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M. Sc. Franco Giovenzana

Gutachter: Prof. Christian Lehn (Betreuer)
Prof. Nicolas Addington
Prof. Paolo Stellari

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Giovenzana, Franco
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Introduction

This thesis concerns the hyperkähler eightfolds constructed by Lehn, Lehn, Sorgen, and van Straten [LLSvS17]. We study its periods, its birational properties and we describe some geometric features.
Hyperkähler manifolds have a rich geometry and constitute a building block of varieties with trivial canonical bundle in light of Beauville-Bogomolov decomposition theorem. Basic examples are K3 surfaces, with which they share many properties, and punctual Hilbert scheme of them. Maybe the most important common feature is the existence of a quadratic form on the second cohomology group, named after Beauville, Bogomolov and Fujiki, which governs (almost) completely their geometry in virtue of the global Torelli theorem ([Ver13, Huy12]).
Other sources of hyperkähler varieties are cubic fourfolds. For what concerns us two will play a major role: The Fano variety of lines ([BD85]) and the LLSvS eightfold. The construction of the latter is involved, here is an upshot: Given a smooth cubic fourfold Y that does not contain any plane, one considers the compactification M of the twisted cubics lying on Y inside the Hilbert scheme of Y with Hilbert polynomial $3t+1$. This is not hyperkähler, but it admits a contraction $u\colon M \to Z$ to the hyperkähler eightfold of dimension 8 [LLSvS17, Theorem B]. It is deformation equivalent to the Hilbert scheme of 4 points on a K3 surface ([AL17, LLMS18, Leh99]).
Our first result is

Theorem 0.0.0.1 (see Theorem 2.0.0.2 below)**.** *Let Y be a smooth cubic fourfold not containing a plane. Let*

$$u\colon M \to Z$$

be the contraction from the ten-dimensional space of generalized twisted cubics on Y to the LLSvS symplectic eightfold. Let $C \subset Y \times M$ be the universal curve. Then the pullback

$$u^* : H^2(Z,\mathbb{Z}) \to H^2(M,\mathbb{Z})$$

is injective, and the map

$$[C]_*\colon H^4(Y,\mathbb{Z}) \to H^2(M,\mathbb{Z}) \tag{0.1}$$

restricts to a Hodge isometry

$$[C]_*\colon H^4_{\mathrm{prim}}(Y,\mathbb{Z}) \xrightarrow{\sim} u^*(H^2_{\mathrm{prim}}(Z,\mathbb{Z})), \tag{0.2}$$

with the intersection pairing on the left-hand side and the opposite of the Beauville–Bogomolov–Fujiki pairing on the right.

We actually get this as a corollary of a more general result regarding the full cohomology $H^2(Z,\mathbb{Z})$ and the Mukai lattice [AT14, Definition 2.2] of the Kuznestov component of the cubic fourfold. See Theorem 2.0.0.3 for the precise statement. As an application we characterise for which cubic fourfolds Y the LLSvS variety is birational to the punctual Hilbert scheme of some K3 surface S or a moduli space of twisted sheaves. Recall that in the moduli space $\mathscr{C}$ of (periods of) smooth cubic fourfolds, Hassett described a family of divisors $\mathscr{C}_d$ indexed by natural numbers d, which are not empty exactly for d satisfying

$$d > 6 \text{ and } d \equiv 0 \text{ or } 2 \pmod 6. \tag{$*$}$$

The cubic fourfolds lying in $\mathscr{C}_d$ for d fulfilling the condition

$$d/2 \text{ is not divisible by 9 or any odd prime } p \equiv 2 \pmod 3, \tag{$**$}$$

have a naturally associated $K3$ surface. Let us consider the weaker condition

$$\text{In the prime factorization of } d/2, \text{ primes } p \equiv 2 \pmod 3 \text{ appear with even exponents.} \tag{$**'$}$$

and the condition

$$d \text{ is of the form } \frac{6n^2+6n+2}{a^2} \text{ for some } n, a \in \mathbb{Z}. \tag{$***'$}$$

Remark 0.0.0.2. The construction of the hyperkähler manifold Z works for any smooth cubic fourfold Y not containing a plane. These are exactly smooth cubic fourfolds whose periods do not lie in the Hassett divisor $\mathscr{C}_8$.

Then we can prove

Theorem 0.0.0.3 (see Theorem 3.1.3.8)**.** *Let Y be a cubic fourfold not containing a plane and let Z be its LLSvS eightfold.*

(a) Z is birational to a moduli space of sheaves on a K3 surface if and only if $Y \in \mathscr{C}_d$ for some d satisfying ($**$).

(b) Z is birational to a moduli space of twisted sheaves on a K3 surface if and only if $Y \in \mathscr{C}_d$ for some d satisfying ($**'$).

(c) Z is birational to the Hilbert scheme of four points on a K3 surface if and only if $Y \in \mathscr{C}_d$ for some d satisfying ($***'$).

This will take up Chapters 2 and 3.

In Chapter 4, we will discuss some other results regarding the geometry of the Fano variety F of lines and of the LLSvS variety Z of a cubic fourfold Y. The geometry of these varieties is strictly related one to the other as Voisin showed by constructing a degree 6 rational map

$$\varphi : F \times F \dashrightarrow Z.$$

The indeterminacy locus of φ is the variety I of incidental lines ([Mur20]) and the blow-up of $F \times F$ in I resolves the indeterminacy of φ ([Che18]). Let $\bar{\varphi} : \mathrm{Bl}_I F \times F \to Z$ be the extension of the Voisin map, let $\pi : \mathrm{Bl}_I F \times F \to F \times F$ be the natural projection with exceptional divisor E. The divisor of twisted cubics lying on singular cubic surfaces forms a divisor D in Z, which has two irreducible components D_α and D_β.

Theorem 0.0.0.4 (see Theorem 4.1.3.1 below)**.** *The image $\bar{\varphi}(E) \subset Z$ is D_α.*

Theorem 0.0.0.5 (see Theorem 4.3.2.1 below)**.** *Let $F \simeq \Delta \subset F \times F$ be the diagonal embedding. Then we have*

$$\bar{\varphi}(\pi^{-1}\Delta) = Y \subset Z$$

where $Y \subset Z$ is the lagrangian embedding described in [LLSvS17, Theorem B].

We hope that such a statement can help in giving a nice geometric description of the Chow ring of Z in terms of the Chow ring of $F \times F$ and providing a new[1] proof of the Beauville-Voisin conjecture [Voi08] for the variety Z.

[1] The conjecture has been already solved for the variety Z by Fu, Laterveer and VIal in [FLV20].

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Generalities

In this thesis, we study the geometry of the hyperkähler eightfold, which parametrises flat families of twisted cubics on a smooth cubic fourfold. We relate their periods and we describe the branched divisor of a resolution of the Voisin map. We start with a brief overview of irreducible holomorphically symplectic manifolds, focusing on examples that are relevant to us.
To deal with cubic fourfolds and models of the LLSvS we will work in different contexts. In Sections 2 and 3 we collect the basic tools and protagonists we are going to encounter later.

1.1 Irreducible holomorphically symplectic manifold

Definition 1.1.0.1. *A compact Kähler manifold X is said to be irreducible holomorphically symplectic manifold if it is simply connected and $H^0(\Omega^2_X, X)$ is generated by a holomorphic 2-form σ_X that is non-degenerate on the tangent space at every point.*

In particular, the holomorphic form σ_X defines a symplectic form on the tangent space at every point, hence the name of the manifolds. It follows that such manifolds are even dimensional.

Example 1.1.0.2. Basic examples are K3 surfaces. Indeed, in dimension 2 these are all. A more elaborated example, historically the first in higher dimension, consists of punctual Hilbert schemes of (projective[1]) K3 surfaces ([Bea83, § 6]).
Other families of irreducible holomorphically symplectic manifolds come from moduli spaces of Gieseker semi-stable sheaves on some K3 surface.

Remark 1.1.0.3. Actually, in this definition the condition of being simply connected can be replaced by the triviality of irregularity [Sch20], so that the similarity with K3 surfaces appears stronger.

If X is irreducible holomorphic manifold of complex dimension n then the form σ_x^n trivialises the canonical bundle of X. Indeed, such varieties first appeared when people started to study varieties with trivial canonical class.

[1] In case the K3 surface is not projective, one considers Douady spaces.

Theorem 1.1.0.4 ([Bea83, §2, Théorèm 2]). *Any compact Kähler manifold with trivial canonical class is isomorphic up to an étale cover to a product of a complex torus, Calabi-Yau manifolds and irreducible holomorphically symplectic manifolds.*

Here Calabi-Yau is used in the stronger sense[2], indeed for it we mean a complex Kähler manifold X with trivial canonical class and $H^0(\Omega_X^k, \mathbb{C}) = 0$ for any k strictly between 0 and the dimension of X. In Definition 1.1.0.1 the term irreducible refers to the fact that X cannot be further decomposed.
The underlying real manifold of an irreducible holomorphically sympelctic manifolds admits more than one Kähler structure, for this reason this class of manifolds is referred to as hyperkähler [Huy99, §1]. In the following we will use both.

1.1.1 Periods

Hyperkähler varieties share many properties with K3 surfaces, one among all: their geometry is governed by an intersection form on the second cohomology group with integral coefficients. Let X be a hyperkähler manifold of complex dimension $2n$ and σ_X a 2-form such that $\int (\sigma\bar{\sigma})^n = 1$, then for any $\alpha \in H^2(X, \mathbb{R})$ the formula

$$q_X(\alpha) = (n/2) \int \alpha^2(\sigma\bar{\sigma})^{n-1} + (1-n) \int \alpha(\sigma)^{n-1}(\bar{\sigma})^n \int \alpha(\sigma)^n(\bar{\sigma})^{n-1}$$

defines a quadratic form on the real cohomology. It has signature $(3, b_2(X)-3)$.

Theorem 1.1.1.1 ((Beauville-Bogomolov-Fujiki)). *There exists a positive constant $c_X \in \mathbb{R}$ such that*

$$q_X(\alpha)^n = c_X \int \alpha^{2n}$$

for any $\alpha \in H^2(X, \mathbb{R})$. In particular, q_X can be rescaled such that it is an integral quadratic primitive form on integral cohomology.

Remark 1.1.1.2. Such a integral quadratic form is unique[3] and is called Beauville-Bogomolov form. The constant c_X is named after Fujiki and it has been computed for the known examples of hyperkähler manifolds ([O'G, Table 1]).
The form q_X is a topological invariant in the sense that it is constant in deformation families.

Example 1.1.1.3. Let S be a K3 surface. For its punctual Hilbert scheme $\mathrm{Hilb}^n(S)$ the Beauville-Bogomolov form was computed by Beauville. Indeed, there exists an isometry

$$H^2(\mathrm{Hilb}^n(S), \mathbb{Z}) = H^2(S, \mathbb{Z}) \oplus \mathbb{Z}[E/2]$$

[2] Rather than a Kähler manifold with trivial Chern class, as defined originally in [CHSW85].
[3] Except when $b_2(X) = 6$, in which case there is a sign ambiguity (cf. [GHJ03, Remark 23.15]).

where E is the exceptional divisor of the Hilbert-Chow morphism $\rho\colon \mathrm{Hilb}^n(S) \to S^{(n)}$ be the Hilbert-Chow morphism mapping a scheme to its support counted with multiplicity. More can be said. The whole integral cohomology ring of a K3 surface S can be endowed with the structure of a lattice, named after Mukai. The vector $v = (1, 0, n-1) \in H^*(S, \mathbb{Z})$ is the Mukai vector of the ideal sheaf defining 4 points on S[4]. Taking advantage of the universal closed subscheme $\Xi \subset S \times \mathrm{Hilb}^n(S)$, O'Grady showed that $v^\perp \subset S \times \mathrm{Hilb}^n(S)$ is Hodge isometric to $H^2(\mathrm{Hilb}^n(S), \mathbb{Z})$ ([O'G97, Main Theorem]). We will get back to this theorem in Proposition 1.2.4.2.

Torelli theorem A global Torelli theorem holds for hyperkähler manifolds ([Ver13, Huy12]), the statement involves the period map and monodromy operators. Markman computed the monodromy group of hyperkähler manifolds of K3$^{[n]}$type, so that the Torelli theorem for this class of varieties assumes a very concrete form.

Theorem 1.1.1.4 ([Mar11, Corollary 9.9]). *If two manifolds X and X' of K3$^{[n]}$type, with $n-1$ a prime power then X and X' are bimeromorphic if and only if there is a Hodge isometry $H^2(X, \mathbb{Z}) \simeq H^2(X', \mathbb{Z})$.*

A version of Torelli theorem holds for general manifolds of K3$^{[n]}$type (see [Mar11, §9]), we have recorded this precise statement, because we will make use of it just for the LLSvS eightfold, which being of K3$^{[4]}$ type fulfils its hypothesis.

1.1.2 Some hyperkähler varieties from cubic fourfolds

There are many hyperkähler manifolds associated to smooth cubic fourfolds, above all K3 surfaces, which we will discuss extensively in Chapter 3.
Apart from that, an example comes from the Fano variety of lines. Indeed, Beauville and Donagi showed that if Y is a smooth cubic fourfold of Pfaffian type, then its Fano variety of lines is isomorphic to $\mathrm{Hilb}^2(S)$ for some K3 surface S ([BD85, Proposition 5]) and with a deformation argument they conclude that the Fano variety of any smooth cubic fourfold is hyperkähler. On every smooth cubic fourfold there is a canonical $(1,3)$-form ω_Y, and the symplectic form σ on the Fano variety is the pullback of followed by the pushforward ω_Y along the canonical projections of the universal family of lines.
After this fundamental result, de Jong and Starr investigated in [dJS04] the moduli space $M_d(Y)$ of rational curves of degree d on a smooth cubic fourfold Y[5] and proved interesting results. Indeed, they considered some desingularisations of compactifications of such spaces, any of these has a natural 2-form σ_d: There exists a well-defined pushforward along the projection from the universal correspondence over the Deligne-Mumford moduli stack parametrising flat families of stable maps from curves to Y of

[4]For the precise definition of Mukai lattice and Mukai vector see Sections 1.2.4 and 1.2.2.
[5]The Fano variety of lines is indeed isomorphic to $M_1(Y)$.

fixed genus and degree ([dJS04, Corollary 4.3]). As before the pullback followed by the pushforward along the projection of the universal family gives a 2-form on the stack that in the end descends to the coarse moduli space ([dJS04, § 3]). The form σ_d may fail to be symplectic.
In particular, σ_2 has generically rank 4, whereas $\dim M_2(Y) = 7$; σ_3 has generically rank 8 whereas $\dim M_3(Y) = 10$. A natural problem is then understanding if there exists a MRC fibration of such spaces that admits a hyperkähler compactification.
Given a general plane quadric Q on a cubic fourfold Y, the residual curve to Q in the linear span of Q intersected with Y has degree 1, hence it is a line. This gives a rational map $M_2(Y) \to F$ which is fibred in $\mathbb{P}^3$. So we have no new example of hyperkähler varieties.
In the beautiful work [LLSvS17] the authors show that the compactfication of twisted cubics in the Hilbert scheme $\mathrm{Hilb}^{3t+1}(Y)$ admits a contraction to a hyperkähler manifold of dimension 8 of $\mathrm{K3}^{[4]}$ type. Its geometry is the subject of the present thesis.
The variety of rational quartics has been studied in [Pet18]; for quintics we point the recent result in [LPZ20].

1.2 Fourier-Mukai functors

Fourier-Mukai functors are the most common operations intervening in the derived world of varieties, here we present the basics following quite closely [Huy06a] for this presentation, which we nevertheless point to for a more thorough treatment.
For a smooth projective variety X we denote by $D^b(X)$ the bounded derived category of coherent sheaves. Let X and Y be smooth projective varieties, we denote the natural projections by

$$q : X \times Y \to X \quad p : X \times Y \to Y.$$

Definition 1.2.0.1. [6] *Given an object $\mathscr{P} \in D^b(X \times Y)$, its associated Fourier-Mukai transform is*

$$\Phi_{\mathscr{P}} : D^b(X) \to D^b(Y),\ E \mapsto p_*(\mathscr{P} \otimes q^*E).$$

The object $\mathscr{P}$ is called kernel of the Fourier-Mukai transform.

Fourier-Mukai functors enjoy many properties, the most important are: their composition is again of Fourier-Mukai type, they admit both right and left adjoint functors. Let us see them.
Let X_1, X_2 and X_3 be smooth projective varieties and $\mathscr{P} \in D^b(X_1 \times X_2)$, $\mathscr{Q} \in D^b(X_2 \times X_3)$, then we define the convolution of the two kernels as

$$\mathscr{Q} \circ \mathscr{P} := \pi_{12*}\left(\pi_{12}^*\mathscr{P} \otimes \pi_{23}^*\mathscr{Q}\right).$$

[6] Fourier-Mukai functors were introduced by Mukai to prove that the derived categories of an abelian variety and its dual are equivalent [Muk81]. The name Fourier is due to their resemblance with the concept of Fourier transform for real vector spaces.

where $\pi_{i,j}: X_1 \times X_2 \times X_3 \to X_i \times X_j$ denote the natural projections

Proposition 1.2.0.2. *The composition*

$$D^b(X_1) \xrightarrow{\Phi_{\mathscr{P}}} D^b(X_2) \xrightarrow{\Phi_{\mathscr{Q}}} D^b(X_3)$$

is isomorphic to the Fourier-Mukai transform with kernel $\mathscr{Q} \circ \mathscr{P}$.

Now denote by ω_X and ω_Y the canonical bundles of two smooth projective varieties X and Y. Given $\mathscr{P} \in D^b(X \times Y)$ we set

$$\mathscr{P}_L := \mathscr{P}^\vee \otimes p^*\omega_Y[\dim(Y)]$$
$$\mathscr{P}_R := \mathscr{P}^\vee \otimes q^*\omega_X[\dim(X)]$$

both in $D^b(X \times Y)$.

Proposition 1.2.0.3. *Then we have* $\Phi_{\mathscr{P}_L} \dashv \Phi_{\mathscr{P}}$ *and* $\Phi_{\mathscr{P}} \dashv \Phi_{\mathscr{P}_R}$ *where*

$$\Phi_{\mathscr{P}_R}, \Phi_{\mathscr{P}_L} : D^b(Y) \to D^b(X).$$

1.2.1 ...in K-theory

Let X be a smooth projective variety and let $K(X)$ be the Grothendieck group associated to the derived category $D^b(X)$, there is a natural map

$$D^b(X) \xrightarrow{[-]} K(X),\ E \mapsto \sum (-1)^i [\mathscr{H}^i(E)].$$

Given a projective morphism $f : X \to Y$ we have induced group homomorphisms

$$f^*: K(Y) \to K(X),\ f_*: K(X) \to K(Y);$$

both compatible with the corresponding derived versions.[7] In complete analogy with the previous paragraph, given an element $e \in K(X)$ we will speak of the associated Fourier-Mukai map

$$\Phi_e^K : K(X) \to K(Y),\ f \mapsto p_*(e \otimes q^* f).$$

Here $p: X \times Y \to X$ and $q: X \times Y \to Y$ are again the natural projections. Clearly, the two notions of Fourier-Mukai transform and map are compatible, i.e. we have the following commutative diagram:

$$\begin{array}{ccc} D^b(X) & \xrightarrow{\Phi_{\mathscr{P}}} & D^b(Y) \\ \downarrow{\scriptstyle [-]} & & \downarrow{\scriptstyle [-]} \\ K(X) & \xrightarrow{\Phi^K_{\mathscr{P}}} & K(Y). \end{array}$$

[7] Since we are dealing with a projective morphism f the pushforward f_* is equal on classes of coherent sheaves to the alternating sum of its higher direct images. If f were not projective, we would need to use cohomology with proper support.

Moreover, the K-group is equipped with the integral bilinear form called Euler pairing, which on any $E, F \in D^b(X)$ is defined to be

$$\chi([E],[F]) := \chi(E,F) = \sum (-1)^i \dim \mathrm{Ext}^i(E,F).$$

Remark 1.2.1.1. It is in general not symmetric, though for a variety with trivial canonical bundle one uses Serre duality to show that it is symmetric, respectively alternating, if the dimension of the variety is even, respectively odd.

From the very definition of it, the right (respectively left) adjoint maps $\Phi^K_{P_R}$ and $\Phi^K_{P_L}$ are right (respectively left) adjoint to Φ^K_P with respect the pairing, more precisely

$$\chi(\Phi(E),F) = \chi(E,\Phi^R(F))$$
$$\chi(\Phi^L(E),F) = \chi(E,\Phi(F)).$$

1.2.2 ...in cohomology.

We can now repeat the same story for rational cohomology and thus define the notion of Fourier-Mukai transform. First, we notice that when we treat two smooth complex manifolds X and Y of dimensions n and m we can define the pushforward along a morphism $f : X \to Y$ with the help of Poincaré duality.

$$f_* \colon H^i(X,\mathbb{Q}) \xrightarrow{\sim} H_{2n-i}(X,\mathbb{Q}) \xrightarrow{f_*} H_{2n-i}(Y,\mathbb{Q}) \xrightarrow{\sim} H^{2m-2n+i}(X,\mathbb{Q}).$$

Further, we endow the cohomology ring with complex coefficients with the bilinear pairing, named after Mukai:

$$\langle v, v' \rangle = \int_X \exp(c_1(X)/2) \cdot (v^\vee \cdot v') \tag{1.1}$$

where the $- \cdot -$ denotes the cup product in the cohomology ring. Here the dual in cohomology is defined for any $v \in H^k(X,\mathbb{C})$ as

$$v^\vee := (i)^k v.$$

Therefore, the pairing is defined on even rational cohomology as well.
In order to compare the cohomology and algebraic K-groups one can do as follows. Given an object $E \in D^b(X)$ one defines the Mukai vector as

$$\mathrm{v}(E) = \mathrm{ch}(E) \cdot \mathrm{td}(X)^{1/2} \in H^{2*}(X,\mathbb{Q}).$$

Then taking advantage of the Riemann-Roch theorem one gets the following.

Proposition 1.2.2.1 ([Huy06a, Corollary 5.29]). *The construction of Fourier-Mukai transforms in cohomology and K–theory are compatible, that is for smooth projective varieties X and Y and an object $\mathscr{P} \in D^b(X \times Y)$ the diagram*

$$\begin{array}{ccc} K(X) & \xrightarrow{\Phi^K_{\mathscr{P}}} & K(Y) \\ \downarrow \mathrm{v} & & \downarrow \mathrm{v} \\ H^*(X,\mathbb{Q}) & \xrightarrow{\Phi^H_{\mathscr{P}}} & H^*(Y,\mathbb{Q}). \end{array} \tag{1.2}$$

is commutative. Moreover, the Mukai vector respects the pairing on the two groups, i.e. for any object $E, F \in D^b(X \times Y)$

$$\chi([E],[F]) = \langle \mathrm{v}(E), \mathrm{v}(F) \rangle.$$

It is good to point out that in general the integral structure of K–groups and cohomology groups may behave very differently. Although the image of the Mukai pairing may be very small, we have though this surprising result.

Proposition 1.2.2.2 ([Huy06a, Proposition 5.44]). *If the Fourier-Mukai functor $\Phi_{\mathscr{P}} : D^b(X) \to D^b(Y)$ is an equivalence, then $\Phi^H_{\mathscr{P}}$ is an isometry on the whole rational cohomology rings.*

1.2.3 Topological K-theory

We now turn to topological K-theory. In the end for the spaces we are interested in (namely K3 surfaces and smooth cubic fourfolds) their topological K-group coincides with the Grothendieck ring of topological $\mathbb{C}$-vector bundles, which is isomorphic via the Mukai vector to the integral cohomology ring. Here we give a brief account, references are [AH62], [AH59] and [AH61].
For a smooth variety X we set:

$$K_{\mathrm{top}}(X) = K_0(X) \oplus K_1(X).$$

where $K_1(X)$ is the kernel of the homomorphism $K_0(X \times S^1) \to K_0(X)$ induced from the embedding $X \to X \times S^1$. Making use of the Bott periodicity theorem, $K_{\mathrm{top}}(X)$ is turned in a graded anti-commutative ring [AH61, Proposition page 203]. The analytification of sheaves on X induces a natural morphism From algebraic K-theory to topological K-theory

$$K(X) \to K_{\mathrm{top}}(X).$$

A map

$$\mathrm{ch} : K_{\mathrm{top}}(X) \to H^*(X,\mathbb{Q})$$

is constructed. It extends the usual Chern character from the Grothendieck group and it is called Chern character, too. It has the following properties:

- The Chern character is a natural transformation between K-theory and cohomology. That means, given a morphism $f : X \to Y$
$$\mathrm{ch} \circ f^* = f^* \circ \mathrm{ch}$$
where the f^* denote the usual pullback in cohomology on one side while in K–theory is given by lifting vector bundles along the morphism.
- It is bijective once tensored with rational numbers. Moreover, it takes $K_0(X) \otimes \mathbb{Q}$ onto $H^{2*}(X, \mathbb{Q})$ and $K_1(X) \otimes \mathbb{Q}$ onto $H^{2*+1}(X, \mathbb{Q})$.
- Provided that integral cohomology is torsion free, the Chern character is injective.

For a pretty wide variety of topological spaces, a continuous map $f : X \to Y$ defines a pushforward $f_* : K_{\mathrm{top}}(X) \to K_{\mathrm{top}}(Y)$, which satisfies a projection formula and a Riemann-Roch-type theorem. In case of a proper morphism between smooth projective varieties f_* is compatible with the usual direct image between algebraic Grothendieck groups [AH62, Theorem 4.2]. With such a pushforward at our disposal, one defines an integral Euler pairing on the topological K–group. Practically, for $\pi : X \to \mathrm{Spec}(\mathbb{C})$ a smooth projective variety and for any $e, f \in K_{\mathrm{top}}(X)$ we set:
$$\chi(e, f) := \pi_*(e^\vee \otimes f) \in K_{\mathrm{top}}(\mathrm{point}) \simeq \mathbb{Z}.$$
Clearly by the very definition this agrees with the Euler pairing on the algebraic K-group and, as discussed above, thanks to the Riemann-Roch theorem the Mukai vector respects the pairings. Having at our disposal the pushforward, we can transpose the notion of Fourier-Mukai transforms in topological K–theory, i.e for an element $e \in K_{\mathrm{top}}(X \times Y)$ we set
$$\begin{aligned} \Phi_e^K : K_{\mathrm{top}}(X) &\to K_{\mathrm{top}}(Y) \\ f &\mapsto p_*(e \otimes q^* f). \end{aligned}$$
In the end having set this general footing, for any smooth projective varieties X, Y and an object $P \in D^b(X \times Y)$ we have the following diagram, where every square is commutative:
$$\begin{array}{ccccccc} D(X) & \longrightarrow & K_{\mathrm{alg}}(X) & \longrightarrow & K_{\mathrm{top}}(X) & \xrightarrow{v} & H^*(X, \mathbb{Q}) \\ \downarrow{\scriptstyle \Phi_P} & & \downarrow{\scriptstyle \Phi_P^K} & & \downarrow{\scriptstyle \Phi_P^K} & & \downarrow{\scriptstyle \Phi_P^H} \\ D(Y) & \longrightarrow & K_{\mathrm{alg}}(Y) & \longrightarrow & K_{\mathrm{top}}(Y) & \xrightarrow{v} & H^*(Y, \mathbb{Q}). \end{array}$$

Notation In the following we will drop much of the notation because it makes the reading much lighter and causes no confusion. Therefore, any Fourier-Mukai map will be denoted by the kernel, dropping all the decorations referring to which context is applied (derived, cohomological or K–theory).
We will not even specify which pushforward functor we are thinking of. In case of confusion we limit ourself to specify if it is derived or not in case of a sheaf.

