result about the relations of the groups $U(F)$ and the cones $C(F)$ for boundary components dominating one another. In order to state it, we introduce the notion of a cone boundary component. We will not need the precise definition but only the following fact. Via the theory of Jordan algebras it can be proved that the closure of a self-adjoint homogeneous cone is the disjoint union of self-adjoint homogeneous cones that are called boundary components. We refer to [AMRT10] Chapter II for a detailed treatment of the subject.

Theorem 4.2.12. *Let F be a boundary component of a hermitian symmetric domain D.*

- *If F is a rational, the group $U(F)$ is defined over $\mathbb{Q}$, i.e. there exists a group $\mathcal{U} \subseteq \mathcal{N}$ defined over $\mathbb{Q}$ such that $U(F) \simeq \mathcal{U}(\mathbb{Q}) \otimes \mathbb{R}$. In particular $U(F)$ is a vector space with the rational structure given by the $\mathbb{Q}$-points of $\mathcal{U}$;*
- *if $F \subseteq \overline{F'}$ is a dominating boundary component, then $U(F') \subseteq U(F)$;*
- *associating to a boundary component F' dominating F the cone $C(F')$ gives a bijection:*

$$\rho : \{F' \text{ boundary component} : \ F \subseteq \overline{F'}\} \to \{C \subseteq \overline{C(F)} : \ C \text{ boundary component}\}$$
$$F' \mapsto C(F');$$

- *if F is rational, the bijection restricts to a bijection between the rational boundary components F' dominating F and the boundary components of $C(F)$ that are rational for the rational structure given by $\mathcal{U}(\mathbb{Q})$.*

Proof. This is [AMRT10] Theorem III.4.8. □

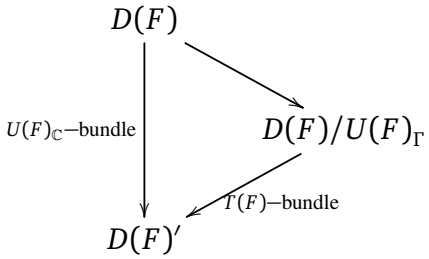
4.3 The construction of a toroidal compactification

In this section we finally present the construction of toroidal compactifications. These are constructed compactifying the partial quotients $D/U(F)$ and then gluing quotients of these partial compactifications together to get a normal compact Hausdorff space.
Given a rational boundary component F, we consider the cone $C(F)$ lying in the vector space $U(F)$, which has a rational structure. By the arithmeticity of Γ the intersection

$U(F)_\Gamma := U(F) \cap \Gamma$ gives an integral structure to $U(F)$. Furthermore, we denote the intersection $\Gamma \cap N(F)$ by $N(F)_\Gamma$. Since the group $N(F)$ acts on $U(F)$ by conjugation preserving the cone $C(F)$, the group $N(F)_\Gamma$ acts on $U(F)_\Gamma$ and we denote by $\overline{N(F)}_\Gamma$ the image in $\mathrm{Aut}(U(F)_\Gamma)$:

$$N(F)_\Gamma := \Gamma \cap N(F) \qquad \overline{N(F)}_\Gamma := \mathrm{Im}(N(F)_\Gamma \to \mathrm{Aut}(U(F)_\Gamma)).$$

Let $p_F : D(F) \to D(F)'$ be the $U(F)_{\mathbb{C}}$ -bundle defined in the previous section. Considering the quotient of $D(F)$ by $U(F)_\Gamma$ gives a torus bundle with respect to the algebraic torus $T(F) := U(F)_{\mathbb{C}}/U(F)_\Gamma$ with lattice of one parameter subgroups $U(F)_\Gamma$.



Recall the Definition 3.1.1 of an admissible decomposition. Let Σ_F be such a decomposition for the cone $C(F) \subseteq U(F)$ that is $\overline{N(F)}_\Gamma$-admissible.
Let $T(F)_{\Sigma_F}$ be the toric variety associated to the fan Σ_F. We consider the contracted product

$$(D(F)/U(F)_\Gamma)_{\Sigma_F} := (D(F)/U(F)_\Gamma) \times^{T(F)} T(F)_\Sigma$$

which comes with a natural map

$$(D(F)/U(F)_\Gamma)_{\Sigma_F} \to D(F)'$$

making it a $T(F)_{\Sigma_F}$-fiber bundle over $D(F)'$.
We define the partial compactification of $D/U(F)_\Gamma$ in the direction of F to be:

$$(D/U(F)_\Gamma)_{\Sigma_F} := \text{interior of the closure of } D/U(F)_\Gamma \text{ inside } (D(F)/U(F)_\Gamma)_{\Sigma_F}$$

This construction has to be carried out for any rational boundary component of D. In order to be able to glue the various spaces arising by this process, compatibility conditions need to be considered in choosing the $\overline{N(F)}_\Gamma$-admissible decompositions. These take into account boundary components dominating one another and boundary components that are identified via Γ.

Remark 4.3.1. If $F_1, F_2 \subseteq \overline{D}$ are two boundary components such that for an element $\gamma \in \Gamma$ the equality $\gamma \cdot F_1 = F_2$ holds, then $U(F_2)_\Gamma = \gamma\,(U(F_1)_\Gamma)\,\gamma^{-1}$. Indeed, we have clearly that $\gamma|_{F_1} : F_1 \to F_2$ is diffeomorphism with inverse $\gamma^{-1} : F_2 \to F_1$. Hence, for any element $x \in F_2$

it follows that $\gamma^{-1}(x) \in F_1$ and since $g \in N(F_1)$ we have that $g\gamma^{-1}(x) \in F_1$ and $\gamma g \gamma^{-1}(x) \in N(F_2)$. In particular, conjugating by γ gives an isomorphism $\gamma \cdot (_) \cdot \gamma^{-1} \colon N(F_1) \to N(F_2)$, which restricts to the centers of the unipotent radicals $U(F_1) \to U(F_2)$ and, since $\gamma \in \Gamma$, it restricts to an isomorphism of the intersection with Γ, proving the claim.

Let β denote the set of rational boundary components of D. The compatibility conditions for the choice of the decompositions lead to the following

Definition 4.3.2. A Γ-admissible collection Σ is a collection of $\overline{N(F)}_\Gamma$-admissible fans Σ_F for the cone $C(F)$, one for each rational boundary component F:

$$\Sigma = \{\Sigma_F\}_{F \in \beta}$$

such that:

- if $F_1 = \gamma \cdot F_2$ for an element $\gamma \in \Gamma$ then $\gamma \Sigma_{F_1} \gamma^{-1} = \Sigma_{F_2}$;
- if $F_1 \subseteq \overline{F}_2$, then:

$$\Sigma_{F_2} = \{\sigma \cap \overline{C(F_2)} \colon \sigma \in \Sigma_{F_1}\}.$$

Remark 4.3.3. In the Definition 3.1.1 it is not required that for a G-fan S for the cone C the rational closure $\overline{C(F)}^r$ is the union of all the cones. Nonetheless this holds true for the Γ_F-fans Σ_F of a Γ-admissible collection $\Sigma = \{\Sigma_F\}_{F \in \beta}$. Indeed, given a rational boundary component F the rational closure $\overline{C(F)}^r$ is the union of the cones associated to the rational boundary components dominating F (see Theorem 4.2.12). So that if F' is such a boundary component, the cone $C(F') = \bigcup_{\sigma \in \Sigma_F} \sigma \cap C(F')$ lies in the support of Σ_F.

The above construction for every boundary component F of D leads us to consider the space

$$\widetilde{D/\Gamma} := \bigsqcup_{F \in \beta} (D/U(F)_\Gamma)_{\Sigma_F}.$$

In order to glue the quotients of the partial compactifications so far constructed we recall the following

Lemma 4.3.4. *Let F, F' be rational boundary components with F' dominating F. Then*

- *we have $U(F') \subseteq U(F) \subseteq N(F')$, in particular $T(F') \subseteq T(F)$ is a subtorus and $U(F)$ acts on $D(F') = N(F') \cdot D$;*
- *the quotient of $(D(F')/U(F')_\Gamma) \times^{T(F')} T(F')_{\Sigma_{F'}}$ by the action of $T(F)$ gives an étale map:*

$$\varphi_{F',F} \colon \left(D/U(F')_\Gamma\right)_{\Sigma_{F'}} \to (D/U(F)_\Gamma)_{\Sigma_F}$$

Proof. The inclusions follow from the descriptions of the Lie algebras of $U(F)$ and $N(F)$ given in [AMRT10] page 143.
For the second part we recall that the contracted product $(D(F')/U(F')_\Gamma) \times^{T(F')} T(F')_{\Sigma_{F'}}$ is the quotient of $(D(F')/U(F')_\Gamma) \times T(F')_{\Sigma_{F'}}$ by the action of $T(F')$ where any $t \in T(F')$ acts by $t \cdot (a,b) = (at, t^{-1}b)$. Equivalently, it is the quotient by the equivalence relation defined by declaring $(at, b) \sim (a, tb)\ \forall t \in T(F')$.
The association $u \cdot (at, b) := (atu, b)$ gives an action of $U(F)_\Gamma$ on $(D(F')/U(F')_\Gamma) \times^{T(F')} T(F')_{\Sigma_{F'}}$:

$$u \cdot (at, b) = (atu, b) = (aut, b) \sim (au, tb) = u \cdot (a, tb)$$

where $tu = ut$ by abelianity of $U(F)$. The quotient is isomorphic to an open of $(D(F)/U(F)_\Gamma) \times^{T(F)} T(F)_{\Sigma_{F'}}$ which is in turn an open of $(D(F)/U(F)_\Gamma) \times^{T(F)} T(F)_{\Sigma_F}$. The restriction of the composition of these morphisms to $D/U(F')_\Gamma$ gives the sought étale map.