1.2.4 O'Grady result revisited.

K-theory of K3 surfaces Let S be a K3 surface, then by explicit computation one gets that the Mukai vector for an object $E \in D^b(S)$ is integral. First one computes the Todd class

$$\mathrm{td}(S) = (1, 0, 2)$$

and then

$$v(E) = \mathrm{ch}(E) \cdot \sqrt{\mathrm{td}(S)} = \big(\mathrm{rk}(E), c_1(E), \mathrm{rk}(E) + c_1^2(E)/2 - c_2(E)\big).$$

Since the intersection form on $H^2(S, \mathbb{Z})$ is even, it is indeed an integral vector. The same holds for a complex vector bundle on S and we have

Proposition 1.2.4.1. *Let S be a K3 surface, then the Mukai vector*

$$\mathrm{v}\colon K_{\mathrm{top}}(S) \to H^*(S, \mathbb{Q})$$

factors through an isometry on the integral cohomology.

Proof. We have seen that the Mukai vector for a K3 surface S is integral. Since the integral cohomology of S is torsion free, it is injective by [AT14, Theorem 2.1, (1)] and, using [AT14, Theorem 2.1, (3)], one easily shows that it is surjective. □

Moreover, since the canonical bundle on S is trivial, the Mukai pairing is symmetric on even cohomology and therefore on the whole cohomology. The pairing can be explicitly so written

$$\langle (v_1, v_2, v_3), (w_1, w_2, w_3) \rangle = v_o w_4 + v_1 w_1 + v_4 w_0$$

for any $(v_0, v_2, v_4), (w_0, w_2, w_4) \in H^*(S, \mathbb{Z})$.
One endows $H^*(S, \mathbb{Z})$ with a Hodge structure by retaining the one on $H^2(S, \mathbb{Z})$ and by imposing $H^0(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})$ to be of weight two. Via the Mukai vector one endows K_{top} with an integral Hodge structure as well.

O'Grady result revisited. We rewrite [O'G97, Main Theorem] in terms of the setting we have introduced.

Proposition 1.2.4.2. *Let S be a K3 surface and $S^{[n]}$ be its Hilbert scheme of n points, then the composition*

$$[I_\xi]^\perp \subset K_{\mathrm{top}}(S) \xrightarrow{I_\Xi^\vee} K_{\mathrm{top}}(S^{[n]}) \xrightarrow{c_1} H^2(S^{[n]}, \mathbb{Z}) \tag{1.3}$$

where I_ξ is the ideal sheaf of four points in S, is a Hodge isometry.

Proof. We consider the map (1.3). By the above discussion on K3 surfaces passing to cohomology is harmless, indeed the Mukai vector takes the class of I_ξ to $v = v(I_\xi) = (1, 0, n-1)$ and takes $[I_\xi]^\perp$ isometrically to $v^\perp$. We are then left to prove that

$$v^\perp \hookrightarrow H^*(S, \mathbb{Z}) \xrightarrow{I_\Xi^\vee} H^*(S^{[n]}, \mathbb{Q}) \xrightarrow{c_1} H^2(S^{[n]}, \mathbb{Z}). \tag{1.4}$$

is an isometry. For every $k \in v^\perp$ we compute

$$\begin{aligned} k \mapsto\ & c_1\left(p_*\left(v(I_\Xi^\vee) \cdot q^* k\right)\right) = \\ & c_1\left(p_*\left(\mathrm{ch}(I_\Xi)^\vee \cdot p^* \mathrm{td}(S^{[n]})^{1/2} \cdot q^* \mathrm{td}(S)^{1/2} \cdot q^* k\right)\right) = \\ & c_1\left(\mathrm{td}(S^{[n]})^{1/2} \cdot p_*\left(\mathrm{ch}(I_\Xi)^\vee \cdot q^* \mathrm{td}(S)^{1/2} \cdot q^* k\right)\right) \end{aligned} \tag{1.5}$$

where the last equality holds because of projection formula. The Todd class of a K3 surface was discussed above, its root is the sum of 1 and the fundamental class of the surface. Since $S^{[n]}$ is hyperkähler, its first Chern class is trivial as well as its canonical bundle and its Todd class does not contribute to the last quantity. Finally, $S^{[n]}$ being a fine moduli space [8], the similitude of I_Ξ is 1[9] (in other words its rank is one) and thus, after the computation (1.5), the map in (1.4) coincides with the O'Grady map, which is known to be an isometry [O'G97, Main Theorem]. □

1.3 Kuznetsov component and its Mukai lattice

The world of cubic fourfolds and K3 surfaces are deeply intertwined: Hassett investigated it from a Hodge theoretic view point. More recently Kuznetsov studied their derived categories and thanks to Addington and Thomas now we can compare the two perspectives. In this section we want to introduce the Kuznetsov component of the derived category of cubic fourfolds (see Theorem 1.3.1.7) and its Mukai lattice in topological K-theory (see Definition 1.3.2.1).

1.3.1 Kuznetsov component

Let $\mathscr{T}$ be a triangulated category and $i\colon \mathscr{B} \subset \mathscr{T}$ a subtriangulated category. For simplicity we will restrict to categories that are $\mathbb{C}$–linear and with finite dimensional Hom vector spaces.

[8] Here, we should more correctly say that $S^{[n]}$ is isomorphic to the moduli space of semistable sheaves with fixed Mukai vector $(1, 0, n-1)$, the isomorphism being given by mapping a subscheme of length n to its ideal sheaves. For an accurate proof of such statement we point to [KPS18, Lemma B.5.6].

[9] If M is a moduli space of semistable sheaves with Hilbert polynomial P on S, a quasi-universal family of semistable sheaves $F \in \mathrm{Coh}(S \times M)$ is called quasi-universal of similitude ρ if for any T-flat family $E \in \mathrm{Coh}(S \times T)$ of semistable sheaves with Hilbert polynomial P there exists a locally free $\mathscr{O}_T$-module V of rank ρ such that $E \otimes p^* V = \Phi^* F$, where $p\colon S \times T \to T$ is the projection and $\Phi\colon T \to M$ is the classifying morphism.

Definition 1.3.1.1. *The category $\mathscr{B}$ is said to be right admissible if the following equivalent conditions are satisfied:*

(i) The inclusion functor i admits a right adjoint funtor $i \dashv R$.

(ii) For every element $T \in \mathscr{T}$ there exists an exact triangle

$$B \to T \to C \to B[1]$$

where B is in an element in $\mathscr{B}$ and C is such that $\mathrm{Hom}_{\mathscr{T}}(B_1, C) = 0$ for every object $B_1 \in \mathscr{B}$.

Specularly one can define the notion of left admissible for a subtriangulated category. If cB is both right and left admissible, then it is called admissible.

To see the equivalence: For the direct implication, one uses the counit morphism and complete it to a triangle to get the exact triangle

$$iR(X) =: B \xrightarrow{counit} T \to C \to B[1].$$

Invoking the faithfulness of Yoneda functor one checks that C satisfies the wanted condition. Conversely, taking advantage of the condition on the Hom sets, one shows that the triangle in condition (ii) is unique, the association $T \mapsto B$ is well-defined and gives rise to a functor $R : \mathscr{T} \to \mathscr{B}$. Applying $\mathrm{Hom}(B_1, -)$ to the triangle, one gets that R is right adjoint.
The two notions of left and right admissible subtriangulated category are strictly bounded together. Indeed, if we set

$$\mathscr{B}^{\perp} = \{T \in \mathscr{T} : \mathrm{Hom}_{\mathscr{T}}(B, T) = 0,\ \forall B \in \mathscr{B}\}$$
$$^{\perp}\mathscr{B} = \{T \in \mathscr{T} : \mathrm{Hom}_{\mathscr{T}}(T, B) = 0,\ \forall B \in \mathscr{B}\}$$

Applying the same ideas above, it is easy to check that $\mathscr{C} := \mathscr{B}^{\perp}$ is left admissible and that $^{\perp}(\mathscr{B}^{\perp}) = \mathscr{B}$. In the end, we have that $\mathscr{B}$ and $\mathscr{C}$ are two subtriangulated categories that generate $\mathscr{T}$ and each is orthogonal (with direction) to the other. We then come to the following more general

Definition 1.3.1.2. *Given a triangulated category $\mathscr{T}$, a semi-orthogonal decomposition*

$$\mathscr{T} = \langle \mathscr{T}_1, ..., \mathscr{T}_n \rangle$$

of $\mathscr{T}$ consists of a sequence of full subtriangulated categories $\mathscr{T}_1, ..., \mathscr{T}_n$ such that

- $\mathrm{Hom}(\mathscr{T}_i, \mathscr{T}_j) = 0$ *for every $i > j$.*

- *For every object E in $\mathcal{T}$ there exists a sequence of morphisms*

$$0 = E_n \to E_{n-1} \to \ldots \to E_1 \to E_0 = E$$

such that $\mathrm{cone}(E_i \to E_{i-1}) \in \mathcal{T}_i$ *for* $1 \leq i \leq n$.

Remark 1.3.1.3.

- As before, taking for any object $E \in \mathcal{T}$ the objects $\mathrm{cone}(E_i \to E_{i-1})$, which appear in the definition, gives rise to functors $\mathcal{T} \to \mathcal{T}_i$, which do not admit in general neither left nor right adjoints.

- The case, where there are just two components, is exactly what has been discussed at the beginning. This definition can be reduced inductively to the case of two components in the sense of the following writing:

$$\mathcal{T} = \langle \mathcal{T}_1, \langle \mathcal{T}_2, \ldots, \mathcal{T}_n \rangle \rangle = \langle \mathcal{T}_1, \langle \mathcal{T}_2, \langle \mathcal{T}_3, \ldots, \mathcal{T}_n \rangle \rangle \rangle = \ldots$$

- In the case a subtriangulated category $\mathcal{B}$ which is both left and right admissible, in short admissible, then we have two semiorthogonal decompositions

$$\mathcal{T} = \langle \mathcal{B}, \mathcal{B}^{\perp} \rangle = \langle {}^{\perp}\mathcal{B}, \mathcal{B} \rangle.$$

Moreover, in the presence of a Serre functor S_X, e.g. in the geometric case of the derived category $\mathcal{T} = D^b(X)$ of a smooth projective variety X, the two left and right orthogonals are equivalent via S_X:

$$S_X : {}^{\perp}\mathcal{B} \simeq \mathcal{B}^{\perp}.$$

A first example of a semi-orthogonal decomposition is that of an orthogonal decomposition, that means two subcategories that generate and are both left and right orthogonal to each other. By a result of Bridgeland this does not appear in the geometric case of an connected smooth variety (cf. [Huy06a, Proposition 3.10]). Relevant examples to us are given by exceptional objects and collections.

Definition 1.3.1.4. *(i) An object $E \in \mathcal{T}$ is exceptional if* $\mathrm{Hom}(E, E[p]) = 0$ *for all integers $p \neq 0$ and* $\mathrm{Hom}(E, E) \simeq \mathbb{C}$.

(ii) A set of objects $\{E_1, \ldots, E_m\}$ in $\mathcal{T}$ is an exceptional collection if E_i is an exceptional object for every i and $\mathrm{Hom}(E_i, E_j[p]) = 0$ *for all p and all $i > j$.*

(iii) An exceptional collection $\{E_1, \ldots, E_m\}$ is said to be full if the smallest subtriangulated category containing the collection is equivalent to $\mathcal{T}$.

The smallest subtriangulated category $\langle E\rangle$ containing an exceptional object E is admissible. Indeed, the right and left adjoints to the inclusion are given by

$$F \mapsto \mathrm{Hom}^\bullet(E,F)\otimes E$$
$$F \mapsto \mathrm{Hom}^\bullet(F,E)^*\otimes E$$

Here, it is fundamental to assume that the category has finite dimensional Hom (vector spaces). Exceptional objects define admissible subcategories, therefore from exceptional collections we get semi-orthogonal decompositions

$$\mathscr{T} = \langle \mathscr{A}, E_1, ..., E_m\rangle = \langle E_1, ..., E_m, \mathscr{B}\rangle$$

where $\mathscr{A} = E_1^\perp \cap ... \cap E_m^\perp$ and $\mathscr{B} = {}^\perp E_1 \cap ... \cap {}^\perp E_m$. Notice that $\mathscr{A}$ and $\mathscr{B}$ are abstractly equivalent but differently embedded. In this case, to pass from one description to another it is rather easy.

Definition 1.3.1.5. *Let E be an exceptional object. The functors*

$$\mathbb{L}_E : F \mapsto \mathrm{cone}(\mathrm{Hom}(E,F)\otimes E \to F)$$
$$\mathbb{R}_E : F \mapsto \mathrm{cone}(F^\vee \to \mathrm{Hom}(E,F)^\vee \otimes E^\vee)$$

are called left and right mutation past E.

These functors have this special property (e.g. [Kuz09, Theorem 2.9,2.10]): If $\mathscr{T} = \langle E_1, ..., E_m\rangle$ is an orthogonal decomposition given by a collection of exceptional objects, then

$$\mathscr{T} = \langle E_1, ...E_{i-1}, \mathbb{L}_{E_i}E_{i+1}, E_i, E_{i+2}, .., E_m\rangle$$

is a semi-orthogonal decomposition as well. Similarly, using the right mutating functor, we get the semi-orthogonal decomposition

$$\mathscr{T} = \langle E_1, ...E_{i-1}, E_i, \mathbb{R}_{E_i}E_{i-1}, E_{i+2}, .., E_m\rangle.$$

As an example to illustrate the theory introduced so far we present the classical result from Beilinson: the semiorthogonal decomposition of $D^b(\mathbb{P}^n)$.

Example 1.3.1.6 ([Huy06a, Corollary 8.29])**.** The derived category of $\mathbb{P}^n$ admits the semiorthogonal decomposition

$$D^b(\mathbb{P}^n) = \langle \mathscr{O}_{\mathbb{P}^n}(-n), ..., \mathscr{O}_{\mathbb{P}^n}\rangle.$$

From the line

$$\mathrm{Hom}^\bullet(\mathscr{O}_{\mathbb{P}^n}(i), \mathscr{O}_{\mathbb{P}^n}(j)) = H^\bullet(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(j-i)) = \begin{cases} \mathbb{C} & if\ i=j, \\ 0 & if\ -n \le j-i < 0 \end{cases}$$

we see that it is an exceptional collection. Using the Beilinson resolution of the diagonal Δ in $\mathbb{P}^n \times \mathbb{P}^n$

$$0 \to \wedge^n(\mathcal{O}(-1) \boxtimes \Omega(1)) \to \wedge^{n-1}(\mathcal{O}(-1) \boxtimes \Omega(1)) \to \ldots \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}_n} \to \mathcal{O}_\Delta \to 0$$

one proves the fullness of the sequence, that is the sequence generates the whole category $D^b(\mathbb{P}^n)$ (cf. [Huy06a, Corollary 8.29]).
Once this decomposition is proven, one mutates $\mathcal{O}_{\mathbb{P}^n}(-n)$ right past through the other sheaves and finds another decomposition

$$D^b(\mathbb{P}^n) = \langle \mathcal{O}_{\mathbb{P}^n}(-n+1), \ldots, \mathcal{O}_{\mathbb{P}^n}, \mathbb{R}_{-n+1}\mathbb{R}_{-n+2}\ldots\mathbb{R}_1\mathcal{O}_{\mathbb{P}^n}(-n+1)\rangle$$

The object $\mathbb{R}_{-n+1}\mathbb{R}_{-n+2}\ldots\mathbb{R}_1\mathcal{O}_{\mathbb{P}^n}(-n+1)$ is now equal to $\mathcal{O}_{\mathbb{P}^n}(1)$, because it is orthogonal to $\mathcal{O}_{\mathbb{P}^n}(k)$ for every $k = -n+1, .., 0$ (cf. Remark 1.3.1.3). So we get that for $i = 1$

$$D^b(\mathbb{P}^n) = \langle \mathcal{O}_{\mathbb{P}^n}(-n+i), \ldots, \mathcal{O}_{\mathbb{P}^n}(i)\rangle \ .$$

is a semiorthogonal decomposition as well. Iterating we prove that these are semiorthogonal decompositions for any $i \in \mathbb{Z}$. So we get the claimed decomposition for $i = 1$, iterating we get the others, too.

Let X be a smooth projective Fano variety of index k, that is the canonical bundle K_X is anti-ample and $-K_X = \mathcal{O}_X(k)$, where $\mathcal{O}_X(1)$ denotes the restriction of the polarisation. Relying on Beilinson's result (cf. Example 1.3.1.6) one proves $\mathcal{O}_X, \ldots, \mathcal{O}_X(k-1)$ is an exceptional collection. Because of Serre duality we cannot go further with $\mathcal{O}_X(k)$ (in the sense that $\mathcal{O}_X, \ldots, \mathcal{O}_X(k-1)$ is not exceptional), nevertheless the collection may not generate, in contrast with the example of $\mathbb{P}^n$ above. It becomes therefore of interest to study the orthogonal complement of the exceptional collection. This has been done in great generality, in the case of cubic fourfolds (which are Fano of index 4) we have the following result by Kuznetsov.

Theorem 1.3.1.7 ([Kuz10]). *Let Y be a smooth cubic fourfold then*

$$\mathcal{A} := \langle \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2)\rangle^\perp$$

is a non-commutative K3 surface, that means that $\mathcal{A}$ is indecomposable, with Serre functor given by the shift by 2 and with Hochschild cohomology and homology of a K3 surface.

The category $\mathcal{A}$ is known as Kuznetsov component. In general, it is not known when it is equivalent to the derived category of an honest K3 surface: According to the Kuznetsov conjecture ([Kuz10, Conjecture 1.1]) it is so, when Y is rational (see [HLS19, § 3] for a more comprehensive treatise).

Remark 1.3.1.8. There is nothing special about the sequence $\mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2)$, indeed one could have chosen another sequence, too. Tensoring with $\mathcal{O}_Y(-1)$ give rises to an autoequivalence of $D^b(Y)$ and we get a new semi-orthogonal decomposition

$$D(Y) = \langle \mathcal{A}', \mathcal{O}_Y(-1), \mathcal{O}_Y, \mathcal{O}_Y(1) \rangle.$$

Applying the right mutation $\mathbb{R}_{\mathcal{O}_Y(-1)}$ we get

$$D(Y) = \langle \mathcal{O}_Y(-1), \mathbb{R}_{\mathcal{O}_Y(-1)}(\mathcal{A}'), \mathcal{O}_Y, \mathcal{O}_Y(1) \rangle = \langle \mathbb{R}_{\mathcal{O}_Y(-1)}(\mathcal{A}'), \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle.$$

In the end we have an autoequivalence

$$\mathcal{A} \xrightarrow{\otimes \mathcal{O}(-1)} \mathcal{A}' \xrightarrow{\mathbb{R}_{\mathcal{O}_Y(1-)}} \mathcal{A}.$$

1.3.2 Topological K–theory of the Kuznetsov component

Addington and Thomas introduced a Mukai lattice for the Kuznetsov component and studied their properties taking advantage of the topological K–theory of cubic fourfolds. We introduce here the definitions and postpone to Chapter 3 the arithmetic properties. Anything herein comes from [AT14].
Let Y be a smooth cubic fourfold and let $\mathcal{A}$ be the associated Kuznetsov component.

Definition 1.3.2.1. *The Mukai lattice of $\mathcal{A}$ is*

$$K_{\mathrm{top}}(\mathcal{A}) = \{k \in K_{\mathrm{top}}(Y) \colon \chi(\mathcal{O}_Y(i), k) = 0 \text{ for } i = 0, 1, 2\}.$$

It is endowed by a Hodge structure by pulling-back the one of weight zero on the total cohomology of the cubic fourfold

$$H^*(Y, \mathbb{Z}) = \bigoplus_i H^{2i}(Y, \mathbb{Z})(i)$$

along the Chern character.

In other words, the group underlying $K_{\mathrm{top}}(\mathcal{A})$ is the right orthogonal complement to the group generated by the classes of $\mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2)$. Thus, we have a natural projection $\mathrm{pr} \colon K_{\mathrm{top}}(Y) \to K_{\mathrm{top}}(\mathcal{A})$. The Euler pairing endows $K_{\mathrm{top}}(\mathcal{A})$ with the structure of a lattice ([Huy18, Proposition 1.20]). We set

$$\lambda_1 := \mathrm{pr}[\mathcal{O}_{\mathrm{line}}(1)] \qquad \lambda_2 := \mathrm{pr}[\mathcal{O}_{\mathrm{line}}(2)]$$

where $\mathrm{pr} \colon K_{\mathrm{top}}(Y) \to K_{\mathrm{top}}(\mathcal{A})$ is the orthogonal projection. The two elements generate a primitive sublattice which is isomorphic to A_2 (e.g. [Huy18, Lemma 1.18]), the orthogonal complement is the primitive cohomology.

Proposition 1.3.2.2 ([AT14, Thm 6]). *The Mukai vector $\mathrm{v} \colon K_{\mathrm{top}}(Y) \to H^*(Y, \mathbb{Q})$ takes $\langle \lambda_1, \lambda_2 \rangle^{\perp} \subset K_{\mathrm{top}}(\mathcal{A})$ isometrically onto $H^4_{\mathrm{prim}}(Y, \mathbb{Z})$.*

We will come back to other properties of the Mukai lattice in Chapter 4, where we will spend some words on the arithmetic properties of $K_{\mathrm{top}}(\mathcal{A})$.

2

The periods of the Lehn–Lehn–Sorger–van Straten variety

This chapter is devoted to compute the periods of the LLSvS variety, or better to relate them to the periods of the associated cubic fourfold; this work appeared in our paper [AG20] in collaboration with Nicolas Addington.
Periods are one of the most important invariants of hyperkähler manifolds, indeed the integral Hodge structure on the second cohomology group governs most of the geometry of such manifolds as expressed in the Torelli theorem.
Therefore, understanding the Hodge structure is of fundamental importance. Various results are known in this direction, of paramount importance for us will be the case of the Fano varieties of lines on a smooth cubic fourfold. Beauville and Donagi [BD85], after having proven that such a variety is hyperkähler, showed that the cubic fourfold Y and its Fano variety of lines F have the same periods, the isomorphism being given by the correspondence of the universal family of lines.

Theorem 2.0.0.1 ([BD85, Proposition 4]). *Let Y be a smooth cubic fourfold, F be its Fano variety of lines and let $\mathbb{L} \subset F \times Y$ be the universal family of lines, with $p : \mathbb{L} \to Y$ and $q : \mathbb{L} \to F$ the natural projections. Then*

$$q_* p^* : H^4_{\text{prim}}(Y, \mathbb{Z}) \to H^2_{\text{prim}}(F, \mathbb{Z}) \tag{2.1}$$

is an isometry.

Our result relates the cohomology of the cubic fourfold with the associated LLSvS variety. We begin with the following statement regarding the primitive cohomology.
Let Y be a smooth cubic fourfold not containing a plane. We consider the contraction

$$u : M \to Z$$

from the ten-dimensional space of generalized twisted cubics on Y to the LLSvS symplectic eightfold (see [LLSvS17, Theorem B]). Let $C \subset Y \times M$ be the universal curve.

Theorem 2.0.0.2 ([AG20, Theorem 1]). *The pullback*

$$u^* : H^2(Z, \mathbb{Z}) \to H^2(M, \mathbb{Z})$$

is injective, and the map

$$[C]_*: H^4(Y,\mathbb{Z}) \to H^2(M,\mathbb{Z}) \tag{2.2}$$

restricts to a Hodge isometry

$$[C]_*: H^4_{\mathrm{prim}}(Y,\mathbb{Z}) \xrightarrow{\sim} u^*(H^2_{\mathrm{prim}}(Z,\mathbb{Z})), \tag{2.3}$$

with the intersection pairing on the left-hand side and the opposite of the Beauville–Bogomolov–Fujiki pairing on the right.

For a statement about the full $H^2(Z,\mathbb{Z})$, we must recall Kuznetsov's K3 category

$$\mathscr{A} = \langle \mathscr{O}_Y, \mathscr{O}_Y(h), \mathscr{O}_Y(2h)\rangle^{\perp} \subset D^b(Y),$$

the Mukai lattice introduced in [AT14, §2],

$$K_{\mathrm{top}}(\mathscr{A}) = \{[\mathscr{O}_Y],[\mathscr{O}_Y(h)],[\mathscr{O}_Y(2h)]\}^{\perp} \subset K_{\mathrm{top}}(Y),$$

and the two special classes $\lambda_1, \lambda_2 \in K_{\mathrm{top}}(\mathscr{A})$, which we discussed in Section 1.3.2.

Theorem 2.0.0.3 ([AG20, Theorem 1]). *We retain the notation of Theorem 2.0.0.2. Let*

$$\Phi^K: K_{\mathrm{top}}(\mathscr{A}) \subset K_{\mathrm{top}}(Y) \to K_{\mathrm{top}}(M)$$

be the map on topological K-theory induced by the Fourier–Mukai kernel $I_C^{\vee}(-3h) \in D^b(Y \times M)$. Then the map

$$c_1 \circ \Phi^K : K_{\mathrm{top}}(\mathscr{A}) \to H^2(M,\mathbb{Z}) \tag{2.4}$$

restricts to a Hodge isometry

$$\langle \lambda_2 - \lambda_1\rangle^{\perp} \xrightarrow{\sim} u^*(H^2(Z,\mathbb{Z})), \tag{2.5}$$

with the Euler pairing on the left-hand side and the opposite of the Beauville–Bogomolov–Fujiki pairing on the right.

One of the main ideas of [AL17] is that Z is a moduli space of complexes in $\mathscr{A}$ containing the projections of $\mathscr{O}_y$ for points $y \in Y$; the K-theory class of these complexes is $\lambda_2 - \lambda_1$, so (2.5) is natural in light of O'Grady's description of the period of a moduli space of sheaves on a K3 surface [O'G97, Main Theorem]. Li, Pertusi, and Zhao developed this idea further using Bridgeland stability conditions in [LPZ18]; note that their class $2\lambda_1 + \lambda_2$ is related to our $\lambda_2 - \lambda_1$ by an autoequivalence of $\mathscr{A}$ (e.g. [Huy17a, Proposition 3.12]). Statements similar to Theorem 2.0.0.3 appear in [LPZ18, Proposition 5.2] and [BLM+19, Theorem 29.2(2)]. We believe nonetheless that our approach is of its own interest as our proofs require less machinery, the statement in Theorem 2.0.0.2 is very classical and explicit.[1]

[1] After the first version of the paper [AG20] was posted, Li, Pertusi, and Zhao informed us that the submitted version of [LPZ18] includes a statement analogous to our Theorem 3(c).

2.1 A first approach to the periods of LLSvS variety

Here we propose a first naive approach to relate the periods of the LLSvS variety to the ones of the cubic fourfolds. This is instructive and will elaborate on this in the next sections, where we get the main result.
The first idea was to look at the correspondence given by the universal curve $C \subset Y \times M$, although it looks very difficult to descend the action on Z, because there is no family of curves on Z that could do the work. Moreover, the family C per se is not smooth, thus it is better to look at the action by cup product of its cohomology class $[C]$ and at the composition in cohomology with rational coefficients:

$$H^4(Y,\mathbb{Q}) \xrightarrow{\mathrm{pr}_Y^*} H^4(Y\times M',\mathbb{Q}) \xrightarrow{-\cup[C]} H^{10}(Y\times M',\mathbb{Q}) \xrightarrow{\mathrm{pr}_{M*}} H^2(M,\mathbb{Q}).$$

This is a natural generalisation of the correspondence used by Beauville and Donagi, because being the universal family of lines $\mathbb{L} \subset Y \times F$ smooth, the morphism (2.1) equals

$$H^4(Y,\mathbb{Q}) \xrightarrow{\mathrm{pr}_Y^*} H^4(Y\times F,\mathbb{Q}) \xrightarrow{-\cup[\mathbb{L}]} H^{8}(Y\times F,\mathbb{Q}) \xrightarrow{\mathrm{pr}_{F*}} H^2(F,\mathbb{Q}).$$

This follows from [Voi02, Exercise 11.1].
The next step would be to find a natural morphism onto the cohomology of Z, however the pushforward u_* along the contraction $u\colon M \to Z$ is useless for us, because of the dimension gap. Indeed, u_* acts on cohomology with a drop by 4 for cohomological degree, thus is trivial on the second degree cohomology of M. On the contrary, it is better to pullback the cohomology along u (u^* is in effect injective (cf. Lemma 2.3.1.1) and at the same time find some geometric object such as a correspondence or a family on the variety $Y \times Z$.
To this end, it turns out to be very useful to look at various geometric interpretations of the variety Z, there are at least three at our disposal: the birational models presented in [Leh18], in [LLMS18] and in [AL17]. [2] The first deals with singular cubic fourfolds, thus not interesting for our purposes; the latter is treated in the next section.
Recall that by construction of Z is obtained in [LLSvS17] in more steps: First one considers the natural morphism

$$M \to \mathrm{Gr}(4,6)$$

from the variety of generalised twisted cubics on Y to the Grassmannian of linear 3-spaces in $\mathbb{P}^5$, which maps any generalised twisted cubic to its linear span. This factors through a smooth variety Z' of dimension 8

$$\begin{array}{ccc} M & & \\ \downarrow{\scriptstyle a} & \searrow & \\ Z' & \xrightarrow{\sigma} & \mathrm{Gr}(4,6). \end{array}$$

[2] At the time of this work the preprint [LPZ18] was not yet available.

The variety Z is obtained from a (divisorial) further contraction $b\colon Z' \to Z$. We denote by u the composition $M \to Z' \to Z$.
The strategy adopted in this section is based on the result of Lahoz, Lehn, Macrì and Stellari, who describe the variety Z' as a moduli space of Gieseker stable sheaves (We were not able to take advantage of their major result about Z (cf. [LLMS18, Theorem B])!). Having this at our disposal we will cook up a correspondence $\phi : H^4(Y.\mathbb{Z}) \to H^2(Z', \mathbb{Z})$ to get the following Theorem 2.1.0.1.
Let us define ϕ. Let U be the quasi-universal family on $Y \times Z'$ and let ρ be its similitude. The Grassmaniann $\mathrm{Gr} := \mathrm{Gr}(4, V)$ parametrises a family of cubic surfaces on Y given by intersecting each $E \in \mathrm{Gr}$ with Y, we call $\mathscr{S}$ the pullback of this family along the morphism $\sigma\colon Z' \to \mathrm{Gr}$. We define

$$F := \mathrm{ch}_3(\mathscr{O}_{\mathscr{S}}) - \frac{\mathrm{ch}_3(U)}{\rho(U)} \in H^6(Y \times Z', \mathbb{Q})$$

and call ϕ the composition in cohomology with rational coefficients

$$H^4(Y, \mathbb{Q}) \xrightarrow{\mathrm{pr}_Y^*} H^4(Y \times Z', \mathbb{Q}) \xrightarrow{-\cup F} H^{10}(Y \times Z', \mathbb{Q}) \xrightarrow{\mathrm{pr}_{Z'*}} H^2(Z', \mathbb{Q}).$$

Recall that by construction of Z we have a birational morphism $b\colon Z' \to Z$. We are going to prove

Theorem 2.1.0.1. *Let Y be any smooth cubic fourfold not containing a plane. The composition*

$$H^4(Y, \mathbb{Q}) \xrightarrow{\phi} H^2(Z', \mathbb{Q})(-1) \xrightarrow{b_*} H^2(Z, \mathbb{Q})(-1)$$

restricts to an isomorphism of rational Hodge structures

$$H^4_{\mathrm{prim}}(Y, \mathbb{Q}) \xrightarrow{\sim} H^2_{\mathrm{prim}}(Z, \mathbb{Q})(-1).$$

Let us explain a little bit. The first thing to notice is that, since our claim regards rational cohomology, there is no harm to consider the Chern character $\mathrm{ch}(\mathscr{O}_C)$ instead of just the cohomology class $[C]$. Indeed, $\mathrm{ch}_3(\mathscr{O}_C)$ is the first non-trivial component of the Chern character and differs from $[C]$ just by a coefficient. Secondly, C and U are strictly connected as is clear from the construction of the isomorphism between Z' and the moduli space.
We thus report briefly on the result contained in [LLMS18]. Let $\mathfrak{J}$ be the moduli space of Gieseker stable sheaves with reduced Hilbert polynomial

$$\frac{3}{2}n(n-1).$$

For generalised twisted cubic C on Y we consider the cubic surface S_C obtained by intersecting the linear span of C with Y. Since the ideal sheaf I_{C/S_C} of C in S_C is Gieseker

stable for any twisted cubic C on Y ([LLMS18, Lemma 1.1]), the ideal $\mathscr{I}$ of the universal curve $\mathscr{C}$ inside the family $\mathscr{S}_M$[3] of surfaces gives a morphism $M \to \mathfrak{J}$. This descends to an isomorphism $Z' \to \mathfrak{J}_1$ on an irreducible component $\mathfrak{J}_1 \subset \mathfrak{J}$ ([LLMS18, Propostion 1.2]. Over the moduli space $\mathfrak{J}$ we have a universal family of sheaves, let U be its restriction on $\mathfrak{J}$. We compare the correspondence given by U on $Y \times Z'$ with the one given by the universal curve C in $Y \times M$. By the discussion above and the universal property of U we have an isomorphism

$$a^*U = I_{C/S} \otimes \mathrm{pr}_M^* E \tag{2.6}$$

for some vector bundle E on M.

Remark 2.1.0.2. By the short exact sequence

$$0 \to \mathscr{I}_{\mathscr{C}/\mathscr{S}} \to \mathscr{O}_{\mathscr{S}} \to \mathscr{O}_{\mathscr{C}} \to 0$$

and the additivity of the Chern character we have

$$\mathrm{ch}(\mathscr{I}_{\mathscr{C}/\mathscr{S}}) = \mathrm{ch}(\mathscr{O}_{\mathscr{S}}) - \mathrm{ch}(\mathscr{O}_{\mathscr{C}}).$$

Moreover, by [Ful84, Example 15.3.1] we get

$$\mathrm{ch}(\mathscr{I}_{\mathscr{C}/\mathscr{S}}) = [\mathscr{S}] + (-6[C] + \mathrm{ch}_3(\mathscr{O}_{\mathscr{S}})) + \text{higher-order terms.} \tag{2.7}$$

Lemma 2.1.0.3. *The diagram*

$$\begin{array}{ccc} H^4_{\mathrm{prim}}(Y,\mathbb{Q}) & \xrightarrow{[\mathrm{ch}_3(\mathscr{O}_{\mathscr{C}})]_*} & H^2_{\mathrm{prim}}(M,\mathbb{Q}) \\ & \searrow^{\phi} & \uparrow{\scriptstyle a^*} \\ & & H^2_{\mathrm{prim}}(Z',\mathbb{Q}) \end{array}$$

is commutative.

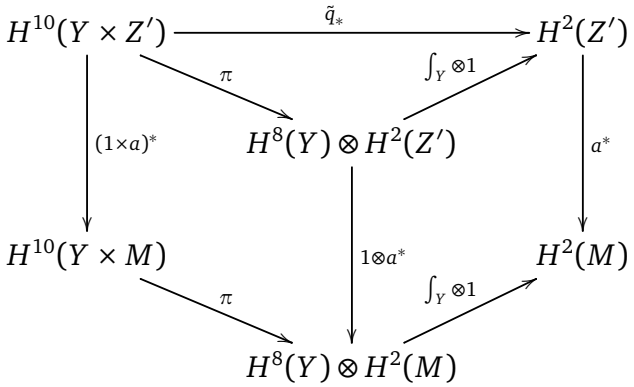
Proof. We break the two maps into pieces

$$\begin{array}{ccccccc} H^4(Y) & \xrightarrow{\tilde{p}^*} & H^4(Z'\times Y) & \xrightarrow{\cup F} & H^{10}(Z'\times Y) & \xrightarrow{\tilde{q}_*} & H^2(Z') \\ \| & & \downarrow{\scriptstyle (1\times a)^*} & & \downarrow{\scriptstyle (1\times a)^*} & & \downarrow{\scriptstyle a^*} \\ H^4(Y) & \xrightarrow{p^*} & H^4(M\times Y) & \xrightarrow{\cup \mathrm{ch}_3(\mathscr{O}_{\mathscr{C}})} & H^{10}(M\times Y) & \xrightarrow{q_*} & H^2(M) \end{array} \tag{2.8}$$

where the claim regards the commutativity of the big outer diagram restricted to primitive cohomology. Here $p\colon M\times Y \to Y$, $q\colon M\times Y \to M$, $\tilde{p}\colon Z'\times Y \to Y$ and $\tilde{q}\colon Z'\times Y \to Z'$ denote the natural projections.

[3] The family $\mathscr{S}_M$ of surfaces is defined by pullback of the family on the Grassmannian, as before we did for the family $\mathscr{S}$ over Z'.

The commutativity of the rightmost square can be explained as follows. Consider the diagram:



where the map π is the projection given by the Künneth formula. The triangle commutes because of the definition of $\tilde{q}_*$ as integration along fibres. The leftmost square commutes by the very definition of pullback on forms.
We come back to our big diagram (2.8). We then shall prove the commutativity of the outer diagram, that is:

$$q_*(\mathrm{ch}_3(\mathscr{O}_{\mathscr{C}})\cup p^*(\alpha)) = a^*\tilde{q}_*\left(\left(\mathrm{ch}_3(\mathscr{O}_{\mathscr{S}_{Z'}})-\frac{\mathrm{ch}_3(U)}{\rho(U)}\right)\cup\tilde{p}^*(\alpha)\right)$$

for any α in $H^4(Y,\mathbb{Z})$. We work on the right hand side:

$$\begin{aligned}
a^*\tilde{q}_*\left(\left(\mathrm{ch}_3(\mathscr{O}_{\mathscr{S}_{Z'}})-\frac{\mathrm{ch}_3(U)}{\rho(U)}\right)\cup\tilde{p}^*\alpha\right) &= q_*(1\times a)^*\left(\left(\mathrm{ch}_3(\mathscr{O}_{\mathscr{S}_{Z'}})-\frac{\mathrm{ch}_3(U)}{\rho(U)}\right)\cup\tilde{p}^*\alpha\right)\\
&= q_*\left((1\times a)^*\left(\mathrm{ch}_3(\mathscr{O}_{\mathscr{S}_{Z'}})-\frac{\mathrm{ch}_3(U)}{\rho(U)}\right)\cup(1\times a)^*\tilde{p}^*\alpha\right)\\
&= q_*\left((1\times a)^*\left(\mathrm{ch}_3(\mathscr{O}_{\mathscr{S}_{Z'}})-\frac{\mathrm{ch}_3(U)}{\rho(U)}\right)\cup p^*\alpha\right)\\
&= q_*\left(\left(\mathrm{ch}_3(\mathscr{O}_{\mathscr{S}_M})-\frac{(1\times a)^*\mathrm{ch}_3(U)}{\rho(U)}\right)\cup p^*\alpha\right)
\end{aligned}$$

where the first equality follows from what just explained.
Next, by the formula (2.7) and the isomorphism (2.6) we rewrite

$$\begin{aligned}
-\frac{(1\times a)^*\mathrm{ch}_3(U)}{\rho(U)}+\mathrm{ch}_3(\mathscr{O}_{\mathscr{S}_M}) &= -\frac{\mathrm{ch}_3(\mathscr{I}_{\mathscr{C}/\mathscr{S}}\otimes pr_M^*E)}{\mathrm{rk}(E)}+\mathrm{ch}_3(\mathscr{O}_{\mathscr{S}_M})\\
&= -\frac{\mathrm{ch}_2(\mathscr{I}_{\mathscr{C}/\mathscr{S}})\cdot\mathrm{ch}_1(pr_M^*E)}{\mathrm{rk}(E)}-\mathrm{ch}_3(\mathscr{I}_{\mathscr{C}/\mathscr{S}})+\mathrm{ch}_3(\mathscr{O}_{\mathscr{S}_M})
\end{aligned}$$

$$= -\frac{[\mathscr{S}]\cdot \mathrm{ch}_1(q^*E)}{\mathrm{rk}(E)} + \mathrm{ch}_3(\mathscr{O}_{\mathscr{C}}).$$

Thus, we have proven that:

$$\begin{aligned} a^*\phi(\alpha) &= q_*\left(\left(-\frac{[\mathscr{S}]\cdot \mathrm{ch}_1(q^*E)}{\mathrm{rk}(E)} + ch_3(\mathscr{O}_{\mathscr{C}})\right)\cup p^*\alpha\right) \\ &= q_*\left(\left(-\frac{[\mathscr{S}]\cdot \mathrm{ch}_1(q^*E)}{\mathrm{rk}(E)}\right)\cup p^*\alpha\right) + q_*\left(\mathrm{ch}_3(\mathscr{O}_{\mathscr{C}})\cup p^*\alpha\right) \\ &= q_*\left(\left(-\frac{[\mathscr{S}])\cdot \mathrm{ch}_1(q^*E)}{\mathrm{rk}(E)}\right)\cup p^*\alpha\right) + [\mathrm{ch}_3(\mathscr{O}_{\mathscr{C}})]_*\alpha \end{aligned}$$

We are left to prove that

$$q_*\left(\left(\frac{[\mathscr{S}]\cdot \mathrm{ch}_1(q^*E)}{\mathrm{rk}(E)}\right)\cup p^*\alpha\right) = [\mathscr{S}]_*\alpha \cup \frac{\mathrm{ch}_1(E)}{\mathrm{rk}(E)} = 0$$

for any $\alpha \in H^4_{\mathrm{prim}}(Y,\mathbb{Q})$ and that is shown in the next lemma. □

Lemma 2.1.0.4. *We have:*

$$\ker\left([\mathscr{S}]_*\colon H^4_{\mathrm{prim}}(Y,\mathbb{Q}) \to H^8(M,\mathbb{Q})\right) = H^4_{\mathrm{prim}}(Y,\mathbb{Q}).$$

Proof. By a standard argument one reduces to prove the statement for a very general cubic fourfold, for which the primitive cohomology has a simple Hodge structure. We sketch here the reduction argument, which will be discussed extensively in Section 2.3.1: Cupping with $[\mathscr{S}]$ gives rise to a morphism of local systems over the space of all smooth cubic fourfolds and so the kernel of $[\mathscr{S}]\cup p^*-$ is a local system, whose rank can be detected on any cubic fourfold.
The class $[\mathscr{S}]$ is algebraic, so that $\ker([\mathscr{S}]\cup p^*-)$ is a Hodge substructure. Now, choosing a certain cubic fourfold Y for which the primitive cohomology is simple, we have just two options: Either the kernel is trivial or it is the all of $H^4_{\mathrm{prim}}(Y,\mathbb{Q})$. Let ω be a $(3,1)$-form spanning the $H^{3,1}(Y)$. Then, $[\mathscr{S}]_*\omega$ is an element in $H^{1,-1}(M) = 0$, therefore trivial. □

Lemma 2.1.0.5. *Assume Y is very general, then:*

$$\phi : H^4(Y,\mathbb{Q}) \to H^2(Z',\mathbb{Q})$$

is injective.

Proof. For a cubic fourfold the Hodge numbers of the middle cohomology are (e.g. [Huy, Theorem 1.15])

$$0 \qquad 1 \qquad 21 \qquad 1 \qquad 0.$$

In particular, there exists a $(3,1)$-form ω that spans $H^{3,1}$. For the very general cubic fourfold Y, the integral Hodge classes $H^{2,2} \cap H^4(Y,\mathbb{Z})$ are spanned over the integers by the square of the class of a hyperplane section h and the primitive part coincides with the smallest Hodge substructure containing ω.
To prove the claim, it is then enough to show that the classes ω and h^2 do not map to zero.
Let us prove that $\phi(\omega) \neq 0$. From Section 4.4 in [LLSvS17] we know that $\sigma_3 := [\mathrm{ch}_3(\mathcal{O}_{\mathscr{C}})]_*(\omega) = a^*\sigma \in H^2(M,\mathbb{C})$, where σ is a generically symplectic form on Z'. Since σ is generically symplectic, it is different from zero and so is $a^*\sigma$, as the pullback morhpism is injective ([Voi02, Lemma 7.28])
Let us prove that $\phi(h^2) \neq 0$. By Lemma 2.1.0.3 it is enough to prove that $[\mathrm{ch}_3(\mathcal{O}_{\mathscr{C}})]_* h^2$ is not null, and since $\mathrm{ch}_3(\mathcal{O}_{\mathscr{C}}) = 3c_3(\mathcal{O}_{\mathscr{C}}) = 6[\mathscr{C}]$ ([Ful84, Example 15.3.1]), it suffices to show $[\mathscr{C}]_* h^2 \neq 0$. The class h^2 is represented by a smooth cubic surface S lying inside Y and by construction we have that $[\mathscr{C}]_*(h^2)$ is represented by the divisorial part D of

$$\{C \in M : C \cap S \neq \emptyset\}.$$

The careful reader may find a complete explanation of the previous statement in the lemma below. Let us see that D is a non-trivial divisor: Every linear 3-space $E \subset \mathbb{P}^5$ intersects the surface S in three points as follows from an easy computation in cohomology[4]. Now, if E is general, then on the smooth cubic surface $E \cap Y$ there is a $\mathbb{P}^1$ of twisted cubics through each point, as it is clear from representing $E \cap Y$ as blow-up of $\mathbb{P}^2$.[5] This proves that $D \to \mathrm{Gr}(4,6)$ is surjective with 1-dimensional generic fibre, thus an effective divisor. □

Lemma 2.1.0.6. *Let S be a smooth surface on Y, which is cohomologous to the square of a hyperplane section h. Then $[\mathscr{C}]_*(h^2)$ is the class of the divisorial part of*

$$\{C \in M : C \cap S \neq \emptyset\} =: K.$$

Proof. First, we notice that K coincides with $q(p^{-1}(S) \cap \mathscr{C})$ so that the cohomological class of K is exactly $[\mathscr{C}]_*(h^2)$, provided that $p^{-1}(S) \cap \mathscr{C}$ represent the cohomological class $[S \times M] \cup [\mathscr{C}]$. For this, it suffices that the intersection has the expected codimension, that is, $\mathrm{codim}(\mathscr{C}) + \mathrm{codim}(S \times M) = 5$ or, equivalently, that each irreducible component of $(S \times M) \cap \mathscr{C}$ has dimension 9. Thus, the lemma is equivalent to

Claim 2.1.0.7. *Each irreducible component of $(S \times M) \cap \mathscr{C}$ has dimension at most 9.*

First, we analyse the restriction of q

$$(S \times M) \cap \mathscr{C} \to K.$$

[4] Indeed in the cohomology of $\mathbb{P}^5$ we have $[S] = 3h^3$, $[E] = h^2$ and $[S] \cdot [E] = 3h^5$ is the class of three points.
[5] Indeed any twisted cubic on a cubic surface S is the pullback of a line along a blow down $S \to \mathbb{P}^2$.

Its fibres are finite except for the two-dimensional family of curves in K that lie on S, in which case the fibre is one-dimensional. Hence, it suffices to show that each irreducible component of K has dimension at most 9. We consider the morphism

$$\pi : K \to \mathrm{Gr}(4,6).$$

Let E be a point in $\mathrm{Gr}(4,6)$. First, we observe that $\pi^{-1}(E)$ is the subset of the Hilbert scheme $\mathrm{Hilb}^{\mathrm{gtc}}(E \cap Y)$ parametrising generalised twisted cubics on $E \cap Y$ with non-trivial intersection with S. We start with the case $\dim E \cap S = 0$:

- If $E \cap Y$ is smooth, then $\pi^{-1}(E)$ is one-dimensional, as discussed above. The Grassmannian $\mathrm{Gr}(4,6)$ has dimension 8 and this determines a dimension 9 component.
- If $E \cap Y$ is normal with rational double points, then $\mathrm{Hilb}^{\mathrm{gtc}}(E \cap Y)$ has dimension 2 ([LLSvS17, Theorem 2.1]). The locus

  $$X := \{E \in \mathrm{Gr}(4,6) : E \cap Y \text{ is singular}\}$$

  of singular cubic surfaces has dimension 7, so that $\pi^{-1}(X)$ has dimension at most 9.
- The locus of E such that $E \cap Y$ has either a simple elliptic singularity or non-normal singularities has dimension at most 4 ([LLSvS17, Proposition 4.2, Proposition 4.3]). Moreover, $\mathrm{Hilb}^{\mathrm{gtc}}(E \cap Y)$ is at most four-dimensional ([LLSvS17, Proposition 2.7, Proposition 2.8, Corollary 3.11]), so that the preimage of this locus has dimension at most 8.

The locus

$$\{E \in \mathrm{Gr}(4,6) : \dim E \cap Y = 1\}$$

has dimension 4. Since $\dim \mathrm{Hilb}^{\mathrm{gtc}}(Y \cap E) \leq 4$, the preimage of such locus has dimension at most 8. Finally, there is one E that contains S, in this case $\mathrm{Hilb}^{\mathrm{gtc}}(E \cap Y)$ has dimension 2. Hence, the claim is proven. □

We are now ready to finalise the proof of Theorem 2.1.0.1. Indeed, from a standard argument[6] it suffices to prove the theorem for just one cubic fourfold. We may assume that Y is very general so that assumptions of Lemma 2.1.0.5 are fulfilled. Moreover, the morphism $b : Z' \to Z$ is a divisorial contraction, so the action of b_* is injective. Indeed, we may assume that the smooth cubic fourfold Y has a simple Hodge structure for primitive cohomology. Hence, the statement follows from the fact that $b_*\sigma'$ is not zero, where σ' is the generically symplectic $(2,0)$–form spanning $H^{2,0}(Z')$ [LLSvS17, Theorem 4.10].

[6] This will be extensively explained in Section 2.3.1 and in greater generality of what we need here, therefore we invite the interested reader to temper their patience for the moment.

Hence we have proven that

$$b_* \circ \phi : H^4_{\text{prim}}(Y, \mathbb{Q}) \to H^2(Z, \mathbb{Q})$$

is injective. Since we are working on vector spaces, we are left to compare Hodge numbers. The cohomology group $H^4_{\text{prim}}(Y, \mathbb{Q})$ has a pure Hodge structure of weight 4, its Hodge numbers are (e.g. [Huy, Theorem 1.15])

$$0 \qquad 1 \qquad 20 \qquad 1 \qquad 0.$$

On the other hand, $H^2_{\text{prim}}(Z, \mathbb{Q})$ is a pure Hodge structure of weight 2 whose Hodge numbers are[7]

$$1 \qquad 20 \qquad 1.$$

The Tate twist takes care of the weight shift and the Theorem 2.1.0.1 is proven.

In order to get a more complete statement regarding the whole integral cohomology and the natural quadratic forms attached to them is much more difficult. The major obstacles arise from the impossibility to get a nice description of the cohomology class of the universal family of curves over M, the appropriate action of b_* in cohomology. It would be meaningful to use the other results in [LLMS18]. The authors prove that Z is isomorphic to a moduli space of tilt stable objects and they even realise the contraction $b\colon Z' \to Z$ as wall crossing for tilt stability.
However, in the next sections, we will succeed in that by looking at the other birational model for the LLSvS eightfold, which for our purposes is easier to handle. We will look at the problem from another perspective, taking into account not only the cohomology of the cubic fourfold but also its K-theory.

2.2 The Addington–Lehn birational model of the LLSvS variety

In the paper [AL17] the authors describe a birational model of the LLSvS variety working with smooth Pfaffian cubic fourfolds. Here we briefly recall their work adding here and there some improvements, which we will need later in the proof of our result.
First we recall the construction of the Pfaffian cubic fourfold Y, the K3 surface X, and the correspondence $\Gamma \subset X \times Y$ in detail. Fix a vector space $V \cong \mathbb{C}^6$ and a generic 6-dimensional subspace $L \subset \Lambda^2 V^*$. The cubic is

$$Y = \left\{ [\varphi] \in \mathbb{P}^5 \;\middle|\; \operatorname{rank}(\varphi) = 4 \right\} = \left\{ [\varphi] \in \mathbb{P}(L) \;\middle|\; \wedge^3 \varphi = 0 \right\},$$

[7] Indeed, the LLSvS variety is deformation equivalent to the Hilbert scheme of four points on a K3 surface (e.g [AL17, Corollary]).

the K3 surface is

$$X = \left\{ [P] \in \mathrm{Gr}(2, V) \;\middle|\; \varphi|_P = 0 \text{ for all } \varphi \in L \right\},$$

and the correspondence is

$$\Gamma = \left\{ ([P], [\varphi]) \in X \times Y \;\middle|\; P \cap \ker \varphi \neq 0 \right\}.$$

For a generic choice of L, both X and Y are smooth, X does not contain a line, and Y does not contain a plane ([Huy, § 6, Lemma 1.28]). Let $0 \to \mathscr{P} \to V_X \to \mathscr{Q} \to 0$ denote the tautological bundle sequence on X, and let $A\colon V_{\mathbb{P}(L^\vee)} \to V_{\mathbb{P}(L^\vee)}^\vee \otimes \mathscr{O}(1)$ denote the tautological skew-symmetric form parametrized by $\mathbb{P}(L^\vee)$. By construction, the restriction of A_φ to any P, $[P] \in X$, vanishes, so A induces a homomorphism $A'\colon \mathscr{P} \boxtimes \mathscr{O} \to \mathscr{Q}^\vee \boxtimes \mathscr{O}(1)$ on $X \times \mathbb{P}(L^\vee)$. Then $\Gamma \subset X \times \mathbb{P}(L^\vee)$ is the subscheme defined by the vanishing of the 2×2-minors of A'. There are natural morphisms $X \xleftarrow{p_X} \Gamma \xrightarrow{p_Y} Y$.
The variety Γ is not the zero locus of a regular section of the A', but the authors in [AL17, §3] prove that the Eagon-Northcott complex of A' is indeed a locally free resolution of the ideal sheaf I_Γ. This turns out to be useful to compute the first Chern class of the structural sheaf $\mathscr{O}_\Gamma$ as we explain in the following.