□

In view of this we define the following equivalence relation:

Definition 4.3.5. Let $x_1, x_2 \in \widetilde{D/\Gamma}$ be two points so that $x_1 \in (D/U(F_1)_\Gamma)_{\Sigma_{F_1}}$ and $x_2 \in (D/U(F_2)_\Gamma)_{\Sigma_{F_2}}$ for boundary components F_1 and F_2. We denote by Λ the relation generated by declaring x_1 equivalent to x_2 if

- there is a rational boundary component F and an element $\gamma \in \Gamma$, such that $F_1 \subseteq \overline{F}$ and $\gamma \cdot F_2 \subseteq \overline{F}$.
- there is a point $x \in (D/U(F)_\Gamma)_{\Sigma_F}$ such that it projects to x_1, respectively to x_2 via the natural map given by Lemma 4.3.4

  $$\varphi_{F,F_1} : \left(D/U(F)_\Gamma\right)_{\Sigma_F} \to \left(D/U(F_1)_\Gamma\right)_{\Sigma_{F_1}},$$

  respectively via

  $$\varphi_{F,F_2} : \left(D/U(F)_\Gamma\right)_{\Sigma_F} \to \left(D/U(\gamma \cdot F_2)_\Gamma\right)_{\Sigma_{(\gamma \cdot F_2)}}.$$

And finally we set:

Definition 4.3.6. The toroidal compactification associated to a Γ-admissible collection Σ is the quotient of $\overline{D/\Gamma}^\Sigma$ by the equivalence just defined:

$$\overline{D/\Gamma}^\Sigma := \widetilde{D/\Gamma}/\Lambda$$

Remark 4.3.7. The transitivity of this relation is not trivial and shown in [AMRT10] Lemma III.5.5.

The main theorem of [AMRT10] asserts:

Theorem 4.3.8 (Theorem III.5.2)**.** *We retain the notation from above. Let* $\Sigma = \{\Sigma_F\}_{F\in\beta}$ *be a* Γ*-admissible collection of fans, then the variety* $\overline{D/\Gamma}^{\Sigma}$ *is the unique Hausdorff analytic variety containing* D/Γ *as an open dense subset and such that for every rational boundary component* F *of* D*, there are jointly surjective*[3] *open analytic morphisms* π_F *making the following diagram commutative:*

$$\begin{array}{ccc} D/U(F)_\Gamma & \longrightarrow & (D/U(F)_\Gamma)_{\Sigma_F} \\ \downarrow & & \downarrow \exists \pi_F \\ D/\Gamma & \longrightarrow & \overline{D/\Gamma}^{\Sigma} \end{array}.$$

Moreover, $\overline{D/\Gamma}^{\Sigma}$ *is an algebraic space.*

Remark 4.3.9. Analogously to what has been done in Definition 4.3.6 several other spaces can be defined. For example if Σ is a Γ-admissible collection we consider the collection

$$\Sigma(1) := \{\Sigma_F(1) | F \text{ is a rational boundary component}\}$$

consisting of the subfans of 1-dimensional cones of the fans Σ_F, defines the open subspace:

$$\widetilde{D/\Gamma}^{\Sigma(1)} := \bigsqcup_{F\in\beta} (D/U(F)_\Gamma)_{\Sigma_F(1)} \subseteq \widetilde{D/\Gamma}.$$

The quotient by the restriction of Λ defines in turn an open subspace:

$$\overline{D/\Gamma}^{\Sigma(1)} \subseteq \overline{D/\Gamma}^{\Sigma}.$$

In the rest of this section we sum up the most important consequences of Definition 4.3.6. Firstly, we show how the boundary of a toroidal compactification looks like mostly following [AMRT10] Section III.6. Secondly, we show what the importance of considering arithmetic quotients by neat arithmetic subgroups is.
Recall the following

Definition 4.3.10. If $\mathcal{G}$ is an algebraic group, a subgroup $\Gamma < \mathcal{G}(\mathbb{C})$ is said to be neat if there exists a faithful representation $\rho : \mathcal{G} \to \mathrm{GL}(n, \mathbb{C})$ such that for all elements $g \in \Gamma$ the subgroup $\mathbb{C}^*$ generated by the eigenvalues of $\rho(g)$ is torsion-free.

Remark 4.3.11. A neat subgroup is torsion-free.

[3] We recall that a collection of morphisms with common target is said to be jointly surjective if the union of their images equals the target space.

We recall first of all that in view of the following, the Baily-Borel compactification is considered to be a minimal compactification:

Proposition 4.3.12 ([AMRT10] Proposition III.5.3)**.** *For any toroidal compactification there is a natural morphism:*

$$\overline{D/\Gamma}^{\Sigma} \to \overline{D/\Gamma}^{BB} \tag{4.3.1}$$

which restricts to the identity on D/Γ.

We recall that $\overline{D/\Gamma}^{BB}$ is constructed as the quotient by Γ of the rational closure $\overline{D}^{rat}$. Given a boundary component $F \subseteq D^{rat}$, the points $x \in \overline{D/\Gamma}^{\Sigma}$ mapped to the cusp $F/\Gamma_F \subseteq \overline{D/\Gamma}^{BB}$ are said to belong to the F-stratum [4].
Finally, the main result of [AMRT10] Section III.6 allows us to give a description of the boundary of a toroidal compactification

Theorem 4.3.13. *Let* $F \subseteq \overline{D}^{rat}$ *be a rational boundary component, let* Γ_F *denote the intersection* $\Gamma_F = \Gamma \cap F$*, then the quotient morphism*

$$\widetilde{D/\Gamma} \to \widetilde{D/\Gamma}/\Lambda$$

restricts to an open map

$$(D/U(F)_\Gamma)_{\Sigma_F} \to \overline{D/\Gamma}^{\Sigma}.$$

In particular $\overline{D/\Gamma}^{\Sigma}$ *is locally isomorphic at all points of the* F*-stratum to the quotient:*

$$(D/U(F)_\Gamma)_{\Sigma(F)} \diagup (\Gamma_F/U(F)_\Gamma)\cdot$$

Corollary 4.3.14. *If* Γ *is assumed to be neat in Theorem 4.3.13, then the map:*

$$(D/U(F)_\Gamma)_{\Sigma_F} \to \overline{D/\Gamma}^{\Sigma}.$$

is étale.

Proof. See [AMRT10], beginning of Section III.7. □

The second point we want to make is that for a domain D a Γ-admissible collection of fans can be used to define toroidal compactifications of the quotients of D by finite index subgroups of Γ. We explore natural morphisms between compactifications arising in this way.

[4] Compare with the definition given in [AMRT10] page 164. From the description of the map $f : \widetilde{D/\Gamma} \to \overline{D/\Gamma}^{BB}$ given in Paragraph III.6.2 (*loc.cit.*) it follows that the two definitions are compatible.

To be more precise, let D be a hermitian symmetric domain isomorphic to $\mathcal{G}(\mathbb{R})/K$ where $\mathcal{G}$ is a semisimple algebraic group defined over $\mathbb{Q}$ and $K \leq \mathcal{G}(\mathbb{R})$ is maximally compact. We stress that the rationality of a boundary component does not depend on the arithmetic group Γ but just on the rational structure of $\mathcal{G}(\mathbb{R})$. In particular, for a finite index subgroup $\Gamma' \leq \Gamma$ the associated Baily-Borel compactifications are quotients of the same object $\overline{D}^{rat}$. This in particular guarantees the existence of a finite morphism:

$$\overline{D/\Gamma'}^{BB} \to \overline{D/\Gamma}^{BB}$$

which is a quotient for Γ/Γ'.
With some more work something similar can be said for toroidal compactifications[5]. We consider for every rational boundary component F the groups $N(F)_\Gamma = N(F) \cap \Gamma$, $N(F)_{\Gamma'} = N(F) \cap \Gamma'$ and their images $\overline{N(F)}_\Gamma$, $\overline{N(F)}_{\Gamma'}$ in $\mathrm{Aut}(U(F))$. In particular we have $\overline{N(F)}_\Gamma \hookrightarrow \overline{N(F)}_{\Gamma'}$.
Given a Γ-admissible collection Σ, this is also Γ'-admissible: Since Σ_F is a $\overline{N(F)}_\Gamma$-admissible fan it is a fortiori a $\overline{N(F)}_{\Gamma'}$-admissible fan; the compatibility conditions for fans of different boundary components follow as well from the inclusion $\Gamma' \leq \Gamma$. Thus, given a Γ-admissible collection of fans we have the two toroidal compactifications $\overline{D/\Gamma}^{\Sigma}$ and $\overline{D/\Gamma'}^{\Sigma}$.
We recall in this lemma a few elementary properties of quotients that are used in the proof of the next proposition

Lemma 4.3.15. *Let G be a group acting on a topological space X, let $K \leq G$ be a subgroup and $\gamma \in G$ be any element. Then, acting by γ induces an automorphism Ψ_K of the quotient of X by K:*

$$\begin{array}{ccc} X & \xrightarrow{\gamma\cdot -} & X \\ \downarrow & & \downarrow \\ X/K & \xrightarrow[\Psi]{} & X/\gamma K\gamma^{-1} \end{array}$$

given by $\Psi_K([x]) = [\gamma x]$.
Moreover, if $K \leq H \leq G$ and $\gamma \in G$ the induced morphisms fit in the following commutative diagram:

$$\begin{array}{ccc} X/K & \xrightarrow{\Psi_K} & X/\gamma K\gamma^{-1} \\ \downarrow & & \downarrow \\ X/H & \xrightarrow[\Psi_H]{} & X/\gamma H\gamma^{-1} \end{array}$$

[5] This would be straightforward if one, as for the Baily-Borel compactification, could describe a toroidal compactification as a quotient of a partial compactification of D rather than gluing partial compactifications of intermediate quotients. Such a thing is claimed in [BJ06] paragraph III.7.21, but to the best of the author's knowledge there is no precise statement in the literature about how to get that.

where the vertical arrows are the natural projections.