Proposition 2.2.0.1. *The first Chern class of $p_{Y*}\mathscr{O}_\Gamma$ is:*

$$c_1(p_{Y*}\mathscr{O}_\Gamma) = -9h \in H^2(Y, \mathbb{Q}).$$

Proof. Applying the Grothendieck–Riemann–Roch formula to the embedding $i\colon Y \hookrightarrow \mathbb{P}^5$, we get

$$i_* \operatorname{ch}(p_{Y*}\mathscr{O}_\Gamma) = \operatorname{ch}(i_* p_{Y*}\mathscr{O}_\Gamma) \cdot \operatorname{td}(\mathscr{O}_{\mathbb{P}^5}(3)). \tag{2.9}$$

The Todd class is:

$$\operatorname{td}(\mathscr{O}_{\mathbb{P}^5}(3)) = 1 + \frac{3}{2}h + \frac{3}{4}h^2 - \frac{9}{80}h^4 \in \mathrm{CH}^*(\mathbb{P}^5).$$

We now deal with the first factor of (2.9). By the commutativity of the following diagram with all natural maps

$$\begin{array}{ccc} \Gamma & \xrightarrow{j} & X \times \mathbb{P}^5 \\ \downarrow{\scriptstyle p_Y} & & \downarrow{\scriptstyle \mathrm{pr}_{\mathbb{P}^5}} \\ Y & \xrightarrow{i} & \mathbb{P}^5. \end{array}$$

it follows that

$$\operatorname{ch}(i_* p_{Y*}\mathscr{O}_\Gamma) = \operatorname{ch}(\mathrm{pr}_{\mathbb{P}^5*}\, j_* \mathscr{O}_\Gamma)$$

and again by Grothendieck-Riemann-Roch

$$\operatorname{ch}(\mathrm{pr}_{\mathbb{P}^5*}\, j_* \mathscr{O}_\Gamma) = \mathrm{pr}_{\mathbb{P}^5*}\left(\operatorname{ch}(j_* \mathscr{O}_\Gamma) \cdot \operatorname{td}(T_{\mathrm{pr}_{\mathbb{P}^5}})\right).$$

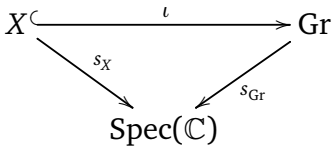
Since $p_{\mathbb{P}^5}$ is the projection of a product, the Todd class appearing in the formula equals the pullback of the Todd class of the $K3$ surface X

$$\mathrm{td}(X) = ([X], 0, 2[Q]) \in \mathrm{CH}^*(X),$$

here $[Q]$ is the class of a point Q on any rational curve on the $K3$ surface.
We set $G = \mathscr{P}|_X \boxtimes \mathscr{O}_{\mathbb{P}^5}$ and $F = \mathscr{Q}|_X \boxtimes \mathscr{O}_{\mathbb{P}^5}(-1)$, then the Eagon-Northcott complex for $I_{\Gamma|X\times\mathbb{P}^5}$ (e.g. [Eis95, Section A2.6]) is the long exact sequence

$$0 \to S^2G^* \boxtimes \wedge^4 F \to S^1G^* \boxtimes \wedge^3 F \to S^0G^* \boxtimes \wedge^2 F \to I_{\Gamma|X\times\mathbb{P}^5} \to 0.$$

With it at our disposal taking advantage of the additivity and multiplicativity property of the Chern character we get an expression of $\mathrm{ch}(I_{\Gamma|X\times\mathbb{P}^5})$ in terms of the Chern roots of $\mathscr{P}$ and $\mathscr{Q}$ restricted to X. In particular it admits a kind of Künneth decomposition in the graded subring $\mathrm{pr}^*_{\mathbb{P}^5}\mathrm{CH}^*(\mathbb{P}^5)\cdot\mathrm{pr}^*_X\mathrm{CH}^*(X)$ of the bigger graded ring $\mathrm{CH}^*(\mathbb{P}^5\times X)$.
The last step is to compute the pushforward along $\mathrm{pr}_{\mathbb{P}^5}$, this corresponds to apply the linear operator $\int_X -$ to the terms in $\mathrm{pr}^*_X\mathrm{CH}^*(X)$. One reduces to a computation in the cohomology ring of the Grassmaniann as follows. We consider the inclusion with structural morphisms



For any sheaf F on Gr we have by projection formula

$$\int_X \mathrm{ch}(\iota^*F) = s_{X*}(\iota^*\mathrm{ch}(F)\cap[X]) = s_{\mathrm{Gr}*}\iota_*(\iota^*\mathrm{ch}(F)\cap[X])$$
$$= s_{\mathrm{Gr}*}(\mathrm{ch}(F)\cap\iota_*[X]) = \int_Y \mathrm{ch}(F)\cap\iota_*[X].$$

Since the $K3$ is a regular section of $L^*\otimes\wedge^2\mathscr{P}$, its fundamental class $[X]$ in the Chow ring of the Grassmaniann equals $c_6(L^*\otimes\wedge^2\mathscr{P})$, which again can be computed as element in the terms of the Chern roots of $\mathscr{P}$ and $\mathscr{Q}$.
Hand computations are lengthy and are reduced just to an elementary use of linear algebra, thus we only include the code of the computations we carried out with the Schubert2 package of Macaulay2 [GS]; the code may be found in the Appendix 5.
The result is

$$i_*\mathrm{ch}(p_*\mathscr{O}_\Gamma) = 12h - 27h^2 + \frac{65}{2}h^3 - \frac{33}{2}h^4 + \frac{19}{8}h^5 \in H^*(\mathbb{P}^5,\mathbb{Q}). \tag{2.10}$$

To extract the first Chern class from the above equation, we observe that, Y being a cubic hypersurface, we have $i_*(h) = 3h^2$. Hence from (2.10) we read off $c_1(p_*\mathscr{O}_\Gamma) = -9h$. □

Let I_Γ be the ideal sheaf defining Γ in $X \times Y$.

Proposition 2.2.0.2. *The Fourier-Mukai functor*

$$D^b(X) \xrightarrow{I_\Gamma^\vee(-2h)[4]} D^b(Y)$$

is fully faithful. Its image is $\mathscr{A} = \langle \mathscr{O}_Y, \mathscr{O}_Y(1), \mathscr{O}_Y(2)\rangle^\perp$.

Proof. This is essentially [AL17, Proposition 3][8]. Therein it is proved that the right adjoint to the Fourier-Mukai functor Ψ with kernel I_Γ defines an equivalence

$$\Psi : D(X) \to \mathscr{A}' = \langle \mathscr{O}_Y(-1), \mathscr{O}_Y, \mathscr{O}_Y(1)\rangle.$$

From Proposition 1.2.0.3 we see that Ψ is isomorphic to the Fourier-Mukai functor with kernel $I_\Gamma^\vee(-3h)[4]$. How to pass from $\mathscr{A}$ and $\mathscr{A}'$ was discussed in Remark 1.3.1.8 and the claim is now clear. □

Lemma 2.2.0.3. *Let C be a generalized twisted cubic on Y, then* $\mathrm{pr}[I_{C/Y}] = \lambda_2 - \lambda_1 \in K_{\mathrm{top}}(\mathscr{A})$. *Here* $\mathrm{pr}\colon K_{\mathrm{top}}(Y) \to K_{\mathrm{top}}(\mathscr{A})$ *is the orthogonal projection.*

Proof. Let y be a point of Y and L be a line through y. Consider the short exact sequence

$$0 \to \mathscr{O}_L(-1) \to \mathscr{O}_L \to \mathscr{O}_y \to 0$$

from [AL17, Lemma 1 b)] and additivity of the projection functor we get the claim. □

In virtue of this Lemma, the mysterious class appearing in Theorem 2.0.0.3 is now clear: We want to use the universal curve as correspondence to relate the cohomology of Y and Z, thus we are interested in complexes which has class $\lambda_2 - \lambda_1$.
In [AL17, Lemma 4] it is proven that Γ is generically 4-to-1 over Y. We improve this as follows:

Lemma 2.2.0.4 ([AG20, Lemma 7])**.** *If X does not contain a (-2)-curve then Γ is flat over Y.*

Proof. Let $\Gamma_\varphi \subset X$ be the scheme-theoretic fiber of Γ over a point $[\varphi] \in Y$. We will argue that Γ_φ is zero-dimensional; then by the proof of [AL17, Lemma 4] it has length 4.
Consider the Schubert cycle

$$\Sigma_\varphi = \left\{ [P] \in \mathrm{Gr}(2, V) \;\middle|\; P \cap \mathrm{rad}\,\varphi \neq 0 \right\}$$

of which Γ_φ is a linear section, and

$$\tilde{\Sigma}_\varphi = \left\{ (l, P) \in \mathbb{P}(\mathrm{rad}\,\varphi) \times \mathrm{Gr}(2, V) \;\middle|\; l \subset P \right\}.$$

[8] A similar statement was originally proven by Kuznetsov [Kuz06, Theorem 2].

We observe that $\tilde{\Sigma}_\varphi$ is a $\mathbb{P}^4$-bundle over $\mathbb{P}(\operatorname{rad}\varphi) = \mathbb{P}^1$. First we claim that the preimage of Γ_φ in $\tilde{\Sigma}_\varphi$ meets each $\mathbb{P}^4$ fiber in at most one point, even scheme-theoretically: to see this, note that the $\mathbb{P}^4$s are embedded linearly in $\mathrm{Gr}(2,V) \subset \mathbb{P}(\Lambda^2 V)$, so if a linear section contained more than one point of a $\mathbb{P}^4$ it would contain a line, contradicting our hypothesis on X.
Next we claim that the preimage of Γ_φ in $\tilde{\Sigma}_\varphi$ meets at most finitely many $\mathbb{P}^4$ fibers. Otherwise it would meet them all, hence would give a section of this $\mathbb{P}^4$-bundle over $\mathbb{P}^1$, hence a smooth rational curve on X, again contradicting our hypothesis.
Thus the preimage of Γ_φ in $\tilde{\Sigma}_\varphi$ is zero-dimensional, so Γ_φ is as well. □

Let $C \subset M \times Y$ be the universal family of generalised twisted cubics and I_C be its ideal sheaf. We consider the convolution

$$T := I_\Gamma \circ I_C(2h) \in D^b(X \times M).$$

and recall some facts proved in [AL17, §3], with a few improvements:

(a) There is a non-empty, Zariski open set $M_0 \subset M$ such that the restriction $T[1]|_{X\times M_0}$ is quasi-isomorphic to an M_0-flat family of ideal sheaves of length-4 subschemes of X. We take M_0 as big as possible.

(b) There is an open set $Z_0 \subset Z$ such that $M_0 = u^{-1}(Z_0)$.

Proof: From [AL17, Proposition 2] we know that M_0 is a union of fibers of u, so $M_0 = u^{-1}(Z_0)$ for some $Z_0 \subset Z$, maybe not open *a priori*. But u is surjective, so $Z \setminus Z_0 = u(M \setminus M_0)$, and u is proper, hence closed, so $Z \setminus Z_0$ is closed.

(c) The classifying map $t' : M_0 \to X^{[4]}$ descends to an open immersion $t : Z_0 \to X^{[4]}$.

Proof: From [AL17, §3] we know that t' descends to a map t that is injective. But an injective holomorphic map between complex manifolds of the same dimension is an open immersion by the proof of [GH94, Prop. on p. 19]; see also [Ros82]. (So the smaller open set $Z_1 \subset Z_0$ of [AL17, §3] is unnecessary.)

The key to the proof of the main theorem of this chapter will be the following fact, whose proof we postpone to the Section 2.2.2

Proposition 2.2.0.5 ([AG20, Proposition 6]). *If the K3 surface X has Picard rank 1, then the open set Z_0 contains Y, and its complement $Z \setminus Z_0$ has codimension at least 2 in Z.*

Thus $M \setminus M_0$ has codimension at least 2 in M as well, because u is a $\mathbb{P}^2$-fibration over $Z \setminus Y$; and thus the inclusions $Z_0 \hookrightarrow Z$ and $M_0 \hookrightarrow M$ induce isomorphisms on H^2.
The reader may wonder why we need Proposition 2.2.0.5, when we can say immediately that because Z and $X^{[4]}$ have nef canonical bundles, the birational map $Z \dashrightarrow X^{[4]}$ is biregular on an open set W containing Z_0 whose complement has codimension at least 2 [KM98, Corollary 3.54]. The reason is that we would not know anything about the restriction of T to $X \times u^{-1}(W)$, which we need in what follows.

2.2.1 Some background on cones of Hyperkähler varieties

In this short intermezzo we recall some known results about the movable cones of some hyperkähler varieties: Bayer and Macrì computed the movable cone of $X^{[n]}$ for a K3 surface X with small Picard number and Boucksom, as a corollary of the Zariski decomposition, showed that for any hyperkähler the dual of the movable cone is equal to the pseudo-effective cone.
Let X be a hyperkähler variety. An integral divisor in $D \in \mathrm{NS}(X)$ is called *movable* if its stable base locus [9] has codimension greater or equal than 2.
With help of the Bridgeland theory on stability conditions for triangulated categories, Bayer and Macrì described the movable cone of Hilbert scheme of points on a $K3$ surface. Let X be a $K3$ surface and $X^{[n]} = \mathrm{Hilb}^n X$ be its Hilbert scheme. Then by the Remark following [Bea83, Proposition 6]:

$$H^2(X^{[n]}, \mathbb{Z}) \simeq H^2(X, \mathbb{Z}) \oplus_\perp \mathbb{Z}B$$

as lattices and where B is the half of the Hilbert-Chow exceptional divisor, whose Beauville-Bogomolov square is $q_{X^{[n]}}(B) = 2(n-1)$. Similarly we have

$$\mathrm{NS}(X^{[n]}) \simeq \mathrm{NS}(X) \oplus_\perp \mathbb{Z}B.$$

Geometrically, the inclusion of the Nerón-Severi group can be described as follows: The symmetric power of every divisor D on X is a divisor $\tilde{D}$ on $X^{(n)}$ and its strict transform along the Hilbert-Chow morphism gives a divisor on the Hilbert scheme. If X has Picard number 1 with polarisation H, then $\tilde{H}$ and B generate the Nerón–Severi and in this case in order to study the movable cone one has to consider the following Pell equation as explained in [BM14a, § 13][10]

$$(n-1)X^2 - dY^2 = 1 \tag{2.11}$$

$$X^2 - d(n-1)Y^2 = 1 \quad \text{with } X \equiv \pm 1 \pmod{n-1}; \tag{2.12}$$

where $2d = H^2$ is the degree of our K3 surface. Thanks to the Dirichlet Unit Theorem, if d is not a square, the last Pell equation has always a solution. Now if (x, y) is a solution to the Pell equation in (2.12) then $(x^2 + d(n-1)y^2, 2xy)$ is a solution to the same Pell equation, so that there always exists a solution to (2.12) (if d is not a square) satisfying the additional congruence condition.[11]

Proposition 2.2.1.1 ([BM14a, Proposition 13.1]). *Assume* $\mathrm{Pic}(X) \cong \mathbb{Z} \cdot H$. *The movable cone of the Hilbert scheme* $M = \mathrm{Hilb}^n(X)$ *has the following form:*

[9] The stable base locus of the divisor is just the intersection of the base loci of all of its multiples.
[10] A minor error contained in this reference was then fixed in [Deb18, Example 3.17]. Obviously, our exposition will take in consideration the correction.
[11] For a more detailed discussion on Pell equations we point to [Deb18, Appendix A]

(a) *If* $d = \frac{k^2}{h^2}(n-1)$*, with* $k, h \geq 1$*,* $(k,h) = 1$*, then*

$$\mathrm{Mov}(M) = \langle \widetilde{H}, \widetilde{H} - \frac{k}{h}B \rangle,$$

where $q(h\widetilde{H} - kB) = 0$*, and it induces a (rational) Lagrangian fibration on* M*.*

(b) *If* $d(n-1)$ *is not a perfect square, and* (2.12) *has a solution, then*

$$\mathrm{Mov}(M) = \langle \widetilde{H}, \widetilde{H} - d\frac{y_1}{x_1(n-1)}B \rangle,$$

where (x_1, y_1) *is the solution to* (2.11) *with* $x_1, y_1 > 0$*, and with smallest possible* x_1*.*

(c) *If* $d(n-1)$ *is not a perfect square, and* (2.11) *has no solution, then*

$$\mathrm{Mov}(M) = \langle \widetilde{H}, \widetilde{H} - d\frac{y_1'}{x_1'}B \rangle,$$

where (x_1', y_1') *is the minimal positive solution of the equation* (2.12) *that satisfies* $x' \equiv \pm 1 \pmod{n-1}$*.*

The effective divisors span a cone in the Néron-Severi group; its closure is called the pseudo effective cone.

Theorem 2.2.1.2. *Let* X *be a hyperkähler variety, then the pseudo-effective cone is dual to the movable cone.*

Proof. This follows from a deep result of Boucksom about the Zariski divisorial decomposition. He defines a transcendental analogue $\mathscr{E}$ of the algebraic pseudo-effective cone ([Bou04, § 2.3]): Indeed its intersection with the Néron-Severi group is the closure of the usual effective cone [Dem92, Proposition 6.1,(vi)]. By ([Bou04, Proposition 4.4]), $\mathscr{E}$ is dual to the modified nef cone. This modified nef cone (introduced in [Bou04, Definition 2.2]) in the case of an algebraic hyperkähler (as in our case) is nothing else than the closure of the birational Kähler cone, whose intersection with the Néron-Severi is the movable cone ([BHT15, Theorem 7] or [Mar11, Lemma 6.22]). □

2.2.2 Proof of Proposition 2.2.0.5

We retain the notation of the previous section. Recall that for a smooth Pfaffian cubic fourfold Y, we have introduced a K3 surface X of degree 14 and the a correspondence $\Gamma \subset Y \times X$.

Lemma 2.2.2.1 ([AG20, Lemma 8]). *Suppose that X has Picard rank 1, so in particular X contains no (-2)-curve and thus Γ induces a regular map $j\colon Y \to X^{[4]}$. Let $\tilde{H}$ and B be the basis for $\mathrm{Pic}(X^{[4]})$, which have been discussed in the previous section. Then*

$$j^*B = 9h$$
$$j^*\tilde{H} = 14h$$

where $h \in \mathrm{Pic}(Y)$ is the hyperplane class.

Proof. By construction of $j\colon Y \to X^{[4]}$ we have a Cartesian diagram

$$\begin{array}{ccc} \Gamma & \longrightarrow & \Xi \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ Y & \xrightarrow{j} & X^{[4]}. \end{array}$$

By [Leh99, Lemma 3.7] we have $B = -c_1(p'_*\mathcal{O}_\Xi)$, and thus by Propostion 2.2.0.1

$$j^*B = -c_1(p_*\mathcal{O}_\Gamma) = 9h \in H^2(Y, \mathbb{Q}).$$

To find $j^*\tilde{H}$, let $H \subset X$ be a hyperplane section, and let $q'\colon \Xi \to X$ be the natural projection. Then $p'_*q'^*\mathcal{O}_H$ is a sheaf supported on the locus of 4-tuples of which one is contained in H, that is, on $\tilde{H}$; and this sheaf has generic rank 1 on its support, so its first Chern class is $\tilde{H}$. Using the resolution

$$0 \to \mathcal{O}_X(-H) \to \mathcal{O}_X \to \mathcal{O}_H \to 0$$

we find that

$$i_* \operatorname{ch}(p_*(\mathcal{O}_\Gamma \otimes q^*\mathcal{O}_H)) = 42h^2 - 91h^3 + 56h^4 - \frac{35}{4}h^5.$$

Our computations by hand are based on the same ideas appearing in the proof of Proposition 2.2.0.1. Again these are quite lenghty, hence we provide the code for the computations with the help of a computer, which we carried out with the computer algebra package Macaulay 2 (The code may be found in the Appendix 5). Thanks to the fact that $i_*h = 3h^2$, we take the coefficient of h^2 and divide by 3 to get

$$j^*(\tilde{H}) = 14h$$

as desired. □

Lemma 2.2.2.2 ([AG20, Lemma 9]). *Suppose that X has Picard rank 1, so again Γ induces a regular map $j\colon Y \to X^{[4]}$. Then every effective divisor on $X^{[4]}$ meets $j(Y)$.*

Proof. First we calculate the movable cone of $X^{[4]}$, applying Bayer and Macrì's result (cf. § 2.2.1) with $d = 7$ and $n = 4$. Then $d(n-1) = 21$ is not a perfect square; and the Pell equation (2.11) $3X^2 - 7Y^2 = 1$ has no solution, as we see by reducing mod 3; so we are in case (c). The minimal positive solution to $X^2 - 21Y^2 = 1$ is $X = 55$, $Y = 12$. This satisfies $X \equiv 1 \pmod 3$ rather than $X \equiv -1 \pmod 3$. Thus the movable cone is spanned by

$$\tilde{H} \qquad \text{and} \qquad 55\tilde{H} - 84B.$$

The same information can in principle be obtained from work of Markman [Mar11], which does not use Bridgeland stability conditions, but [BM14b] was easier to understand for the author.
Next we need to know that if D is an effective divisor on $X^{[4]}$ and M is movable, then $q(D, M) \geq 0$, where q is the Beauville–Bogomolov–Fujiki pairing. This follows from Theorem 2.2.1.2. In fact we do not need the full strength of Boucksom's result: Indeed, since M is moving we can find representatives of D and M (or better one of its multiples) so that their intersection is of codimension 2 and the computation of $q(D, M)$ reduces to the value of a volume form on their intersection, which is positive (more details can be found in the proof of [BHT15, Theorem 7][12]).
Now if an effective divisor D on $X^{[4]}$ does not meet $j(Y)$, then $j^*(D) = 0$, so from Lemma 2.2.2.1 we know that D is a multiple of $9\tilde{H} - 14B$. But we have

$$q(\tilde{H}, \tilde{H}) = 14 \qquad q(\tilde{H}, B) = 0 \qquad q(B, B) = -6,$$

and thus

$$q(9\tilde{H} - 14B, \tilde{H}) = 126 \qquad q(9\tilde{H} - 14B, 55\tilde{H} - 84B) = -126,$$

so no multiple of $9\tilde{H} - 14B$ pairs non-negatively with both walls of the movable cone. □

We are now ready to complete the

Proof of Proposition 2.2.0.5. From [AL17, §3] we know that the open immersion $t : Z_0 \to X^{[4]}$, restricted to $Y_0 := Y \cap Z_0$, agrees with the map $j : Y_0 \to X^{[4]}$ induced by I_Γ. If X has Picard rank 1, then $Y_0 = Y$ by Lemma 2.2.0.4. The open immersion $t : Z_0 \to X^{[4]}$ gives a birational map $Z \dashrightarrow X^{[4]}$. Because Z and $X^{[4]}$ are minimal models, this birational map is biregular on an open set W containing Z_0 whose complement has codimension at least 2 in both Z and $X^{[4]}$. If $Z \setminus Z_0$ has any component of codimension 1, then so does $W \setminus Z_0$, and taking its closure in $X^{[4]}$ we get an effective divisor that does not meet $j(Y)$, contradicting Lemma 2.2.2.2. □

[12] The assumption of uniruledness for the divisor is not needed in the argument.

2.3 Proof of the Theorem 2.0.0.3 for a very general Pfaffian cubic

2.3.1 Reductions

Lemma 2.3.1.1 ([AG20, Lemma 5]). *Let $u : M \to Z$ be the morphism above, then the pullback in cohomology*

$$u^* : H^2(Z,\mathbb{Z}) \to H^2(M,\mathbb{Z})$$

is injective.

Proof. From the universal coefficient theorem we get a short exact sequence:

$$0 \to \mathrm{Ext}^1(H_1(Z,\mathbb{Z}),\mathbb{Z}) \to H^2(Z,\mathbb{Z}) \to \mathrm{Hom}(H_2(X,\mathbb{Z}),\mathbb{Z}) \to 0$$

The first object is trivial, because Z being hyperkähler we have $H_1(Z,\mathbb{Z}) = \pi_1(Z,\mathbb{Z}) = 0$. Hence $H^2(Z,\mathbb{Z})$ is torsion free and it suffices to show the statement with rational coefficients. The morphism a is surjective, thus the claim is an application of [Voi02, Lemma. 7.28].
Alternatively, we could say that u factors as the blowup $Z' \to Z$ along $Y \subset Z$, and a $\mathbb{P}^2$-fibration $M \to Z'$, both of which induce injections on cohomology. □

The variety of twisted cubics M was defined as an irreducible component of the Hilbert scheme $\mathrm{Hilb}^{3t+1}(Y)$, therefore it carries a universal family C of generalized twisted cubics in Y, which is the pullback of the universal closed subscheme Ξ on the Hilbert scheme:

$$\begin{array}{ccccc} Y & \longleftarrow & \Xi & \longleftarrow & C \\ & & \downarrow & & \downarrow \\ & & \mathrm{Hilb}^{3t+1}(Y) & \hookleftarrow & M. \end{array}$$

In particular the morphism $C \to M$ is flat.
In order to use the birational model of Addington-Lehn for Z we reduce to the case of Pfaffians.

Lemma 2.3.1.2. *If the statement of Theorem 2.0.0.3 holds for a smooth cubic Pfaffian cubic not containing a plane, then the theorem holds.*

Proof. The strategy is to set everything in family and unravel the topological nature of the claim. Thus, we consider in the projective space $\mathbb{P}(H^0(\mathbb{P}^5,\mathcal{O}_{\mathbb{P}^5}(3))$ the open set $\mathscr{U}$ of smooth cubic fourfolds not containing a plane [Voi86, §1, Lemme 1]. Over $\mathscr{U}$ lives the family $\pi_1 : \mathscr{Y} \to \mathscr{U}$ of smooth cubic fourfolds; further, since the construction of M and

Z works in family, we get smooth families over $\mathscr{U}$:

$$\begin{array}{ccccc} \mathscr{Y} & & \mathscr{M} & \longrightarrow & \mathscr{Z} \\ & \searrow & \downarrow & \swarrow & \\ & & \mathscr{U}. & & \end{array} \tag{2.13}$$

In addition, we get the flat family $\mathscr{C} \subset \mathscr{Y} \times_{\mathscr{U}} \mathscr{M}$. Thanks to Ehresmann's theorem we find open sets U_i in $\mathscr{U}$, over which the families $\mathscr{Y}, \mathscr{M}$ and $\mathscr{Z}$ are trivialised as $\mathscr{C}^\infty$-fibre bundles and we can choose U_i so that the families are simultaneously trivialised.
The sheaves on $\mathscr{U}$ whose stalks are cohomology (respectively topological K–theory) rings of the fibres are local systems of lattices for each family.
At the same time the Fourier-Mukai functor involved and the basic operations as push-forward and pullback give rise to morphisms of local systems. Let us discuss this in detail.
Let $f : \mathscr{X} \to \mathscr{Y}$ be a projective morphism between two smooth families of projective varieties over a common base B

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{f} & \mathscr{Y} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & B & \end{array} \tag{2.14}$$

and we consider the constant sheaves $\underline{\mathbb{Z}}_{\mathscr{X}}$ and $\underline{\mathbb{Z}}_{\mathscr{Y}}$ on the two families. We start with the pullback in cohomology. Every constant $\mathbb{Z}$-valued function from $\mathscr{Y}$ determines by precomposing with f a constant function on $\mathscr{Y}$, which corresponds to the natural morphism of sheaves

$$\underline{\mathbb{Z}}_{\mathscr{Y}} \to f_* \underline{\mathbb{Z}}_{\mathscr{X}}.$$

Fibrewise it is the pullback f^* on the 0-degree cohomology

$$H^0(Y_b, \underline{\mathbb{Z}}_{\mathscr{Y}}) = (q_* \underline{\mathbb{Z}}_{\mathscr{Y}})_b \to (q_* f_* \underline{\mathbb{Z}}_{\mathscr{X}})_b = (p_* \underline{\mathbb{Z}}_{\mathscr{X}})_b = H^0(X_b, \underline{\mathbb{Z}}_{\mathscr{X}}).$$

In general, we consider the Leray spectral sequence relative to the Diagram (2.14)

$$E_2^{p,q} = R^p q_* R^q f_* \underline{\mathbb{Z}}_{\mathscr{X}} \Rightarrow R^{p+q} p_* \underline{\mathbb{Z}}_{\mathscr{Y}}.$$

The morphism we are looking for is the composition by proper base change

$$R^p q_* \underline{\mathbb{Z}}_c Y \to R^p q_* R^0 f_* \underline{\mathbb{Z}}_{\mathscr{X}} = E_2^{p,0} \to R^p p_* \underline{\mathbb{Z}}_X$$

where the second morphism is the edge morphism from the first quadrant spectral sequence

$$E_2^{p,0} \overset{edge\ map}{\twoheadrightarrow} E_\infty^{p,0} \hookrightarrow R^p p_* \underline{\mathbb{Z}}_X.$$

As we already pointed it out, for projective smooth varieties the pushforward in cohomology is just given by the action of the pullback twisted by Poincaré duality.
The last thing that has to be discussed is the cup product by the class of the curves $[\mathscr{C}_b] \in H^6(Y_b \times M_b, \mathbb{Z})$. Since the family $\mathscr{C} \to \mathscr{U}$ is flat, it gives rise to a flat section of the local system of $H^6(Y_b \times M_b, \mathbb{Z})$, so that the cup product is a morphism of local systems. □

2.3.2 Proof of Theorem 2.0.0.3 for a very general cubic

For the reader's convenience we reproduce here the statement of the theorem.

Theorem 2.0.0.3. *Let*

$$\Phi^K : K_{\mathrm{top}}(\mathscr{A}) \subset K_{\mathrm{top}}(Y) \to K_{\mathrm{top}}(M)$$

be the map on topological K-theory induced by the Fourier–Mukai kernel $I_C^\vee(-3h) \in D^b(Y \times M)$. *Then the map*

$$c_1 \circ \Phi^K : K_{\mathrm{top}}(\mathscr{A}) \to H^2(M, \mathbb{Z}) \tag{2.4}$$

restricts to a Hodge isometry

$$\langle \lambda_2 - \lambda_1 \rangle^\perp \xrightarrow{\sim} u^*(H^2(Z, \mathbb{Z})), \tag{2.5}$$

with the Euler pairing on the left-hand side and the opposite of the Beauville–Bogomolov–Fujiki pairing on the right.

The strategy is to compare the morphism (2.4) of Theorem 2.0.0.3 to the composition:

$$[I_\xi]^\perp \subset K_{\mathrm{top}}(X) \xrightarrow{I_\Xi^\vee} K_{\mathrm{top}}(X^{[4]}) \xrightarrow{c_1} H^2(X^{[4]}, \mathbb{Z}),$$

which is known to be a Hodge isometry; here $\xi \subset X$ is a subscheme of length 4 and $\Xi \subset X \times X^{[4]}$ is the universal subscheme.

We consider the functor $D^b(M) \to D^b(Y)$ induced by $I_C \otimes \mathscr{O}_Y(2h)$, the functor $D^b(Y) \to D^b(X)$ induced by I_Γ, their composition

$$D^b(M) \xrightarrow{I_C(2h)} D^b(Y) \xrightarrow{I_\Gamma} D^b(X), \tag{2.15}$$

and its left adjoint

$$D^b(X) \xrightarrow{I_\Gamma^\vee[2]} D^b(Y) \xrightarrow{I_C^\vee(-5h)[4]} D^b(M), \tag{2.16}$$

which it will later be convenient to rewrite as

$$D^b(X) \xrightarrow{I_\Gamma^\vee(-2h)[2]} D^b(Y) \xrightarrow{I_C^\vee(-3h)[4]} D^b(M). \tag{2.17}$$

We post-compose with the restriction

$$D^b(X) \xrightarrow{I_\Gamma^\vee(-2h)[2]} D^b(Y) \xrightarrow{I_C^\vee(-3h)[4]} D^b(M) \xrightarrow{i^*} D^b(M_0). \tag{2.18}$$

The first composition (2.15) is induced by T, so the left adjoint (2.16) or (2.17) is induced by $T^\vee[2]$, so the restriction (2.18) is induced by $T^\vee|_{X\times M_0}[2]$.
Because $T[1]|_{X\times M_0}$ is an M_0-flat family of ideal sheaves with classifying map $t' : M_0 \to X^{[4]}$, there is a line bundle $\mathscr{L}$ on M_0 such that

$$T[1]|_{X\times M_0} = (1\times t')^* I_\Xi \otimes \mathscr{L},$$

where again $\Xi \subset X \times X^{[4]}$ is the universal subscheme. So the composition (2.18) agrees with

$$D^b(X) \xrightarrow{I_\Xi^\vee[2]} D^b(X^{[4]}) \xrightarrow{t'^*} D^b(M_0) \xrightarrow{\otimes\mathscr{L}^\vee} D^b(M_0).$$

Because $t' = t\circ u$, this is

$$D^b(X) \xrightarrow{I_\Xi^\vee[2]} D^b(X^{[4]}) \xrightarrow{t^*} D^b(Z_0) \xrightarrow{u^*} D^b(M_0) \xrightarrow{\otimes\mathscr{L}^\vee} D^b(M_0).$$

Passing to topological K-theory, and taking first Chern classes where shown, we get a diagram

$$\begin{array}{ccccccccc} K_{\mathrm{top}}(X) & \xrightarrow{I_\Xi^\vee} & K_{\mathrm{top}}(X^{[4]}) & \xrightarrow{t^*} & K_{\mathrm{top}}(Z_0) & \xrightarrow{u^*} & K_{\mathrm{top}}(M_0) & \xrightarrow{\cdot[\mathscr{L}^\vee]} & K_{\mathrm{top}}(M_0) \\ & & \downarrow{\scriptstyle c_1} & & \downarrow{\scriptstyle c_1} & & \downarrow{\scriptstyle c_1} & & \downarrow{\scriptstyle c_1} \\ & & H^2(X^{[4]},\mathbb{Z}) & \xrightarrow[\sim]{t^*} & H^2(Z_0,\mathbb{Z}) & \xrightarrow{u^*} & H^2(M_0,\mathbb{Z}) & \dashrightarrow & H^2(M_0,\mathbb{Z}), \end{array} \tag{2.19}$$

where there is no well-defined map to fill in the dashed arrow, and the injectivity of the lower u^* follows from Lemma 2.3.1.1 and Proposition 2.2.0.5.
The map $D^b(X^{[4]}) \xrightarrow{I_\Xi} D^b(X)$ takes the the skyscraper sheaf of a point to I_ξ, where $\xi \subset X$ is a subscheme of length 4, so its left adjoint id the Fourier-Mukai functor with kernel $I_\Xi^\vee[2]$ and its induced map on K-theory

$$K_{\mathrm{top}}(X) \xrightarrow{I_\Xi^\vee} K_{\mathrm{top}}(X^{[4]})$$

takes classes in $[I_\xi]^\perp$ to classes of rank 0. Indeed, let k any element in $[I_\xi]^\perp$, then

$$0 = \chi(k,[I_\xi]) = \chi(k,\Phi^K_{I_\Xi}(O_{\mathrm{pt}})) = \chi(\Phi^K_{I_\Xi^\vee}(k),\mathscr{O}_{\mathrm{pt}}) = \mathrm{rk}(\Phi^K_{I_\Xi^\vee}(k)) \tag{2.20}$$

where the second equality holds because of [Huy06a, Prop. 5.4].
Thus, in the big diagram (2.19), if we replace $K_{\mathrm{top}}(X)$ with the subspace $[I_\xi]^\perp$ then multiplying by $[\mathscr{L}^\vee]$ does not affect the first Chern class, so we can forget the last column of the diagram.

The composition

$$[I_\xi]^\perp \subset K_{\mathrm{top}}(X) \xrightarrow{I_\Xi^\vee} K_{\mathrm{top}}(X^{[4]}) \xrightarrow{c_1} H^2(X^{[4]},\mathbb{Z})$$

is a Hodge isometry by O'Grady's classic calculation (cf. Propostion 1.2.4.2). Moreover the map t'^* is an isomorphism on H^2, so the big diagram (2.19) gives a Hodge isometry from $[I_\xi]^\perp \subset K_{\mathrm{top}}(X)$ to $u^*(H^2(Z_0,\mathbb{Z})) \subset H^2(M_0,\mathbb{Z})$. Because $M \setminus M_0$ and $Z \setminus Z_0$ have codimension at least 2, this is the same as $u^*(H^2(Z,\mathbb{Z})) \subset H^2(M,\mathbb{Z})$.
If we take (2.18), pass to topological K-theory, and include first Chern classes, we get a diagram

$$\begin{array}{ccccccc} K_{\mathrm{top}}(X) & \xrightarrow{I_\Gamma^\vee(-2h)} & K_{\mathrm{top}}(Y) & \xrightarrow{I_C^\vee(-3h)} & K_{\mathrm{top}}(M) & \xrightarrow{i^*} & K_{\mathrm{top}}(M_0) \\ & & & & \downarrow{\scriptstyle c_1} & & \downarrow{\scriptstyle c_1} \\ & & & & H^2(M,\mathbb{Z}) & \xrightarrow[\sim]{i^*} & H^2(M_0,\mathbb{Z}). \end{array} \tag{2.21}$$

As discussed in Proposition 2.2.0.2 the Fourier-Mukai functor

$$D^b(X) \xrightarrow{I_\Gamma^\vee(-2h)[4]} \mathscr{A}$$

is an equivalence.
We claim that the induced map on topological K-theory takes $[I_\xi]$ to our class $\lambda_2 - \lambda_1$. To see this, first observe that the functor $D^b(Y) \xrightarrow{I_\Gamma(-h)} D^b(X)$ takes the skyscraper sheaf of a point $\mathscr{O}_y$ to some I_ξ, so its right adjoint $D^b(X) \xrightarrow{I_\Gamma^\vee(-2h)[4]} D^b(Y)$ takes I_ξ to the projection of $\mathscr{O}_y$ into $\mathscr{A}$. In K-theory we have $[\mathscr{O}_y] = [\mathscr{O}_\ell(2)] - [\mathscr{O}_\ell(1)]$, where ℓ is a line on Y, so the class of the projection of $\mathscr{O}_y$ is $\lambda_2 - \lambda_1$.
Thus, in our second big diagram (2.21), the first map takes $[I_\xi]^\perp$ isometrically to $(\lambda_2 - \lambda_1)^\perp \subset K_{\mathrm{top}}(\mathscr{A}) \subset K_{\mathrm{top}}(Y)$. We conclude that the map (2.4) takes $(\lambda_2 - \lambda_1)^\perp$ isometrically to $u^*(H^2(Z,\mathbb{Z}))$, as desired.
This concludes the proof of the main theorem.

2.3.3 Proof of Theorem 2.0.0.2 for a very general cubic

For the reader's convenience we reproduce here the statement of the theorem.

Theorem 2.0.0.2. *The pullback*

$$u^* : H^2(Z,\mathbb{Z}) \to H^2(M,\mathbb{Z})$$

is injective, and the map

$$[C]_* : H^4(Y,\mathbb{Z}) \to H^2(M,\mathbb{Z}) \tag{2.2}$$

restricts to a Hodge isometry

$$[C]_*: H^4_{\mathrm{prim}}(Y,\mathbb{Z}) \xrightarrow{\sim} u^*(H^2_{\mathrm{prim}}(Z,\mathbb{Z})), \tag{2.3}$$

with the intersection pairing on the left-hand side and the opposite of the Beauville–Bogomolov–Fujiki pairing on the right.

By our discussion in §2.3.1, to prove Theorem 2.0.0.2 it is enough to show that for a single cubic Y, the map (2.2) takes $H^4_{\mathrm{prim}}(Y,\mathbb{Z})$ into $u^*(H^2_{\mathrm{prim}}(Z,\mathbb{Z}))$ and respects the pairings. We deduce this from Theorem 2.0.0.3 when Y is not in any Noether–Lefschetz divisor.
Because Y is Noether–Lefschetz general, the transcendental lattice of $\langle \lambda_2 - \lambda_1\rangle^\perp \subset K_{\mathrm{top}}(\mathscr{A})$ is $\langle \lambda_1, \lambda_2\rangle^\perp$. The transcendental lattice of $H^2(Z,\mathbb{Z})$ is at least contained in $H^2_{\mathrm{prim}}(Z,\mathbb{Z})$, so the Hodge isometry (2.5) takes $\langle \lambda_1, \lambda_2\rangle^\perp$ into $u^*(H^2_{\mathrm{prim}}(Z,\mathbb{Z}))$; but they are primitive sublattices of the same rank, so (2.5) gives an isomorphism between them. So we need to compare the action of (2.4) on $\langle \lambda_1, \lambda_2\rangle^\perp$ with the action of (2.2) on $H^4_{\mathrm{prim}}(Y,\mathbb{Z})$. In the diagram (1.2), we know from [AT14, Prop. 2.3] that v takes $\langle \lambda_1, \lambda_2\rangle^\perp \subset K_{\mathrm{top}}(\mathscr{A})$ isometrically to $H^4_{\mathrm{prim}}(Y,\mathbb{Z}) \subset H^*(Y,\mathbb{Q})$. To finish proving Theorem 2.0.0.2, it remains to show that the maps

$$H^4_{\mathrm{prim}}(Y,\mathbb{Z}) \to H^2(M,\mathbb{Q})$$

induced by $v(I_C^\vee(-3h)) \in H^*(Y\times M,\mathbb{Q})$ and $[C] \in H^6(Y\times M,\mathbb{Z})$ are the same. To see this, observe that

$$\begin{aligned} \mathrm{ch}(I_C) &= 1 - [C] + \text{higher-order terms},\\ \mathrm{ch}(I_C^\vee) &= 1 + [C] + \text{higher-order terms}. \end{aligned}$$

The action of the map induced by $v(I_C^\vee(-3h))$ on an element $\alpha \in H^4_{\mathrm{prim}}(Y)$ can be described as follows: First we multiply α by $(\mathrm{td}\, Y)^{1/2}$ and $\mathrm{ch}(\mathscr{O}_Y(-3h))$, which are polynomials in the hyperplane class $h \in H^2(Y,\mathbb{Z})$ and thus have no effect on α. Then we map to $H^*(M,\mathbb{Q})$ using $\mathrm{ch}(I_C^\vee)$, which yields $[C]_*\alpha +$ higher-order terms. Then we multiply by $(\mathrm{td}\, M)^{1/2}$, which does not affect the leading term $[C]_*\alpha$. Then we take the degree-2 part, leaving $[C]_*\alpha$ as desired.

In the end we record the following corollary, that can be easily deduced from the above result.

Corollary 2.3.3.1. *The Beauville-Bogomolov square of the natural polarization of Z is* 2.

Proof. Via the isometry (2.3) we can identify the polarization of Z with the element $\lambda_1+\lambda_2$, which is perpendicular to $\lambda_2-\lambda_1$ in $A_2 = \langle \lambda_2, \lambda_1\rangle$. Its square is therefore 2. □

3

Birational models of the LLSvS variety

As an application of Theorem 2.0.0.3 from the previous chapter, we characterise for which cubic fourfolds Y the LLSvS variety is birational to the punctual Hilbert scheme of some K3 surface or isomorphic to a moduli space of (twisted) sheaves on a K3 surface (see Theorem 3.1.3.8 below).
In order to do so we have to unravel the deep and mysterious bond entangling cubic fourfolds and K3 surfaces, and more in general hyperkähler varieties, as initiated in Hassett's thesis and then developed by many other people. Both objects (cubic fourfolds and hyperkähler varieties) have a very rich geometry, what makes this an active and wide field of research with many directions. We focus on what we think is necessary not only to understand, but also to appreciate the topic. The author had indeed the pleasure to attend the conference "School on Birational Geometry of Hypersurfaces" in Gargnano del Garda[1] and indeed our exposition is much indebted to the lecture notes [Huy18], which we recommend for a more comprehensive treatment of the subject. Important references are [Huy], [MS18], and all the original references therein, above all [Has18].

3.1 Cubic fourfolds and associated K3s

3.1.1 The periods of K3 surfaces

K3 surfaces have a very rich geometry and in the last decades were a hot topic of research. We report here quickly the results we will need in the following.
Recall that complex K3 surfaces are defined as complex compact manifolds of dimension 2 with trivial canonical bundle and trivial irregularity. A classical deep result is that these surfaces are always Kähler, so that their cohomology has Hodge decomposition. However, not all of them are projective.
The Hodge diamond of a K3 surface looks like:

$$\begin{array}{ccccc} & & 1 & & \\ & & & & \\ 1 & & 20 & & 1 \\ & & & & \\ & & 1 & & \end{array}$$

[1] https://sites.google.com/site/gargnano2018/

where we have suppressed the odd cohomology because it is trivial.
The integral second cohomology group of a K3 surface S equipped with the intersection form is an even unimodular lattice of signature $(3,19)$ and thus it is isomorphic to

$$H^2(S,\mathbb{Z}) \simeq U^{\oplus 3} \oplus E_8^{\oplus 2}(-1) =: \Lambda$$

where U is the hyperbolic plane and E_8 the lattice associated to the corresponding Dynkin diagram.
Any primitive element $\ell \in \Lambda$ with square $\ell^2 = d \in 2\mathbb{Z}$ lies in the same orbit for the action of $O(\Lambda)$ ([Huy16, Corollary 14.1.10]), its orthogonal complement is of fixed isomorphic type

$$\ell^\perp \simeq U^{\oplus 2} \oplus E_8^{\oplus 2}(-1) \oplus \mathbb{Z}(-d) \simeq (e + \frac{d}{2} f)^\perp =: \Lambda_d \subset \Lambda.$$

where e and f are generators of a copy of U.
For K3 surfaces one can reconstruct the isomorphism type of the surface just from the Hodge structure on the second cohomology group, in other words a Torelli-type theorem holds in this context. More precisely:

Theorem 3.1.1.1 ([Huy06b, Theorem 1.3]). *Given two K3 surfaces S and S'.*

$$S \simeq S' \iff \textit{There exists a Hodge isometry } H^2(S,\mathbb{Z}) \simeq H^2(S',\mathbb{Z}).$$

In order to understand if two K3 surfaces are derived equivalent, one has to look at a larger lattice. As we have seen in Chapter 1, the whole integral cohomology can be equipped with the Mukai pairing, which agrees with the intersection form on $H^2(S,\mathbb{Z})$. This can be computed as:

$$\langle (v_0,v_2,v_4),(w_0,w_2,w_4)\rangle = v_0w_4 + v_2w_2 + v_4w_0$$

and thus

$$H^*(S,\mathbb{Z}) = U^{\oplus 4} \oplus E_8^{\oplus 2} =: \tilde{\Lambda}.$$

We endow $H^*(S,\mathbb{Z})$ with a pure (effective) Hodge structure of weight 2 by imposing the $(2,0)$-part equal to $H^{2,0}(S,\mathbb{Z})$.
Now we can state the following result known as derived Torelli theorem.

Theorem 3.1.1.2 ([Huy06b, Theorem 5.13]). *Let S and S' be two K3 surfaces, then*

$$D^b(S) \simeq D^b(S') \iff \textit{There exists a Hodge isometry } H^*(S,\mathbb{Z}) \simeq H^*(S',\mathbb{Z}).$$

Twisted K3 surfaces

Definition 3.1.1.3. *A twisted K3 surface (S,α) consists of a K3 surface S and an element $\alpha \in H^2(S,\mathcal{O}_S^*)_{\text{tor}}$. An isomorphism of twisted K3 surfaces (S,α) and (S',α') is an isomorphism $f : S \xrightarrow{\sim} S'$ such that $f^*\alpha' = \alpha$.*

Remark 3.1.1.4. Note that since we are working with complex K3 surfaces, the torsion part of the group $H^2(S, \mathcal{O}_S^*)$ is isomorphic to the Brauer group of isomorphism classes of sheaves of Azumaya algebras.

These data determine, as sketched here (for details [HS05, § 2]), a Hodge structure of $K3$-type on the Mukai lattice $H^*(S, \mathbb{Z})$ and a derived category of twisted sheaves, for which a Torelli theorem holds.
Given α as in the definition, we can consider a Čech 2-cocycle $\{\alpha_{ijk}\}_{ijk} \in \Gamma(U_{ijk}, \mathcal{O}_S^*)$ and *twisted* sheaves (E_i, φ_{ij}) where E_i are coherent sheaves on the open sets U_i and $\varphi_{ij} \colon E_i|_{U_{ij}} \xrightarrow{\sim} E_j|_{U_{ij}}$ are isomorphisms that satisfy a cocycle condition twisted by α:

$$\varphi_{ii} = id, \qquad \varphi_{ij} = \varphi_{ji}^{-1}, \qquad \varphi_{ijk} \circ \varphi_{jki} \circ \varphi_{kij} = \alpha_{ijk}.$$

Such sheaves form an abelian category $\mathrm{Coh}(S, \{\alpha_{ijk}\}_{ijk})$, of which we consider the bounded derived category $D^b(S, \{\alpha_{ijk}\}_{ijk})$. The construction of the latter does not depend actually on the choice of the representing cocycle but just on the class α.
The exponential sequence gives the surjection $H^2(S, \mathbb{R}) \twoheadrightarrow H^2(S, \mathcal{O}_S^*)$, but since our object of interest is the torsion part, we note that rational cohomology factors through

$$H^2(S, \mathbb{Q}) \twoheadrightarrow H^2(S, \mathcal{O}_S^*)_{\mathrm{tor}}.$$

We consider a lifting $B \in H^2(S, \mathbb{Q})$ of α along this morphism and the unique Hodge structure $\tilde{H}(S, \alpha, \mathbb{Z})$ of K3 type given by imposing $\tilde{H}^{2,0}(S, \alpha, \mathbb{Z}) = \mathbb{C}\exp(B)\sigma = \mathbb{C}(\sigma + \sigma \wedge B)$ in $H^*(S, \mathbb{Z})$. Here σ is a non-trivial holomorphic 2-form on S. The Hodge structure is not independent of the choice of B, but to two different choices of the B-field correspond isometric Hodge structures.
A Torelli theorem holds for twisted K3 surfaces as well.

Theorem 3.1.1.5 ([Huy06b, Theorem 6.2]). *Let (S, α) and (S', α') be two twisted K3 surfaces, then the derived categories $D^b(S, \alpha)$ and $D^b(S', \alpha')$ are equivalent if and only if there exists a Hodge isometry $\tilde{H}(S, \alpha, \mathbb{Z}) \simeq \tilde{H}(S', \alpha', \mathbb{Z})$ preserving the natural orientation.*

3.1.2 Cubic fourfolds

The Hodge diamond of a smooth cubic fourfold is

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & & 1 & & & \\ 0 & 1 & & 21 & & 1 & 0 \\ & & & 1 & & & \\ & & & 1 & & & \end{array}$$

where we left out the odd cohomology, because it is trivial. The middle cohomology group is torsion free and, equipped with the intersection form, is a unimodular odd lattice of signature (2,21), thus

$$H^4(Y,\mathbb{Z}) \simeq (1)^{\oplus 21} \oplus (-1)^{\oplus 2} =: \tilde{\Gamma}$$

where $(1)^{\oplus n}$, respectively $(-1)^{\oplus n}$, denotes the lattice of rank n, whose quadratic form is represented by the identity matrix, respectively by minus the identity matrix ([Has18, Proposition 2.1.2]). Its primitive cohomology, which is the orthogonal complement of the square of the hyperplane class, is

$$H^4_{\text{prim}}(Y,\mathbb{Z}) \simeq U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus A_2 =: \Gamma.$$

The square of the hyperplane class h can be realised as the element $(1,1,1) \in (1)^{\oplus 3} \subset \tilde{\Gamma}$, for details we refer to [Huy, page 23].
For cubic fourfolds, a Torelli theorem holds [Voi86] which says that two smooth cubic fourfolds are isomorphic exactly when they have isometric Hodge structures on the primitive middle cohomology. Later we will see a formulation in terms of the period map in Theorem 3.1.2.3.

Moduli spaces and the period map The structure of the coarse moduli space and the period domain for cubic fourfolds has been investigated by Hassett.

The former is constructed with standard GIT technology as the quotient of the open locus $\mathcal{U} \subset |\mathcal{O}_{\mathbb{P}^5}(3)|$ of smooth cubic fourfolds modulo the action of the projective linear group PGL(6):

$$M := \mathcal{U} /\!/ \text{PGL}(6).$$

It is a smooth quasi-projective variety of dimension 20.
In contrast, one can consider the classifying space of Hodge structures on the abstract lattice Γ of cubic fourfold type

$$D := \{x \in \mathbb{P}(\Gamma \otimes \mathbb{C}) |\ (x,x)^2 = 0,\ (x,\bar{x}) > 0\},$$

the so-called period domain. Indeed for a smooth cubic fourfold we have seen that $H^{3,1}$ is spanned by a unique $(3,1)$-form ω, which satisfies $\omega \wedge \omega = 0$ for type reasons, and $\omega \wedge \bar{\omega}$ is a volume form, therefore positive. Given $H^{1,3}(Y)$ one reconstructs the whole Hodge structure using the relations

$$H^{3,1} = \overline{H^{1,3}}$$
$$H^{2,2} = (H^{3,1} \oplus H^{1,3})^{\perp}.$$

In virtue of the Baily-Borel theorem one can then show that the quotient of D by the action of the orthogonal group

$$\tilde{O}(\Gamma) := \{g \in O(\tilde{\Gamma}) |\ g((1,1,1)) = (1,1,1)\}$$

is a smooth quasi-projective variety $\mathscr{C}$ ([Has18, Proposition 2.2.1]), which is actually isomorphic to $D/O(\Gamma)$ as well. [2]
Any cubic forufold Y with a so-called marking, that is an isomorphism the choice of an isomorphism $\phi : H^4(Y,\mathbb{Z}) \xrightarrow{\sim} \tilde{\Gamma}$, determines a point in D, namely $\phi(H^{1,3}(Y))$. Thanks to Voisin Torelli theorem it descends to an injective map, which is actually is an open immersion of analytic spaces [Voi86]. By applying results by Borel, Hassett proved that the period map is actually algebraic [Has18, Proposition 2.2.3] and the complement is defined by algebraic equations [Has18, Corollary 2.2.4].
The theorem has been improved and the complement has been described, as we will see in Theorem 3.1.2.3.