Proof. Let $k \in K$ and $y := kx \in [x] \in X/K$ be a representative of an element in the quotient, then:

$$\gamma x = \gamma k^{-1}\gamma^{-1}\gamma k x = (\gamma k^{-1}\gamma^{-1})\gamma y$$

shows that $\gamma y \in [\gamma x]_{\gamma K \gamma^{-1}}$. In particular the association $[x] \mapsto [\gamma x]$ is well-defined and gives a morphism. □

Proposition 4.3.16. *Let $\Gamma' \lhd \Gamma \leq \mathcal{G}(\mathbb{R})$ be two arithmetic subgroups, let Σ be a Γ-admissible collection of fans, then the group Γ/Γ' acts on $\overline{D/\Gamma'}^{\Sigma}$ with quotient morphism:*

$$\Psi \colon \overline{D/\Gamma'}^{\Sigma} \to \overline{D/\Gamma}^{\Sigma}$$

Proof. We describe first the finite morphism $\Psi \colon \overline{D/\Gamma'}^{\Sigma} \to \overline{D/\Gamma}^{\Sigma}$ and then describe the action of Γ/Γ' that induces it.
Let β denote the set of rational boundary components of D, we consider the two sets

$$\widetilde{D/\Gamma} := \bigsqcup_{F \in \beta} (D/U(F)_\Gamma)_{\Sigma_F} \qquad \text{and} \qquad \widetilde{D/\Gamma'} := \bigsqcup_{F \in \beta} (D/U(F)_{\Gamma'})_{\Sigma_F}.$$

For every $F \in \beta$ we consider the quotient morphism $f_F \colon (D/U(F)_{\Gamma'})_{\Sigma_F} \to (D/U(F)_\Gamma)_{\Sigma_F}$ given by the action of $U(F)_\Gamma/U(F)_{\Gamma'}$ and we define the map f to be:

$$f := \sqcup_\beta f_F \colon \widetilde{D/\Gamma'} \to \widetilde{D/\Gamma}.$$

We claim that the map f descends to the level of toroidal compactifications. We denote by $\sim'$, respectively by $\sim$, the equivalence relation of Definition 4.3.2 on $\widetilde{D/\Gamma'}$ respectively on $\widetilde{D/\Gamma}$. We show that for all $a, b \in \widetilde{D/\Gamma}$, from $a \sim' b$ it follows $f(a) \sim f(b)$. Give such a pair of elements $a \sim' b$ let, as in Definition 4.3.2, $\gamma \in \Gamma'$, $F, F_1, F_2 \in \beta$, $x \in (D/U(F)_{\Gamma'})_{\Sigma_F}$ such that $F_1 \subseteq \bar{F}$, $\gamma F_2 \subseteq \bar{F}$ and $\varphi_{F,F_1}(x) = a$ and $\varphi_{F,\gamma F_2}(x) = b$. The following diagram:

$$\begin{array}{ccc} D/U(F)_{\Gamma'} & \xrightarrow{f_F} & D/U(F)_\Gamma \\ \downarrow{\scriptstyle \varphi^{\Gamma'}_{F,F_1}} & & \downarrow{\scriptstyle \varphi_{F,F_1}} \\ D/U(F_1)_{\Gamma'} & \xrightarrow{f_{F_1}} & D/U(F_1)_\Gamma \end{array}$$

is commutative because the maps involved are induced by the actions of the groups $U(F)_\Gamma/U(F)_{\Gamma'}$, $U(F_1)_\Gamma$, $U(F_1)_\Gamma/U(F_1)_{\Gamma'}$ and $U(F_1)_{\Gamma'}$, in particular $\varphi^\Gamma_{F,F_1}(f_F(x)) = f_{F_1}(a)$. For the same reasoning $\varphi^\Gamma_{F,F_2}(f_F(x)) = \gamma f_{F_2}(b)$. From Lemma 4.3.15, it follows that $\gamma f_{F_2}(b) = f_{\gamma F_2}(\gamma b)$ and this shows that f induces a well-defined map $\overline{D/\Gamma'}^{\Sigma} \to \overline{D/\Gamma}^{\Sigma}$. For the action of Γ/Γ' instead, let $g \in \Gamma$, $a \in (D/U(F)_{\Gamma'})_{\Sigma_F}$ so that $ga \in (D/U(gF)_{\Gamma'})_{\Sigma_{gF}}$, then the rule $g \cdot [a]_{\sim'} := [ga]_{\sim'}$ is well-defined: let a, b, x, γ be as above, then by Lemma 4.3.15 $\varphi_{gF,gF_1}(gx) = ga$; a similar reasoning apply to x and γb. The restriction of this action to $U(F) \cap \Gamma$ on $(D/U(F)_{\Gamma'})_{\Sigma_F}$ induces f_F and the claim is shown.

□

4.4 The perfect cone compactification

We devote an entire section, although extremely short, to define the perfect cone compactification and to underline the special property of it that makes it so interesting: it can be defined for every locally symmetric variety. Indeed, given a hermitian symmetric domain D and an arithmetic subgroup $\Gamma \leq \mathrm{Aut}(D)$, denote by β the set of rational boundary component of D, then the collection

$$\Sigma_p^\Gamma := \{\Sigma_{p,U(\ell)_\Gamma}(C(F)) : F \in \beta\}$$

where $\Sigma_{p,U(F)_\Gamma}(C(F))$ is the the perfect cone decomposition of Definition 3.1.4 for the cone $C(F)$ with respect the integral structure $U(F)_\Gamma$, is Γ-admissible.

Definition 4.4.1. The toroidal compactification associated to the Γ-admissible collection Σ_p^Γ is called the perfect cone compactification and it is denoted:

$$\overline{D/\Gamma}^{\Sigma_p^\Gamma}$$

Remark 4.4.2. The projectivity criterion stated in Section IV.2.1 of [AMRT10] applies to the perfect cone decomposition, thus the perfect cone compactification is projective.

Corollary 4.4.3. *Let D be a hermitian symmetric domain and $\Gamma' \leq \mathrm{Aut}(D)$ be a neat normal subgroup, then the perfect cone compactification*

$$\overline{D/\Gamma'}^{\Sigma_p^{\Gamma'}}$$

has canonical singularities.

Proof. As recalled in Theorem 4.3.13 $\overline{D/\Gamma'}^{\Sigma_p^{\Gamma'}}$ is locally isomorphic at all points of the F-stratum to the quotient:

$$(D/U(F)_\Gamma)_{\Sigma_p^{\Gamma'}(F)} \Big/ \left(N(F)_\Gamma/U(F)_\Gamma\right).$$

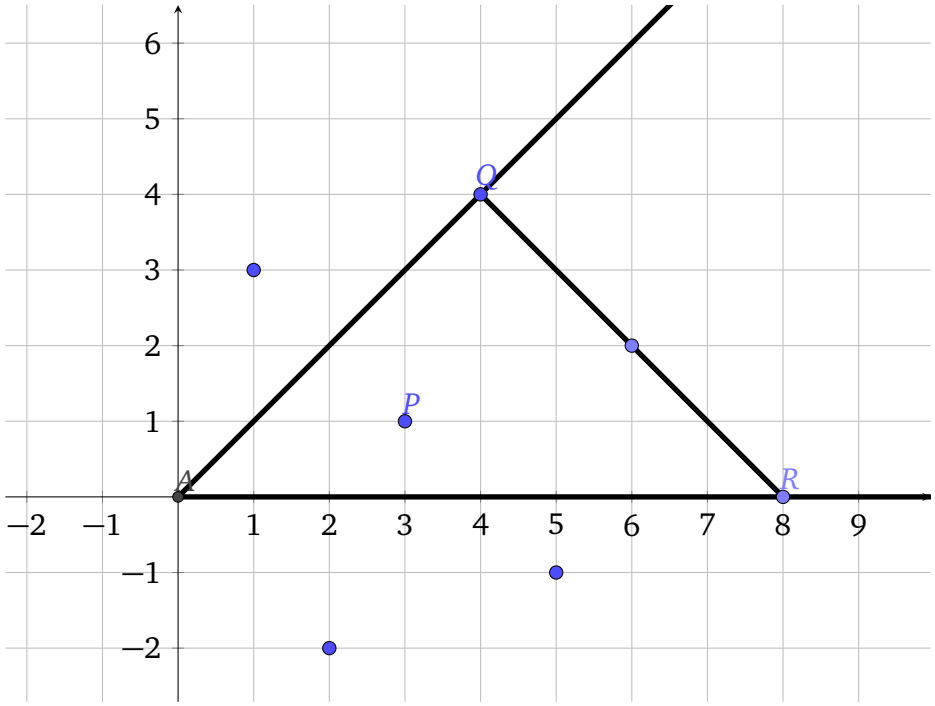
Since Γ' is neat the group $N(F)_\Gamma/U(F)_\Gamma$ acts freely and the perfect cone compactification is (analytically) locally isomorphic to $(D/U(F)_\Gamma)_{\Sigma_p^{\Gamma'}(F)}$. Thanks to Theorem 3.3.3 the latter has canonical singularities. □

As seen in the previous section, if $\Gamma' \leq \Gamma$ is an arithmetic subgroup, a Γ-admissible decomposition is also Γ'-admissible. In light of the latest corollary one might wonder if also the singularities of the toroidal compactification

$$\overline{D/\Gamma'}^{\Sigma_p^{\Gamma}}$$

are canonical. The following example suggests that might not be the case.

Example 4.4.4. We consider the smooth cone $\sigma = \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}(e_1 + e_2) \subseteq \mathbb{R}^2$. It is rational both for the lattice $\Gamma := \mathbb{Z}^2$ and for its sublattice lattice $\Gamma' := \langle 3e_1 + e_2, e_1 + 3e_2 \rangle$. Let $T(\Gamma) := \mathbb{C}^2/\Gamma$ and $T(\Gamma') := \mathbb{C}^2/\Gamma'$ be the associated tori, then the toric variety $T(\Gamma)_\sigma$ is smooth, while the toric variety $T(\Gamma')_\sigma$ does not have canonical singularities as it is shown by the following picture.



Here are drawn the rays delimiting the cone σ and some points of the lattice Γ' are displayed in purple. The points Q and R are the primitive rays of σ (with respect the lattice Γ'). The point P lies in the inner part of the polytope Π_σ (with respect to the lattice Γ') and thanks to Proposition 3.3.1 the singularities of $T(\Gamma')_\sigma$ are not canonical.

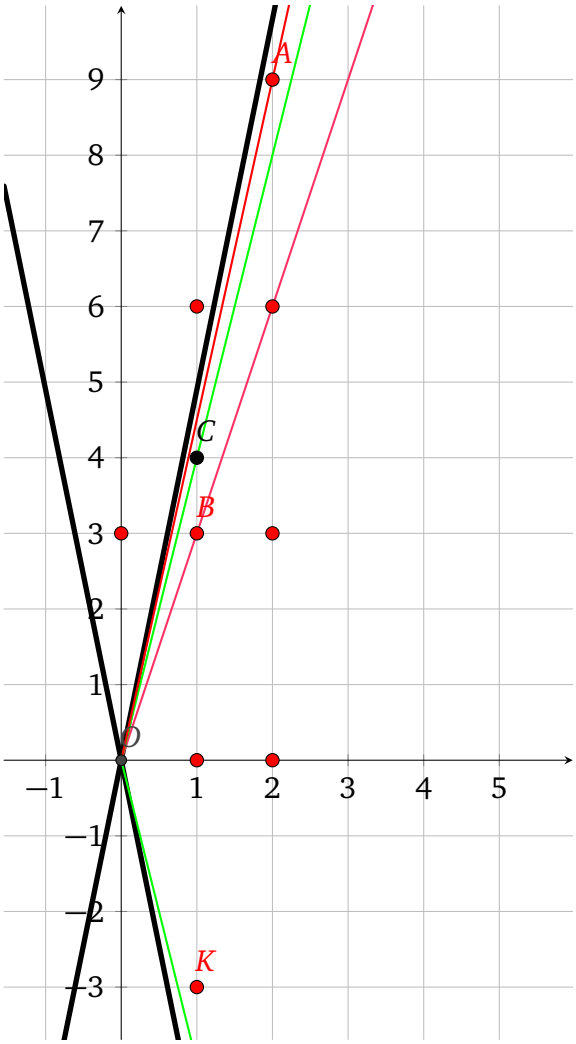
Finally one might wonder if there is a relation between the "perfect" cone compactifications:

$$\overline{D/\Gamma'}^{\Sigma_p^{\Gamma'}} \qquad \text{and} \qquad \overline{D/\Gamma'}^{\Sigma_p^{\Gamma}}.$$

Unfortunately the following example shows that in general Σ_p^Γ does not consist of fans that are refinements of $\Sigma_p^{\Gamma'}$, in other words, if F is a rational boundary component of D, then $\Sigma_{p,U(F)_\Gamma}(C(F))$ is not a refinement of $\Sigma_{p,U(F)_{\Gamma'}}$.