Hassett divisors We introduce now some divisors on the moduli spaces of cubic fourfolds. Usually they are named after Hassett, who was the first to define them and investigate them systematically in his PhD thesis [Has18].
A cubic fourfold Y is said to be special of discriminant d if there exists a primitive rank 2 sublattice K in $H^{2,2}(Y,\mathbb{Z})$ that contains the square of the hyperplane section h^2 and has discriminant d. The choice of such a sublattice K is called a labelling. Special cubic fourfolds of discriminant d, or better their periods, determine a divisor $\mathscr{C}_d$ in $\mathscr{C}$, which Hassett proved to be non-empty exactly when d satisfies:

$$d > 6 \text{ and } d \equiv 0 \text{ or } 2 \pmod 6. \qquad (*)$$

Example 3.1.2.1. As smooth cubic fourfolds satisfy the integral Hodge conjecture ([Voi07]), finding a labelling amounts to find a surface, that is not cohomologous to a complete intersection. We present these classical examples:

- Cubic fourfolds containing a plane are exactly the special cubic fourfolds of discriminant 8 . Indeed, let Y be a smooth cubic fourfold and E a projective plane sitting in Y. Then the self-intersection product of E on Y is equal to the second Chern class $c_2(\mathscr{N}_{E|Y})$ of the normal bundle. Using the normal short exact sequence

$$0 \to \mathscr{T}_E \to \mathscr{T}_Y|_E \to \mathscr{N}_{E|Y} \to 0$$

[2] Indeed any isometry g of Γ extends to an isometry of $\tilde{\Gamma}$ mapping $(1,1,1) \mapsto \pm(1,1,1)$ according to whether the action on the discriminant group $A_\Gamma \simeq \mathbb{Z}/3\mathbb{Z}$ is $\pm$id. Then $\tilde{O}(\Gamma)$ is the kernel of the natural map $O(\Gamma) \to O(A_\Gamma)$ which has in index 2 in $O(\Gamma)$, but $-$id acts trivially on D and the two quotients are the same.

we can express $\langle E, E\rangle$ in terms of the cohomology of E and get rid of the embedding. More precisely:

$$\langle E, E\rangle = 6h^2 + 3hK_E + K_E^2 - \chi_E.$$

where χ_E denotes the topological Euler characteristic of E. The intersection product restricted to the lattice $K = \langle H^2, E\rangle$ has therefore Gram matrix

$$K = \begin{array}{c|cc} & h^2 & E \\ \hline h^2 & 3 & 1 \\ E & 1 & 3 \end{array}.$$

where obviously $1 = \langle E, h^2\rangle$ is the degree of E. Hence the discriminant is 8 and we have proven that cubic fourfolds containing a plane lie in $\mathscr{C}_8$. For the other inclusion we refer to [Voi86, §3].

- Similarly one shows that the divisor $\mathscr{C}_{14}$ is the closure of smooth cubic fourfolds containing a smooth rational normal scroll of degree 4.

Remark 3.1.2.2. A priori, for smooth cubic fourfolds two labellings K and K' with same discriminant d may differ, nonetheless Hassett exploited the action of the orthogonal group of Γ to get that, loosely speaking, there is an isometry of $\tilde{\Gamma}$ fixing $(1,1,1)$ taking K to K' (cf. [Huy18, Lemma 1.4].

The Torelli theorem has been further improved and in this set up we have the following theorem:

Theorem 3.1.2.3 ([Laz10, Loo09]). *The period map induces an algebraic open embedding, which is an isomorphism onto the complement of $\mathscr{C}_2$ and $\mathscr{C}_6$:*

$$\mathscr{P} : M \xrightarrow{\sim} \mathscr{C} \setminus (\mathscr{C}_2 \cup \mathscr{C}_6)$$

In light of this theorem it is customary, as we will do, to identify isomrphism classes of cubic fourfolds with their periods.

3.1.3 Associated K3 surfaces

There is an important link between K3 surfaces and cubic fourfolds based on their cohomology lattices. Let Y be a smooth cubic fourfold and S be a K3 surface. As we have seen, $H^2(S,\mathbb{Z})$ and $H^4(Y,\mathbb{Z})_{\text{prim}}$ look similar in the sense that they have the same rank, signature and, after a Tate-twist, the same Hodge numbers. Nevertheless, they are still very different because the first lattice is even and the second is odd. The way out comes

to relate primitive cohomology for K3 surfaces with the cohomology of special cubic fourfolds. If d is an even number satisfying

$$d/2 \text{ is not divisible by 9 or any odd prime } p \equiv 2 \pmod 3, \qquad (**)$$

then and only then there exists a K3 surface S with a polarisation of degree d such that

$$\Lambda \supset \Lambda_d \simeq \Gamma_d \subset \Gamma.$$

This is the content of [Has00, Proposition 5.1.4] and [Huy18, Lemma 1.10], it is an application of Nikulin's results on primitive embeddings of even lattices in connection with the Torelli theorems.

Remark 3.1.3.1. Behind this purely lattice-theoretic reasoning there is a nice geometric interpretation for the mysterious bond between K3 surfaces and cubic fourfolds. If Y is a rational cubic fourfold so that there exists a birational map $\rho : Y \dashrightarrow \mathbb{P}^4$, then one can resolve ρ as a composition of blow-ups and blow-downs along smooth centres by an application of the Weak Factorisation Theorem ([Wo03, AKMWo02]). In this way one cooks up a bunch of K3 surfaces $S_1, ..., S_m$ and $T_1, ..., T_n$ such that there is a Hodge isometry

$$H^4(Y, \mathbb{Z}) \oplus H^2(S_1, \mathbb{Z}) \oplus ... \oplus H^2(S_m, \mathbb{Z}) = H^2(T_1, \mathbb{Z}) \oplus ... \oplus H^2(T_n, \mathbb{Z}).$$

This idea was in effect fundamental in the milestone work of Griffiths and Clemens on the non-rationality of the cubic threefold. In this context curves, thus the Torelli-theorem for curves, were playing a major role. (For more details we point to [Has16, § 3.1])

Example 3.1.3.2. This principle is well illustrated in the case of a cubic fourfold Y, which contains two different projective planes P_1 and P_2. Such a cubic fourfold is known to be rational, indeed there is a natural birational map:

$$P_1 \times P_2 \dashrightarrow Y$$

which associates to every pair of points (P_1, P_2) the third point lying on the intersection of Y and the line through P_1 and P_2. The indeterminacy locus of such a map is a K3 surface S, the blow-up in S resolves the map.

To put everything on a general footing we need to consider larger lattices. For a K3 surface we have already discussed the Mukai lattice $\tilde{H}(S, \mathbb{Z})$. For a cubic fourfold Y we have introduced in Chapter 1 the lattice $K_{\text{top}}(\mathscr{A}_Y)$, now we discuss its arithmetic properties.
We have already discussed the classes λ_1, λ_2 and that their orthogonal complement is Hodge isometric to the primitive cohomology of the cubic fourfold (see Proposition 1.3.2.2). The Mukai lattice $K_{\text{top}}(\mathscr{A})$ of a cubic fourfold is isomorphic as abstract lattice to $\tilde{\Lambda}$ ([Huy18, Lemma 1.18]).

In this way, $H_{\text{prim}}(Y,\mathbb{Z})$ is naturally embedded in a lattice abstractly isomorphic to $\tilde{\Lambda}$ specularly as for a K3 surface S the lattice $H^2(S,\mathbb{Z})$ is embedded in $\tilde{H}(S,\mathbb{Z})$.

Definition 3.1.3.3 ([Huy18, Definition 1.23]). *Let Y be a smooth cubic fourfold and let S be a K3 surface*

- *S and Y are said to be associated if there exists a Hodge isometry*
$$\tilde{H}(S,\mathbb{Z}) \simeq K_{\text{top}}(\mathscr{A}).$$
- *Let L be a polarisation on S. The polarised K3 surface (S,L) and Y are said to be associated if there exists an embedding*
$$H^2(S,\mathbb{Z})_{\text{prim}} \hookrightarrow H^4(Y,\mathbb{Z})_{\text{prim}} \subset K_{\text{top}}(\mathscr{A}).$$
- *Let α be an element in $H^2(S,\mathscr{O}_S^*)_{\text{tor}}$. The twisted K3 surface (S,α) and Y are said to be associated if there is a Hodge isometry*
$$\tilde{H}(S,\alpha,\mathbb{Z}) \simeq K_{\text{top}}(\mathscr{A}).$$

A comment is in order.

Remark 3.1.3.4. If the polarised K3 (S,L) is associated to a cubic fourfold Y, so is S. Indeed, one can extend any isometry between the primitive cohomology to the whole cohomology group $\tilde{H}$. Moreover, we point out that if a K3 surface S is associated to a cubic fourfold Y, then it is projective. Indeed, we can find in $H^{1,1}(S,\mathbb{Z}) = H^{1,1}(Y,\mathbb{Z})$ a positive class, which in turn is a polarisation on S. Further, if a K3 surface S is associated to a cubic fourfold, then one can find a K3 surface S' with a polarisation L, so that the polarised K3 surface (S',L) is associated to Y (cf. [Huy18, Proposition 1.25]). In virtue of this observation, later we will treat just polarised K3 surfaces.

Example 3.1.3.5. It is not always obvious to find geometrically associated $K3$ surfaces. For a Pfaffian smooth cubic fourfold Y we have seen in Chapter 2 the construction of a $K3$ surface X of degree 14. X is indeed associated to Y [3]. Thus, all Pfaffian cubics lie in the divisor $\mathscr{C}_{14}$ and form a dense locus in it, though not open as recently discovered in [BRS15, Theorem 3.7].

Let us consider the condition:

In the prime factorization of $d/2$,
primes $p \equiv 2 \pmod 3$ appear with even exponents. $(**')$

Theorem 3.1.3.6 ([Huy18, Corollary 2.17]). *Let Y be a smooth cubic fourfold.*

- *Y has an associated K3 surface S if and only if Y is in $\mathscr{C}_d$ for some d satisfying $(**)$.*
- *Y has an associated twisted K3 surface (S,α) if and only if Y is in $\mathscr{C}_d$ for some d satisfying $(**')$.*

[3] Indeed, Beauville and Donagi showed that $H^4(Y,\mathbb{Z})_{\text{prim}} \simeq H^2(F,\mathbb{Z})_{\text{prim}}$, where $F \simeq X^{[2]}$, so that $H^2(X,\mathbb{Z})_{\text{prim}} \subset H^2(F,\mathbb{Z})_{\text{prim}} \simeq H^4(Y,\mathbb{Z})_{\text{prim}}$.

Hyperkähler varieties associated to cubic fourfolds Another fine invariant associated to any cubic fourfold is given by its Fano variety of lines. Indeed, one can even prove the Voisin Torelli theorem for cubic fourfold using the Torelli theorem of hyperkähler manifolds as in [Cha12].
The key point is the result from Beauville-Donagi which says that Fano variety are indeed hyperkähler. The original argument by Beauville and Donagi was to show that the Fano of the Pfaffian is in effect the punctual Hilbert scheme of the naturally associated K3 and concluded that every Fano is hyperkähler of $K3^{[2]}$–type. The question was settled by Addington in [Add16], who introduced the condition

$$d \text{ is of the form } \frac{2n^2+2n+2}{a^2} \text{ for some } n, a \in \mathbb{Z}. \tag{$***$}$$

and showed the following (the twisted result was proven by Huybrechts).

Theorem 3.1.3.7 ([Add16, Theorem 2], [Huy17a, Theorem 1.3]). *Let Y be a smooth cubic fourfold not containing a plane and F its Fano variety of lines.*

(a) F is birational to a moduli space of sheaves on a K3 surface if and only if $Y \in \mathscr{C}_d$ for some d satisfying ($**$).

(b) F is birational to a moduli space of twisted sheaves on a K3 surface if and only if $Y \in \mathscr{C}_d$ for some d satisfying ($**'$).

(c) F is birational to the Hilbert scheme of two points on a K3 surface if and only if $Y \in \mathscr{C}_d$ for some d satisfying ($***$).

Let us introduce a new condition

$$d \text{ is of the form } \frac{6n^2+6n+2}{a^2} \text{ for some } n, a \in \mathbb{Z}. \tag{$***'$}$$

On top of this result, we have

Theorem 3.1.3.8 ([AG20, Theorem 3]). *Let Y be a cubic fourfold not containing a plane — that is, not lying in $\mathscr{C}_8$ — so the symplectic eightfold Z is defined.*

(a) Z is birational to a moduli space of sheaves on a K3 surface if and only if $Y \in \mathscr{C}_d$ for some d satisfying ($**$).

(b) Z is birational to a moduli space of twisted sheaves on a K3 surface if and only if $Y \in \mathscr{C}_d$ for some d satisfying ($**'$).

(c) Z is birational to the Hilbert scheme of four points on a K3 surface if and only if $Y \in \mathscr{C}_d$ for some d satisfying ($***'$).

The key ingredient for such theorems is naturally the Torelli theorem for hyperkähler varieties and in particular for $K3^{[n]}$-varieties.

Remark 3.1.3.9. The number of polarisations of an associated K3 to a cubic fourfold may be the maximum possible, that is 20 [Awa20]. For d satisfying (∗∗) but not (∗∗∗) the Fano variety is not isomorphic to the punctual Hilbert scheme of the associated K3s. On the other hand if a $K3$ surface S with Picard number one is associated to a cubic fourfold in $\mathscr{C}_d$ for some d divisible by 6 and satisfying (∗∗), then there exist exactly one other associated K3 surface S'. However, it may happen that $\mathrm{Hilb}^2(S)$ is not birational to $\mathrm{Hilb}^2(S')$: the precise condition was found by Emma Brakkee [Bra18, Corollary 4.9]. It would be useful to have similar results about the Hilbert scheme of 4 points.

Remark 3.1.3.10. It is worthwhile to mention another side of the story that we have not touched in our presentation of cubic fourfolds. The subject expounded here has been investigated from another perspective and with a different machinery and technology, the Bridgeland stability conditions on the Kuznetsov component $\mathscr{A}$ of the derived category of a cubic fourfold. For a survey we point to [MS18].
The condition (∗∗) is very important in light of the wild open conjecture, usually attributed to Hassett:

Conjecture 3.1.3.11. *Let Y be a smooth cubic fourfold. Then Y is rational if and only if Y lies in $\mathscr{C}_d$ for some d satisfying* (∗∗).

This is equivalent to having an associated K3 surface (in sense of Definition 3.1.3.3) and at the same time to that $\mathscr{A}$ is equivalent to the derived category of a cubic fourfold [MS18, Theorem 3.7].

Remark 3.1.3.12. We have the following chain of implications ([Huy18, Remark 1.9]):

$$(***) \Rightarrow (**) \Rightarrow (**') \Rightarrow (*).$$

Moreover, (∗∗∗′) is strictly stronger than (∗∗) but incomparable to (∗∗∗): the first few ds satisfying (∗∗∗) are 14, 26, 38, 42, 62, and 86, whereas the first few ds satisfying (∗∗∗′) are 14, 38, 62, 74, and 86.

3.2 Proof of Theorem 3.1.3.8

We adapt an argument developed in [Add16] and [Huy17b, Proposition 4.1] and polished in [Huy18, §3.2], staying especially close to the latter reference.
From Verbitsky's hyperkähler Torelli theorem [Ver13, Huy12, Loo19] and Markman's work of the monodromy of manifolds of $K3^{[n]}$-type [Mar11, §9], we know that if $n-1$ is a prime power, then two manifolds M and M' of $K3^{[n]}$-type are birational (or bimeromorphic) if and only if there is a Hodge isometry $H^2(M,\mathbb{Z}) \cong H^2(M',\mathbb{Z})$. If $n-1$ has

two or more prime factors there is a more subtle statement, but this does not concern us since the eightfold Z is of K3$^{[4]}$-type.

As we have seen in Proposition 3.1.3.6, a cubic fourfold Y is in $\mathscr{C}_d$ for some d satisfying (∗∗) if and only if the Mukai lattice $K_{\mathrm{top}}(\mathscr{A})$ is Hodge-isometric to the Mukai lattice $\tilde{H}(S,\mathbb{Z})$ of a K3 surface, and similarly with (∗∗′) and the Mukai lattice $\tilde{H}(S,\alpha,\mathbb{Z})$ of a twisted K3 surface.

If $\varphi : K_{\mathrm{top}}(\mathscr{A}) \to \tilde{H}(S,\mathbb{Z})$ is a Hodge isometry, let $v = \varphi(\lambda_2 - \lambda_1)$, which is a primitive vector satisfying $v^2 = 6$. Then for a v-generic polarization $h \in \mathrm{Pic}(S)$, the moduli space $M := M_h(v)$ of h-stable sheaves with Mukai vector v is a smooth variety of K3$^{[4]}$-type, and satisfies $H^2(M,\mathbb{Z}) \cong v^\perp \subset \tilde{H}(S,\mathbb{Z})$. Thus

$$H^2(Z,\mathbb{Z}) \cong \langle \lambda_2 - \lambda_1 \rangle^\perp \cong v^\perp \cong H^2(M,\mathbb{Z}).$$

Conversely, any Hodge isometry $\langle \lambda_2 - \lambda_1 \rangle^\perp \cong v^\perp$ extends to a Hodge isometry $K_{\mathrm{top}}(\mathscr{A}) \to \tilde{H}(S,\mathbb{Z})$ taking $\lambda_2 - \lambda_1$ to $\pm v$ by [Nik79, Corollary 1.5.2]: the discriminant group of the lattice $v^\perp$ is $\mathbb{Z}/6$ whose automorphism group is ± 1. With twisted K3 surfaces the argument is similar. This proves parts (i) and (ii) of Theorem 3.1.3.8.

To prove part (iii), by [Add16, Proposition 5] or by the proof of [Huy18, Proposition 3.4(i)] we see that Z is birational to $S^{[4]}$ for some K3 surface S if and only if the algebraic lattice $K_{\mathrm{num}}(\mathscr{A}) \subset K_{\mathrm{top}}(\mathscr{A})$ contains a copy of the hyperbolic plane U which contains $\lambda_2 - \lambda_1$; or equivalently, if and only if there there is a vector $w \in K_{\mathrm{num}}(\mathscr{A})$ with $\chi(w,w) = 0$ and $\chi(w, \lambda_2 - \lambda_1) = 1$.

Remark 3.2.0.1. Huybrechts [Huy18] and many other authors define the Mukai pairing on $K_{\mathrm{top}}(\mathscr{A})$ as having the opposite sign from the Euler pairing, in order to agree with the Beauville–Bogomolov pairing on H^2 of various hyperkähler varieties, whereas Addington and Thomas [AT14, Add16] define it having the same sign as the Euler pairing. In effect, the first convention regards $K_{\mathrm{top}}(\mathscr{A})$ as a Hodge structure of weight 2, the second as a Hodge structure of weight 0. In the calculation that follows, to avoid confusion, we only use Euler pairing and do not mention the Mukai pairing.

If there is such a w, let $L = \langle \lambda_1, \lambda_2, w \rangle \subset K_{\mathrm{num}}(\mathscr{A})$, and let $n = \chi(w, \lambda_1)$, so the Gram matrix of the Euler pairing on L is

$$\begin{pmatrix} -2 & 1 & n \\ 1 & -2 & n+1 \\ n & n+1 & 0 \end{pmatrix}$$

Thus $\mathrm{disc}(L) = 6n^2 + 6n + 2$. Let M be the saturation of L in $K_{\mathrm{num}}(\mathscr{A})$, let a be the index of L in M, and let $d = \mathrm{disc}(M)$; then $a^2 d = \mathrm{disc}(L)$, so $d = (6n^2 + 6n + 2)/a^2$. By [AT14, Proposition 2.5] we have $Y \in \mathscr{C}_d$.

Conversely, suppose that $Y \in \mathscr{C}_d$ for some d of the form $(6n^2+6n+2)/a^2$. We claim that $a \equiv 1 \pmod 6$: to see this, observe that $6n^2+6n+2$ is the norm of the primitive vector $(n,-n-1)$ in the A_2 lattice, hence satisfies (**) by [Huy18, Proposition 1.13(iii)], so a must be a product of primes $p \equiv 1 \pmod 3$. Thus $d \equiv 2 \pmod 6$. Write $d = 6k+2$ and $a = 3m+1$. By [Add16, Lemma 9] there is an element $\tau \in K_{\mathrm{num}}(\mathscr{A})$ such that $\langle \lambda_1, \lambda_2, \tau \rangle$ is a primitive sublattice on which the Gram matrix of the Euler pairing is

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 2k \end{pmatrix}.$$

Then the vector

$$w := (m-n)\lambda_1 + (2m-n)\lambda_2 + a\tau$$

satisfies $\chi(w,w) = 0$ and $\chi(w, \lambda_2 - \lambda_1) = 1$, as desired.

4

The geometry of Z

The results and discussion presented here originated from an attempt to verify a conjecture of Voisin regarding hyperkähler varieties in the special of the LLSvS eightfold and to understand the geometry of the Voisin rational map

$$F \times F \dashrightarrow Z.$$

In the course of this project the problem was solved by Fu, Laterveer and Vial ([FLV20, Corollary 1]) and several discoveries were made around the Voisin map. Therefore, in order to get a better picture we will place all of it in a wider context.

4.1 Context

4.1.1 The Voisin conjecture on the Chow ring of hyperkähler varieties

Among all the invariants attached to an algebraic variety, the Chow ring is one of the most difficult to understand. Even for a surface it can be quite hard to describe as a result by Mumford [Mum68] tells us.

Theorem 4.1.1.1. *Let S be a surface. If $H^{2,0} \neq 0$, then there does not exist a curve $j: C \hookrightarrow S$ such that $j_*: \mathrm{CH}_0(C) \to \mathrm{CH}_0(S)$ is surjective.*

The converse to this statement is known as the Bloch conjecture.
For K3 surfaces, Beauville and Voisin [BV04, Theorem 1] described a subring with remarkable properties. Indeed, they proved that the degree 2 part of the subring generated by the Picard group of a K3 surface S is the span of the class of any point on some rational curve on S.
This result can be rephrased by saying that any polynomial relation holding for the first Chern classes of line bundles in cohomology already holds at the level of Chow theory. Beauville conjectured a similar statement [Bea07], which was further slightly generalised by Voisin in

Conjecture 4.1.1.2. *[Voi08] Let X be an algebraic hyperkähler variety. Then any polynomial cohomological relation*

$$P([c_1(L_j)],[c_i(T_X)]) = 0 \text{ in } H^{2k}(X,\mathbb{Q}), L_j \in \mathrm{Pic}(X)$$

already holds at the level of Chow groups:

$$P([c_1(L_j)],[c_i(T_X)]) = 0 \text{ in } \mathrm{CH}^k(X)\otimes\mathbb{Q}.$$

She then verified the conjecture in various cases including some Hilbert schemes of points on K3 surfaces and the Fano variety of lines on a cubic fourfold. In a further investigation of the Bloch-Beilinson conjectural filtration for hyperkähler varieties, she proposes a way of how to get control of the Chow ring of the LLSvS variety Z by reducing to the known results about the Fano variety of lines F. Indeed, she constructs ([Voi16, Proposition 4.8]) a rational 6-to-1 map

$$\varphi : F \times F \dashrightarrow Z$$

and she suggests to study the uniruled branch divisor of a resolution of ϕ.

4.1.2 Known results

As already said, the major result regarding these questions is for sure the verification of the Voisin conjecture for the LLSvS variety. This was achieved in a series of papers by Fu, Laterveer, Shen and Vial ([FLV20] [FLV19] [FLVS17]).

About the Voisin map these various results were achieved by different people. Muratore determined the indeterminacy locus of the map:

Theorem 4.1.2.1. *[Mur20, Theorem 1.2] The indeterminacy locus of the Voisin rational map coincides with the variety I of incidental lines. It is irreducible and of dimension 6.*

Chen described its resolution of the indeterminacy locus in terms of extensions of sheaves.

Theorem 4.1.2.2. *[Che18, Theorem 1.2] The blow-up of $F \times F$ along the reduced scheme structure of I resolves the Voisin map.*

4.1.3 Our results

Our main result regards the branch divisor of a resolution of the Voisin map.
Twisted cubics have Hilbert polynomial $3t+1$, their flat degenerations are called generalised twisted cubics. Generalised twisted cubics on a singular cubic surface of type A_1 fall in three different types, which we are going to call α, β, and γ. They determine two divisors D_α, D_β, on Z, see Definition 4.2.1.3.

Theorem 4.1.3.1 ([Gio20, Theorem 1]). *The divisor $D \subset Z$ of generalised twisted cubics lying on singular cubic surfaces has two irreducible components*

$$D = D_\alpha \cup D_\beta.$$

The first component coincides with the uniruled branch divisor of a resolution of the Voisin map.

Besides this, we have some other partial results on the Voisin map. The proof of Theorem 4.1.3.1 is, roughly speaking, based on understanding the fibres of the blow-up morphism over $I \setminus \Delta$ where Δ is the image of the diagonal embedding $F \hookrightarrow F \times F$. Pushing the investigation further, we study the fibres over Δ to get

Theorem 4.1.3.2. *Let $\pi\colon \mathrm{Bl}_I(F\times F) \to F\times F$ be the blow-up in I with the reduced structure and $\bar{\varphi}\colon \mathrm{Bl}_I(F\times F) \to Z$ the resolution of indeterminacy of the Voisin map, then*

$$\bar{\varphi}(\pi^{-1}(\Delta)) \subset Y.$$

Here $Y \subset Z$ is the lagrangian embedding of the cubic fourfold.

We determine the singular locus of the variety of incidental lines.

Organisation of the chapter Section 2 is devoted to the proof of our result about the branch divisor of a resolution of the Voisin map, whereas in Section 3 we discuss the results regarding the Voisin map and some possible paths to be investigated in the future.

4.2 The divisor of singular cubic surfaces on the LLSvS eightfold

4.2.1 Basic properties of the LLSvS variety

First, we recall some details of the construction of the variety Z from [LLSvS17] that we have not yet seen. Let $Y \subset \mathbb{P}(V) \simeq \mathbb{P}^5$ be a smooth cubic fourfold not containing any plane and let M be the variety of generalised twisted cubics on Y, that is, the irreducible component of $\mathrm{Hilb}^{3t+1}(Y)$ containing smooth twisted cubics. For any curve $C \in M$ its linear span $E := \langle C \rangle$ is a $\mathbb{P}^3$ that cuts Y in the cubic surface $S_C := E \cap Y$. These associations give rise to the diagram

$$\begin{array}{ccc} M & \longrightarrow & \mathbb{P}(S^3\mathscr{W}^*) \\ & \searrow^{\sigma} & \downarrow\uparrow s \\ & & \mathrm{Gr}(V,4) \end{array}$$

where $\mathscr{W}$ is the universal quotient bundle over the Grassmannian $\mathrm{Gr}(V,4)$ and the section s is given by intersecting Y with a $\mathbb{P}^3$. The morphism σ factors through a smooth irreducible variety Z' of dimension 8:

$$\begin{array}{ccc} M & & \\ \downarrow a & \searrow^{\sigma} & \\ Z' & \xrightarrow{b} & \mathrm{Gr}(V,4) \end{array}$$

where a is an étale locally trivial $\mathbb{P}^2$-fibration and b is generically finite [LLSvS17, Theorem B]. It is finite over the open set $U_{ADE} \subset Z'$ of surfaces that are either smooth or have ordinary double points. For the complement we have the following estimate [LLSvS17, Corollary 3.11, Proposition 4.2, Proposition 4.3]:

$$\dim(Z' \setminus U_{ADE}) \leq 6. \tag{4.1}$$

The irreducible holomorphically symplectic variety Z is obtained by the contraction $\pi : Z' \to Z$ of the irreducible divisor $D_{nCM} \subset Z'$ of families of curves that are not arithmetically Cohen-Macaulay [LLSvS17, Theorem 4.11].