Example 4.4.5. We consider the open convex cone $C := \{(x, y) \in \mathbb{R}^2 : y^2 < 24x^2\}$. We consider the standard lattice $\Gamma := \mathbb{Z}^2$ and the sublattice $\Gamma' := \langle e_1, 3e_2 \rangle_\mathbb{Z}$. For an interpretation in term of toroidal compactifications of the Hilbert modular surface we refer

to [AMRT10] Section I.5. The following image represents some cones from the perfect cone fans $\Sigma_{p,\Gamma}(C)$ and $\Sigma_{p,\Gamma'}(C)$.



The relevant points belonging to the lattice Γ' are depicted in red. The half-lines in green are the rays of a cone in $\Sigma_{p,\Gamma}(C)$ while the red half-lines are rays in the fan $\Sigma_{p,\Gamma'}(C)$.

Indeed, we consider the set $K_{p,\Gamma'}$=convex-hull$(\overline{C}\cap\Gamma'\setminus\{0\})$, which is locally polyhedral. Two facets of $K_{p,\Gamma'}$ are given by the edge AB and the edge BK, hence by definition $\sigma_1 := \mathbb{R}_{\geq 0}AB$ and $\sigma_2 := \mathbb{R}_{\geq 0}BK$ are cones in $\Sigma_{p,\Gamma'}$. An analogous reasoning show that the ray on which the edge OC lies is in $\Sigma_{p,\Gamma}$. By the relative position of these cones we conclude that the fans cannot be refinement of each other.

4.5 The case of D_L

Let (L, S) be an even non-degenerate lattice with a symmetric bilinear form S of signature $(2, n)$ with $n \geq 3$. In this section we analyse the data necessary to define a toroidal compactification for an arithmetic quotient D_L/Γ.
We start by discussing its boundary components. These are determined in [Wol72] page 354. The boundary components have dimension 0 and 1. If $\ell \in L_{\mathbb{R}}$ is an isotropic vector, then the point $F_\ell := \mathbb{P}(\mathbb{C}\ell) \in D_L^\vee$ is a 1-dimensional boundary component. By Theorem 4.1.2 the automorphism group of D_L acts transitively on boundary components of the same dimension, hence 0-dimensional boundary components correspond to isotropic lines in $L_{\mathbb{R}}$. The normaliser group of F_ℓ is given by the stabiliser $N(\ell) = \mathrm{Stab}_{\mathrm{SO}(L_{\mathbb{R}})}(\ell)$. In particular, since the integral structure of $\mathrm{SO}(L_{\mathbb{R}})$ is induced by the one of L, the boundary component F_ℓ is rational if and only if a multiple of ℓ is in $L_{\mathbb{Q}}$, so that rational 0-dimensional boundary components of D_L are in bijection with isotropic lines in L.
On the other hand, if $I \subseteq L_{\mathbb{R}}$ is an isotropic plane, then $F_I := \mathbb{P}(I) \cap \overline{D}_L$ is a 1-dimensional boundary component isomorphic to the upper half-plane $\mathbb{H}$, and by the previous reasoning 1-dimensional boundary components are in bijection with isotropic planes in $L_{\mathbb{R}}$. In particular, the normaliser group of F_I is given by $N(F_I) = \{g \in \mathrm{SO}(L_{\mathbb{R}}) : g(I) = I\}$, and clearly it follows that F_I is rational if and only if $I \cap L_{\mathbb{Q}}$ has dimension 2. The compact dual of F_I is the Riemann sphere $\mathbb{P}(I \otimes \mathbb{C})$ which lies naturally in $\mathbb{P}(L_{\mathbb{C}})$. Its intersection with each connected component of $\tilde{D}_L$ is isomorphic to $\mathbb{H}$, while $\mathbb{P}(I \otimes \mathbb{R}) \subseteq \mathbb{P}(L_{\mathbb{C}})$ consists of 0-dimensional boundary components.
The centre of the unipotent radical of the normalisers of the boundary components has been determined in [Sat80] page 117. If F_I is a 1-dimensional boundary component, then the centre of the unipotent radical $U(F_I)$ is 1-dimensional[6], i.e. isomorphic to $\mathbb{R}$. The cone $C(F_I) \subseteq U(F_I)$ is isomorphic to $\mathbb{R}_{>0}$, in particular if Σ is a Γ-admissible collection, there is no choice to be done for the fan $\Sigma_{F(I)}$: it consists of the unique cone $\mathbb{R}_{\geq 0} \simeq \overline{C(F_I)} \subseteq U(F_I) \simeq \mathbb{R}$. For 0-dimensional boundary components, things get more interesting. The centre of the unipotent radical has dimension n. We conclude this chapter with a description of the tube domain realisation of D_L projecting along 0-dimensional boundary component and we get back to describe $U(F_\ell)$ in the next chapter.
Let $\ell \in L$ be a primitive isotropic vector and let

$$Q = \{[x] \in \mathbb{P}(L \otimes \mathbb{C}) : S(x, x) = 0\}$$

be the quadric defined by S which is the compact dual of D_L.
In particular, $[\ell] = \mathbb{C}l \in \mathbb{P}(L \otimes \mathbb{C})$ and the projective tangent space at $[\ell]$ is given by:

$$\mathbb{P}T_{[\ell]}Q = \{[x] \in \mathbb{P}(L \otimes \mathbb{C}) : (\ell, x) = 0\} \ = \mathbb{P}(\ell^\perp \otimes \mathbb{C})$$

[6] An explicit description is given in [GHS07], Lemma 2.25

The intersection of this hyperplane with Q gives the singular quadric

$$\widetilde{Q} := \mathbb{P}T_{[\ell]}Q \cap Q = \{[x] \in \mathbb{P}(L \otimes \mathbb{C}) : S(x,\ell) = 0, S(x,x) = 0\}.$$

Let $\ell^\perp \subseteq L$ denote the orthogonal of ℓ in L. Since ℓ is isotropic we have that $\ell \in \ell^\perp$ and S induces naturally a form on the quotient $\ell^\perp/\ell$:

$$\begin{aligned} \overline{S} \colon \left(\ell^\perp \big/ \ell\right) \times \left(\ell^\perp \big/ \ell\right) &\to \mathbb{Z} \\ (a + \mathbb{Z}\ell, b + \mathbb{Z}\ell) &\mapsto S(a,b) \end{aligned}$$

which in turn defines the quadric $\overline{Q} := \{[x] \in \mathbb{P}(\ell^\perp/\ell \otimes \mathbb{C}) : \bar{S}(x,x) = 0\}$. The projection from the point $[\ell]$ on $\mathbb{P}(L_{\mathbb{C}}/\ell)$ gives an isomorphism from $Q \setminus \mathbb{P}(\ell^\perp \otimes \mathbb{C})$ onto $\mathbb{P}(L_{\mathbb{C}}/\ell) \setminus \mathbb{P}(\ell^\perp_{\mathbb{C}}/\ell)$, while $\widetilde{Q} \setminus [\ell]$ is realised as a cone over the quadric $\overline{Q}$:

$$\begin{array}{ccccc} \widetilde{Q} \setminus [\ell] & \longrightarrow & \mathbb{P}(L_{\mathbb{C}}) & \longleftarrow & Q \setminus \mathbb{P}(\ell^\perp_{\mathbb{C}}) \\ \Big\downarrow \mathbb{A}^1\text{–bundle} & & \Big\downarrow \pi & & \Big\downarrow 1{:}1 \\ \overline{Q} & \hookrightarrow & \mathbb{P}\left(L_{\mathbb{C}}/\ell\right) & \hookleftarrow & \mathbb{P}\left(L_{\mathbb{C}}/\ell\right) \setminus \mathbb{P}\left(\ell^\perp_{\mathbb{C}}/\ell\right) \simeq \mathbb{A}^n_{\mathbb{C}} \end{array}$$

The restriction of the projection π gives an isomorphism of D_L onto an open subset of the affine space $\mathbb{P}(L_{\mathbb{C}}/\ell) \setminus \mathbb{P}(\ell^\perp_{\mathbb{C}}/\ell)$, indeed:

Lemma 4.5.1. *If $0 \neq \ell \in L$ is isotropic, then $D_L \cap \mathbb{P}(\ell^\perp_{\mathbb{C}}) = \emptyset$.*

Proof. Let $[v] \in D_L \cap \mathbb{P}(\ell^\perp)$ be a point in the intersection, i.e. $0 \neq v = x + iy \in L \otimes \mathbb{C}$ such that $S(v,l) = S(v,v) = 0$ and $S(v,\bar{v}) > 0$. As already seen, from $0 = S(x+iy, x+iy) = S(x,x) + S(y,y) + 2iS(x,y)$ it follows that $S(x,x) = S(y,y)$ and $S(x,y) = 0$. This together with $0 < S(x+iy, x-iy) = S(x,x) + S(y,y) = 2S(x,x)$ shows that $\langle x,y\rangle_{\mathbb{R}}$ is a positive definite plane in $L \otimes \mathbb{R}$ and its orthogonal complement $\langle x,y\rangle^\perp$ is negative definite because $\operatorname{sign}(L) = (2,n)$. On the other hand, $S(\ell,v) = 0$ means in particular that the isotropic vector ℓ lies in $\langle x,y\rangle^\perp$ which is a contradiction. □

We now describe the image of D_L in the affine space $A_\ell := \mathbb{P}(L_{\mathbb{C}}/\ell) \setminus \mathbb{P}(\ell^\perp_{\mathbb{C}}/\ell)$. Thanks to the bilinear form S, pairing with the vector ℓ gives the coordinate $S(\bullet,\ell)$ on $\mathbb{P}(L_{\mathbb{C}}/\mathbb{C}\ell)$ and we fix it to be equal to 1 in order to give coordinates on the affine space where it is not zero. We treat real and imaginary part separately, so that

$$W_\ell := \{x \in L_{\mathbb{R}} : (x,\ell) = 1\}/\mathbb{R}\ell$$

is a real affine space with translations given by $\ell^\perp_{\mathbb{R}}/\mathbb{R}\ell$ and we can identify the affine space A_ℓ with:

$$\{z \in \ell^\perp_{\mathbb{C}} : (z,\ell) = 1\} = \{z = x + \mathbb{R}\ell + i(y + \mathbb{R}\ell) \in \ell^\perp_{\mathbb{R}}/\mathbb{R}\ell + i\left(\ell^\perp_{\mathbb{R}}/\mathbb{R}\ell\right) : (x,\ell) = 1\ (y,\ell) = 0\}$$

$$= W_l + i\left(\ell_{\mathbb{R}}^{\perp}/\mathbb{R}\ell\right)$$

Via this we finally get a realisation as tube domain:

Proposition 4.5.2. *The projection $\pi : \mathbb{P}(L\otimes\mathbb{C}) \to \mathbb{P}(L\otimes\mathbb{C}/\mathbb{C}\ell)$ restricts to an isomorphism:*

$$\tilde{D}_L \xrightarrow{\sim} \{z = x + iy \in A_l : S(y,y) > 0\}.$$

In particular, each connected component of $\tilde{D}_L$ is realised as tube domain inside A_ℓ.