Notation The points of the variety M are called generalised twisted cubics. In light of the construction above, points of the variety Z' correspond to families of flat degenerations of twisted cubics on Y, while each point on Z determines a class of equivalence of generalised twisted cubics, the equivalence classes being given by the fibers of the morphism $u\colon M \to Z$. In the following we will therefore speak of "families" and "classes of generalised twisted cubics" for the points of Z' and Z.

The divisor of singular cubic surfaces The projection $\mathbb{P}(S^3\mathscr{W}^*) \to \mathrm{Gr}(V,4)$ is equivariant for the natural action of PGL(6). The locus $D_{\mathrm{sing}} \subset \mathbb{P}(S^3\mathscr{W}^*)$ of all singular cubic surfaces in $\mathbb{P}(V)$ surjects onto the Grassmannian, the fiber over a point W_0 is the divisor $D_{\mathrm{sing},W_0} \subset \mathbb{P}(S^3W_0^*)$ of singular cubic surfaces in $\mathbb{P}(W_0)$. Furthermore, D_{sing,W_0} is irreducible [Huy, Theorem 2.2] and locally stratified depending on the singularity type of the parametrised surfaces, the A_1 locus forming an open set in D_{sing,W_0}. Since D_{sing} coincides with the orbit of D_{sing,W_0} under the action of PGL(6) we conclude that D_{sing} is an irreducible divisor and the A_1-locus is open in it. Pulling back this divisor along $s\circ b : Z' \to \mathbb{P}(S^3\mathscr{W}^*)$ and taking advantage of both the finiteness of $s\circ b$ on the ADE-locus and the estimate (4.1) we get the following

Proposition 4.2.1.1 ([Gio20, Proposition 2]). *Let $D' \subset Z'$ be the image under a of the locus of curves lying on singular cubic surfaces. Then D' is a divisor. Moreover, the preimage under $s \circ b$ of singular cubic surfaces of type A_1 is a dense open set in D'.*

Analogously, singular cubic surfaces determine a divisor $D^{\mathrm{Gr}} := s^{-1}(D_{\mathrm{sing}})$ in $\mathrm{Gr}(V,4)$ and a divisor $D := \pi(D')$ in Z. In light of this proposition, when discussing the irreducible components of D' it will suffice to treat only the A_1 locus.

Twisted cubics on A_1-singular cubic surfaces While twisted cubics on smooth cubic surfaces are a classical subject of study, twisted cubics on surfaces with ordinary double points are well explained in [LLSvS17, § 2.1], whose content we briefly recall here and then make explicit in the specific case of A_1 singularities. For basic facts on the root

lattice E_6 in connection with the geometry of the cubic surface we point to [Dol12, §9], [DPT80], for lines on singular cubic surfaces [BW79].
Given a singular cubic surface S with ordinary double points, its minimal resolution $\tilde{S}$ is a weak del Pezzo surface and the orthogonal complement $K_{\tilde{S}}^{\perp} \subset \mathrm{Pic}(\tilde{S})$ is a lattice of type E_6. The exceptional divisor of the resolution $r : \tilde{S} \to S$ consists of (-2)-curves, which form a subset of the root system $R := \{\alpha : \alpha^2 = 0\} \subset K_{\tilde{S}}^{\perp}$ and generate a subroot system $R_0 \subset R$. Let $W(R_0)$ be the Weyl group generated by reflections of elements in R_0, then [LLSvS17, Theorem 2.1] gives a description of the Hilbert scheme with the reduced structure of generalised twisted cubics on S:

$$\mathrm{Hilb}^{\mathrm{gtc}}(S)_{\mathrm{red}} \simeq R/W(R_0) \times \mathbb{P}^2. \tag{4.2}$$

For any $\alpha \in R \setminus R_0$ and for any curve $C \in |\alpha - K_{\tilde{S}}|$ the image $r(C)$ is a generalised twisted cubic on S. Conversely, the pullback of any aCM-curve on S lies in such a linear system [LLSvS17, Proposition 2.2, Proposition 2.5, Proposition 2.6]. On the other hand, roots in R_0 correspond to families of nCM curves. We now want to make (4.2) explicit for A_1-singular surfaces.
Let $S \subset \mathbb{P}^3$ be a singular cubic surface of type A_1 with singular point P, then its minimal resolution $\tilde{S}$ is the blow-up with center P, whose exceptional divisor is a (-2)-curve Γ. The linear system $|-\Gamma - K_{\tilde{S}}|$ realises $\tilde{S}$ as a blow-up of $\mathbb{P}^2$ in 6 points $P_1, \ldots, P_6$ lying on a quadric Q, which is exactly the image of Γ. In a picture:

$$\begin{array}{ccc} & \tilde{S} & \\ {\scriptstyle \pi}\swarrow & & \searrow{\scriptstyle r} \\ \mathbb{P}^2 & \overset{\rho}{\dashrightarrow} & S \subset \mathbb{P}^3. \end{array}$$

Here ρ is the rational map given by the linear system of cubics passing through the 6 points.
The Picard group of $\tilde{S}$ is generated by the pullback H of the hyperplane class of $\mathbb{P}^2$ and the 6 exceptional divisors $E_1, \ldots, E_6$. The surface S contains 21 lines: 6 lines are the image $\overline{E}_i$ of the exceptional divisors, they pass through the singular point; moreover, the line trough any two distinct points P_i, P_j is mapped via ρ to the unique line R_{ij} on S intersecting both $\overline{E}_i$ and $\overline{E}_j$ not in P.
The canonical bundle $K_{\tilde{S}}$ has class $-3H + E_1 + E_2 + E_3 + E_4 + E_5 + E_6$, its orthogonal complement $K_{\tilde{S}}^{\perp} \subset \mathrm{Pic}(\tilde{S})$ is a root lattice of type E_6. It has 72 roots:

$$\begin{aligned} \alpha_{ij} &= E_i - E_j, \text{ for } i \neq j \\ \pm\beta_{ijk} &= \pm(H - E_i - E_j - E_k), \text{ for } i,j,k \text{ pairwise distinct;} \\ \pm\gamma &= \pm(2H - E_1 - E_2 - E_3 - E_4 - E_5 - E_6). \end{aligned}$$

We are going to call the roots $\alpha_{i,j}$ of type α, the roots $\pm\beta_{i,j,k}$ of type β, and the roots $\pm\gamma$ of type γ. The unique effective root is γ, the reflection with center γ fixes the roots of type α, whereas its action on roots of type β is:

$$\pm\beta_{ijk} \mapsto \mp\beta_{lmn}$$

with $\{i,j,k,l,m,n\} = \{1,2,3,4,5,6\}$. Hence according to [LLSvS17, Theorem 2.1] the Hilbert scheme with the reduced structure is the disjoint union of 51 copies of $\mathbb{P}^2$

$$\mathrm{Hilb}^{\mathrm{gtc}}(S)_{\mathrm{red}} \simeq \bigsqcup_{\substack{i,j \\ i\neq j}} |\alpha_{ij} - K_{\tilde{S}}| \sqcup \bigsqcup_{\substack{i,j,k \\ \text{pairwise} \\ \text{distinct}}} |\beta_{ijk} - K_{\tilde{S}}| \sqcup |-\gamma - K_{\tilde{S}}|.$$

The curves parametrised by $|-\gamma - K_{\tilde{S}}|$ are not arithmetically Cohen Macaulay and will play no role in the following. For any curve C in $|\alpha_{ij} - K_{\tilde{S}}|$ or in $|\beta_{ijk} - K_{\tilde{S}}|$ the schematic image $r(C)$ is a generalised twisted cubic on S.
An ordered pair $(\overline{E}_i, \overline{E}_j)$ of distinct lines through the singular point determines the family of generalised twisted cubics on S corresponding to the linear system $|E_i - E_j - K_{\tilde{S}}|$. In contrast any triple $(\overline{E}_i, \overline{E}_j, \overline{E}_k)$ of pairwise distinct lines through the singular point determines the family of generalised twisted cubic on S corresponding to the linear system $|H - E_i - E_j - E_k - K_{\tilde{S}}|$.
We recapitulate what so far has been discussed in the following

Proposition/Definition 4.2.1.2 ([Gio20, Proposition 3]). *Let S be a singular cubic surface of type A_1, let P be the singular point.*

(i) The surface S has 6 distinguished lines $\overline{E}_1, .., \overline{E}_6$, they are the only lines through the singular point. For each pair $(\overline{E}_i, \overline{E}_j)$ there exists a unique line R_{ij} meeting $\overline{E}_i$ (respectively $\overline{E}_j$) in a point different from P.

(ii) Each ordered pair $(\overline{E}_i, \overline{E}_j)$ determines the family of twisted cubics of type α corresponding to the linear system $|E_i - E_j - K_{\tilde{S}}|$. We call such a family of type α.

(iii) Each triple $(\overline{E}_i, \overline{E}_j, \overline{E}_k)$ determines the family of twisted cubics of type β corresponding to the linear system $|H - E_i - E_j - E_k - K_{\tilde{S}}|$.[1] We call such a family of type β.

(iv) The linear systems $|-\Gamma - K_{\tilde{S}}|$ determines a family of generalised twisted cubics on S, which are not arithmetically Cohen-Macaulay. We call these curves of type γ.

The general element of the divisor D' is a family of generalised twisted cubics lying on a singular surface of type A_1 and can be of type α, β or γ.

[1] The element H is well determined, because $H = -\Gamma - K_{\tilde{S}}$, where Γ is the unique (-2)-curve of the unique minimal resolution $\tilde{S} \to S$.

Definition 4.2.1.3. *We define D'_α, D'_β, respectively D'_γ, as the closure of the sets in Z' of families of generalised twisted cubics of type α, β, respectively γ lying on A_1-singular cubic surfaces. We call D_α, respectively D_β, the image $\pi(D'_\alpha)$ in Z, respectively $\pi(D'_\beta)$.*

Proposition 4.2.1.4 ([Gio20, Proposition 5])**.** *The closed set D'_γ is an irreducible divisor in Z'.*

Proof. The divisor D'_γ parametrises families of generalised twisted cubics which are not arithmetically Cohen-Macaulay, it was already considered in [LLSvS17, Proposition 4.5], where it is shown to be irreducible. □

In the next sections we show the irreducibility of D'_α, D_α (Corollary 4.2.2.3) and of D'_β, D_β (Corollary 4.2.2.5).

4.2.2 The irreducible components of D

The irreducible component D_α Let Y be a smooth cubic fourfold not containing a plane and let F be its Fano variety of lines, which is an irreducible holomorphically symplectic variety of dimension 4. We recall that Voisin [Voi16, Proposition 4.8] constructed a degree 6 rational map

$$\varphi : F \times F \dashrightarrow Z.$$

The construction goes as follows. Let (l, l') be a general point in $F \times F$. It corresponds to a pair of non-coplanar lines L, L'[2]. After having chosen a point x on L, one takes the residual conic Q_x to L' of the intersection $Y \cap \langle x, L' \rangle$. The union of Q_x and L determines then the class of a generalised twisted cubic on the cubic surface $S := \langle L, L' \rangle \cap Y$. In other words, the lines L and L' determine the family $|\mathcal{O}_S(-K_S + L - L')|$ of generalised twisted cubics on S.
The indeterminacy locus of φ coincides with the variety I of incident lines, which is irreducible of dimension 6, [Mur20, Theorem 1.2, Lemma 2.3]. The branch divisor of a resolution of φ is a uniruled divisor, as remarked in [Voi16, Remark 4.10], for details see [Mur20, Lemma 4.4]. We denote by $\mathrm{Gr} := \mathrm{Gr}(V, 4)$ parametrises 3-dimensional projective spaces in $\mathbb{P}^5 = \mathbb{P}(V)$. We consider the rational maps

$$\varphi' := \pi^{-1} \circ \varphi : F \times F \dashrightarrow Z'$$
$$\psi := b \circ \pi^{-1} \circ \varphi : F \times F \dashrightarrow \mathrm{Gr}$$

where $\pi : Z' \to Z$ is the morphism discussed above and π^{-1} is its rational inverse map. As natural resolution of the indeterminacy locus of ψ, we consider the closure Γ of its graph with projections $p : \Gamma \to F \times F$, $q : \Gamma \to \mathrm{Gr}$. We study its points by taking flat limits along curves λ on $F \times F$ through points where the rational map is not defined.

[2]Here and in the following we will use small letters, as l, to denote lines as points of F and capital letters, as L, to the denote lines as subschemes of Y.

Lemma 4.2.2.1 ([Gio20, Lemma 6]). *For the preimage of the graph over I we have:*

$$q(p^{-1}(I)) = D^{\mathrm{Gr}}.$$

Moreover, for the general point (l,l') in I we have:

$$p^{-1}(l,l') = \{E \in \mathrm{Gr} : \langle L, L'\rangle \subset E \subset T_y Y\} \simeq \mathbb{P}^1$$

where y is the unique intersection point of L and L'.

Proof. Let $i = (l,l')$ be a general point in I and $\lambda \subset F \times F$ a smooth curve in an affine open intersecting I exactly in i. By the properness of the Grassmannian we get a morphism $\lambda \to \mathrm{Gr}$ and the curve parametrises a family $f : \mathscr{S} \to \lambda$ of cubic surfaces contained in Y. Since singular cubic surfaces determine a divisor in the Grassmannian, we may assume that $\mathscr{S}_t$ is smooth for every $i \neq t \in \lambda$. By [Har77, III, Theorem 10.2] the smoothness of the morphism f is equivalent to the smoothness of the surface $\mathscr{S}_i$.
We claim that the surface $\mathscr{S}_i$ is not smooth. Suppose to the contrary that the family were smooth, then the Picard groups $\mathrm{Pic}(\mathscr{S}_t) = H^2(\mathscr{S}_t, \mathbb{Z})$ would glue together in the local system $R^2 f_* \mathbb{Z}$ on λ. Every point $t \in \lambda$ parametrises a pair of lines (l_t, l'_t), which are disjoint for $t \neq i$. Taking their intersection product in $\mathrm{Pic}(\mathscr{S}_t)$ we then get

$$0 = L_t \cdot L'_t = L_i \cdot L'_i = 1.$$

Hence, we conclude that $\mathscr{S}_i$ is singular.
This shows the factorisation $p^{-1}(I) \to D^{\mathrm{Gr}} \subset \mathrm{Gr}$. Choosing λ accurately, one proves that the morphism is dominant onto D^{Gr} and thus surjective. Indeed, let $E \in D^{\mathrm{Gr}}$ be a $\mathbb{P}^3$ that contains two distinct incident lines L and L' meeting in a point y, in which the cubic surface $E \cap Y$ is singular. We consider the following diagram involving the tangent space $T_y Y$ of Y at y, the normal bundle $N_{L|Y}$ of L in Y and its stalk $N_{L|Y}(y)$ at the point y with the natural maps:

$$\begin{array}{ccc} & & T_y Y \\ & & \downarrow \\ H^0(L, N_{L|Y}) & \xrightarrow[ev]{} & N_{L|Y}(y). \end{array}$$

The general line L is of type I [CG72, Definition 6.6], that is, $N_{L|Y} \cong \mathscr{O}_L^{\oplus 2} \oplus \mathscr{O}_L(1)$, thus we may assume that the evaluation map ev is surjective. Let e be a vector in $T_y E$ not contained in $T_y\langle L, L'\rangle$. The image of e under $T_y E \subset T_y Y \to N_{L|Y}(y)$ lifts to a vector $\tilde{e} \in H^0(L, N_{L|Y})$, which corresponds to a deformation of L and is represented by a curve λ' in F through l. If we set $\lambda = \lambda' \times \{l'\} \subset F \times F$, then the limit $\mathbb{P}^3$ computed along λ coincides with E.
The first assertion is now proven.

The limit surface $\mathscr{S}_i$ is in effect singular in the intersection point y of L_i and L_i'. Indeed, we may assume $\mathscr{S}_i$ has one A_1-singularity in the point P. In virtue of [Rie92, §2], after restricting to an analytic neighbourhood of i, there is a diagram

$$\begin{array}{ccc} T & \xrightarrow{r} & \mathscr{S} \\ \downarrow & & \downarrow \\ (\lambda', i') & \xrightarrow{f} & (\lambda, i) \end{array}$$

where $f : (\lambda', i') \to (\lambda, i)$ is a finite Galois cover mapping i' to i and where $T \to \lambda'$ is a family of smooth surfaces such that $T_t \to \mathscr{S}_{f(t)}$ is an isomorphism for any $i' \neq t \in \lambda'$ and $T_{i'} \to \mathscr{S}_i$ is the minimal resolution of $\mathscr{S}_i$. The surfaces T_t are isomorphic to blow-ups of $\mathbb{P}^2$ in 6 points, which are in general position for any $t \neq i'$, hence the groups $\mathrm{Pic}(T_t)$ form a local system over all of λ'. The latter becomes trivial, after further shrinking λ' to a contractible neighbourhood of i'. The lines $L_{f(t)} \subset \mathscr{S}_{f(t)} \simeq T_t$ form a flat family over $\lambda' \setminus \{i'\}$. Taking its closure we find a curve X in $T_{i'}$, which completes the family to a flat family over all λ' and corresponds to a section of the local system of Picard groups. The support of X can be either the strict transform $\tilde{L}$ of the line L or the union of $\tilde{L}$ and the (-2)-curve Γ, which arises as exceptional divisor of the resolution of $\mathscr{S}_i$; the second case can happen only if the singular point P lies on L. Thus, we have

$$X = a\tilde{L} + b\Gamma$$

for some nonnegative integer numbers a and b. Analogously, for the flat limit X' of the family $\{L_t'\}_{t\neq i}$ we get

$$X' = a'\tilde{L}' + b'\Gamma. \tag{4.3}$$

where $\tilde{L}'$ is the strict transform of L'.
Since X, respectively X', is the flat limit of the lines $\{L_t\}_{t\neq i}$, respectively $\{L_t\}_{t\neq i}$, we have

$$\begin{aligned} X^2 = L_t^2 &= -1 \\ X \cdot K_{\tilde{S}} = L_t \cdot K_{\mathscr{S}_t} &= 1 \\ (X')^2 = (L_t')^2 &= -1 \\ X' \cdot K_{\tilde{S}} = L_t' \cdot K_{\mathscr{S}_t} &= 1 \\ X \cdot X' &= 0. \end{aligned}$$

From the first 4 conditions we get

$$a = a' = 1;\ b, b' \in \{0, 1\}.$$

The last condition excludes the case $b \neq b'$, so that

$$X = \tilde{L} + b\Gamma \qquad X' = \tilde{L}' + b'\Gamma \qquad \text{with } b = b' \in \{0,1\}. \tag{4.4}$$

From both the remaining cases (i.e $b = b' = 1$ or $b = b' = 0$) we get that $P \in L \cap L'$, as desired.

□

Let λ be a curve as the one in the proof above. Since Z' is proper the Voisin map extends to a well-defined morphism

$$\phi'_\lambda \colon \lambda \to Z'.$$

We are interested in the image of i, which is represented by a family of generalised twisted cubics. By the previous lemma we know that any such curve lies on a singular surface.

Lemma 4.2.2.2 ([Gio20, Lemma 7]). *For the general point $i \in I$ and the general curve λ the limit twisted cubic $\phi'_\lambda(i)$ is of type α.*

Proof. We may assume that the limit family of twisted cubics lies on a singular surface S_i of type A_1 with one singularity at the point of intersection of L, L'. The point $\phi'_\lambda(i)$ is represented by a family of generalised twisted cubics on S_i, which in turn corresponds to a linear system A on the minimal resolution $\tilde{S}$ of S_i. In contrast, for any other point $t \neq i$ in λ the image $\phi'_\lambda(t)$ consists of the family of curves in $|\mathcal{O}_{S_t}(-K_{S_t} + L_t - L'_t)|$. After passing to a Galois cover of an analytic neighbourhood of i in λ as before, we see that the linear system A is equal to

$$|\mathcal{O}_{\tilde{S}}(-K_{\tilde{S}} + X - X')| = |\mathcal{O}_{\tilde{S}}(-K_{\tilde{S}} + \tilde{L} - \tilde{L}')|$$

where X and X' are the flat limits computed in (4.4). Thus, the limit family $\phi'_\lambda(i)$ is of type α. □

In terms of the geometry of Z we have thus proven the following.

Corollary 4.2.2.3 ([Gio20, Corollary 8]). *The closed set D'_α is an irreducible uniruled divisor in Z'. Its image D_α in Z coincides with the branch locus of a resolution of the Voisin map.*

The irreducible component D_β We consider the variety of triples of lines with non-trivial common intersection:

$$I_3 := \{(l_1, l_2, l_3) \in F \times F \times F : L_1 \cap L_2 \cap L_3 \neq \emptyset\}.$$

Lemma 4.2.2.4 ([Gio20, Lemma 9]). *The variety I_3 is irreducible of dimension 7.*

Proof. Let $\mathbb{L} \subset F \times Y$ be the universal family of lines on Y parametrised by F, its threefold product fits in the diagram

$$\begin{array}{ccc} \mathbb{L} \times \mathbb{L} \times \mathbb{L} & \xrightarrow{p} & Y \times Y \times Y \\ \downarrow q & & \\ F \times F \times F. & & \end{array}$$

where p and q denote the natural projections. The variety I_3 is the image via q of $J := p^{-1}(\Delta)$, where $Y \simeq \Delta \subset Y \times Y \times Y$ is the diagonal embedding. Since J is locally cut out by 8 equations, all its irreducible components have dimension greater than or equal to 7. The restriction of q to J is birational and just contracts the diagonal embedding of $\mathbb{L}$ in J:

$$\{(y_1, l_1, y_2, l_2, y_3, l_3) \in \mathbb{L} \times \mathbb{L} \times \mathbb{L} : y_1 = y_2 = y_3 \text{ and } l_1 = l_2 = l_3\} \subset J$$

which has dimension 5. Thus, all irreducible components of I_3 have dimension at least 7. Via the projection $F \times F \times F \to F \times F$ onto the first two factors the variety I_3 is fibred over the irreducible variety I of dimension 6 ([Mur20, Lemma 2.3]):

$$p_{12} : I_3 \to I.$$

We study its fibres.

- if (l, l') lies on the diagonal of $F \times F$, that is $L = L'$, then its preimage is the variety F_L of lines intersecting L.
- In contrast, if $(l, l') \in I$ is a point such that $L \cap L' = \{y\}$ then the fibre $p_{12}^{-1}(l, l')$ is the variety C_y of lines through y.

The variety C_y of lines through a general point y is a curve, which is in general irreducible. When it is not a curve, it must be a surface and this is the case just for a finite number of points [CS09, Proposition 2.4].
The variety F_L admits a rational map to L well-defined away from $l \in F_L$:

$$F_L \dashrightarrow L, \; r \mapsto R \cap L.$$

The fibre over a point $y \in L$ is the variety C_y, thus F_L is a surface. It follows that I_3 is irreducible. □

A general triple (l_1, l_2, l_3) in I_3 spans a $\mathbb{P}^3$ which intersects the cubic fourfold Y in a singular cubic surface of type A_1: the singular point being the unique common intersection point of the three lines. According to our discussion in the previous section, this data determines the class of a generalised twisted cubic of type β (cf. Proposition 4.2.1.2). We have therefore constructed a rational map

$$\rho : I_3 \dashrightarrow Z'.$$

which is dominant onto D'_β. As immediate consequence we get

Corollary 4.2.2.5 ([Gio20, Corollary 10]). *The closed set D'_β in Z' as well as its image D_β in Z is an irreducible divisor.*

Questions Uniruled ample divisors on hyperkähler varieties have been studied by Charles, Mongardi and Pacienza [CMP19] to generalise the results by Beauville and Voisin and to tackle the Beauville-Voisin conjecture and more in general the conjectural existence of the Bloch-Beilinson filtration.
As we have seen, D_α is a uniruled divisor and it is ample too, because the very general LLSvS eightfold has Picard number one, thus D_α needs to be linearly equivalent to a multiple of the polarisation H:

$$D \in |nH| \text{ for some } n.$$

It would be interesting to compute exactly the integer n. Notice that, knowing n, we would get the Beauville-Bogomolov square of D_α, as we know that the Beauville-Bogomolov square of H is 2 by Corollary 2.3.3.1.
In contrast, it is not clear weather D_β is uniruled or not as the natural morphism of I_3 over I is fibred in genus 5 curves.

4.3 Other results

4.3.1 The singular locus of I

Let Y be a cubic fourfold, F its Fano variety of lines. Set I to be the variety of incident lines:

$$I = \{(l, l') \in F \times F : L \cap L' \neq \emptyset\}.$$

Here again we we use small letters, as l, to denote lines as points of F and capital letters, as L, to the denote lines as subschemes of Y. Given a line L on Y, its normal bundle $N_{L|Y}$ can be of two types, namely ([CG72, Definition 6.6]):

(I) $N_{L|Y} \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$

(II) $N_{L|Y} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}(1)$.

Keeping in mind the distinction in type (I) and (II), we define the variety:

$$\Delta^{(II)} = \{(l, l') \in F \times F : L = L', L \text{ of type (II)}\}.$$

Remark 4.3.1.1. The lines of type I and II are characterised as follows. The linear space

$$\cap_{P \in L} T_P Y$$

is a 2-dimensional (respectively 3-dimensional) vector space if L is of type I (respectively II).

Theorem 4.3.1.2. *The singular locus of I is $\Delta^{(II)}$.*

To prove the theorem, we will need to deal with the auxiliary variety

$$W = \{(y, l, l') \in Y \times I : y \in L \cap L'\}.$$

Let $\pi : W \to I$ be the natural projection. If we denote by $\mathbb{L} \subset F \times Y$ the universal family of lines, then W is isomorphic to the fibre product $\mathbb{L} \times_Y \mathbb{L}$.
A line has Hilbert polynomial $t+1$ and F is isomorphic to the Hilbert scheme of Y with Hilbert polynomial $t+1$. Analogously, $\mathbb{L}$ is isomorphic to the Hilbert scheme $\mathrm{Hilb}^{1,t+1}(Y)$ that parametrises flag of schemes in Y with Hilbert polynomials $(1, t+1)$[3]. Thus, one sees that W is isomorphic to the fibre product

$$\mathrm{Hilb}^{1,t+1}(Y) \times_{\mathrm{Hilb}^1(Y)} \mathrm{Hilb}^{1,t+1}(Y).$$

Lemma 4.3.1.3. *The singular locus of W is $\pi^{-1}(\Delta^{(II)}) = \Delta^{(II)} \times_I W$.*

Proof. We compute the dimension of the tangent space at every point $w \in W$. Thanks to the description above, this is an easy task once we employ deformation theory. Indeed, generalising the formula for the tangent space of a flag Hilbert scheme (cf. [Ser06, Remark 4.5.4 (ii)]), one gets that the tangent space at the point $w = (y, l, l') \in W$ equals the fibre product

$$T_w W = H^0(N_{L|Y}) \times_{N_{L|Y}(y)} T_y Y \times_{N_{L'|Y}(y)} H^0(N_{L'|Y})$$

taken over the diagram

$$\begin{array}{ccc} & & H^0(L', N_{L'|Y}) \\ & & \downarrow{\scriptstyle \phi'} \\ & T_y Y & \xrightarrow[\psi']{} N_{L'|Y}(y) \\ & \downarrow{\scriptstyle \psi} & \\ H^0(L, N_{L|Y}) & \xrightarrow[\phi]{} N_{L|Y}(y) & . \end{array}$$

Here the maps ϕ, ϕ' are given by evaluating global sections at the point y, while ψ, ψ' are the natural quotients maps. We observe that $H^0(L, N_{L|Y})$ is always a vector space of dimension 4 and so $N_{L|Y}(y)$ is of dimension 3; on the other hand, the rank of ϕ is 3 or 4, according to whether L is of type I or II. Moreover, the kernel of ψ coincides with the direction spanned by the line L.
We proceed with a distinction in cases:

[3] We point to [Ser06, § 4.5] for a general reference for flag Hilbert schemes.