Proof. By the above, any point $[p] \in \mathbb{P}(L\otimes\mathbb{C}) \setminus \mathbb{P}(\ell_{\mathbb{C}}^{\perp})$ is the class of a point $p = \lambda + x + iy$ with $\lambda \in \mathbb{C}$, $x, y \in \ell_{\mathbb{R}}^{\perp}$ and $(x,\ell) = 1$, $(y,\ell) = 0$. Moreover $[p] \in D_L$ if and only if $S(p,p) = 0$ and $S(p,\bar{p}) > 0$. From the former we get:

$$0 = S(p,p) = 2\mathrm{Re}(\lambda)S(\ell,x) + S(x,x) - S(y,y) + i\left(2\mathrm{Im}(\lambda) + 2S(x,y)\right).$$

In particular, $S(y,y) = 2\mathrm{Re}(\lambda) + S(x,x)$. The second condition reads

$$0 < S(p,\bar{p}) = S(\lambda\ell + x + iy, \bar{\lambda}\ell + x + iy) = 2\mathrm{Re}(\lambda)S(x,\ell) + S(x,x) + S(y,y) = 2S(y,y)$$

This concludes the proof. □

Remark 4.5.3. Since the identification is given by projecting from the point $[\ell] \in \mathbb{P}(L\otimes\mathbb{C})$, the boundary component in the tube domain $W_l + i\left(\ell^{\perp}/\ell\right)_{\mathbb{R}}$ can be thought as of the "point lying at infinity".

A trivialisation of the affine space W_ℓ is not canonical but corresponds to the choice of a vector ℓ' that pairs with ℓ to 1, indeed for every such ℓ'

$$W_\ell \to \left(\ell^{\perp}/\ell\right)_{\mathbb{R}}$$
$$x \mapsto x - \ell'$$

trivialises W_ℓ.

Remark 4.5.4. Moreover the choice of such a vector ℓ' allows us to identify the vector space $\left(\ell^{\perp}/\ell\right)_{\mathbb{R}}$ with a subspace of $L\otimes\mathbb{C}$ preserving the rational structure. Indeed, the morphism

$$\pi : \langle \ell, \ell' \rangle_{\mathbb{Q}}^{\perp} \to \ell_{\mathbb{Q}}^{\perp}/\ell$$
$$a \mapsto a + \mathbb{Q}\ell$$

is well-defined since if $a \in \langle \ell, \ell' \rangle_{\mathbb{Q}}^{\perp}$, a fortiori $a \in \ell^{\perp}$ and it is easily seen to be an isomorphism: If $b \in \ell^{\perp}/\ell$ is any element, let $\tilde{b}$ be any lifting with $(\tilde{b}, \ell') = q \in \mathbb{Q}$, then $\tilde{b} - q\ell$ is a lifting of b in $\langle \ell, \ell' \rangle_{\mathbb{Q}}^{\perp}$. This shows that π is a surjective morphism between vector spaces and for dimension reasons it is an isomorphism.

Finally, let $D_\ell := \left(\ell^{\perp}/\ell\right)_{\mathbb{R}} + iC$ be a tube domain where $C \subseteq \left(\ell^{\perp}/\ell\right)_{\mathbb{R}}$ is a connected component of the cone of vectors of positive length, then

$$\varphi_{\ell'} : D_\ell \to D_L$$
$$v \mapsto v + \ell' - \frac{1}{2}\left((v,v) + (\ell',\ell')\right)\ell$$

gives an isomorphism between D_ℓ and D_L depending on ℓ'.

5

Main result

This chapter is devoted to our main result. If our discussion discussion has so far dealt with any arithmetic subgroup acting on a type IV domain, now we specialise to a particular arithmetic group that now we introduce. We continue the above notation, that is (L, S) denotes an even non-degenerate lattice with a symmetric bilinear form S of signature $(2, n)$ with $n \geq 3$. We denote by D_L the domain $\{[x] \in \mathbb{P}(L \otimes \mathbb{C}) \mid S(x,x) = 0,\ S(x,\bar{x}) > 0\}^+$, where the symbol $+$ denotes the choice of a connected component. We denote by $O^+(L \otimes \mathbb{R})$ the subgroup of $O(L \otimes \mathbb{R})$ fixing it and the intersection with the orthogonal group of the lattice L by:

$$O^+(L) := O(L) \cap O^+(L \otimes \mathbb{R}).$$

Recall that a lattice Λ has an injection of finite index in the dual $\Lambda^\vee := \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$, the finite quotient group $A_\Lambda := \Lambda^\vee / \Lambda$ is called discriminant group. The pairing of Λ induces a pairing on the dual with values in $\mathbb{Q}$ which in turn gives a $\mathbb{Q}/\mathbb{Z}$-valued form on A_Λ. If Λ is even, the form on $\Lambda^\vee$ yields:

$$q : A_\Lambda \to \mathbb{Q}/2\mathbb{Z}.$$

We denote by $O(A_\Lambda)$ the group of automorphisms of A_Λ preserving this form and by $\widetilde{O}(\Lambda)$ the group

$$\widetilde{O}(\Lambda) := \ker\left(O(\Lambda) \to O(A_\Lambda)\right)$$

known as the stable orthogonal group. Our main result concerns the group

$$\widetilde{O}^+(L) := O^+(L) \cap \widetilde{O}(L)$$

and the toroidal compactification of the quotient $D_L / \widetilde{O}^+(L)$ defined by the collection of perfect cone decompositons Σ_p (see Definition 3.1.4 and Definition 4.4.1).

In what follows, given an arithmetic group Γ we denote by Σ a Γ-admissible collection and by Σ_p the collection introduced in Chapter 3. Here is a brief outline of the chapter. In the first section we present some special features of the presentation of the boundary of

$\overline{D/\Gamma}^{\Sigma}$ given in Theorem 4.3.13 when one deals with the arithmetic group $\widetilde{O}^+(L)$. Section 5.2 combines the description given in Section 5.1 with the results obtained in Chapter 3 about Σ_p to give a proof of the main result. Finally, Section 5.3 shows how to apply the main result to a case of interest: a certain moduli space of polarised K3 surfaces with ADE singularities.

5.1 Local description of the boundary of $\overline{D_L/\widetilde{O}^+(L)}^{\Sigma}$

The set-up has already been developed in the previous section. Let $[\ell] \in \check{D}_L$ be a 0-dimensional rational boundary component of D_L, where $\ell \in L$ is a primitive isotropic vector. As seen in the previous chapter the unipotent radical $U(\ell)$ of the normaliser of $[\ell]$ is isomorphic to the real vector space $\left(\ell^\perp/\ell\right)_{\mathbb{R}}$. The choice of an element $\ell' \in L \otimes \mathbb{Q}$ such that $(\ell, \ell') = 1$ gives an isomorphism of D_L with the tube domain: $D_\ell = U(\ell) + iC(\ell) \subseteq U(\ell)_{\mathbb{C}}$ where $C(\ell)$ corresponds to a connected component of the cone of positive length vectors in $(\ell^\perp/\ell)_{\mathbb{R}}$:

$$\begin{aligned} \varphi_{\ell'} : D_\ell &\to D_L \\ z &\mapsto z + \ell' - \frac{1}{2}((z,z) + (\ell',\ell'))\ell. \end{aligned}$$

Let $N(\ell)$ be the stabiliser of ℓ in $O^+(L_{\mathbb{R}})$, it acts on D_L and hence on the tube domain D_ℓ via the isomorphism $\varphi_{\ell'}$. Its action on it fixes the "point at infinity" corresponding to the boundary component $[\ell]$. To be more precise, because the tube domain realisation is $N(\ell)$-equivariant (for details, see [AMRT10] page 153), $N(\ell)$ acts continuously on the closure $\overline{D_\ell}$ in the natural topology of the affine space $U(\ell)_{\mathbb{C}}$. Let $\mathrm{Aut}(U(\ell) + iC(\ell))$ be the group of automorphisms of D_ℓ preserving $\overline{D_\ell}$, by Theorem 3.0.1 it fits in the following exact sequence:

$$0 \to U(\ell) \to \mathrm{Aut}(U(\ell) + iC(\ell)) \xrightarrow{g} \mathrm{Aut}(U(\ell), C(\ell)) \to 1 \tag{5.1.1}$$

The image of $N(\ell)$ via g is given by the induced action on $(\ell^\perp/\ell)_{\mathbb{R}}$: Indeed, if $\gamma \in N(\ell) \leq O(L_{\mathbb{R}})$, then since γ fixes ℓ it preserves the orthogonal $\ell^\perp$, too, and induces an automorphism on the quotient by ℓ which preserves the bilinear form inherited from L, i.e. $g(\gamma) \in O((\ell^\perp/\ell)_{\mathbb{R}})$. Moreover, since γ acts on D_L, it preserves the tube domain D_ℓ and in particular the cone $C(\ell)$. We denote by $O^+(\ell^\perp/\ell)_{\mathbb{R}})$ the group[1]:

$$O^+((\ell^\perp/\ell)_{\mathbb{R}}) := \{g \in O(L_{\mathbb{R}}) : g \text{ preserves the connected cone } C(\ell)\}$$

and by $O^+((\ell^\perp/\ell)_{\mathbb{Q}})$ the intersection with the group $O(L_{\mathbb{Q}})$.

[1] We notice that previously we defined $O^+(L_{\mathbb{R}})$ for a lattice of signature $(2, n)$. But here the lattice $(\ell^\perp/\ell)_{\mathbb{R}}$ has signature $(1, n)$.

We recall that, since the boundary component $[\ell]$ is rational, the real groups $U(\ell)$ and $N(\ell)$ come with the rational structures

$$U(\ell)_{\mathbb{Q}} := U(\ell) \cap O(L_{\mathbb{Q}}) \qquad \text{and} \qquad N(\ell)_{\mathbb{Q}}$$

where $U(\ell)_{\mathbb{Q}}$ is a vector space over $\mathbb{Q}$ of the same dimension of $U(\ell)$. We give now a description of the unipotent radical $U(\ell)$. For this we consider the following transformations:

Definition 5.1.1. For any $y \in (\ell^{\perp}/\ell)_{\mathbb{Q}}$ the transformation:

$$\begin{aligned} E_{y,\ell} \colon L_{\mathbb{Q}} &\to L_{\mathbb{Q}} \\ x &\mapsto x - (\tilde{y}, x)\ell + (\ell, x)\tilde{y} - \frac{1}{2}(y, y)(\ell, x)\ell \end{aligned}$$

where $\tilde{y}$ is a lift of y, is called an Eichler transvection. We denote by

$$\mathscr{E}_{\ell,\mathbb{Q}} := \{E_{y,\ell} : y \in (\ell^{\perp}/\ell)_{\mathbb{Q}}\}$$

the set of all Eichler transvections.

Remark 5.1.2. We collect in this remark a few properties of Eichler transformations:

- The definition of $E_{y,\ell}$ is independent of the choice of the lifting of y. Indeed, if $\check{y} = \tilde{y} + n\ell$ are two different liftings of y, then:

$$\begin{aligned} & x - (\check{y}, x)\ell + (\ell, x)\check{y} - \frac{1}{2}(y, y)(\ell, x)\ell \\ &= x - (\tilde{y}, x)\ell - (n\ell, x)\ell + (\ell, x)\tilde{y} + (\ell, x)n\ell - \frac{1}{2}(y, y)(\ell, x)\ell \\ &= x - (\tilde{y}, x)\ell + (\ell, x)\tilde{y} - \frac{1}{2}(y, y)(\ell, x)\ell \end{aligned}$$

- For any $y \in (\ell^{\perp}/\ell)_{\mathbb{Q}}$ we have $E_{y,\ell}(\ell) = \ell$, indeed $(\tilde{y}, \ell) = 0$ because $\tilde{y} \in \ell^{\perp}$ and $(\ell, \ell) = 0$ because ℓ is isotropic;

- given $z, y \in (\ell^{\perp}/\ell)_{\mathbb{Q}}$, then it holds $E_{z,\ell} \circ E_{y,\ell} = E_{z+y,\ell}$. Indeed, for any $x \in L_{\mathbb{Q}}$:

$$\begin{aligned} E_{z,\ell}\left(E_{y,\ell}(x)\right) &= E_{z,\ell}(x) - (\tilde{y}, x)E_{z,\ell} + (\ell, x)E_{z,\ell}(\tilde{y}) - \frac{1}{2}(y, y)(\ell, x)E_{z,\ell}(\ell) \\ &= x - (\tilde{z}, x)\ell + (\ell, x)\tilde{z} - \frac{1}{2}(z, z)(\ell, x)\ell - (\tilde{y}, x)\ell \\ &\quad + (\ell, x)\tilde{y} - (\ell, x)(\tilde{z}, \tilde{y})\ell + (\ell, x)(\ell, \tilde{y})\tilde{z} - \frac{1}{2}(z, z)(\ell, \tilde{y})\ell \\ &\quad - \frac{1}{2}(y, y)(\ell, x)\ell \\ &= x - (\tilde{z} + \tilde{y}, x)\ell + (\ell, x)(\tilde{z} + \tilde{y}) - \frac{1}{2}(y + z, y + z)(\ell, x)\ell \\ &= E_{y+z,\ell}(x). \end{aligned}$$

- any Eichler transvection induces the identity on $(\ell^\perp/\ell)_\mathbb{Q}$. Indeed, if $x \in \ell^\perp$, then $(x,\ell) = 0$ and

$$E_{y,\ell}(x) = x - (\tilde{y},\ell)\ell \equiv x \pmod{\ell}.$$

In light of this, associating to an element an Eichler transvection defines a morphism:

$$E : \ell^\perp_\mathbb{Q} \to N(\ell)_\mathbb{Q}$$
$$x \mapsto E_{x,\ell}.$$

Corollary 5.1.3. *Considering the inclusion $N(\ell)_\mathbb{Q} \hookrightarrow \mathrm{Aut}(U(\ell)+iC(\ell))$ the exact sequence (5.1.1) gives the following one:*

$$0 \to \mathscr{E}_{\ell,\mathbb{Q}} \to N(\ell)_\mathbb{Q} \xrightarrow{\pi} O^+((\ell^\perp/\ell)_\mathbb{Q}) \to 1.$$

Moreover, the choice of a vector $\ell' \in L_\mathbb{Q}$ such that $(\ell,\ell') = 1$ induces a section $s\colon O^+((\ell^\perp/\ell)_\mathbb{Q}) \to N(\ell)_\mathbb{Q}$, which in turn gives an isomorphism:

$$\varphi_{\ell'} : N(\ell)_\mathbb{Q} \to O^+(U(\ell)_\mathbb{Q}) \ltimes U(\ell)_\mathbb{Q}.$$

Proof. From the previous discussion, the group $\mathscr{E}_{\ell,\mathbb{Q}}$ embeds in $N(\ell)_\mathbb{Q}$ and maps to the identity via π, in particular it is a subgroup of $U(\ell)_\mathbb{Q}$. Since $\mathscr{E}_{\ell,\mathbb{Q}}$ and $U(\ell)_\mathbb{Q}$ are both vector spaces isomorphic to $(\ell^\perp/\ell)_\mathbb{Q}$, they coincide. The isomorphism observed in Remark 4.5.4 gives a splitting $L_\mathbb{Q} \simeq \langle \ell,\ell'\rangle_\mathbb{Q} \oplus (\ell^\perp/\ell)_\mathbb{Q}$ and the section s can be defined by extending every element $g \in O^+((\ell^\perp/\ell)_\mathbb{Q})$ to the morphism $\tilde{g} := \mathrm{id}_{\langle \ell,\ell'\rangle_\mathbb{Q}} \oplus g \in O(L_\mathbb{Q})$. Since $\pi(g)$ preserves the cone $C(\ell)$, the morphism $\tilde{g}$ acts on D_L, i.e. $\tilde{g} \in O^+(L_\mathbb{Q})$, hence $\tilde{g} \in N(\ell)_\mathbb{Q}$. This shows that π is onto $O^+((\ell^\perp/\ell)_\mathbb{Q})$, that the exact sequence of the claim is split and hence proves the desired isomorphism. □

So far we have identified the group $U(\ell)_\mathbb{Q}$ with several others, we recall them for clarity's sake. Indeed, from now on we will use them interchangeably according to the context:

$$U(\ell)_\mathbb{Q} = \mathscr{E}_{\ell,\mathbb{Q}} \xleftarrow{E} (\ell^\perp/\ell)_\mathbb{Q} \xleftarrow{\mathrm{pr}} \langle \ell,\ell'\rangle^\perp_\mathbb{Q} \left(\subseteq L_\mathbb{Q}\right)$$

where pr is the projection of Remark 4.5.4, which depends on the choice of ℓ'.
We turn now to the intersection with an arithmetic group. For this, let $\Gamma \leq O^+(L_\mathbb{Q})$ be arithmetic, we treat the following subgroups introduced in Chapter 4

$$N(\ell)_\Gamma := N(\ell) \cap \Gamma \qquad U(\ell)_\Gamma := U(\ell) \cap \Gamma \qquad \overline{N(\ell)}_\Gamma := {}^{N(\ell)_\Gamma}\!/_{U(\ell)_\Gamma}.$$

Remark 5.1.4. Comparing the notation just introduced with the one of our main source ([AMRT10] page 159), we notice that in the exact sequence (loc. cit.)

$$1 \to \Gamma'_F \to \Gamma_F \to \overline{\Gamma}_F \to 1,$$

in our case thanks to Corollary 5.1.3, $\Gamma'_F = U(F)_\Gamma$ so that our definitions coincide with the ones of our main source.

Since the action of $N(\ell)$ on $(\ell^\perp/\ell)_\mathbb{R}$ is given by conjugation on $U(\ell)$, it follows that the restriction to $N(\ell)_\Gamma$ preserves $U(\ell)_\Gamma$, so that $\overline{N(\ell)}_\Gamma \subseteq O^+(U(\ell)_\Gamma)$. In particular, $N(\ell)_\Gamma$ is realised as a subgroup of:

$$N(\ell)_\Gamma \hookrightarrow \pi^{-1}\left(O^+(U(\ell)_\Gamma)\right) \simeq O^+(U(\ell)_\Gamma) \ltimes U(\ell)_\mathbb{Q}.$$

so that the group $\overline{N(\ell)}_\Gamma$ is identified with a subgroup of

$$\pi^{-1}(O^+(U(\ell)_\Gamma)/U(\ell)_\Gamma \simeq O^+(U(\ell)_\Gamma) \ltimes U(\ell)_\mathbb{Q}/U(\ell)_\Gamma.$$

In analogy with the notation of Section 4.3, we denote by $T(\ell) := U(\ell)_\mathbb{C}/U(\ell)_\Gamma$ the torus with one parameter subgroup lattice $U(\ell)_\Gamma$. The automorphism group of the torus $T(\ell)$ is $\mathrm{Aut}(T(\ell)) = \mathrm{GL}(U(\ell)_\Gamma \ltimes U(\ell)_\mathbb{C}/U(\ell)_\Gamma$. In particular, the group $\overline{N(\ell)}_\Gamma$ acts on $T(\ell)$ via the embedding:

$$\overline{N(\ell)}_\Gamma \leq O^+(U(\ell)_\Gamma) \ltimes U(\ell)_\mathbb{Q}/U(\ell)_\Gamma \leq \mathrm{GL}(U(\ell)_\Gamma) \ltimes U(\ell)_\mathbb{C}/U(\ell)_\Gamma.$$

Given an element $g \in N(\ell)_\Gamma$, we denote by $(\gamma, t) \in \mathrm{GL}(U(\ell)_\Gamma) \ltimes U(\ell)_\mathbb{C}/U(\ell)_\Gamma$ the corresponding element in $\mathrm{Aut}(T(\ell))$, where $\gamma = \pi(g)$ and $t = [\tilde{g}(\ell') - \ell'] \in (\ell^\perp/\ell)_\mathbb{Q}$ for any lift $\tilde{g} \in N(\ell)_\Gamma$ thanks to the next lemma.