- $w=(y,l,l')$ with L and L' both of type I. Any element in T_wW consists of a choice of a vector v in T_yY (4 degrees of freedom) and the liftings of $\psi(v)$ (respectively $\psi'(v)$) both 1 degree of freedom. Thus in this case $\dim T_wW=6$.
- $w=(y,l,l')$ with L of type I and L' of type II. Any element in T_wW consists of a choice of a vector $v\in H^0(N_{L'|Y})$ (4 degrees of freedom), a choice of a lifting v_1 of $\phi'(v)$ along ψ' (1 degree of freedom) and the choice of a lifting of $\psi(v_1)$ along ϕ (again 1 degree of freedom). Hence $\dim T_wW=6$.
- $w=(y,l,l')$ with L different from L', both of type II. A tangent vector to w consists of a choice of a vector $v\in\psi^{-1}(\mathrm{im}\phi)\cap\psi'^{-1}(\mathrm{im}\phi')=:X$. A quick computation tells us that X can have dimension either 3 or 2, but the former case never happens. Indeed $\psi^{-1}(\mathrm{im}\phi)$ is the tangent space of the unique $E\simeq\mathbb{P}^3$ containing L and tangent to Y at any of its point. Clearly E and E' are different. Therefore dimension of T_wW is of dimension 6.
- $w=(y,l,l')$ with $L=L'$ of type II. We choose an element $v\in\psi^{-1}(\mathrm{im}\phi)\cap\psi'^{-1}(\mathrm{im}\phi')$ for which we have 3 degrees of freedom and the two liftings, for each we have 2 degrees of freedom. Thus $\dim T_wW=7$.

Since W is irreducible of dimension 6, we conclude that the singular locus consists of the preimage of $\Delta^{(II)}$ along π as desired. □

We are now ready to prove that the singular locus of the variety of incidental lines is the surface of lines of type II diagonally embedded in $F\times F$.

Proof of Theorem 4.3.1.2. Since the morphism $\pi:W\to I$ is an isomorphism off $\Delta\subset I$, from the above lemma we deduce:

$$I^{sing}\subset\Delta. \tag{4.5}$$

Thus we just need to analyse the points on the diagonal. We are going to prove:

$$\dim_{\mathbb{C}}(T_PI)=6;\ \forall P\in\Delta\setminus\Delta^{(II)} \tag{4.6}$$

$$\dim_{\mathbb{C}}(T_PI)=8;\ \forall P\in\Delta^{(II)}. \tag{4.7}$$

Intuitively, vectors in T_PI with $P=(l,l')$ parametrise deformations of L and L' that still meet in some point. This amounts to finding a point $y\in L\cap L'$ such that y admits a deformation with L and, at the same time, with L', thus for any such vector $v\in T_PI$ there exists a $y\in L\cap L'$ so that $(y,l,l')\in W$ deforms in W along v.
In other words, T_PI contains the subspace in $T_P(F\times F)$ generated by the images of T_QW along $d_Q\pi$ for all points lying over P

$$T_PI\supset\sum_{Q\in\pi^{-1}(P)}d_Q\pi(T_QW).$$

The differential $d_Q\pi : T_QW \to T_PI \subset T_P(F \times F)$ is the projection

$$H^0(N_{L|Y}) \times_{N_{L|Y}(y)} T_yY \times_{N_{L'|Y}(y)} H^0(N_{L'|Y}) \longrightarrow H^0(N_{L|Y}) \times H^0(N_{L'|Y}).$$

Assuming now that $P \in \Delta \backslash \Delta^{(II)}$, choosing isomorphisms $L \simeq \mathbb{P}^1$ and $N_{L|Y} \simeq \mathscr{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}^1}(1)$, we can identify $H^0(N_{L|Y}) = H^0(\mathscr{O}_{\mathbb{P}^1}, \mathbb{P}^1)^{\oplus 2} \oplus H^0(\mathscr{O}_{\mathbb{P}^1}(1), \mathbb{P}^1) = \mathbb{C}^{\oplus 2} \oplus \mathbb{C}\langle x_0, x_1\rangle$. Writing $v = (a, b, q) \in H^0(N_{L|Y})$, where a, b are global section of the trivial line bundle and q is a global section of the Serre twist, we have:

$$\begin{aligned} T_PI &= \{(v, v') \in H^0(N_{L|Y})^{\oplus 2} : \exists x \in \mathbb{P}^1 \text{ s.th. } v(x) = v'(x)\} \\ &= \{\big((a, b, q), (a', b', q')\big) : \exists x \in \mathbb{P}^1 \text{ s.th. } a = a', b = b', q(x) = q'(x)\} \\ &\simeq \mathbb{C}^6 \end{aligned}$$

where the last isomorphism holds because the condition on the linear forms q and q' is vacuous.
Proceeding analogously for $(l, l) = P \in \Delta^{(II)}$, we have $N_{L|Y} = \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2}$ and we write $v = (q, r) \in H^0(\mathscr{O}_{\mathbb{P}^1}(1), \mathbb{P}^1) = \mathbb{C}\langle x_0, x_1\rangle$ where q and r are linear forms:

$$\begin{aligned} T_PI &= \langle\{(v, v') \in H^0(N_{L|Y})^{\oplus 2} : \exists x \in \mathbb{P}^1 \text{ s.th. } v(x) = v'(x)\}\rangle \\ &= \langle\{\big((q, r), (q', r')\big) : \exists x \in \mathbb{P}^1 \text{ s.th. } q(x) = q'(x), r(x) = r'(x)\}\rangle. \end{aligned}$$

It is easy to see that T_PI in this case contains two distinct vector subspaces of dimension 6. Indeed, given a quadratic form r and a point $x \in \mathbb{P}^1$ it is always possible to find r' such that $r(x) = r'(x)$; thus given any triple $((q, r), (q', -))$ we can always find a quadratic form r' so that $((q, r), (q', r'))$ is in T_PI. Applying the same reasoning to triples $(q, -), (q', r')$, we find the second vector space of dimension 6.
The claim is now proven. □

4.3.2 The image of Δ

We consider the blow-up X of $F \times F$ along I with the reduced structure. As explained in the introduction of the present chapter (cf. Theorem 4.1.2.2) this resolves the indeterminacy of the Voisin map.

$$\begin{array}{ccccc} X & & Z' & \xrightarrow{\sigma} & \mathrm{Gr}(4,6) \\ & \searrow^{\bar\varphi} & \downarrow b & & \\ \downarrow \pi & & & & \\ F \times F & \overset{\varphi}{\dashrightarrow} & Z & & \end{array}$$

Theorem 4.3.2.1. *Let $\bar\varphi : X \to Z$ the resolution of indeterminacy of the Voisin map, then*

$$\bar\varphi(\pi^{-1}(\Delta)) \subset Y.$$

Here $Y \subset Z$ is the lagrangian embedding of the cubic fourfold coming from the LLSvS construction.

Proof. First of all, since the singular locus of I coincides with $\Delta^{(II)}$, the normal bundle $N_{I|F\times F}$ is a vector bundle of rank 2 outside $\Delta^{(II)}$ and we interpret the fibres of π over $\delta \in \Delta \setminus \Delta^{(II)}$ as $\mathbb{P}(N_{I|F\times F}(\delta)) \simeq \mathbb{P}^1$. Thus $\pi^{-1}(I)$ is smooth away from $\pi^{-1}(\Delta^{(II)})$.
Secondly, $\pi^{-1}(\Delta)$ is irreducible, as follows from the description of the blow-up given in [Che18]: its fibres over $\Delta \setminus \Delta^{(II)}$ are $\mathbb{P}^1$ and over $\Delta^{(II)}$ are $\mathbb{P}^2$. Thus $\pi^{-1}(\Delta \setminus \Delta^{(II)})$ is irreducible and its closure must contain $\pi^{-1}(\Delta^{(II)})$.
Looking for a contradiction we assume that $\bar{\varphi}(\pi^{-1}(\Delta)) \not\subset Y$, so that the restriction of the composition

$$\eta : X \to Z \dashrightarrow Z' \to \mathrm{Gr}.$$

gives a nontrivial rational map $\pi^{-1}(\Delta) \dashrightarrow \mathrm{Gr}$. We claim that such a map cannot be well defined on $\pi^{-1}(\Delta)$. More precisely we pose

Claim 4.3.2.2. *The point $\eta(\delta)$ corresponds to a linear projective 3-space $E_{\eta(\delta)}$ that contains the line L_δ. Further any point in L_δ lies in the singular locus of $E_{\eta(\delta)} \cap Y$.*

This leads to the desired contradiction, because there exists no such $\mathbb{P}^3$ for a line of type I (cf. Remark 4.3.1.1).
In order to prove the claim, we consider a general point $\delta \in \pi^{-1}(\Delta)$ and its image $\eta(\delta)$ and we deduce the properties in the claim by looking at $\eta(\delta)$ as limit of various curves lying inside $\pi^{-1}(I)$, similarly as we did in the previous section. For any point t in the blow-up X, its image $\pi(t)$ parametrises two lines, which we denote by L_t, L'_t; in case t lies in $\pi^{-1}(\Delta)$ there is actually just one line.
Now for a general point $\delta \in \pi^{-1}(\Delta)$ and for any point $y \in L_\delta$, we consider the variety C_y of lines through y (which is a curve for the general $y \in Y$) and the curve

$$\lambda'_y = \{l_\delta\} \times C_y \subset F \times F.$$

Such a curve passes through (l_δ, l_δ) and we choose a local lifting λ_y passing through δ. By construction of λ_y and Lemma 4.2.2.1 the point y is singular in $E_{\eta(t)} \cap Y$ for any $\delta \neq t \in \lambda$[4], thus y must be contained in $E_{\eta(\delta)} \cap Y$ and in its singular locus.
Hence the claim is now proven and so is the theorem. □

Remark 4.3.2.3. There is a natural anti-symplectic involution σ on the LLSvS variety Z and there is the natural involution s on $F \times F$ given by switching the two factors. They fit in a commutative diagram

$$\begin{array}{ccc} F \times F & \overset{\varphi}{\dashrightarrow} & Z \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \sigma} \\ F \times F & \overset{\varphi}{\dashrightarrow} & Z. \end{array}$$

[4]Here, $E_{\eta(t)}$ denotes the projective 3-space corresponding to $\eta(t)$.

The involution s extends to an involution $\bar{s}$ on the blow-up X. To see this[5], let Σ be the cyclic group of two elements, we denote by $\alpha\colon \Sigma\times F\times F\to F\times F$ the action of Σ on $F\times F$ via σ. Notice that I is invariant under the action of Σ and that the preimage of I under α is $\Sigma\times I$. By the universal property of the blow-up (e.g. [Har77, § II, Corollary 7.15]) we get a morphism $\Sigma\times X\to X$, by assumption this induces an action over the complement of I. Since any two morphism which are equal on a dense open subset must coincide, the action extend to the all of X.
In the end, we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\bar{\varphi}} & Z \\ \downarrow{\scriptstyle \bar{s}} & & \downarrow{\scriptstyle \sigma} \\ X & \xrightarrow{\bar{\varphi}} & Z. \end{array}$$

We have that the fixed locus of $\bar{s}$ maps to the fixed locus of σ, the previous lemma shows indeed that the fixed locus of $\bar{s}=\pi^{-1}(\Delta)$ lands via $\bar{\varphi}$ in Y, which is just one of the two components of the fixed locus of σ.
It would be interesting to give a geometric description of the other component, maybe taking advantage of the divisor D_β and its symmetries given by switching lines.

[5]The argument here is taken by https://mathoverflow.net/questions/122922/group-actions-on-blow-ups.

5

Appendix

5.1 Code for Γ

This is the code used in Chapter 2 to compute the Chern character of $p_* q^* \mathcal{O}_X$.

```
loadPackage "Schubert2"
P5 = abstractProjectiveSpace' 5
R = intersectionRing P5
P5xG26 = flagBundle({2,4}, P5)
P5xX = sectionZeroLocus(6*OO_P5xG26(1))
P = P5xG26.Bundles_0 ** OO_P5xX -- tautological subbundle
Q = P5xG26.Bundles_1 ** OO_P5xX -- quotient bundle
Gam = degeneracyLocus(1, dual Q ** OO_P5(1), P)
p = Gam/P5 -- the projection map
ch(p_* OO_Gam) * todd(OO_P5(3))
```

The output for the last line is:

$$\frac{19}{8}h^5 - \frac{33}{2}h^4 + \frac{65}{2}h^3 - 27h^2 + 12h.$$

To compute the Chern character of $p_* q^* \mathcal{O}_X(H)$.

```
OP5xH = OO_P5xX - OO_P5xX(-1) -- this is q^* O_H in the paper
ch(p_*(OO_Gam ** OP5xH)) * todd(OO_P5(3))
```

The output for the last line is:

$$-\frac{35}{4}h^5 + 56h^4 - 91h^3 + 42h^2.$$

5.2 Cubic surfaces

In the present section we describe a code to find the ideal of a family of cubic surfaces and the families of lines lying on the cubic surfaces!

The Voisin map $\varphi : F \times F \dashrightarrow Z$ is morally of local nature in the sense that can be understood at the level of cubic surfaces linear sections of Y. Here we present a piece of code we wrote in the attempt to study the map and in particular to determine the image of the branch divisor of a resolution of such a map. Once we understood that singular cubic surfaces were the key to solve the problem, the code turned out to be useless, because it could not take into account the monodromy of the local system of Picard groups of the surfaces. We hope nonetheless it might turn useful for further studies with the opportune modifications and it gave us an idea of the general picture.

To study the indeterminacy of the Voisin map, our idea was to find the ideal of a family of smooth cubic surfaces and of two families of lines such that they are disjoint except over the special point. (Un)luckily our idea was flawed and the code turned out useless for our purposes. [1]

5.2.1 Generalities on cubic surfaces

Cubic surfaces are a classical topic of study, for a modern reference we point to the book [Dol12], in particular §9. Here we give a very brief presentation in order to revise the bit of theory necessary to explain the strategy of our code.

Smooth cubic surfaces S are rational and admit 72 descriptions as blow up of $\mathbb{P}^2$ in 6 points in general position. In particular if $\tilde{S} \to \mathbb{P}^2$ is such a blow up, the canonical model S of $\tilde{S}$ is a cubic surface isomorphic to S and we have a diagram

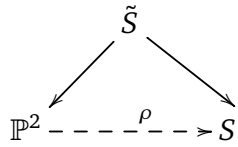
$$\begin{array}{ccc} & \tilde{S} & \\ \swarrow & & \searrow^{\simeq} \\ \mathbb{P}^2 & \overset{\rho}{\dashrightarrow} & S \subset \mathbb{P}^3. \end{array} \tag{5.1}$$

For cubic surfaces with ADE-singularities the situation is similar. Indeed, let $S \subset \mathbb{P}^3$ be a cubic surface with ordinary double points, then its minimal resolution $\tilde{S} \to S$ can be described in several ways[2] as blow-up of $\mathbb{P}^2$ in 6 points lying in quasi-general position (cf. [DPT80, Surfaces de del Pezzo III, § 2, Definition 1]), i.e. no line passes through 4 points of them. On the other hand, given 6 points in $\mathbb{P}^2$ lying in quasi-general position,

[1] The impossibility to find such a family lead us to upgraded this misconception to the proof of Theorem 4.1.3.1

[2] The exact number depends on the type of singularity.

the canonical model of the blow-up is a cubic surface. We get a diagram



(5.2)

In both diagrams (5.1) and (5.2) the horizontal birational map ρ is given by the linear system of cubics through the 6 blown-up points. In particular the surface S is the closure of the image of ρ.
The number of lines on a smooth cubic surface is 27. These are the strict transform (along the blow-up $\pi : \tilde{S} \to S$) of the 15 lines through two different blown-up points, the strict transform of the 6 quadrics passing through 5 of the 6 blown-up points, and the image of the 6 exceptional divisors. For an ADE-surface S the number of lines is smaller, as the 27 $\mathbb{P}^1$s with degree 1 with respect to the anti-canonical bundle that we find on $\tilde{S}$ as before may get contracted along the canonical morphism $\tilde{S} \to S$. In both cases the lines on the cubic surface are birational transforms along ρ of the above mentioned lines and quadrics. With some reasoning we can describe the image of the exceptional divisors via ρ, too.
These observations are the base of the following code.

5.2.2 Explanation of the code

We want to write a piece of code that gives (the ideal of) a family of cubic surfaces and (the ideals of) the family of lines on it. We shall present such a family as canonical models of the blow-up in 6 points moving in a fixed $\mathbb{P}^2$, thus the family given by the closure of the image of the map given by the linear system of cubics through the 6 moving points. Let us pick 5 fixed points

$$P_1 = (1:0:0) \quad P_1 = (0:1:0) \quad P_1 = (0:0:1) \quad P_4 = (1:1:1) \quad P_6 = (1:2:3)$$

The sixth point is moving and we parametrise it as $P_5 = (1 : a_1 : a_2)$ for $(a_1, a_2) \in \mathbb{A}^2$. [3] As generators of the space of cubics through the points we may choose 4 reducible cubics K_1, K_2, K_3, and K_4, which are union of lines, each through 2 distinct points. If we set x_1, x_2, x_3 for the coordinates of the $\mathbb{P}^2$ and y_1, y_2, y_3, y_4 for the coordinates of $\mathbb{P}^3$, then the ideal of the family of surfaces is given by the elimination of the ideal

$$(y_1 - K_1(\underline{x}), y_2 - K_2(\underline{x}), y_3 - K_3(\underline{x}), y_4 - K_4(\underline{x}))$$

with respect to a global ordering on $\mathbb{C}[\underline{x}, \underline{y}]$ with $y_i > x_j$ for any i and j.

[3] Initially we wrote the code letting two points move $P_5 = (1 : a_1 : a_2)$ and $P_6 = (1 : a_3 : a_4)$, but our computer could not support the complexity of the computations. Thus we adapted the code to just one moving point. This is the reason of the funny notation for the points.
Indeed, having two points moving is quite useless, because flat limits exists on curves, but not on surfaces. P_5 is actually moving on a $\mathbb{P}^1$!

```
LIB "primdec.lib";
LIB "elim.lib";
// we set the ring
ring R=0,(x(1..3),y(1..4),a(1..2)),(dp(3),wp(2,2,2,2,1,1));

// consider the points P1...P6 given by (1,0,0), (0,1,0), (0,0,1),
//(1,1,1), (1,a,b), (1,c,d).
// we want to construct 4 linearly independent cubics that go ////
//through these 6 points. We do so by taking reducible cubics given
//by lines through the points.
// We start with the equation of some lines.

poly R12 = x(3);
poly R13 = x(2);
poly R14 = x(2)-x(3);
poly R16 =  3*x(2) - 2*x(3);
poly R23 = x(1);
poly R24 = x(1) - x(3);
poly R25 = a(2)*x(1) - x(3);
poly R34 = x(1) - x(2);
poly R36 = 2*x(1) - x(2);
poly R45 = (a(2) - a(1))*x(1) + (1 - a(2))*x(2)+ (a(1) - 1)*x(3);
poly R56 = (a(1)*3 - a(2)*2)*x(1) + (a(2) - 3)*x(2) + (2 - a(1))*x(3);

// we define the maximal ideals of the points;

ideal M1 = x(2), x(3);
ideal M2 = x(1), x(3);
ideal M3 = x(1), x(2);
ideal M4 = x(1) - x(2), x(2) - x(3);
ideal M5 = a(1)*x(1) - x(2), a(2)*x(1) - x(3);
ideal M6 = 2*x(1) - x(2), 3*x(1) - x(3);


// We check that the lines go through the points.

size(reduce(R12,intersect(M1,M2))); // check R12 goes through P1 and P2
size(reduce(R12,intersect(M1,M2)));
size(reduce(R13,intersect(M1,M3)));
size(reduce(R14,intersect(M1,M4)));
```

```
size(reduce(R16,intersect(M1,M6)));
size(reduce(R23,intersect(M2,M3)));
size(reduce(R24,intersect(M2,M4)));
size(reduce(R25,intersect(M2,M5)));
size(reduce(R34,intersect(M3,M4)));
size(reduce(R36,intersect(M3,M6)));
size(reduce(R45,intersect(M4,M5)));
size(reduce(R56,intersect(M5,M6)));
// here we get 0 everywhere, that means, e.g. in the last case that
// $R56$ is in $M5\cap M6$.

// Equations of some cubics through all the 6 points
poly K1 = R12*R34*R56;
poly K2 = R13*R24*R56;
poly K3 = R12*R45*R36;
poly K4 = R14*R25*R36;

//check that the cubics go through the six points.
// we define the ideal of the uninon of the six points;
ideal MM = intersect(M1,M2,M3,M4,M5,M6);
MM = std(MM);
ideal KK = K1,K2,K3,K4;
KK = std(KK);
size(reduce(KK,MM)); // output is ZERO iff KK is in MM

// Finally we get the equations of the cubic, by the elimination trick.
ideal I = y(1)-K1, y(2)-K2, y(3)-K3, y(4)-K4;
// std(I);   // and here I FINALLY get a homogeneous 3-degree
//polynomial in y's, with coefficient in a_i
ideal C = std(I)[1];


// we compute the discriminant locus of the family of conics
ideal t = diff(C,y(1)), diff(C,y(2)), diff(C,y(3)), diff(C,y(4));
t = t + C;
ideal disc = eliminate(t+(y(1)-1),y(1)*y(2)*y(3)*y(4));
disc = intersect(disc, eliminate(t+(y(2)-1),y(1)*y(2)*y(3)*y(4)));
disc = intersect(disc, eliminate(t+(y(3)-1),y(1)*y(2)*y(3)*y(4)));
disc = intersect(disc, eliminate(t+(y(4)-1),y(1)*y(2)*y(3)*y(4)));
disc = std(disc);
```

```
ideal Ldisc = diff(C,y(1)), diff(C,y(2)), diff(C,y(3)), diff(C,y(4));
Ldisc = Ldisc + C;
ideal t = y(1), y(2), y(3), y(4);
Ldisc = sat(Ldisc,t)[1];
Ldisc = eliminate(Ldisc, y(1)*y(2)*y(3)*y(4));
Ldisc = std(Ldisc);

size(reduce(disc,Ldisc)); // the output is 0
size(reduce(Ldisc,disc)); // the output is 0
// we conclude the two ideals are equal and (hopefully)
// the computation is right

////////////////////////////////////
// 1st Line: strict transform of R12
////////////////////////////////////

// The strict transform of R12 can be computed just as preimage
// of its ideal.

ideal G1 = R12, I;
G1 = eliminate(G1,x(1)*x(2)*x(3));
G1 = std(G1);
G1; // here I get two linear equations: a line in P3!

// We check that the line lies on the cubic.
size(reduce(C,G1)); // here I get 0


////////////////////////////////////
// 2nd Line: strict transform of Q5
////////////////////////////////////

//Let us compute now the strict transform of the conic through 5
//points, this should be a line!
// Here in particular Q5 is the conic passing through all points, but P5.
poly Q5 = x(2)*x(3) - 4*x(1)*x(3) + 3*x(1)*x(2);

// we check that Q5 goes through the points

size(reduce(Q5, intersect(M1, M2, M3, M4, M6))); // I get 0
```

```
//we compute the strict transform of Q5, named G2
ideal G2 = Q5, I;
G2 = eliminate(G2, x(1)*x(2)*x(3));
G2 = std(G2);

// we check that the line G2 lies on the surface
size(reduce(C,G2)); //here I get 0

////////////////////////////
// 4th family: strict transform of a blown-up point and all the others
//////////////////////////////////

// The exceptional divisor is the projective space of the tangent
//space of the blown-up point.
// We compute it as the preimage of the square of the maximal ideal
//of the point.
ideal t = M1*M1;
t = t + I;
std(t)[1];
std(t)[2];
std(t)[3];

ideal E1 = std(t)[1], std(t)[2];
////////////////////////////////////////////////////////////
// The other exceptional divisors
////////////////////////////////////////////////

// 2nd exceptional divisor
ideal t = M2*M2;
t = t + I;
ideal E2 = std(t)[1], std(t)[2];

// 3rd exceptional divisor
ideal t = M3*M3;
t = t + I;
ideal E3 = std(t)[1], std(t)[2];

// 4th exceptional divisor
ideal t = M4*M4;
t = t + I;
ideal E4 = std(t)[1], std(t)[2];
```

```
// 5th  exceptional divisor
ideal t = M5*M5;
t = t + I;
ideal E5 = std(t)[1], std(t)[2];

// 6th exceptional divisor
ideal t = M6*M6;
t = t + I;
ideal E6 = std(t)[1], std(t)[2];

// check that the Ei's do not intersect, unless (possibly) over the
// discriminant locus

// primdecGTZ(E1+E2);
// primdecGTZ(E1+E3);
// primdecGTZ(E1+E4);
// primdecGTZ(E1+E5);
// primdecGTZ(E1+E6); // => E1 does not intersect the others
// primdecGTZ(E2+E3);
// primdecGTZ(E2+E4);
// primdecGTZ(E2+E5);
// primdecGTZ(E2+E6);// => E2 does not intersect the others
// primdecGTZ(E3+E4);
// primdecGTZ(E3+E5);
// primdecGTZ(E3+E6);// => E3 does not intersect the others
// primdecGTZ(E4+E5);
// primdecGTZ(E4+E6);//=> E4 does not intersect the others
// primdecGTZ(E5+E6);//=> E5 does not intersect the others
```

Theses

- Hyperkähler varieties represent a class of varieties in the realm of algebraic geometry with a rich geometry. We study the example of the LLSvS variety Z which is constructed as a contraction of the variety parametrising generalised twisted cubics on a smooth cubic fourfold Y. In Theorem 2.0.0.3 we show that the periods of Z and Y are the same.
- The LLSvS variety Z of a cubic fourfold Y is deformation equivalent to the Hilbert scheme of 4 points of some K3 surface. In Theorem 3.1.3.8 we determine a condition for which the variety Z is birational to such a Hilbert scheme.
- Uniruled divisors drew a lot of attention in the study of Hyperkähler manifolds. In Theorem 4.1.3.1 we give a description the uniruled divisor given by the branch locus of a resolution of the Voisin map: We show that it coincides with an irreducible component of the divisor determined by singular cubic surfaces on Y.
- Moreover, we show that the divisor determined by singular cubic surfaces on Y has an other irreducible component and we give a parametrisation of the latter.

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