Lemma 5.1.5. *The isomorphism $\varphi_{\ell'}$ has the form*

$$\begin{aligned} \varphi_{\ell'} : N(\ell)_\mathbb{Q} &\xrightarrow{\sim} O^+(U(\ell)_\Gamma) \ltimes U(\ell)_\mathbb{Q} \\ g &\mapsto \left(\pi(g),\, t := g(\ell') - \ell' + \mathbb{Q}\ell\right) \end{aligned}$$

where $U(\ell)_\mathbb{Q}$ is identified with $(\ell^\perp/\ell)_\mathbb{Q}$.

Proof. First of all, we observe that an Eichler transvection $E_{m,\ell}$ is determined by its value on ℓ', indeed:

$$E_{m,\ell}(\ell') - \ell' = \ell' - (\tilde{m}, \ell')\ell + (\ell, \ell')\tilde{m} - \frac{1}{2}(m,m)(\ell,\ell')\ell - \ell' \equiv m \pmod{\ell}$$

Secondly, in order to determine t we recall that $g = s(\pi(g)) \circ t$ where $s : O^+(\ell^\perp/\ell) \to N(\ell)$ is the section given by Corollary 5.1.3. Hence, by $t = g \circ s(\pi(g))^{-1}$ we can recover the element $m \in (\ell^\perp/\ell)_\mathbb{Q}$ for which $t = E_{m,\ell}$ holds:

$$u(\ell') - \ell' = g \circ s(\pi(g))^{-1}(\ell') - \ell' = g(\ell') - \ell' =: \tilde{m} \in \ell^\perp_\mathbb{Q}.$$

This shows the claim. □

Finally, we analyse the case when Γ is chosen to be the group $\widetilde{O}^+(L)$:

Proposition 5.1.6. *If $\Gamma = \widetilde{O}^+(L)$ and $\ell' \in L^\vee$, then $\overline{N(\ell)}_\Gamma = \overline{N(\ell)}_{\widetilde{O}^+(L)}$ is identified with a subgroup of* $\mathrm{GL}(U(\ell)_\Gamma)$ *via $\varphi_{\ell'}$, i.e.:*

$$\varphi_{\ell'}\left(\overline{N(\ell)}_{\widetilde{O}^+(L)}\right) \leq \mathrm{GL}(U(\ell)_\Gamma) \ltimes 1 \leq GL^+(U(\ell)_\Gamma) \ltimes U(\ell)_{\mathbb{Q}}.$$

Proof. If $\ell' \in L^\vee$, then for any $g \in \overline{N(\ell)}_{\widetilde{O}^+(L)}$ we have that $\tilde{g}(\ell') - \ell' \in \ell^\perp \cap L$ so that it belongs to $\ell^\perp/\ell \subseteq (\ell^\perp/\ell)_{\mathbb{Q}}$. In order to conclude we show that via the isomorphism $U(\ell)_{\mathbb{Q}} \simeq (\ell^\perp/\ell)_{\mathbb{Q}}$ of Corollary 5.1.3, the lattice $U(\ell)_{\widetilde{O}^+(L)}$ corresponds to $\ell^\perp/\ell$.
If $x \in \ell^\perp/\ell$, let $\tilde{x}$ be a lift, then $\forall y \in L^\vee$:

$$E_{\ell,x}(y) - y = -(\tilde{x}, y)\ell + (\ell, y)\tilde{x} - \frac{1}{2}(x,x)(\ell,y)\ell \in L$$

as $(\tilde{x}, y)$, (l, y), $1/2(x,x)(\ell,y) \in \mathbb{Z}$, i.e. $E(\ell^\perp/\ell) \subseteq U(\ell)_\Gamma$.
On the other hand, let $x \in (\ell^\perp/\ell)_{\mathbb{Q}}$ be such that $E_{x,\ell} \in \widetilde{O}^+(L)$, i.e.:

$$\forall y \in \ell^\perp \cap L \qquad E_{\ell,x}(y) = y - (\tilde{x}, y)\ell \equiv y \pmod{L}.$$

Equivalently $(\tilde{x}, y)\ell \in L$ for all $y \in \ell^\perp \cap L^\vee$. By primitivity of ℓ this means that $(\tilde{x}, y) \in \mathbb{Z}\ \forall y \in \ell^\perp \cap L^\vee$. Since $\pi(\ell^\perp \cap L) = \left(\ell^\perp/\ell\right)^\vee$ this shows that $x \in \left(\left(\ell^\perp/\ell\right)^\vee\right)^\vee = \ell^\perp/\ell$ and we are done. □

5.2 Proof of the main result

In what follows we denote by Σ a Γ-admissible collection, which defines a toroidal compactification $\overline{D_L/\Gamma}^\Sigma$.
We consider the following divisor:

Definition 5.2.1. Given a Γ-admissible collection Σ, we denote by Δ the closure in $\overline{D_L/\Gamma}^\Sigma$ of the branch divisor of $D_L \to D_L/\Gamma$.

The following result describes the branch divisor of the arithmetic quotient:

Theorem 5.2.2 ([GHS07] Theorem 2.1)**.** *Let $\Gamma \leq O^+(L)$ be a subgroup of finite index. Then the branch divisor of $D_L \to D_L/\Gamma$ is the union of the images of the divisors that are fixed by reflections in Γ.*

Moreover, the divisor Δ admits another description:

Theorem 5.2.3 ([GHS07] Theorem 1.1)**.** *Let $\Gamma \leq O^+(L)$ be a subgroup of finite index, and $\Gamma' \lhd \Gamma$ a neat normal subgroup of finite index, let*

$$\Psi : \overline{D_L/\Gamma'}^\Sigma \to \overline{D_L/\Gamma}^\Sigma$$

be the morphism described in Proposition 4.3.16. Then Ψ has ramification index 1 along each divisorial component and the branch divisor coincides with Δ. In particular, the branch divisor of Ψ does not have support on the boundary of $\overline{D_L/\Gamma}^{\Sigma}$.

Proof. See [GHS07], proof of Theorem 1.1, item (i) and item (ii). □

Remark 5.2.4. In light of these theorems we see that Definition 5.2.1 agrees with the definition of Δ given in the introduction. The quotient morphism by $\widetilde{O}^+(L)$ factors through the quotient of a subgroup Γ'

$$D_L \xrightarrow{a} D_L/\Gamma' \xrightarrow{b} D_L/\widetilde{O}^+(L).$$

If Γ' is neat, then a has no branch divisor so that Δ coincides with the closure of the branch divisor of b.
Secondly, from Theorem 5.2.3 the closure of the branch divisor of b coincides with the one of its extension to the toroidal compactifications given by a $\widetilde{O}^+(L)$-admissible decomposition.

In the proof of the main theorem we need the following elementary lemmas:

Lemma 5.2.5. *If $r \in N(\ell) \cap \widetilde{O}^+(L)$ is a reflection, then its image $\bar{r} \in \overline{N(\ell)}_{\widetilde{O}^+(L)}$ is a reflection.*

Proof. Since r has order 2, it does not lie in the torsion free group $U(l) \cap \widetilde{O}^+(L)$ and $\bar{r}$ has order 2, too. Let $H \subseteq L$ be the hyperplane pointwise fixed by r, then $\bar{r}$ fixes $(H \cap \ell^\perp)/\ell$ and is therefore a reflection. □

Lemma 5.2.6. *Let N be a free abelian group, $\Sigma \subseteq N_{\mathbb{R}}$ a fan and $T \subseteq T_\Sigma$ the associated toric variety. Let $G \leq \mathrm{GL}(N)$ be a finite group acting on T_Σ, let $x \in T_\Sigma$ be any point and $\sigma \in \Sigma$ the cone in whose orbit x lies, then*

$$\mathrm{Stab}_G(x) \subseteq \mathrm{Stab}_G(\sigma).$$

Proof. Since $T \lhd \mathrm{Aut}(T)$ for any $g \in G$ and any $t \in T$ there exists an element $t' \in T$ such that $gt = t'g$. If $y \in O(\sigma)$ is an element in the orbit of σ, then $y = t \cdot x$ for some $t \in T$ and hence for any element $g \in \mathrm{Stab}_G(x)$ one has

$$g \cdot y = gt \cdot x = t'g \cdot x = t' \cdot x \in O(\sigma).$$

This shows the claim. □

We focus now on the collection Σ_p obtained by considering perfect cone decompositions (see Definition 3.1.4 and Definition 4.4.1). Finally, we can state our main theorem:

Theorem 5.2.7. *We assume that*

$$\text{every reflection } s \in \overline{N(\ell)} \text{ admits a lifting to a reflection in } N(\ell)_{\widetilde{O}^+(L)}. \tag{5.2.1}$$

Then the pair

$$\left(\overline{D_L/\widetilde{O}^+(L)}^{\Sigma_p}, \frac{1}{2}\Delta\right)$$

has klt singularities.

Remark 5.2.8. In the next section we give a concrete example for which the theorem holds.

Proof. Thanks to [Mat02], Proposition 4-4-4 the notion of klt singularities is local in the analytic category, therefore we may prove the statement for an open cover in the analytic topology.
In order to improve the readability we make use of the following notation: $X := \overline{D_L/\widetilde{O}^+(L)}^{\Sigma_p}$ and if $\Sigma_p(1)$ denotes the fan consisting of the 1-dimensional cones of Σ_p, we denote the associated partial compactification by $X^{(1)} := \overline{D_L/\widetilde{O}^+(L)}^{\Sigma_p(1)}$, which is an open subset of X containing all the points lying over the 1-cusps.
Let $\Gamma' \leq \widetilde{O}^+(L)$ be a neat[2] normal subgroup of finite index we denote by $Y := \overline{D_L/\Gamma'}^{\Sigma_p}$ the toroidal compactification and by $Y^{(1)} := \overline{D_L/\Gamma'}^{\Sigma_p(1)}$ the partial compactification defined by $\Sigma_p(1)$. The latter is a smooth open subset of Y. The ramification formula for the quotient morphism $\Psi\colon Y \to X$ described in Proposition 4.3.16 reads:

$$K_{Y^{(1)}} = \Psi^*\left(K_{X^{(1)}} + \frac{1}{2}\Delta|_{X^{(1)}}\right)$$

Hence the pair $\left(X^{(1)}, \frac{1}{2}\Delta|_{X^{(1)}}\right)$ has klt singularities thanks to Lemma 3.2.9.
We are left to show that any point lying over a 0-cusp has a neighbourhood U such that the pair $(U, \frac{1}{2}\Delta|_U)$ has klt singularities.
Let $\ell \in L$ be a primitive isotropic vector and let $p \in X$ be a point in the associated 0-stratum. Let $T(\ell)_{\Sigma_p(\ell)}$ be the toric variety defined by the perfect cone fan. We denote by $X_{\Sigma_p(\ell)}$ the open part of the closure of $D_\ell/(U(\ell)_{\widetilde{O}^+(L)})$ in it. Let $V \subseteq T(\ell)_{\Sigma_p(\ell)}$ be an open subset such that its image in the quotient by $\overline{N(\ell)}_{\widetilde{O}^+(L)}$ is isomorphic to an open neighbourhood of p. Let $x \in V$ be a lift of p and $G := \mathrm{Stab}_{\overline{N(\ell)}(x)}$ be its stabiliser. Then there exists a G-invariant open neighbourhood $U_x \subseteq V$ such that its image in the quotient is isomorphic to U_x/G. By Lemma 5.2.6 the group G acts on the toric variety T_σ where σ lies in the torus orbit of $\sigma \in \Sigma$, thus $(U_x \cap T_\sigma)/G \subseteq T_\sigma/G$ is open. Let R be the ramification divisor of the quotient morphism $\pi\colon T_\sigma \to T_\sigma/G$, thanks to Proposition 5.1.6 the

[2] see Definition 4.3.10.

group G acts via group morphisms on T_σ, hence by Proposition 3.4.7 it follows that the pair $\left((U_x \cap T_\sigma)/G, \frac{1}{2}\pi(R)|_{(U_x\cap T_\sigma)/G}\right)$ has klt singularities.
Finally, because of Lemma 5.2.5 and the assumption (5.2.1), the reflections in $\overline{N(\ell)}$ and in $N(\ell)\cap\widetilde{O}^+(L)$ are in bijection. Hence if $p \in U_p \subseteq X$ is isomorphic to $(U_x\cap T_\sigma)/G$, it follows that the pair $(U_p, \frac{1}{2}\Delta|_{U_p})$ has klt singularities and the claim is shown. □

One of the most important properties of klt singularities is that they are rational and, in particular, Cohen-Macaulay. We recall this notion:

Definition 5.2.9. A normal variety X is said to have rational singularities if there exists a resolution of singularities $f : X' \to X$ such that the higher direct images of $\mathcal{O}_{X'}$ vanish:

$$\forall i > 0 \qquad R^i f_* \mathcal{O}_{X'} = 0.$$

We have the following

Corollary 5.2.10. *We assume that*

$$\text{every reflection } s \in \overline{N(\ell)} \text{ admits a lifting to a reflection in } N(\ell)_{\widetilde{O}^+(L)}. \tag{5.2.1}$$

Then the space $\overline{D_L/\widetilde{O}^+(L)}^{\Sigma_p}$ *has rational singularities and is, in particular, Cohen-Macaulay.*

Proof. If a pair (X, Δ) has canonical singularities, then X has rational singularities, see [KM98], Theorem 5.22 and references therein. □

5.3 An application: the moduli space of polarised K3 surfaces

We recall here the construction of the moduli space of polarised K3 surfaces with ADE singularities and explain how Theorem 5.2.7 can be applied to it.
A complex K3 surface is a compact connected complex manifold of dimension 2 such that

$$\Omega_X^2 \cong \mathcal{O}_X \text{ and } H^1(X, \mathcal{O}_X) = 0.$$

The group $H^2(X, \mathbb{Z})$ equipped with the intersection form is an even unimodular lattice of rank 22 of signature $(3, 19)$ isomorphic to $\Lambda_{K3} := U^{\oplus 3} \oplus E_8^{\oplus 2}(-1)$ where U is the hyperbolic lattice and E_8 is the positive definite lattice associated to the E_8 Dynkin diagram (see Proposition 1.3.5 of [Huy16]).
Let L be a line bundle on a K3 surface X. Recall that L is big and nef if and only if $(L.C) \geq 0$ for all closed curves $C \subset X$, and $L^2 > 0$ (see for example Proposition 2.61 of [KM98]). A quasi-polarised K3 surface is a pair (X, L) where X is a K3 surface and L is a primitive big and nef line bundle on it.

By fixing an isomorphism of lattices $\varphi : H^2(X,\mathbb{Z}) \xrightarrow{\sim} \Lambda_{K3}$, a polarisation of degree $2d$ (i.e. $L^2 = 2d$) gives an element $h = \varphi([L]) \in \Lambda_{K3}$ whose orthogonal complement $h^\perp$ is isomorphic to $\Lambda_{2d} := U^{\oplus 2} \oplus E_8^{\oplus 2}(-1) \oplus \langle -2d \rangle$. We consider the period domain

$$\mathscr{D}_{2d} := \{x \in \mathbb{P}(\Lambda_{2d} \otimes \mathbb{C}) |\ (x,x) = 0 \text{ and } (x,\bar{x}) > 0\}^+$$

where the sign "+" denotes a connected component.
Denote by

$$O(\Lambda_{K3}, h) := \{g \in O(\Lambda_{K3}) : g(h) = h\} \leq O(\Lambda_{K3})$$

the subgroup of automorphisms of Λ_{K3} fixing h. The coarse moduli space of quasi-polarised K3 surfaces $\mathscr{F}_{2d}$ is given by the quotient of $\mathscr{D}_{2d}$ by the action of the finite index subgroup of $O(\Lambda_{K3}, h)$ fixing the connected component $\mathscr{D}_{2d}$, see [Huy16], Remark 6.4.5.
The disadvantage of working with the moduli functor of quasi-polarised K3 surfaces is that it is not separated; therefore it is more convenient to treat the moduli functor of polarised K3 surfaces with ADE singularities, which is separated and whose coarse moduli space can be identified with $\mathscr{F}_{2d}$ (for more details see Section 5.1.4 and Remark 6.4.5 of [Huy16]).
The quotient $\mathscr{F}_{2d}$ is a quasi-projective variety of dimension 19. We can state the following:

Theorem 5.3.1. *Let $\overline{\mathscr{F}_{2d}}^p$ be the perfect cone compactification of the coarse moduli space of polarised K3 surfaces with ADE singularities and let Δ be the closure in it of the branch divisor of the morphism $\pi : \mathscr{D}_{2d} \to \mathscr{F}_{2d}$. If d is square-free, the pair $(\overline{\mathscr{F}_{2d}}^p, \frac{1}{2}\Delta)$ has klt singularities.*

Proof. Thanks to Corollary 1.5.2 of [Nik80], there is an isomorphism $O(\Lambda_{K3}, h) \simeq \widetilde{O}(\Lambda_{2d})$. Thanks to [Sca87], Theorem 4.0.1 , if d is square free, there is one $\widetilde{O}^+(\Lambda_{2d})$ orbit of primitive isotropic vectors. Thus, given $\ell \in \Lambda_{2d}$ primitive isotropic there exists an element $\ell' \in \Lambda_{2d}$ giving a splitting $\Lambda_{2d} \simeq U \oplus \left(\ell^\perp/(\mathbb{Z}\ell)\right)$ with $\langle \ell, \ell' \rangle \simeq U$. Hence the condition (5.2.1) is fulfilled and the claim now follows by direct application of Theorem 5.2.7. □

As a direct application of Corollary 5.2.10 we get:

Corollary 5.3.2. *If d is a square-free integer, then the perfect cone compactification $\overline{\mathscr{F}_{2d}}^p$ of the coarse moduli space of polarised K3 surfaces with ADE singularities has rational singularities and, in particular, is Cohen-Macaulay.*

Theses

As tradition dictates, we shall present the key results of this dissertation in the form of the following theses:

- If L is a free abelian group of finite rank and $C \subseteq L \otimes \mathbb{R}$ is a self-adjoint homogeneous cone, the perfect cone decomposition consists of a fan carrying an action of $\mathrm{GL}(L)$. In Theorem 3.3.3 we prove that the toric varieties associated to the cones of the perfect cone decomposition have canonical singularities, which is a class of singularities of primary importance in the study of higher dimensional algebraic varieties.

- We recall that Kawamata-log-terminal (klt) singularities is a class of singularities, which is very important in the study of the Minimal Model Program. In Theorem 3.4.7, we prove that certain finite quotients of toric varieties have klt singularities. More precisely, let T be a torus with associated 1-parameter subgroup lattice N. Let $T \hookrightarrow X$ be a toric variety with canonical singularities acted on by a finite group $G < GL(N)$. We show that the pair $(X/G, \frac{1}{2}\pi(R))$ has klt singularities, where $\pi : X \to X/G$ denote the quotient morphism and R its ramification divisor.

- Arithmetic quotients of type IV domains are deeply studied in algebraic geometry. Among the known compactifications of such a quotient, toroidal ones enjoy many properties. If L is a non-degenerate lattice of signature $(2, n)$ for an integer $n \geq 3$ and D_L is the associated type IV domain, in Theorem 5.2.7 we consider the singularities of the perfect cone compactification $\overline{D_L/\widetilde{O}^+(L)}^{\Sigma_p}$: Under the technical assumption (5.2.1) we show that the pair

$$\left(\overline{D_L/\widetilde{O}^+(L)}^{\Sigma_p}, \frac{1}{2}\Delta\right)$$

  has klt singularities, where $\widetilde{O}^+(L)$ is the stable orthogonal group and Δ is the branch divisor of the quotient of D_L by $\widetilde{O}^+(L)$.

- The coarse moduli space $\mathscr{F}_{2d}$ of quasi-polarised K3 surfaces of degree $2d$ can be identified with the quotient $\mathscr{D}_{2d}/\widetilde{O}^+(\Lambda_{2d})$ where Λ_{2d} is the lattice $U^{\oplus 2} \oplus E_8^{\oplus 2}(-1) \oplus \langle -2d \rangle$ and $\mathscr{D}_{2d}$ is the associated type IV domain. In Theorem 5.3.1 we show that if d

is square-free, then the pair

$$(\overline{\mathscr{F}_{2d}}^{p}, \frac{1}{2}\Delta)$$

has klt singularities, where $\overline{\mathscr{F}_{2d}}^{p}$ is the perfect cone compactification and Δ is the branch divisor of the quotient of $\mathscr{D}_{2d}$ by $\widetilde{O}^{+}(\Lambda_{2d})$.

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